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Connectivity and Cycles

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Connectivity plays a critical role in the existence of paths and cycles in graphs. We present recent extensions of classical results on the relationship between connectivity and properties of paths, cycles and 2-factors in graphs. We also explore these connectivity-cycle relationships for special classes of graphs, such as regular, planar and claw-free graphs.

1. Introduction

Connectivity is a key graphical parameter in conditions that imply the existence of paths, cycles and 2-factors in graphs. Conditions in which connectivity plays a major role in the existence of such subgraphs are discussed. In some cases, and for some classes of graphs, connectivity conditions alone imply the existence of paths, cycles and 2-factors. For example, it is well known that every 2-connected graph with at least three vertices has a cycle. In fact, by the classical result of Menger [86] (see Chapter 1), every pair of vertices of G lies on a common cycle. Using Menger's theorem and induction, Dirac [32] generalized this result.

Theorem 1.1 *If G is a k-connected graph with $k \geq 2$ and order $n \geq 3$, then every set of k vertices lies on a cycle.*

This result is sharp, as the k connected graph $K_k + (\overline{K}_{k+1} \cup K_{n-2k-1})$ shows: there is no cycle that contains the $k+1$ independent vertices in the $\overline{K}_{k+1}$ part of the graph. Watkins and Mesner [107] characterized those k-connected graphs in which there are $k+1$ vertices that do not lie on a cycle – they are those graphs with a cutset of k vertices whose removal results in a graph with more than k components. The graph just described is one example of this class of exceptions.

It is clear from the previous observations that larger connectivity implies longer cycles, and the nature of this relationship has been investigated. Dirac [31] gave a lower bound on the length of the longest cycle in terms of the minimum degree.

Theorem 1.2 *Every 2-connected graph of order n and minimum degree δ contains a cycle of length at least* $\min\{2\delta, n\}$.

An immediate consequence of this is that if $k \leq \frac{1}{2}n$, then any k-connected graph of order n has a cycle of length at least $2k$, and is Hamiltonian if $k \geq \frac{1}{2}n$. Also, this result is sharp, since for $n \geq 2k$, the k-connected graph $K_k + \overline{K}_{n-k}$ contains no cycle of length longer than $2k$.

Egawa, Glass and Locke [34] gave a common generalization of Theorems 1.1 and 1.2.

Theorem 1.3 *If G is a k-connected graph with $k \geq 2$, minimum degree δ, and order n, then every set of k vertices is on a cycle of length at least* $\min\{2\delta, n\}$.

The existence of disjoint cycles was considered by Corradi and Hajnal [29], who proved the following result.

Theorem 1.4 *For $k \geq 2$, if G is a graph with minimum degree at least $2k$ and order at least $3k$, then G contains k vertex-disjoint cycles.*

This result is sharp in that the graph $K_{2k-1} + \overline{K}_{n-2k+1}$ has minimum degree $2k-1$ (in fact, its connectivity is $2k-1$) yet it does not contain k disjoint cycles.

There are many generalizations of these classical results on connectivity and cycles in graphs. In the next section, these are explored and expanded to include paths, cycles with special properties and 2-factors. Surveys on cycles and paths in graphs include Bondy [16], Gould [46] and Faudree [40].

2. Generalizations of classical results

A generalization of Dirac was considered by Kaneko and Saito [67]. For $r \geq s$, a graph satisfies property $P(r, s)$ if for any set R of r vertices, there is a cycle C such that $|C \cap R| = s$. Thus, Dirac's theorem implies that every k-connected graph satisfies $P(k, k)$. Kaneko and Saito proved that every k-connected graph satisfies $P(k+t, k)$

if $t \leq \frac{1}{4}(-1+\sqrt{8k+9})$. This was improved recently by Kawarabayashi [71] in the following result.

Theorem 2.1 *Let G be a k-connected graph with $k \geq 3$. Then for any set S of s vertices with $k \leq s \leq \frac{3}{2}k$, G contains a cycle with precisely k vertices of S.*

A natural question arises: *Can vertices be replaced by edges in this result?* The k edges would then have to form a union of disjoint paths, which we call a *path system*. It is straightforward to verify that each pair of edges in a 2-connected graph lies on some cycle. However, this cannot be extended to the 3-connected case, since it is possible for three edges to form an edge-cut in a graph, and thus there cannot be a cycle containing all three.

Lovász [79] conjectured that this structure – an odd number of edges forming a path system that is also an edge-cut – is the only exception to the existence of a cycle. This was shown to be the case for graphs of low connectivity.

Theorem 2.2 *For $2 \leq k \leq 7$ and any k-connected graph G, if S is a set of k edges that form a path system, then there is a cycle that contains the edges of S, unless k is odd and the set S is an edge-cut of G.*

The case $k=3$ was proved by Lovász [80], the case $k=4$ by Erdős and Győri [38], and the case $k=5$ by Sanders [97]. The cases $k=6$ and 7 were established by Kawarabayashi [70], who also outlined a proof of Lovász's conjecture and said that details would be provided in a series of three papers. Additionally, he showed that if the set of edges is not on a single cycle, then two cycles will do.

Theorem 2.3 *If G is a k-connected graph with $k \geq 2$, and if S is a set of k edges that form a path system, then S is contained either in a single cycle or in the union of two cycles, unless k is odd and the set S is an edge-cut of G.*

The following result of Häggkvist and Thomassen [49] lends additional support to Lovász's conjecture and proves a conjecture of Woodall [110].

Theorem 2.4 *If G is a k-connected graph with $k \geq 2$ and if S is a set of $k-1$ edges that form a path system, then there is a cycle in G that contains the edges of S.*

Very high connectivity is needed to guarantee the existence of a 2-factor in a graph, since the nearly regular complete bipartite graph $K_{\lfloor \frac{1}{2}(n-1)\rfloor, \lceil \frac{1}{2}(n-1)\rceil}$ has no 2-factor and is $\lfloor \frac{1}{2}(n-1)\rfloor$-connected. However, $2k$-connectedness in a sufficiently large graph does imply the existence of k disjoint cycles. This was established by Corradi and Hajnal [29], using only the additional assumption of minimum degree $2k$ (Theorem 1.4), a result that was extended by Egawa [33].

