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CONNECTIVITY AND CYCLES IN GRAPHS

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1 INTRODUCTION

Connectivity paired with other graphical properties, such as minimal degree, maximal degree, neighborhood conditions, and forbidden subgraphs, have been explored extensively in terms of implying cycles in graphs. However, appropriate connectivity alone implies the existence of specified paths and cycles in graphs. It is well known that any 2-connected graph G of order at least 3 has a cycle. In fact by the classical result of Menger [68], any pair of distinct vertices of G lie on a common cycle. Using Menger's Theorem, Dirac generalized this in [24].

Theorem 1.1 (Dirac [24]) *If G is a k -connected graph with $k \geq 2$ and order $n \geq 3$, then any set of k vertices lie on a common cycle.*

This result is also sharp. If $H_1 \cup H_2$ denotes the graph obtained from vertex disjoint copies of H_1 and H_2 and $H_1 + H_2$ is obtained from $H_1 \cup H_2$ by adding all edges between the two graphs H_1 and H_2 , then in the k -connected graph $G = K_k + (\overline{K}_{k+1} \cup K_{n-2k-1})$ there is no cycle that contains the $k + 1$ independent vertices in the " \overline{K}_{k+1} " part of G . Watkins and Mesner in [81] characterized the k -connected graphs that have $k + 1$ vertices that do not lie on a common cycle. They are those with a cutset of k vertices whose removal results in a graph with strictly more than k -components. The graph G just described is one example of this class of exceptions for $k + 1$ vertices.

A natural question that arises: Can the k vertices be replaced by k edges? Clearly the k edges would have to induce a graph that is the vertex disjoint union of paths, which we will call a *path system*, for otherwise it could not be a subgraph of a cycle. It is easy to verify that any pair of edges in a 2-connected graph are on a common cycle. However this cannot be extended to the 3-connected case. This occurs

because 3 edges that form an edge cut in a graph, even if they induce a path system, cannot be on a common cycle. This led to the following conjecture of Lovász in [63].

Conjecture 1.1 (Lovász [63]) *If G is a k -connected graph with $k \geq 2$ and S is a set of k edges that form a path system, then there is a cycle that contains the edges of S , unless k is odd and the set S is an edge cut of G .*

There is support for this conjecture. The case $k = 3$ was proved by Lovász in [64]. The case $k = 4$ was verified by P. L. Erdős and Győri in [28], and Sanders verified the case $k = 5$ in [75]. The cases $k = 6, 7$ was proved by Kawarabayashi by using the following result proved in [56]

Theorem 1.2 (Kawarabayashi [56]) *If G is a k -connected graph with $k \geq 2$ and S is a set of k edges that form a path system, then there is one or two cycles that contain the edges of S , unless k is odd and the set S is an edge cut of G .*

The following result of Häggkvist and Thomassen [37] lends additional strong support to Conjecture 1.1 and confirms the slightly weaker conjecture proposed by Woodall in [83].

Theorem 1.3 (Häggkvist and Thomassen [37]) *If G is a k -connected graph with $k \geq 2$ and S is a set of $k - 1$ edges that form a path system, then there is a cycle that contains the edges of S .*

A proof of Conjecture 1.1 is not yet in print, but it has been verified by Kawarabayashi with the first part of the lengthy proof to appear in [57].

Clearly from the previous results, higher connectivity will imply longer cycles, and the nature of this relationship has been addressed. In [23] Dirac gave a lower bound on the length of the cycle in terms of the minimum degree.

Theorem 1.4 (Dirac [23]) *If G is a 2-connected graph of order n and minimum degree δ , then G contains a cycle of length at least $\min\{2\delta, n\}$.*

An immediate consequence of this is that if $k \leq n/2$, then any k -connected graph G of order n will have a cycle of length at least $2k$, and G will be hamiltonian if $k \geq n/2$. Also, this result is sharp, since for $n \geq 2k$, the k -connected graph $G = K_k + \overline{K}_{n-k}$ does not contain a cycle of length longer than $2k$. Egawa et.al. in [27] gave a common generalization of Theorem 1.1 and Theorem 1.4 of Dirac.

Theorem 1.5 (Egawa, Glas, Locke [27]) *If G is a k -connected graph with $k \geq 2$, minimum degree δ , and of order n , then for any set of k vertices there is a cycle of length at least $\min\{n, 2\delta\}$ that contains the k vertices.*

The relationship between connectivity and cycles and paths in graphs is much richer than the classical and basic results that have been mentioned. Some of the richness of this relationship will be explored in the sections that follow.

2 CYCLES IN CLAW-FREE GRAPHS

In general, k -connectivity does not imply the existence of cycles longer than $2k$, but for special classes of graphs, such as claw-free graphs, much stronger statements can be made. A graph is *claw-free* if it does not contain an induced copy of the graph $K_{1,3}$, which is called a *claw*. The class of claw-free graphs has been extensively studied with a wide range of motivations. Considerable activity resulted from the following conjecture in a paper by Matthews and Sumner in [66].

Conjecture 2.1 (Matthews and Sumner [66]) *Every 4-connected claw-free graph is hamiltonian.*

This conjecture is related to a classical conjecture of Chvátal concerning general graphs and the concept of *toughness*. Toughness is a concept designed to measure the resistance of a graph to be torn into components with the deletion of vertices, and is formally defined as follows.

Definition 2.1 (Chvátal [21]) *The toughness $\tau(G)$ of a graph G is the minimum of the ratio $|S|/\omega(G - S)$, where S is a cut set of vertices of G and $\omega(G - S)$ is the number of components of the graph $G - S$.*

Clearly any hamiltonian graph is 1-tough, since the deletion of m vertices will always leave a graph with at most m components. However, 1-tough is not sufficient to imply a graph is hamiltonian. For example, the Petersen graph is $3/2$ -tough, but it is not hamiltonian. This led to the question of whether $\tau(G) > 3/2$ is sufficient to imply hamiltonian. In [21] Chvátal conjectured that there exists a t_0 such that any t_0 -tough graph is hamiltonian. In 1985 Enomoto, Jackson, Katerinis, and Saito in [26] showed that t_0 would have to be at least 2 to imply hamiltonicity. This led to the following conjecture, which is generally attributed to Chvátal and the paper [21], but it was not explicitly stated there.