Theorem 2.5 *For each $k \geq 3$, every graph with minimum degree at least $2k$ and sufficiently many vertices, contains k disjoint cycles all of the same length.*

A natural question that arises from Theorem 2.5 is: *What connectivity is needed to guarantee the existence of k disjoint cycles that 'separate' any specified set of k vertices?* More precisely, given any set X of k vertices in a graph G, *what is the minimum connectivity that guarantees that there is a set of k disjoint cycles each with precisely one vertex of X?*

At the other extreme is the intersection of longest cycles in a graph. It is easy to see that any two longest cycles in a 2-connected graph share at least two vertices. For example, observe that if C and C' are vertex-disjoint cycles, then 2-connectivity implies the existence of vertex-disjoint paths P and P' between C and C^*. The union of these paths and cycles contains a longer cycle than either C or C'. The same argument also applies if the cycles share precisely one vertex. This is the 2-connected case of a conjecture attributed by Grötschel [47] to Scott Smith.

Conjecture A *For $k \geq 2$, any two longest cycles in a k-connected graph intersect in at least k vertices.*

This conjecture cannot be sharpened, since the longest cycles in the graph $K_k + \overline{K}_{n-k}$ are of length $2k$, and for $n \geq 3k$ some pairs of longest cycles intersect in just k vertices. The results in [47] imply that this conjecture holds for $k \leq 5$, and a comment is made as to the conjecture having been verified for $k \leq 10$. S. Burr and T. Zamfirescu (unpublished) showed that every pair of longest cycles in a k-connected graph meet in at least $\sqrt{k} - 1$ vertices. This bound was improved by Chen, Faudree and Gould [23].

Theorem 2.6 *For $k \geq 2$, every pair of longest cycles in a k-connected graph intersect in at least $(k/(4\sqrt[3]{4} + 3))^{3/5}$ vertices.*

If all of the longest cycles are considered, then it is possible for their intersections to be empty. This question was explored by Jendrol and Skupień [61], who proved the following result.

Theorem 2.7 *For $m \geq 7$, there is a 2-connected graph with m longest cycles whose intersection is empty, but in which every set of $m - 1$ longest cycles has non-empty intersection.*

The relationship between connectivity and cycles and paths in graphs is much richer than the classical results and their extensions discussed here. Some of the richness of this relationship is explored in the following sections, on special classes of graphs. The next section is devoted to the relationships between cycle lengths and path lengths.

3. Relative lengths of paths and cycles

It is reasonable to expect that if a 2-connected graph has long paths then it must have a long cycle. To explore this relationship, we let $l_k(p)$ be the smallest integer p for

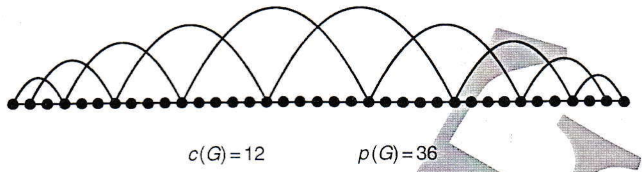

Fig. 1.

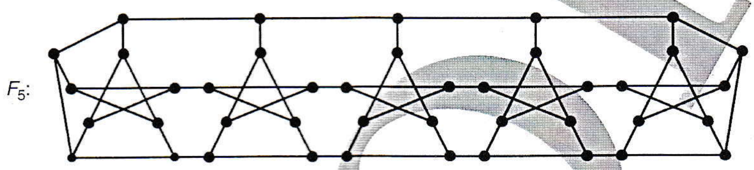

Fig. 2.

which every k-connected graph with a path of length p contains a cycle of length at least $l_k(p)$. In his classic paper [31], Dirac proved the following result, and provided examples verifying the sharpness (see Fig. 1).

Theorem 3.1 $l_2(p) \geq 2\sqrt{p}$, *and this bound is sharp.*

Bondy and Locke [18] proved corresponding results for 3-connected graphs, and they also considered the special class of cubic graphs. We let $l_3^*(p)$ be the smallest integer p for which every 3-connected cubic graph with a path of length p contains a cycle of length at least $l_3^*(p)$.

Theorem 3.2 *With the above notation,*

$$\frac{2}{5}p + 2 \leq l_3(p) \leq \frac{1}{2}p + O(p^{\alpha}),\ \textit{for}\ \alpha < 1,\ \textit{and}\ \frac{2}{3}p + 2 \leq l_3^*(p) \leq \frac{7}{8}p + 3.$$

Graphs for which the lower bounds in both results are sharp involve the concept of a 'vine'. This is a series of overlapping paths attached to a fixed path, and an example is shown in Fig. 1. The upper bound for $l_3^*(p)$ is achieved by the graph F_5 in Fig. 2, which is derived from the Petersen graph. The general graph F_m (the case $m = 5$ is Fig. 2) has a Hamiltonian path of length $8m + 1$, but has no cycle longer than $7m + 2$.

Bondy and Locke [18] gave a general upper bound for $l_k(p)$ and conjectured the nature of $l_k(p)$ as $k \to \infty$. The upper bound was later generalized by Locke [78], as follows.

Theorem 3.3 $l_k(p) \leq p(2k - 4)/(3k - 4)$.

Conjecture B *There exists a sequence of constants $c_3, c_4, \ldots$ converging to 1, for which $l_k(p) \geq c_k p$ for all k and p.*

In graphs with many edges, the lengths of the longest cycle and the longest path are essentially the same. We define a graph to be *cycle-tight* if the order of a longest path is at most 1 more than the order of a longest cycle. The following result of Liu, Lu and Tian [77] presents some of these graphs – here, $\sigma_4(G)$ is the minimum sum of the degrees of any four independent vertices.

Theorem 3.4 *Every 3-connected graph of order n with $\sigma_4(G) \geq \frac{1}{3}(4n+5)$ is cycle-tight, and this bound is sharp.*

Corollary 3.5 *Every graph G of order n with $\kappa(G) \geq \lceil \frac{1}{3}n \rceil$ is cycle-tight.*

4. Regular graphs

Regularity plays a strong role in forcing the circumference of a graph to be large. Jackson [58] gave a minimum-degree condition in a regular graph G of order n that guarantees Hamiltonicity (see part (a) of the following theorem) that is significantly less than the degree condition $\delta(G) \geq \frac{1}{2}n$ of Dirac [31]. The extensions (b) and (c) of this result are due to Zhu, Liu and Yu [114].

Theorem 4.1 *An r-regular 2-connected graph G of order n is Hamiltonian if any of the following conditions hold:*

(a) $n \leq 3r$;
(b) $n = 3r + 1$, *and G is not the Petersen graph;*
(b) $n = 3r + 2$ *or* $3r + 3$, *and* $r \geq 6$.

We turn now to results on the circumference $c(G)$ of a graph G. There are also stronger versions for regular graphs of Dirac's circumference result [31]. In the following theorem, the first part is due to Fan [39] and the second part is due to Aung [4].

Theorem 4.2 *Let G be an r-regular graph of order n.*

- *If G is 3-connected, then* $c(G) \geq \min\{3r, n\}$.
- *If G is 4-connected, then* $c(G) \geq \min\{4(r-1), \frac{1}{2}(n+3r-2)\}$.

For claw-free regular graphs, there is a still stronger result, as proved by Li [76].

Theorem 4.3 *If G is a 2-connected claw-free r-regular graph of order n, then* $c(G) \geq \min\{4r-2, n\}$.

For positive integers $r \geq 3$ and $s \leq r-3$, the graph $K_2 + (s-1)K_{r+1}$ has a spanning subgraph that is 2-connected and r-regular, and has order $n = (s-1)(r+1)+2$ and circumference $2r+4$ (which equals $2(n+s-3)/(s-1)$). This example was the basis for a conjecture of Bondy [15], and led to the following result of Wei [108].

Theorem 4.4 *If G is a 2-connected r-regular graph of sufficiently large order $n \le sr$ with $s \ge 3$, then $c(G) \ge 2(n+s-3)/(s-1)$.*

In general, for $r \ge 3$, the combination of being r-connected and r-regular is not enough to guarantee that a graph be Hamiltonian; Meredith [87] was the first to construct a family of such graphs. This family led Jackson and Parsons [60] to the following upper bound on the circumference.

Theorem 4.5 *For $r \ge 3$, there exists a number $\varepsilon(r)$ between 0 and 1 for which, if n is sufficiently large, there is an r-connected r-regular graph of order n and circumference less than $n^{\varepsilon(r)}$.*

In fact, there is a wide gap between the best-known upper bounds $n^{\varepsilon(r)}$ and the lower bounds (linear in r) for the circumference of such graphs. However, every r-connected r-regular graph has a 2–factor, as the following results show. In many cases these graphs are *2-factorable* – that is, their edge-sets can be partitioned into 2-factors. Petersen [93] investigated this class of graphs, and showed in particular that while not all 3-connected cubic graphs are Hamiltonian, all have a 2-factor.

Theorem 4.6 *Every cubic graph without a cut-edge is the edge-disjoint union of a 1-factor and a 2-factor.*

Rosenfeld [94] showed that the number of components in 2-factors of 3-connected cubic graphs is not bounded.

Theorem 4.7 *There are arbitrarily large 3-connected cubic graphs G in which every 2-factor has at least $\frac{1}{10}n$ components, where n is the order of G. For 3-connected cubic planar graphs, the number of components is at least $\frac{1}{28}n$.*

The results of Petersen [93] apply in general to k-connected k-regular graphs.

Theorem 4.8 *Let G be a k-connected k-regular graph with $k \ge 2$.*

- *If k is even, then G is 2-factorable.*
- *If k is odd, then G is the union of a 1-factor and $\frac{1}{2}(k-1)$ 2-factors.*

In a k-connected graph, any k vertices lie on a common cycle (by Theorem 1.1). It is natural to investigate whether adding a regularity condition increases the number of vertices that are always in a common cycle. With that in mind, we make the following definition. For integers k and r with $2 \le k \le r$, let $g(k,r)$ be the largest integer l for which every collection of l vertices in an r-regular k-connected graph lies on some cycle.

The case $r=k=2$ is trivial, since every such graph is a cycle. Holton and Plummer [57] gave examples showing that $g(2,r)=2$ for all r. The cases $r=3$ and 4 are shown in Fig. 3, and these give a clear pattern for the general case. Note that the Petersen graph shows that $g(3,3) \le 9$, and infinitely many examples can be obtained

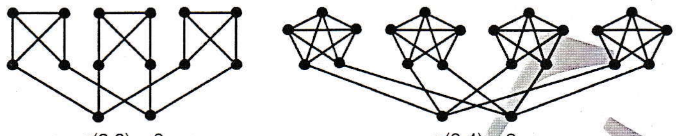

Fig. 3.

by inflating its vertices. Holton *et al.* [56] showed that equality holds here., Much is known about the cases when $r > k$, as a result of examples given by Holton [52].

Theorem 4.9

- $g(2, r) = 2$ *for all* $r \geq 2$.
- $g(3, 3) = 9$.
- $g(k, k+1) = k$ *if* k *is even.*
- $g(k, r) = k$ *if* $r \geq k+2$.

Ellingham, Holton and Little [35] proved an extension of the second part of Theorem 4.9, showing that each set of ten vertices in a 3-connected cubic graph lies on a cycle unless the graph can be contracted to the Petersen graph. They also showed that, for any set of five vertices and one edge in a 3-connected cubic graph, there is a cycle that contains the vertices and avoids the edge. Aldred [1] proved that in every 3-connected cubic graph each set of up to 13 vertices is contained in some path.

For $k \geq 4$, Theorem 4.9 leaves unsettled only the cases of $g(k, k)$ for all k, and $g(k, k+1)$ for k odd. In the latter case, Holton [52] proved that $g(k, k+1) \geq k+2$, but an upper bound was not given. Much attention has been given to the difficult case of $g(k, k)$. Holton [52] and Kelmans and Lomonosov [73] proved that $g(k, k) \geq k+4$ for $k \geq 3$, but this bound is not exact for $k = 3$, since $g(3, 3) = 9$. Meredith [87] proved that $g(k, k) \leq 10k - 11$. However, for k even, McCuaig and Rosenfeld [85] gave examples showing that $g(k, k) \leq 6k - 4$ for $k \equiv 0 \pmod 4$, and that $g(k, k) \leq 8k - 5$ for $k \equiv 2 \pmod 4$. The results for $r = k$ and $r = k+1$ when k is odd are summarized in the following theorem. There is a general cyclability survey by Bau and Holton [7].

Theorem 4.10 *For* $r \geq k \geq 3$,

- $k+4 \leq g(k, k) \leq 6k-4$ *if* $k \equiv 0 \pmod 4$;
- $k+4 \leq g(k, k) \leq 8k-5$ *if* $k \equiv 2 \pmod 4$;
- $k+4 \leq g(k, k) \leq 10k-11$ *if* $k \geq 5$ *is odd*;
- $k+2 \leq g(k, k+1)$ *if* k *is odd.*

Saito [96] addressed the issue of the lengths of cycles containing specified sets of vertices, which was not considered in determining $g(k, r)$.

Theorem 4.11 *Let G be a k-connected graph of order at least $2k$ and circumference c.*

- *For $m < k$, every set of m vertices is contained in a cycle of length at least $2m + c(k-m)/k$.*
- *If G is k-regular, then every set of k vertices is contained in a cycle of length at least $\frac{1}{3}c/k + \frac{2}{3}(k+2)$.*
- *If G is planar and 3-connected, then every set of three vertices is contained in a cycle of length at least $\frac{1}{4}c + 3$.*

5. Bipartite graphs

Frequently the same minimum degree and connectivity guarantee longer cycles in bipartite graphs than in graphs in general, and in some cases the guaranteed length is doubled. The following result of Bauer *et al.* [10] illustrates this. A cycle C in a graph is *dominating* if each edge of the graph has at least one of its endpoints on C.