Conjecture 2.2 *Every 2-tough graph is hamiltonian.*

Conjecture 2.2 is related to Conjecture 2.1 of Matthews and Sumner, since there is a relationship between connectivity $\kappa(G)$ of a graph and the toughness $\tau(G)$. In general, $\tau(G) \leq \kappa(G)/2$, since the deletion of a set of cut vertices will leave at least 2 components. However, in claw-free graphs the deletion of any minimal cut will result in precisely 2 components. This implies that if G is a claw-free graph, then

$\tau(G) = \kappa(G)/2$. Therefore, in the class of claw-free graphs Conjecture 2.2 is equivalent to Conjecture 2.1. Note that Bauer, Broersma, and Veldman showed in [6] that Conjecture 2.2 is not true in general, and in fact showed the existence of non-hamiltonian graphs with $\tau(G) = 9/4$.

A special proper subclass of claw-free graphs is the class of line graphs. Given a graph G , the *line graph* $L(G)$ of G is the graph whose vertices are the edges of G and two vertices in $L(G)$ are adjacent if the corresponding edges in G are incident. Any line graph $L(G)$ is claw-free, since an edge e of G has only two endvertices and so at most two edges incident to e can be independent in $L(G)$. In [78] the following conjecture was introduced by Thomassen.

Conjecture 2.3 (Thomassen [78]) *Every 4-connected line graph is hamiltonian.*

Clearly Conjecture 2.1 is at least as strong as Conjecture 2.3, since all line graphs are claw-free graphs. However, we now know that these two conjectures are equivalent. This is a consequence of a closure concept for claw-free graphs introduced by Ryjáček in [73], which is similar in form and application to the closure introduced by Bondy and Chvátal in [12].

Let G be a claw-free graph. If v is a vertex of G , then the neighborhood $N(v)$ of v has independence number 2. Thus, $N(v)$ induces a connected graph or is the disjoint union of 2 complete components. If $N(v)$ is connected, then the “local closure” at v is the graph obtained by replacing $N(v)$ by a complete graph with the same vertex set. Recursively doing this until every vertex of G has a neighborhood that is either complete or the disjoint union of 2 complete graphs yields a graph $cl(G)$, which is called the *closure* of G . See Figure 2.1 for an example of a graph G in which the Ryjáček closure $cl(G)$ is the complete graph and this closure is obtained by 2 local closures. The graph $cl(G)$ is unique and it is claw-free. In [73] Ryjáček proved the following about $cl(G)$, where $c(G)$ denotes the circumference of G , the length of the longest cycle in G , and $p(G)$ denotes the length of the longest path in G .

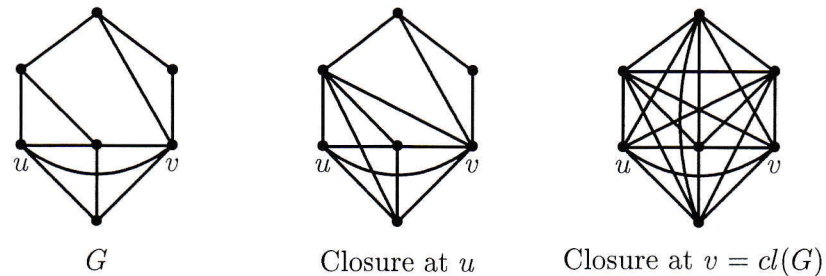


Figure 2.1: Closure

Theorem 2.1 (Ryjáček [73]) *Let G be a claw-free graph. Then*

- (i) *the closure $cl(G)$ is well-defined,*
- (ii) *there is a triangle-free graph H such that $cl(G) = L(H)$, and*
- (iii) $c(G) = c(cl(G))$
- (iv) $p(G) = p(cl(G))$ (see [16]).

The closure concept is a powerful tool in studying the cycle structure of a claw-free graph, and an immediate consequence of this closure concept is the determination of the circumference of a claw-free graph can be reduced to determining the circumference of an appropriate line graph of a triangle-free graph. In particular, this implies that Conjecture 2.1 is equivalent to Conjecture 2.3, and questions concerning the hamiltonicity in claw-free graphs can be reduced to the smaller class of line graphs, and in fact to line graphs of triangle-free graphs.

Cycles in line graphs have been studied extensively with one of the useful tools being a result of Harary and Nash-Williams in [38], which gives a necessary and sufficient condition for hamiltonicity of the line graph $L(G)$ in terms of a dominating eulerian subgraph of the graph G . An *eulerian subgraph* H is one that contains a closed trail using each edge of the graph H precisely one time, and it is *dominating* if all edges of G are incident to an edge of H .

Theorem 2.2 (Harary and Nash-Williams [38]) *Let G be a graph with at least 3 edges. Then, $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

Using Theorem 2.2 and variations of the same concept for other hamiltonian properties, Zhan in [86] proved the following hamiltonian result for line graphs. This was also done independently by Jackson [49].

Theorem 2.3 (Jackson and Zhan [49] and [86]) *Every 7-connected line graph is hamiltonian.*

An immediate consequence of Theorem 2.3 and the closure concept of Ryjáček used in the proof of Theorem 2.1 is that every 7-connected claw-free graph is hamiltonian. The gap between the strongest result and the equivalent Conjectures 2.1 and 2.3 is the difference between the 7-connectivity that follows from Theorem 2.3 and the 4-connectivity of Conjecture 2.1. The more general Conjecture 2.2 dealing with toughness, which is also equivalent to Conjectures 2.1 and 2.3 in the class of line graphs, is not true for general graphs. As was mentioned earlier Bauer et. al. exhibited graphs in [6] with toughness exceeding

2, in fact $9/4$, that are not hamiltonian. It is still not known if sufficiently large toughness implies the existence of a hamiltonian cycle.

There are infinite families of 3-connected claw-free graphs that are not hamiltonian. In [66] a 3-connected claw-free graph L of order 20 is exhibited (see Figure 2.2) that is not hamiltonian. The graph L is obtained by starting with the Petersen graph P_{10} , subdividing the edges of a perfect matching of P_{10} , and then letting $L = L(P_{10})$. In [66] Matthews and Sumner also showed that the graph L is the 3-connected claw-free non-hamiltonian graph of smallest order.

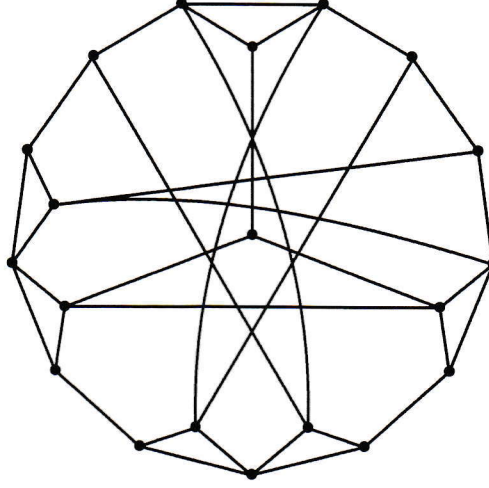


Figure 2.2: L

More generally, the circumference $c(G)$ of 3-connected claw-free graphs has been explored. The following result by Jackson and Wormald in [52] gives a lower bound on the circumference of 3-connected $K_{1,d}$ -free graphs in general and claw-free graphs in particular. Also, there are infinite families of examples obtained by taking the inflations of appropriate 3-regular graphs (replacing a vertex by a triangle) that imply that the Jackson-Wormald bound is of the correct order of magnitude, although the best value of the constant c is not known.

Theorem 2.4 (Jackson, Wormald [52]) *Let G be a 3-connected $K_{1,d}$ -free graph with n vertices. Then G contains a cycle of length at least n^c where $c = (\log_2 6 + 2 \log_2(2d - 1))^{-1}$.*

Inflations of appropriate 3-regular graphs, in particular the Petersen graph P_{10} , gives examples of graphs that have restrictions on the number of vertices that can be commonly on a cycle. This type of result was investigated by Győri and Plummer in [35] and they proved the following result.

Theorem 2.5 (Győri and Plummer [35]) *Let G be a 3-connected claw-free graph with a set S of at most 9 vertices. Then G contains a cycle that contains all of the vertices of S .*

This result in Theorem 2.5 is sharp. The graph P^* in Figure 2.3, which is an inflation of P_{10} , substantiates this, since there is no cycle that contains any set of 10 vertices that are chosen by selecting precisely one vertex from each triangle. If such a cycle existed in P^* , then P_{10} would be hamiltonian.

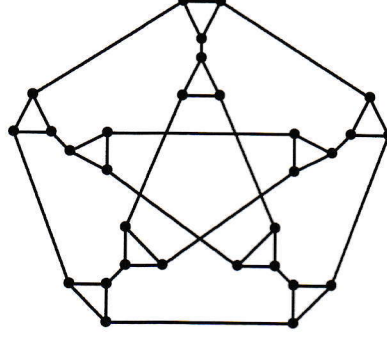


Figure 2.3: P^*

The circumference of 2-connected claw-free graphs was investigated by Broersma et. al. in [18]. They proved upper and lower bounds on $c(G)$ using the relationship between the toughness and connectivity of a superclass of claw-free graphs, namely $K_{1,r}$ -free graphs for any fixed $r \geq 3$.

Theorem 2.6 (Broersma, Van den Heuvel, Jung, and Veldman [18]) *If G is a 2-connected $K_{1,r}$ -free graph on n vertices with circumference $c(G)$, then*

$$c(G) \geq 4 \log_{r-1}(n) - A = \frac{4}{\log(r-1)} \cdot \log(n) - A,$$

where A is an appropriate absolute constant.

In [18] there is a construction that shows that for every $r \geq 3$, and sufficiently large n , there exists a 2-connected $K_{1,r}$ -free graph $H_{r,n}$ on n vertices such that

$$c(H_{r,n}) < \begin{cases} \frac{4}{\log(r-2)} \cdot \log(n) + 4 & \text{if } r \geq 3, \\ 8 \log(n+6) - 8 \log(3) - 2 & \text{if } r = 3. \end{cases}$$

Hence, the order of magnitude of the lower bound in Theorem 2.6 is best possible.

Related to Conjecture 2.1 of Matthews and Sumner is the following question of Jackson and Wormald in [52].

Question 2.1 (Jackson and Wormald [52]) For $r \geq 4$, is every r -connected $K_{1,r}$ -free graph hamiltonian.

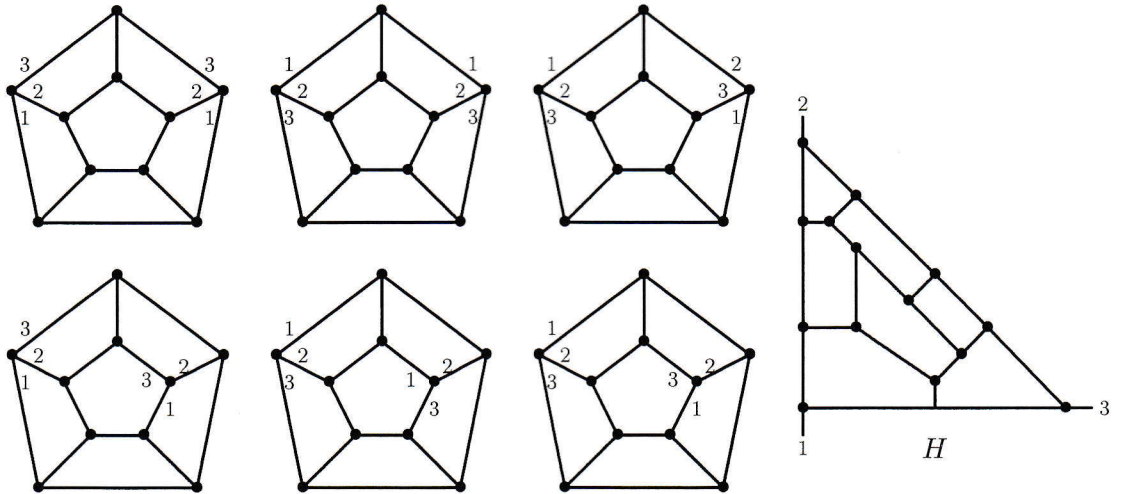
It is not known if sufficiently large connectivity implies that a $K_{1,r}$ -free graph is hamiltonian for $r \geq 4$.

3 CYCLES IN PLANAR GRAPHS

All planar graphs are at most 5-connected, since the average degree in a planar graphs is less than 6. However, connectivity has a substantial impact on the size of cycles in planar graphs. The following result of Tutte in [79] gives a sufficient condition in terms of connectivity for a planar graph to be hamiltonian.

Theorem 3.1 (Tutte [79]) Every 4-connected planar graph is hamiltonian.

Planar graphs that are 3-connected are not necessarily hamiltonian. Non-hamiltonian examples of 3-connected cubic planar graphs of order 38 are exhibited by Holton and McKay in [43]. They show that all 3-connected cubic graphs with at most 36 vertices are hamiltonian, and Figure 3.1 exhibits the 6 non-hamiltonian 3-connected cubic planar graphs of order 38 that are not cyclically 4-connected. A graph is *cyclically k -connected* if there is no cut set with less than k vertices with each component containing a cycle, so cyclically 3-connected cubic graphs are those with no nontrivial cutsets.