Theorem 5.1 *If G is a 2-connected triangle-free graph of order n, circumference c and minimum degree δ, then either $c \geq \min\{n, 4\delta\}$ or every longest cycle is dominating.*

The following result on bipartite graphs was proved by Jackson [59].

Theorem 5.2 *Let G be a 2-connected $r \times s$ bipartite graph with $r \leq s$, let k and l be the minimum degrees in the partite sets of orders r and s, respectively, and let c be the circumference of G. Then,*

- $c \geq 2\min\{s, k+l-1, 2k-2\}$;
- $c \geq 2\min\{r, 2k-2\}$ *if* $l = k$;
- $c \geq 2\min\{r, 2k-1\}$ *if* $r = s$ *and* $l = k$.

Note that the last part of Theorem 5.2 implies that a 2-connected balanced bipartite graph with minimum degree k in each part is either Hamiltonian or has a cycle of length at least $4k-2$. This is an example of the cycle length in a bipartite graph being approximately twice the length in the class of all graphs.

More generally, Wang [105] considered an Ore-type condition (on the minimum sum of degrees of non-adjacent vertices) in bipartite graphs. This resulted in the following corollary.

Theorem 5.3 *If G is a 2-connected $r \times s$ bipartite graph with minimum degree δ, then its circumference is at least $2\min\{r, s, 2\delta - 1\}$.*

Further, Wang [106] considered disjoint cycles in balanced bipartite graphs and established a minimum degree condition for a balanced bipartite graph to have a 2-factor. He also showed that the minimum-degree condition is sharp.

Theorem 5.4 *Every* 2*-connected* $r \times r$ *bipartite graph with* $r \geq 2k+1$ *and minimum degree* $\delta > s \geq k \geq 2$ *has* k *disjoint cycles of total length at least* $\min\{2r, 4s\}$.

6. Claw-free graphs

In this section we consider the impact of connectivity on the circumference of claw-free graphs– that is, graphs without $K_{1,3}$ as an induced subgraph. Additional information on these graphs appears in a survey by Faudree, Flandrin and Ryjáček [41]. Much of the research on connectivity and claw-free graphs was motivated by the following conjecture by Matthews and Sumner [84].

Conjecture C *Every* 4*-connected claw-free graph is Hamiltonian.*

Related to this conjecture is a classical conjecture of Chvátal [26] on *toughness*, where the toughness $\tau(G)$ is the minimum ratio of the order of a cutset and the number of components left after the deletion of the cutset.

Conjecture D *There is a* t *such that every* t_0*-tough graph is Hamiltonian.*

In general, $\tau(G) \leq \frac{1}{2}k(G)$, since the deletion of a set of cutvertices leaves at least two components. However, in claw-free graphs the deletion of any minimal cut results in precisely two components, so $\tau(G) = \frac{1}{2}k(G)$ for claw-free graphs. Thus, for claw-free graphs, Conjecture **D** when $t_0 = 2$ is equivalent to Conjecture **C**. Results of Enomoto *et al.* [36] showed that $t_0 \geq 2$ would be needed to imply Hamiltonicity, and more recently Bauer, Broersma and Veldman [9] exhibited an example of non-Hamiltonian graphs with $\tau(G) = (9/4 - \epsilon)$ for any $\epsilon > 0$. Bauer, Broersma and Schmeichel [8] have a survey on toughness.

A special subclass of claw-free graphs is the class of line graphs, and Thomassen [102] conjectured the following:

Conjecture E *Every* 4*-connected line graph is Hamiltonian.*

Since line graphs are claw-free, Conjecture implies conjecture E. However, we now know that the two are equivalent. This is a consequence of a closure concept for claw-free graphs introduced by Ryjáček [95], an operation that is similar in form and application to the closure operation introduced by Bondy and Chvátal [17].

In a claw-free graph, the neighbourhood $N(v)$ of a vertex has independence number at most 2. Thus, the induced graph $\langle N(v) \rangle$ is either connected or the union of two complete components. If it is connected, then the 'local closure' at v is the graph obtained by replacing $N(v)$ by a complete graph with the same vertex set. Doing this recursively until every vertex has a neighbourhood that is either complete or the union of two complete graphs yields a graph $\mathrm{cl}(G)$ called the *Ryjáček closure*. Fig. 4 shows a graph G for which $\mathrm{cl}(G)$ is the complete graph; it is the result of taking two local closures. Clearly, the graph $\mathrm{cl}(G)$ is always claw-free and, as Ryjáček [95] showed, the operation is well defined. As before,

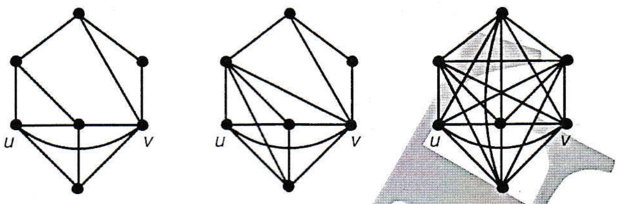

Fig. 4.

$c(G)$ denotes the circumference of the graph G and $p(G)$ is the order of a longest path (see [19]).

Theorem 6.1 *Let G be a claw-free graph. Then*

- *there is a triangle-free graph whose line graph is* $\mathrm{cl}(G)$;
- $c(G) = c(\mathrm{cl}(G))$ *and* $p(G) = p(\mathrm{cl}(G))$.

The Ryjáček closure is a useful tool in the study of cycles in claw-free graphs. For example, the determination of the circumference of a claw-free graph can be reduced to considering an appropriate line graph. Specifically, this implies that Conjectures C and E are equivalent.

Cycles in line graphs have been studied extensively, with one of the useful tools being a result of Harary and Nash-Williams [50], which gives a necessary and sufficient condition for the Hamiltonicity of the line graph $L(G)$ in terms of a dominating Eulerian subgraph of the graph G. An Eulerian subgraph H in a graph G is *dominating* if every edge of the graph has at least one of its endpoints in H.

Theorem 6.2 *The line graph $L(G)$ of a graph G with at least three edges is Hamiltonian if and only if G has a dominating Eulerian subgraph.*

Using this theorem, Zhan [112] proved the following result.

Theorem 6.3 *Every 7-connected line graph is Hamiltonian.*

An immediate consequence of Theorem 6.3 and the Ryjáček closure is that every 7-connected claw-free graph is Hamiltonian. Thus, the gap is between 4 and 7 for the connectivity sufficient to guarantee Hamiltonicity.