At each labeled vertex place a copy of graph H

Figure 3.1: 6 non-hamiltonian graphs of order 36

It is not even true in general that a 3-connected planar graph G of order n will have a cycle of length ϵn for a positive number ϵ . By successively inserting a vertex of degree 3 inside each face of a planar

graph starting with a 3-connected planar graph such as K_4 , Moon and Moser in [70] exhibited 3-connected planar graphs with a circumference of at most n^d for $d = \log_3 2$. In fact, the graphs of Moon and Moser are triangulations, so this kind of example cannot be improved. They also conjectured that if G is a 3-connected planar graph on n vertices, then $c(G) = \Omega(n^{\log_3 2})$. A series of lower bounds for $c(G)$ of a 3-connected planar graph G have been given. Barnette in [7] showed that $c(G) \geq \sqrt{\log n}$, Clark in [22] improved this bound to $e^{\sqrt{\log n}}$, and Jackson and Wormald verified in [51] that $c(G) \geq c_1 n^d$ for some positive constant c_1 and for $d = .207248898$. In addition, Gao and Yu in [32] improved the lower bound for $c(G)$ to $c_2 n^{0.4}$ for some positive number c_2 , and extended the result to not just planar graphs but all graphs embeddable in the projective plane, the torus, or the Klein bottle. Finally, Chen and Yu in [20] verified the conjecture of Moon and Moser with the following result.

Theorem 3.2 (Chen and Yu [20]) *Let G be a 3-connected graph with n vertices that is embeddable in the sphere, the projective plane, the torus, or the Klein bottle. Then $c(G) = \Omega(n^{\log_3 2})$.*

The complete bipartite graph $K_{2,n}$ is planar, 2-connected, and all cycles have length 4, and so 2-connectivity does not imply the existence of long cycles in planar graphs. If other parameters are considered, such as the toughness, then more can be said about the circumference. Note that the toughness of $K_{2,n}$ is $2/n$ and this approaches 0 as $n \rightarrow \infty$. If a bound is placed on the number of components after the deletion of a cutset with 2 vertices, more can be said about the length of longest cycles. The next result of Böhme et. al. in [15] gives a lower bound on $c(G)$ for a 2-connected planar graph G in terms of the toughness $\tau(G)$.

Theorem 3.3 (Böhme, Broersma, and Veldman [15]) *If G is a 2-connected planar graph with bounded toughness, then $c(G) \geq d \log n$ for some constant d depending on $\tau(G)$.*

Other conditions, like restrictions on the maximum degree will imply the existence of longer cycles, since the diameter of the graph will be a function of the order, and each pair of vertices of a 2-connected graph is contained on a cycle.

4 CYCLES IN BIPARTITE GRAPHS

It is generally accepted that the same degree and connectivity conditions applied to a bipartite graph, as compared to a general graph, will give longer cycles, and in fact in many cases cycles approximately twice as long will be implied. Examples of this type of result will be described. Let $B(a, b, k)$ denote a bipartite

graph with parts having $a \geq 2$, $b \geq k \geq 2$ vertices and with the minimum degree of the vertices in the “ a ” part being k . In [47] Jackson proved the following.

Theorem 4.1 (Jackson [47])

- (i) *If a graph $B(a, b, k)$ satisfies $a \leq k$ and $b \leq 2k - 2$, then there is a cycle of length $2a$.*
- (ii) *If a graph $B(a, b, k)$ satisfies $b \leq \lceil a/(k - 1) \rceil(k - 1)$, then there is a cycle of length at least $2k$.*

In the previous theorem no connectivity condition is assumed, and the result is sharp, since an obvious 1-connected bipartite graph will give this. In a general graph with no restrictions on the connectivity, a minimum degree of k implies only a cycle of length at least $k + 1$.

If the bipartite graph is assumed to be 2-connected, then more can be said. Let $B(a, b, k, \ell)$ denote a bipartite graph with parts having a and b vertices and with corresponding minimum degrees k and ℓ respectively. For 2-connected graphs Jackson proved the following in [48].

Theorem 4.2 (Jackson [48])

- (i) *If $B(a, b, k, k)$ is 2-connected, then there is a cycle of length at least $2 \min\{a, b, 2k - 2\}$.*
- (ii) *If a graph $B(a, b, k, \ell)$ with $a \leq b$ is 2-connected, then there is a cycle of length at least $2 \min\{b, k + \ell - 1, 2k - 2\}$.*
- (iii) *If a graph $B(a, a, k, k)$ is 2-connected, then there is a cycle of length at least $2 \min\{a, 2k - 1\}$.*

The third part of Theorem 4.2 gives that a 2-connected balanced bipartite graph with minimum degree k will either be hamiltonian or have a cycle of length at least $4k - 2$. This is approximately twice what is true in the class of general graphs.

5 CYCLES IN REGULAR GRAPHS

As noted earlier a classical result of Dirac ([23]) implies that a 2-connected graph of order n with minimum degree δ will have a cycle of length at least $\min\{2\delta, n\}$, and the length of the cycle is best possible. An immediate consequence of this is that if G is a graph of order n with $\delta(G) \geq n/2$, then G is hamiltonian. In the class of regular graphs this result was strengthened by Jackson in [46].

Theorem 5.1 (Jackson [46]) *If G is a 2-connected r -regular graph of order n with $r \geq n/3$, then G is hamiltonian.*

The Petersen graph P_{10} is a 3-regular graph of order 10 that is not hamiltonian, and this graph is an example that shows that Theorem 5.1 cannot be improved. However, P_{10} is the only such graph. It was shown by Y. Zhu et. al. in [84] that all 2-connected r -regular graphs of order $3r + 1$ are hamiltonian except for P_{10} , and a simpler proof of the same result was given by Bondy and Kouider in [13]. Also, Y. Zhu et.al. in [85] showed for $r \geq 6$, any 2-connected r -regular graph on at most $3r + 3$ vertices is hamiltonian. It was conjectured by Häggkvist (see [46]) that for $r \geq 4$, any 3-connected r -regular graph of order at most $4r$ is hamiltonian. Jackson et. al. in [50] provided some support for this conjecture by showing that if G is a 3-connected graph of order at most $4r$ that is r -regular for $r \geq 63$, then every longest cycle in G is dominating.

The more general result of Dirac has an analogue for regular graphs that was proved by Jung in [55] and Fan in [30].

Theorem 5.2 (Jung [55], Fan [30]) *If G is a 3-connected r -regular graph of order n , then $c(G) \geq \min\{3r, n\}$.*

In this same vein the following result for 4-connected regular graphs was proved by Aung in [4].

Theorem 5.3 (Aung [4]) *If G is a 4-connected r -regular graph of order n , then $c(G) \geq \min\{4r - 4, \frac{1}{2}(n + 3r - 2)\}$.*

In the class of claw-free graphs regularity also plays a strong role and forces the circumference to be larger. MingChu Li considered regular 2-connected claw-free graphs in [62] and proved the following.

Theorem 5.4 (Li [62]) *If G is a 2-connected claw-free r -regular graph of order n , then $c(G) \geq \min\{4r - 2, n\}$.*

The previous results are of the same nature of a more general conjecture made by Bondy in [10].

Conjecture 5.1 (Bondy [10]) *If G is a 2-connected d -regular graph of order $n \leq rd$ with $r \geq 3$ an integer and n sufficiently large, then $c(G) \geq 2n/(r - 1)$.*

This conjecture is motivated by the following examples. For integers $d \geq 3$ and $r \leq d - 3$, consider the graph $H = K_2 + ((r + 1)K_{d+1})$. There is a spanning subgraph G of H that is 2-connected, d -regular, of order $n = (r + 1)(d + 1) + 2$, and with $c(G) = 2d + 2 = 2n/(r - 1) + 2(r - 3)/(r - 1)$. The following result of Wei in [82] verifies Conjecture 5.1 when r is an integer and shows that the previous examples are maximal. It was noted in [82] that if r is not an integer, then the conjecture is not true for some small values of r .

Theorem 5.5 (Wei [82]) *If G is a 2-connected d -regular graph of order $n \leq rd$ with $r \geq 3$ an integer and n sufficiently large, then $c(G) \geq 2n/(r-1) + 2(r-3)/(r-1)$.*

An upper bound on the length of longest cycles in r -regular and r -connected graphs has been explored for many years. G.H.J. Meredith in [69] constructed the first family of non-hamiltonian r -connected r -regular graph for $r \geq 3$. This same class was studied by Brad Jackson and T. D Parsons in [53], where they proved the following upper bound on the circumference.

Theorem 5.6 (Jackson and Parsons [53]) *For $r \geq 3$ there is a positive $\epsilon = \epsilon(r) < 1$ such that for n sufficiently large, there is an r -connected r -regular graph of order n with circumference less than n^ϵ .*

There is a wide gap between the best upper bounds ($n^{\epsilon(r)}$) and the lower bounds (linear in r) for the circumference of an r -connected and r -regular graph of order n .

For the special class of graphs that are the intersection of planar, regular, and bipartite classes, there is an interesting conjecture by Barnette in [8] that is still open.

Conjecture 5.2 (Barnette [8]) *Every 3-connected cubic bipartite planar graph is hamiltonian.*

The conjecture is known to be true for graphs with at most 64 vertices ([42]). The conjecture has also been verified for infinite classes of graphs. For example, Goodey ([33]) showed that 3-connected planar bipartite graphs with only faces that are squares or hexagons are hamiltonian. This was generalized by Feder and Subi ([31]) to include the larger class of graphs that when the faces are 3-colored, two of the color classes contain only squares and hexagons and each face in the third class is surrounded by an even number of squares.

6 CYCLABILITY OF REGULAR GRAPHS

The classical result of Dirac (Theorem 1.1) implies that in a k -connected graph any set of k vertices are on a common cycle. It is natural to ask if adding a regularity condition will increase the number of vertices that are in a common cycle. That is the motivation for the following definition.

Definition 6.1 *For integers $r \geq k \geq 2$, let $g(k, r)$ denote the largest integer m such that any collection of m vertices in an r -regular k -connected graph is contained in a common cycle.*

The case $r = k = 2$ is trivial, since any such graph is hamiltonian. For $r > k = 2$, it is not difficult to show that $g(k, r) = 2$, and in fact in Holton and Plummer in [45] describe classes of examples that

imply this. See Figure 6.1 for examples that verify that $g(2, 3) = g(2, 4) = 2$ and give the pattern for more general classes of examples that imply $g(2, k) = 2$ for $k \geq 2$.

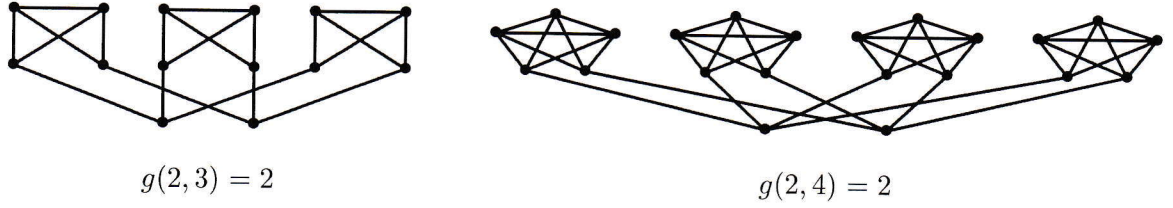


Figure 6.1: Cyclability for 2-connected regular graphs

In the remainder of this section we will just consider the cases $k \geq 3$. The most difficult case in determining $g(k, r)$ is when $r = k$. It was proved independently by Holton in [39] and Kelmans and Lomonosov in [58] that $g(k, k) \geq k + 4$ for $k \geq 3$. It is most unlikely that this lower bound is the correct bound, and in fact it is known that it is not correct for $k = 3$, which has been shown to be 9. For k even, McCuaig and Rosenfeld in [67] exhibited examples that implied that $g(k, k) \leq 6k - 4$ for $k \equiv 0 \pmod{4}$ and $g(k, k) \leq 8k - 5$ for $k \equiv 2 \pmod{4}$. This improved the general upper bound of $g(k, k) \leq 10k - 11$ of Meredith in [69] when k is even.

When $r > k$, much more is known about the function $g(k, r)$. For k odd Holton proved that $g(k, k+1) \geq k + 2$ in [39], and this was also proved but not published by Kelmans and Lomonosov. No reasonable linear upper bounds for $g(k, k+1)$ when k is odd are known, since the proof of the upper bound given in [45] has a gap that has not been repaired. In [39] examples were described that imply $g(k, r) < k + 1$ for k even and $r \geq k + 1$ or if k is odd and $r \geq k + 2$.

The special case of 3-regular and 3-connected graphs has received considerable attention. The Petersen graph P_{10} is 3-regular and 3-connected but is not hamiltonian. Thus, $g(3, 3) \leq 9$. There are arbitrarily large 3-connected 3-regular graphs obtained from P_{10} by inflating vertices, which have 10 vertices that are not on a common cycle. With this background, Holton et. al. in [44] proved the following lower bound for $g(3, 3)$.