The latest result on connectivity and claw-free graphs is by Kaiser and Vrána [66], who show that 5-connectedness with a minimum-degree condition is sufficient for being Hamiltonian.

Theorem 6.4 *Every 5-connected claw-free graph with minimum degree at least 6 is Hamiltonian (and in fact Hamiltonian-connected).*

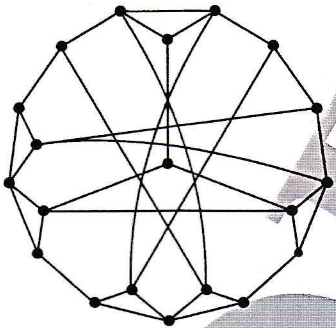

Fig. 5.

Generalizing Conjecture C, Jackson and Wormald [62] asked whether, for $r \geq 4$, every r-connected $K_{1,r}$-free graph is Hamiltonian. It is not even known whether some assumption of sufficiently large connectivity is enough.

If the Hamiltonian condition is relaxed to the existence of a 2-factor, then an assumption of being 4-connected and claw-free was shown by Choudum and Paulraj [25] to be sufficient.

Theorem 6.5 *Every 4-connected claw-free graph has a 2-factor.*

Yoshimoto [111] proved the following result on the number of components in a 2-factor in a claw-free graph.

Theorem 6.6 *Let G be a claw-free graph of order n and minimum degree 4. If each edge of G lies on a triangle, then G has a 2-factor with no more than $\frac{1}{4}(n-1)$ components, and this bound is sharp.*

We note that if Conjecture C is true then, for 4-connected graphs, there is a 2-factor with just one component. The question still remains as to the connectivity needed to ensure that any k vertices of a claw-free graph can be separated by k disjoint cycles. Connectivity of at least $2k$ is needed, since the k vertices could all be in the closed neighbourhood of one of the vertices.

There are infinite families of non-Hamiltonian 3-connected claw-free graphs. One example, due to Matthews and Sumner [84], is the graph in Fig. 5; it is the result of subdividing the edges of a perfect matching in the Petersen graph and then taking the line graph.

Jackson and Wormald [62] found a lower bound on the circumference of 3-connected $K_{1,r}$-free graphs in general, and claw-free graphs in particular.

Theorem 6.7 *Every 3-connected $K_{1,r}$-free graph of order n has a cycle of length at least n^{ε} where $\varepsilon = (\log_2 6 + 2\log_2(2r-1))^{-1}$.*

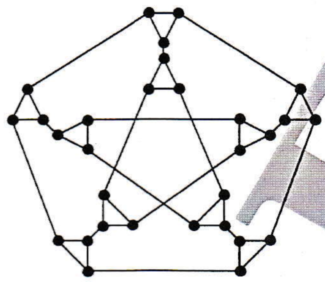

Fig. 6.

By taking the inflations of appropriate 3-regular graphs (replacing vertices by triangles), we deduce that this bound is of the correct order of magnitude. Whether there is a better value of the constant ε is not known.

Although not every 3-connected claw-free graph is Hamiltonian, Jackson and Yoshimoto [64] showed when the minimum degree is greater than 3, such a graph does have a 2-factor.

Theorem 6.8 *Every 3-connected claw-free graph G with $\delta(G) \geq 4$ has a 2-factor with at most $\frac{2}{15}n$ components.*

Inflations of certain cubic graphs, such as the Petersen graph, are examples of graphs with restrictions on the number of vertices that can be together on a cycle. This topic was investigated by Győri and Plummer [48], who proved the following theorem.

Theorem 6.9 *If G is a 3-connected claw-free graph, then every set of up to nine vertices lies on a cycle in G.*

This result is sharp, as the graph in Fig. 6 shows. This is an inflation of the Petersen graph, and since the Petersen graph is not Hamiltonian, there is no cycle that contains any set of ten vertices with one from each triangle.

The existence of disjoint cycles in 3-connected claw-free graphs, and in particular 2-factors, has been established. However, little is known about the existence of disjoint cycles that separate specified vertices. For example, in the above graph there are pairs of vertices that are not separated by two cycles. Thus, no set of two or more vertices can always be separated by disjoint cycles in 3-connected claw-free graphs, unless some conditions are placed on the vertices.

The circumference of 2-connected claw-free graphs was investigated by Broersma *et al.* [21]. In proving their results, they used the relationship between toughness and connectivity.

Theorem 6.10 *If G is a 2-connected claw-free graph of order n and circumference c, then there is a constant C for which*

$$c(G) \geq \frac{4\log n}{\log 2} - C.$$

Furthermore, there exists a 2-connected claw-free graph of order n and circumference less than $8\log(n+6) - 8\log 3 - 2$.

This theorem is a special case of their more general result on $K_{1,r}$-free graphs.

Theorem 6.11 *If G is a 2-connected $K_{1,r}$-free graph of order n and circumference c, then there is a constant C_r such that*

$$c(G) \geq \frac{4\log n}{\log(r-1)} - C_r.$$

They also gave a construction that shows that, for given $r \geq 4$ and sufficiently large n, there exists a 2-connected $K_{1,r}$-free graph of order n whose circumference is less than $4\log n/(\log(r-2)+4)$. Hence, the order of magnitude of the lower bound in Theorem 6.11 is correct.

Jackson and Yoshimoto [63] proved that every 2-connected claw-free graph with a sufficiently high minimum degree has a 2-factor, even though not all such graphs are Hamiltonian.

Theorem 6.12 *Every 2-connected claw-free graph G with $\delta(G) \geq 4$ has a 2-factor with at most $\frac{1}{4}(n+1)$ components.*

We conclude this section with some results relating cycles and connectivity in certain families of graphs.

A graph is *locally connected* if the neighbourhood of each vertex is connected. Local connectivity in claw-free graphs implies the existence of many cycles, a fact first observed by Oberly and Sumner [92].

Theorem 6.13 *Every connected locally connected claw-free graph with at least three vertices is Hamiltonian.*

A graph is *cycle-extendable* if each cycle can be extended to a cycle with one more vertex, and is *fully cycle-extendable* if each vertex is also on a triangle. The Oberly–Sumner result was extended by Hendry [51].

Theorem 6.14 *Every connected locally connected claw-free graph with at least three vertices is fully cycle-extendable.*

A graph G of order n is *panconnected* if, between each pair of vertices v and w of G and for each l satisfying $d(v,w) \leq l \leq n-1$, there is a v–w path of length l. Stronger local connectivity conditions were shown to imply panconnectedness by Kanetkar and Rao [69].

Theorem 6.15 *Every connected locally 2-connected claw-free graph is panconnected.*

Chartrand, Gould and Polimeni [22] gave a local connectivity condition that implies the existence of many long cycles.