Theorem 6.1 (Holton, McKay, Plummer, and Thomassen [44]) *If G is a 3-connected cubic graph, then there is a cycle containing any specified set of 9 vertices.*

Shortly after the appearance of Theorem 6.1 it was shown by Ellingham, Holton and Little in [25] that any set of 10 vertices in a 3-connected 3-regular graph G is contained in a common cycle unless the graph G can be contracted to a P_{10} . Also in [25] as well as in [40] the related questions of including specified

vertices but excluding specified edges was considered. In [1] Aldred considered placing specified vertices on paths, where he showed that if S is a set of at most 13 vertices in a 3-connected 3-regular graph G , then there is a path containing the set S .

If the condition of planarity is added, then more can be proved. Aldred, Bau, Holton, and McKay proved the following in [2].

Theorem 6.2 (Aldred, Bau, Holton, and McKay [2]) *If G is a 3-connected cubic planar graph and S is any set of at most 23 vertices of G , then there is a cycle containing S .*

The conditions in Theorem 6.2 are sharp, since Holton exhibited in [41] an example of a 3-connected cubic planar graph with a set of 24 vertices that are not contained in any cycle of the graph.

In general the only open questions involving the values of $g(k, r)$ are for $k \geq 4$ and when $r = k$ or when $r = k + 1$ and k is odd. The following result summarizes what is known about $g(k, r)$ for $k \geq 3$.

Theorem 6.3 *For $r \geq k \geq 3$ the bounds on $g(k, r)$ are the following:*

- (i) $g(3, 3) = 9$
- (ii) $k + 4 \leq g(k, k) \leq 6k - 4$ if $k \equiv 0 \pmod{4}$
- (iii) $k + 4 \leq g(k, k) \leq 8k - 5$ if $k \equiv 2 \pmod{4}$
- (iv) $k + 4 \leq g(k, k) \leq 10k - 11$ if $k \geq 5$ is odd
- (v) $k + 2 \leq g(k, k + 1)$ if k is odd
- (vi) $g(k, r) = k$ if $(r \geq k + 2)$ or $(r = k + 1 \text{ with } k \text{ even})$.

The function $g(k, r)$ deals with the existence of cycles containing specified vertices, but is not concerned with the length of the cycle containing the set of vertices. This question was addressed by Saito in [74], where the following was proved.

Theorem 6.4 (Saito [74]) *Let G be a k -connected graph of order at least $2k$ with circumference ℓ . Then,*

- (i) *for any set of $m < k$ vertices of G , there is a cycle of length at least $((k - m)/k)\ell + 2m$ containing the m vertices,*
- (ii) *for any set of k vertices of G , there is a cycle of length at least $(1/3k)\ell + \frac{2}{3}(k + 2)$ containing the k vertices, if G is k -regular, and*
- (iii) *for $k = 3$ and any set of 3 vertices of G , there is a cycle of length at least $(\ell/4) + 3$ containing all of them, if G is planar.*

7 ORDERED GRAPHS

For a positive integer k , a graph G is k -ordered if for every ordered set $\{x_1, x_2, \dots, x_k\}$ of k distinct vertices of G , there exists a cycle that encounters the vertices in the designated order. This concept, which is stronger than the cyclability condition of the previous section, was introduced by Ng and Schultz in [72]. It is natural to ask what “connectivity” conditions imply k -ordered. The concept of k -ordered is related to several other “connectivity” properties, in particular to linkage. A graph G is k -linked if given any collection of k pairs of vertices $L = \{\{x_i, y_i\} : 1 \leq i \leq k\}$, there are k vertex disjoint paths (except possibly for endvertices) P_i ($1 \leq i \leq k$) such that P_i is a path from x_i to y_i .

Clearly k -linked implies k -ordered, since given an ordered k -set $S = \{x_1, x_2, \dots, x_k\}$ in a k -linked graph, the set of k paths associated with consecutive pairs of vertices in S with x_1 considered as following x_k form a cycle that contains the set S in the proper order. Also, $2k$ -ordered implies k -linkage, since if a set of vertices $S_{2k} = \{x_1, y_1, x_2, y_2, \dots, x_k, y_k\}$ are placed in order on a cycle C , there is a collection of k vertex disjoint subpaths of the cycle C between the pairs $\{x_i, y_i\}$ for $(1 \leq i \leq k)$.

It has been known for some time that sufficient connectivity would imply linkage. Independently, Jung in [54] and Larman and Mani in [60] proved that if a graph G is $2k$ -connected and contains a subdivision of a K_{3k} , then G is k -linked. This fact along with the result of Mader in [65] that if $\delta(G) \geq 2\binom{3k}{2}$, then G contains a subdivision of K_r , gives an upper bound of $2\binom{3k}{2}$ on the connectivity that implies k -linkage. However, a much sharper bound was proved by Bollobás and Thomason in [9].

Theorem 7.1 [9] *If G is a graph with $\kappa(G) \geq 22k$, then G is k -linked.*

It is very likely that the connectivity needed to imply k -linkage is significantly less than $22k$. Consider the example $F_1 = K_{3k-1} - (kK_2)$, which appears in Figure 7.1. The “dotted” edges represent missing edges of F_1 . The graph F_1 is not k -linked, because if the k pairs $\{(x_i, y_i) : 1 \leq i \leq k\}$ associated with the k missing edges of F_1 are selected, there is an insufficient number of additional vertices to form the required k paths. Since $\kappa(F_1) = 3k - 3$, this leads to the following question.

Question 7.1 *For $k \geq 3$, if $\kappa(G) \geq 3k - 2$, then is G k -linked?*

There is a corresponding graph and question for k -ordered graphs. Consider the example $F_2 = K_{2k-1} - (C_k)$ which appears in Figure 7.2. Note that $\kappa(F_2) = 2k - 4$ and it is not k -ordered, because if the set of k vertices on the “missing” cycle are chosen in the natural order, then it is not possible to choose a cycle C in F_2 that contains this set of vertices. This leads to the following question.

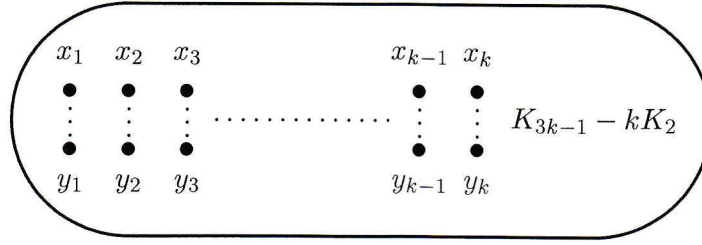


Figure 7.1: F_1

Question 7.2 For $k > 4$, if $\kappa(G) \geq 2k - 3$, then is G k -ordered?

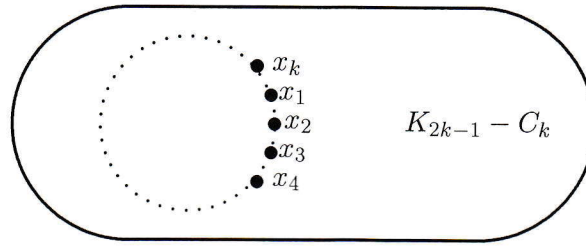


Figure 7.2: F_2

In [54] Jung gave an upper bound for the connectivity need to imply 2-linked.

Theorem 7.2 [54] If $\kappa(G) \geq 6$, then G is 2-linked.

The previous result is sharp. The graph F_3 in Figure 7.3 is a 5-regular 5-connected planar graphs that are not 2-linked, since there is no linkage for the pairs $\{x_1, x_3\}$ and $\{x_2, x_4\}$. Also, the 5-connected graph F_3 in Figure 7.3 is not 4-ordered, since there is no cycle containing the ordered set $\{x_1, x_3, x_2, x_4\}$. Little is known about the minimal connectivity that implies 4-ordered, but it is natural to ask the following specialized question.

Question 7.3 If G is a 6-connected graph, is G 4-ordered?

The complete graph K_{2k} is k -linked and $\kappa(K_{2k}) = 2k - 1$. Any graph G with $\kappa(G) \leq 2k - 2$ cannot be k -linked, since if the collection of k -pairs contains the minimum cutset S and the two vertices of some pair are in different components of $G - S$, then there cannot be the path system required for k -linkage. Thus k -linkage implies $(2k - 1)$ -connected. For the same reason, k -ordered implies $(k - 1)$ -connected, and the graph K_k is k -ordered with $\kappa(K_k) = k - 1$. Recall that the graph F_2 is not k -ordered, but it is

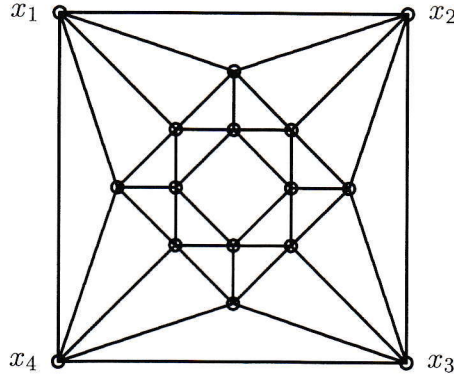


Figure 7.3: F_3

$(k-1)$ -linked. Also, for $n \geq 2k$ and k even the complete bipartite graph $K_{k,n-k}$ is k -orderable, but it is not $(k/2 + 1)$ -linked. We summarize the known results about the relationship between linkage, ordered, and connectivity in the following theorem.

Theorem 7.3 For $k \geq 3$,

- (1) k - linked $\implies (2k-1)$ - connected, but k - linked $\not\Rightarrow (2k)$ - connected.
- (2) k - ordered $\implies (k-1)$ - connected, but k - ordered $\not\Rightarrow (k)$ - connected.
- (3) k - linked $\implies k$ - ordered, but k - linked $\not\Rightarrow (k+1)$ - ordered.
- (4) k - ordered $\implies \lfloor k/2 \rfloor$ - linked, but k - ordered $\not\Rightarrow (\lfloor k/2 \rfloor + 1)$ - linked.
- (5) $(22k)$ - connected $\implies k$ - linked, but $(3k-3)$ - connected $\not\Rightarrow (k)$ - linked.
- (6) $(22k)$ - connected $\implies k$ - ordered, but $(2k-4)$ - connected $\not\Rightarrow (k)$ - ordered.

It would be of interest to know the sharpest relationships between linkage, ordered, and connectivity.

8 NUMBER OF CYCLES

The minimum number of different cycle lengths that any k -connected must have is not difficult to determine. The complete graph K_{k+1} is k -connected and has only $k-1$ different cycle lengths. Also, the complete bipartite graph $K_{k,n-k}$ for $n \geq 2k$ is k -connected but also has only $k-1$ different cycle lengths, namely $4, 6, \dots, 2k$. It is easy to see that any non-complete k -connected graph G will have at least $k-1$ different cycle lengths. Take a longest path P in G and consider an endvertex x of P . There will be at least $k-1$ chords of P emanating from x , and each chord will give a different cycle length. Thus k -connectivity implies $k-1$ different cycle lengths, and this bound is sharp.

Counting the number of difference cycles, allowing for a duplication in length, is a much more difficult problem. Let $c_n(k)$ denote the maximum number m such that every k -connected graph of order n will have at least m different cycles. The function $c_n(k)$ was investigated by Knor in [59]. By considering trees, cycles, wheels, and the complete bipartite graphs $K_{k,n-k}$ for $n \geq k$, it follows immediately that $c_n(1) = 0$, $c_n(2) = 1$, $c_n(3) = O(n^2)$, and $c_n(k) = O(n^k)$ if $k \geq 4$. The subclass of k -connected graphs with minimum degree $\delta \geq k$ was also investigated. Let $c_n(k, \delta)$ denote the corresponding maximum number m such that every k -connected graph with minimum degree δ and order n will have at least m different cycles. The following was proved by Knor in [59].

Theorem 8.1 (Knor [59])

- (i) $c_n(3) = \Theta(n^2)$.
- (ii) $c_n(2, \delta) = \Theta(n^2)$ if $\delta \geq 3$.
- (iii) $c_n(3, \delta) = \Theta(n^3)$ if $\delta \geq 5$.

It is conjectured in [59] that if some regularity restraints are placed on k -connected graphs with minimum degree $\delta > k$, such as a bound on the maximum degree, then the number of cycles will be $\Theta(n^k)$ for $k > 2$.

For cubic graphs more can be said about the number of cycles. In [5] Barefoot et. al. showed that every 2-connected cubic graph of order n will have at least $(n^2 + 14)/8$ cycles, and this bound is sharp. They also conjectured that if the cubic graph is 3-connected then the number of cycles will be at least superpolynomial in n and for some graphs it will be subexponential in n . This was verified by Aldred and Thomassen in [3], where they showed that every 3-connected cubic graph of order n has at least $2^{n^{0.17}}$ cycles and one cannot be assured as many as $2^{n^{0.95}}$ cycles. This implies the minimum number of different cycle lengths in 3-connected cubic graphs is not bounded, as it is for general 3-connected graphs, but it is in fact a function of the order n of the graph.