Theorem 6.16 *If G is a connected locally k-connected claw-free graph, then the removal of any set of fewer than k vertices leaves a Hamiltonian graph.*

An immediate consequence of this theorem is that every connected locally $(3r-1)$-connected claw-free graph has a 2-factor with precisely r components. However, there are many questions left unanswered on what connectivity and local connectivity is required to guarantee the existence of cycles of prescribed lengths, 2-factors and disjoint cycles that separate specified vertices.

7. Planar graphs

Connectivity has an especially significant impact on the orders of cycles in planar graphs. One of the earliest results illustrating this was a theorem of Tutte [104].

Theorem 7.1 *Every 4-connected planar graph is Hamiltonian.*

This has led to considerable investigations into more general results. In our discussion, it is convenient to have some additional definitions. A graph G of order n is *pancyclic* if it has cycles of all lengths from 3 to n, and is *4-almost pancyclic* if it has cycles of all these lengths except 4. For $r \le n$, G is *r-ordered Hamiltonian* if, for any r vertices, there is a Hamiltonian cycle with those vertices in the given order.

Before moving on to the broader 3-connected case, we consider 4-connected graphs. Bondy [14] conjectured that every such planar graph is either pancyclic or nearly so.

Conjecture F *Every 4-connected planar graph contains cycles of every length, except possibly for one even length.*

A related conjecture was made by Malkevitch [83], who exhibited a 4-connected graph with cycles of all lengths, except 4 (see Fig. 7).

Conjecture G *Every 4-connected planar graph with a 4-cycle is pancyclic.*

In support of these conjectures, it has been shown that every 4-connected planar graph G of large order n has a cycle of each length from $n-7$ to $n-1$. For length $n-1$, this follows from a result of Tutte. A series of authors extended the result, and a summary can be found in [30] by Cui, Hu and Wang.

In connection with Malkevitch's conjecture, Trenkler [103] found those values of n for which there is a planar graph of order n that is 4-connected and

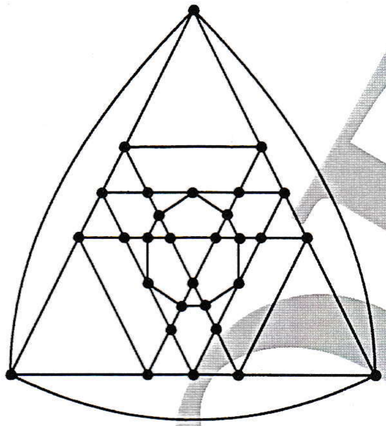

Fig. 7.

4-almost pancyclic. It follows from his result that there is such a graph of each order $n \geq 48$.

Theorem 7.2 *There exists a 4-connected 4-almost pancyclic planar graph of order n if and only if one of the following holds:*

- *for* $n \equiv 0$ *(mod 3)*, $n = 30$ *or* $n \geq 36$;
- *for* $n \equiv 1$ *(mod 3)*, $n \geq 46$;
- *for* $n \equiv 2$ *(mod 3)*, $n = 44$ *or* $n \geq 50$.

Sanders [98] showed that each pair of edges is on a Hamiltonian cycle, a consequence of the following theorem.

Theorem 7.3 *For any pair of vertices v and w and any edge $e \neq vw$ in a 4-connected planar graph, there is a Hamiltonian v–w path containing e.*

This is a generalization of the result of Thomassen [101] that in a 4-connected planar graph, each pair of vertices is joined by a Hamiltonian path. Goddard [44] proved that more can be said when the graph triangulates the plane.

Theorem 7.4 *Every 4-connected maximal planar graph is 4-ordered Hamiltonian.*

The 'double pyramid' graph $H_n = \overline{K}_2 + C_{n-2}$ is 4-connected and planar but not 5-ordered, showing that Theorem 7.4 cannot be extended. Likewise, since H_n has some path systems of three edges that do not lie on a Hamiltonian cycle, Sanders' result cannot be extended. We also note that every 4-connected planar graph of order at least 6 has a pair of disjoint cycles, since it has a cycle disjoint from any given triangle. The graph H_n also shows that there are 4-connected planar graphs of arbitrarily large order that do not have three disjoint cycles.

We now turn to 3-connected planar graphs. Although not all of them are Hamiltonian, Holton and McKay [55] showed that every 3-connected cubic graph of order 36 or less is Hamiltonian, and found those of order 38 that are not. In fact, there is no positive constant C for which every 3-connected planar graph G of order n has circumference $c(G) \geq Cn$.

Moon and Moser [89] showed that if we start with a 3-connected planar graph (such as K_4) and successively insert a vertex of degree 3 inside each face, we get a 3-connected planar graph whose circumference is at most $n^{\log_3 2}$. They conjectured this order of magnitude for the circumference of 3-connected planar graphs, and this was proved by Chen and Yu [24], not only for the plane, but also for the other three surfaces of non-negative Euler characteristic.

Theorem 7.5 *If G is a 3-connected graph of order n that is embeddable in the sphere, the projective plane, the torus or the Klein bottle, then $c(G) = \Omega(n^{\log_3 2})$.*

More is known about cycles of 3-connected cubic planar graphs. Aldred *et al.* [2] verified the existence of cycles containing arbitrary sets with up to 23 vertices.

Theorem 7.6 *In every 3-connected cubic planar graph, each set of up to 23 vertices is contained in some cycle.*

The bound in Theorem 7.6 is sharp, since Holton [53] exhibited a 3-connected cubic planar graph with a set of 24 vertices that is not contained in any cycle. With a bipartite restriction added to the class of 3-connected cubic planar graphs, Barnette [6] made the following conjecture, which is still open.

Conjecture H *Every 3-connected cubic bipartite planar graph is Hamiltonian.*

This conjecture was shown to be true by Holton, Manvel and McKay [54] for graphs with up to 64 vertices, and has also been verified for some infinite classes of graphs. For example, Goodey ([45]) showed that every 3-connected bipartite graph which can be embedded in the plane with every face either a quadrilateral or a hexagon is Hamiltonian.

The complete bipartite graph $K_{2,n}$ is planar and 2-connected, and all cycles have length 4, so 2-connectedness does not imply the existence of long cycles in planar graphs. If other parameters are considered, such as the toughness, then more can be said about the circumference. Note that the toughness of $K_{2,n}$ is $2/n$ and this approaches 0 as $n \to \infty$. If a bound is placed on the toughness, then more can be said about the length of longest cycles. The next result of Böhme, Broersma and Veldman [11] gives a lower bound for the circumference of 2-connected planar graphs in terms of the toughness.

Theorem 7.7 *If G is a 2-connected planar graph of toughness m, then there is a constant d (depending on m) for which $c(G) \geq d \log n$.*

8. The Chvátal–Erdős condition

A classical paper of Chvátal and Erdős [27] on Hamiltonian properties explored implications of the relationship between the connectivity $\kappa(G)$ and the independence number $\alpha(G)$ of a graph G. Because of their results, we say that G satisfies the *Chvátal–Erdős condition* if $\kappa(G) \geq \alpha(G)$.

Theorem 8.1 *Let G be a graph of order $n \geq 3$.*

- *If $\kappa(G) \geq \alpha(G) - 1$, then G has a Hamiltonian path.*
- *If $\kappa(G) \geq \alpha(G)$, then G has a Hamiltonian cycle.*
- *If $\kappa(G) \geq \alpha(G) + 1$, then G is Hamiltonian-connected.*

This theorem has given rise to many generalizations. For example, using classical results from Ramsey theory, Flandrin *et al.* [43] proved the following variation of the second part.

Theorem 8.2 *Every graph of sufficiently large order (depending on the independence number) that satisfies the Chvátal–Erdős condition is pancyclic.*

Häggkvist and Thomassen [49] showed that an appropriate Chvátal–Erdős condition implies the existence of a Hamiltonian cycle containing predetermined disjoint paths.

Theorem 8.3 *If G satisfies the Chvátal–Erdős condition, then each set of disjoint paths of length at most $\kappa(G) - \alpha(G)$ is contained in a Hamiltonian cycle.*

Wei and Zhu [109] showed that triangle-free graphs satisfying the Chvatál–Erdős condition also satisfy a strong version of panconnectedness.

Theorem 8.4 *Let G be a triangle-free graph of order n other than C_5 or $K_{n/2,n/2}$ that satisfies the Chvatál–Erdős condition. Then,*

- *if $4 \leq i \leq n$, each edge is in a cycle of length i;*
- *if $4 \leq i \leq n-1$, each pair of vertices is connected by a path of length i.*

Many of the degree conditions that guarantee that a graph is Hamiltonian also guarantee the existence of a 2-factor with a specified number of cycles. It seems natural to conjecture that the Chvátal–Erdős condition also guarantees this. The case of two cycles was proved by Kaneko and Yoshimoto [68].

Theorem 8.5 *Every 4-connected graph G other than K_5 that satisfies the Chvatál–Erdős condition has a 2-factor with two cycles.*

9. Ordered graphs

In this section, we consider two other concepts and examine how they are related to each other and to connectivity.

A graph G is *k-ordered* if every ordered set of k vertices lies on a cycle in the designated order. This concept, introduced by Ng and Schultz [91], is considerably stronger than cyclability. (In Section 7 we considered the particular case in which the cycle was specified to be Hamiltonian.) Recall also that a graph G is *k-linked* if, given any collection of k pairs of vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$, there are k internally disjoint paths P_i for which P_i is an x_i–y_i path. The following theorem gives some elementary relationships between these two concepts.

Theorem 9.1 *Let $k \geq 1$. Then,*

(a) *every k-linked graph is k-ordered;*
(b) *every $2k$-ordered graph is k-linked.*

Proof (a) Let G be a k-linked graph, and let $X = (x_1, x_2, \ldots, x_k)$ be an ordered set of k of its vertices. Let $Y = (y_1, y_2, \ldots, y_k)$ with $y_i = x_{i+1}$ for $i \leq k-1$ and $y_k = x_1$. Then from the definition of k-linked, there exist internally disjoint x_i–y_i paths, and their union is a cycle containing the vertices of X in the given order.

(b) This follows from the fact that a cycle containing the vertices of the ordered set $(x_1, y_1, x_2, y_2, \ldots, x_k, y_k)$ of $2k$ vertices contains the required k paths for a k-linkage. ■

It has been known for some time that sufficiently high connectivity implies k-linkage, due to results of Jung [65] and Larman and Mani [75]. The much sharper bound that every $22k$-connected graph is k-linked was proved by Bollobás and Thomason [13], and this was improved substantially by Thomas and Wollan [100].

Theorem 9.2 *Every $10k$-connected graph is k-linked.*

It is likely that the connectivity needed to imply k-linkage is significantly less than $10k$. On the other hand, it needs to be at least $3k-1$, as is shown by the graph obtained from K_{3k-1} by deleting k independent edges, which has connectivity $3k-2$ but is not k-linked. In [100] the stronger result was proved that every $2k$-connected graph of order n and at least $10kn$ edges is k-linked. The following was also conjectured, and it was noted that the conjecture is true for $k \leq 3$.

Conjecture I *Every $2k$-connected graph of order n with at least $(2k-1)n - \frac{1}{2}(3k+1)k + 1$ edges is k-linked.*

Corresponding questions can also be posed for k-ordered graphs. The graph $H_2 = K_{2k-1} - C_k$ has connectivity $2k-4$, but is not k-ordered, because if the k vertices on the 'missing' cycle are chosen in the natural order, then there is no cycle

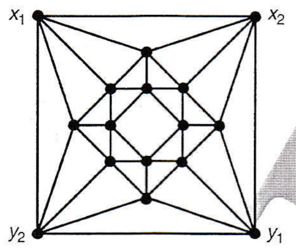

Fig. 8.

meeting our requirement. Thus, for $k \geq 5$, at least $2k - 3$-connectedness is required to imply k-ordered.

Jung [65] found a bound on the connectivity that guarantees that a graph is 2-linked.

Theorem 9.3 *Every* 6*-connected graph is* 2*-linked.*

This result is sharp. The graph H_3 in Fig. 8 is a 5-regular 5-connected planar graph that is not 2-linked, since there is no linkage for the pairs (x_1, y_1) and (x_2, y_2). Also, it is not 4-ordered, since there is no cycle containing the ordered set (x_1, y_1, x_3, y_2).

The least connectivity that implies 4-orderedness is not known, but the following is an interesting question: *Is every* 6*-connected graph* 4*-ordered?*

No graph G with connectivity $2k - 2$ can be k-linked, since if the collection of k-pairs contains a minimum cutset S and the two vertices of some pair are in different components of $G - S$, then there cannot exist the path system required for k-linkage. Thus, every k-linked is $(2k - 1)$-connected, but may not be $2k$-connected, as K_{2k} shows. Likewise, every k-ordered graph is $(k - 1)$-connected, and Mészáros [88] exhibited, for k odd, an infinite family of $(k - 1)$-regular graphs that are k-ordered Hamiltonian. Recall that the graph H_2 is $(k - 1)$-linked, but not k-ordered. Also, for $n \geq 2k$ and k even, the complete bipartite graph $K_{k,n-k}$ is k-ordered but is not $(\frac{1}{2}k + 1)$-linked. We summarize the known results about the relationship between linkage, ordered and connectivity in the following theorem.