9 RELATIVE LENGTH OF PATHS AND CYCLES

It is natural to expect that long paths in a graph G will imply long cycles if the graph G is at least 2-connected. In order to express the relationship between the longest path in a graph and the longest cycle, let $\ell_k(p)$ be the largest integer m such that any k -connected graph G that has a path of length p also contains a cycle of length at least m . In the classical paper [23] Dirac studied this relation for 2-connected

graphs and proved that $\ell_2(p) \geq 2\sqrt{p}$. This result was proved independently by Voss in [80]. The family of examples illustrated by the specific example in Figure 9.1 verifies that the result of Dirac cannot be improved for 2-connected graphs.

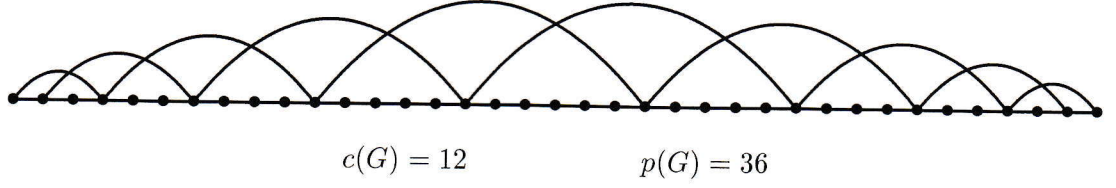


Figure 9.1: $G: c(G) = 2\sqrt{p(G)}$

Later Bondy and Locke considered 3-connected graphs in [14] and proved that

$$\frac{2p}{5} + 2 \leq \ell_3(p) \leq \frac{p}{2} + O(p^\alpha),$$

where $\alpha = \log_3 2$. They also considered the relative length of longest paths and cycles in 3-connected 3-regular graphs. Let $\ell_k^*(p)$ denote the number corresponding to $\ell_k(p)$, when only k -connected and k -regular graphs are considered. For this class of graphs it was shown in [14] that

$$\frac{2p}{3} + 2 \leq \ell_3^*(p) \leq \frac{7p}{8} + 3.$$

The lower bound proofs in both of the previous results use the concept of “vines”, which is a series of overlapping paths (normally just edges) attached to a fixed path. The graph G in Figure 9.1 is an example of a vine. The upper bound for $\ell_3^*(p)$ results from the class of graphs illustrated by the graph F_5 in Figure 9.2, which comes from the Petersen graph P_{10} . In general, F_m has a hamiltonian path of length $8m + 1$, but there is no cycle longer than $7m + 2$.

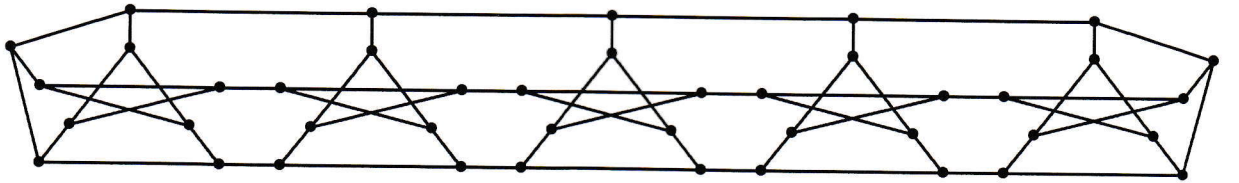


Figure 9.2: $F_5: p(F_5) = 41, c(F_5) = 37$

Also in [14] Bondy and Locke gave a general upper bound by proving that

$$\ell_k(p) \leq \frac{k-2}{k-1} \ell + O(\ell).$$

Then, Locke in [61] generalized the previous results by showing that

$$\ell_k(p) \leq \frac{2k-4}{3k-4} \ell.$$

Bondy and Locke also made the following conjecture in [14].

Conjecture 9.1 (Bondy and Locke [14]) *There exists a sequence of constants $c_3, c_4, \dots, c_k, \dots$ such that $\lim_{k \rightarrow \infty} c_k = 1$ and $\ell_k(p) \geq c_k p$ for all k and p .*

10 INTERSECTION OF LONGEST CYCLES

If C and C^* are two cycles of maximum length in a 2-connected graph G , then these cycles will share at least 2 vertices. This is easy to see; for example, assume the cycles are vertex disjoint. Since G is 2-connected, there are 2 vertex disjoint paths P and P^* between the cycles C and C^* . Using the paths P and P^* along with appropriate longest subpaths of C and C^* between the endvertices of the paths P and P^* on the cycles, a longer cycle can be shown to exist. This yields a contradiction. The same argument also applies if the cycles share precisely one vertex. This is the 2-connected case of a conjecture that is attributed to Scott Smith in a paper of Grötschel in [36].

Conjecture 10.1 (Smith [36]) *In a k -connected graph for $k \geq 2$, any two longest cycles intersect in at least k vertices.*

The conjecture cannot be improved, since the longest cycles in the graph $K_k + \overline{K}_{n-k}$ are of length $2k$, and for $n \geq 3k$ some pairs of longest cycles will intersect in just k vertices. The results in [36] imply that Conjecture 10.1 is valid for $k \leq 5$, and a comment is made in that paper that the conjecture has been verified for $k \leq 10$. The cases $k = 6, 7$ are verified by Stewart and Thompson in [76]. S. Burr and T. Zamfirescu verified in an unpublished manuscript that if G is a k -connected graph then every pair of longest cycles meet in at least $\sqrt{k} - 1$ vertices. This bound was improved by Chen et. al in [19] by showing that any two longest cycles intersect in at least $ck^{3/5}$ vertices, where $c = 1/(\sqrt[3]{256} + 3)^{3/5}$. Conjecture 10.1 of Smith was recently verified in a paper by Tan and Tang in [77], although some mathematicians have questioned whether this proof is valid.

Theorem 10.1 (Tan and Tang [77]) *If G is a k -connected graph for $k \geq 2$, then any two longest cycles intersect in at least k vertices and their intersection is a cut set of G .*

There are many interesting questions to explore involving the relationship between cycles and paths in graphs and the connectivity of the graph. Some areas of investigations have been outlined, some substantive results have been stated, and many open questions have been posed. For more detailed discussions of results

on paths and cycles see a chapter by Bondy in the “Handbook of Combinatorics” (see [11]), and survey articles by Faudree, Flandrin, and Ryjáček in [29] on claw-free graphs, Gould in [34] on cycles in graphs, and Broersma and Ryjáček in [17] on closure concepts.

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