Theorem 9.4 *Let $k \geq 3$. Then,*

- *being k-linked implies being $(2k - 1)$-connected, but does not imply being $2k$-connected;*
- *being k-ordered implies being $(k - 1)$-connected, but does not imply being k-connected;*
- *being k-linked implies being k-ordered, but does not imply being $(k + 1)$-ordered;*
- *being $2k$-ordered implies being k-linked, but does not imply being $(k + 1)$-linked;*
- *being $10k$-connected implies being k-linked, but being $(3k - 3)$-connected does not;*

- *being* $10k$*-connected implies being* k*-ordered, but being* $(2k - 4)$*-connected does not.*

It would be of interest to know the sharpest relationships between the pairs of these parameters. With girth conditions placed on the graph, some sharp results between connectivity and linkage have been established. Improving a result of Mader [82], Kawarabayshi [72] showed the following:

Theorem 9.5 *Let* $k \geq 1$*. Then,*

- *for* $k = 4$ *or* 5*, every* $2k$*-connected graph of girth at least* 19 *is* k*-linked;*
- *for* $k \neq 4$ *or* 5*, every* $2k$*-connected graph of girth at least* 11 *is* k*-linked.*

More is known about the relationships between connectivity and linkage in the case of chordal graphs (that is, those graphs with no induced cycle of length greater than 3). In particular, Böhme, Gerlach and Stiebitz [12] proved the following result.

Theorem 9.6 *A chordal graph of order at least* $2k$ *is* k*-linked if and only if it is* $(2k - 1)$*-connected.*

Connectivity conditions for $K_{1,r}$-free graphs – in particular, claw-free graphs – were established by Faudree *et al.* [42].

Theorem 9.7 *For* $r \geq 3$*, every* $((2r - 2)(k - 1) + 1)$*-connected* $K_{1,r}$*-free graph is* k*-linked.*

Corollary 9.8 *Every* $(4k - 3)$*-connected claw-free graph is* k*-linked.*

10. Numbers of cycles

To conclude this chapter we look at how the number of different cycle lengths and the total number of cycles vary with the connectivity.

Theorem 10.1 *For* $k \geq 1$*, the minimum number of different lengths of cycles in any* k*-connected graph is* $k - 1$*.*

Proof We first observe that the cycles in every k-connected graph have at least $k - 1$ different lengths. This follows from the fact that if P is a longest path in a k-connected graph G, then the chords of P from the first vertex form cycles of $k - 1$ different lengths. That this bound is sharp follows from the fact that the complete bipartite graph $K_{k,k}$ is k-connected and has only cycles of the $k - 1$ lengths $4, 6, \ldots, 2k$. ∎

If additional requirements are made, then there may be cycles of other lengths. An example of this is the following result of Erdős *et al.* [37].

Theorem 10.2 *Let $\lambda_g(k)$ be the minimum number of different lengths of cycles in any graph G of girth g and minimum degree $\delta(G) \geq k$ (or $\kappa(G) \geq k$). Then,*

- $\lambda_5(k) \geq \frac{1}{4}(k^2 - k - 2)$;
- *there exists a positive constant C_7 for which $\lambda_7(k) \geq C_7 k^{5/2}$;*
- *there exists a positive constant C_9 for which $\lambda_9(k) \geq C_9 k^3$;*
- *for $t \geq 3$, there exists a positive constant C_{4t-1} for which $\lambda_{4t-1}(k) \geq C_{4t-1} k^{t/2}$.*

In fact, using only the average degree, Sudakov and Verstraëte [99] strengthened this result with information about the distribution of the cycle lengths.

Theorem 10.3 *There is a constant C for which every k-connected graph of girth g has a set of at least $Ck^{(g-1)/2}$ consecutive even cycle lengths.*

We now turn to the total number of cycles, rather than the number of lengths of cycles. Let $\psi_n(k)$ be the minimum number of different cycles in any k-connected graph of order n. By considering trees, cycles, wheels and regular complete bipartite graphs, we see that $\psi_n(1) = 0$, $\psi_n(2) = 1$, and $\psi_n(3) \leq c_3 n^2$. Given a function $f(n)$ and a family of graphs $\mathcal{F}$, we say that $\psi_n(k)$ is *$f(n)$-bound* on $\mathcal{F}$ if there exist constants A and B for which $\psi_n(k)$ lies between $Af(n)$ and $Bf(n)$, for all graphs of order n in $\mathcal{F}$. Knor [74] proved the following results.

Theorem 10.4 *Let $\psi_n(k)$ be the minimum number of cycles in any k-connected graph of order n. Then,*

- $\psi_n(2) = 1$, *and is n^2-bounded on the family of graphs of minimum degree at least 3;*
- $\psi_n(3)$ *is n^2-bounded, and is n^3-bounded on the family of graphs of minimum degree at least 5.*

Knor also conjectured that $\psi_n(3)$ is n^k-bounded on the family of graphs with connectivity k, minimum degree δ and maximum degree δ for which $k < \delta < \Delta$.

Clark and Entringer [28] determined the minimum number of cycles in graphs with small connectivity, in terms of the cycle rank of the graph.

Theorem 10.5 *Let $\rho_k(r)$ be the minimum number of cycles in any k-connected graph with cycle rank r. Then,*

- *for $r \geq 0$, $\rho_1(r) = r$;*
- *for $r \geq 1$, $\rho_2(r) = \frac{1}{2}r(r+1)$;*
- *for $r \geq 3$, $\rho_3(r) = r^2 - r + 1$.*

For cubic graphs, more is known. For example, Barefoot, Clark and Entringer [5] discovered the following results for the minimum number of cycles in cubic graphs of low connectivity.

Theorem 10.6 *Let $\nu_k(n)$ be the minimum number of cycles in any k-connected cubic graph of order n. Then,*

- *for $n \geq 14$, $\nu_1(n) = 3\lfloor \frac{1}{4}n \rfloor + 8$;*
- *for $n \geq 8$, $\nu_2(n) = \lceil \frac{1}{8}n(n+14) \rceil$.*

They also conjectured that, for greater connectivity, the number $\nu_k(n)$ is superpolynomial in n, which follows from the next result of Aldred and Thomassen [3].

Theorem 10.7 *The minimum number $\nu_3(n)$ of cycles in a 3-connected cubic graph of order n satisfies*

$$2^{n^{0.17}} < \nu_3(n) < 2^{n^{0.95}}.$$

This implies that the minimum number of different cycle lengths in 3-connected cubic graphs is not bounded, as it is for 3-connected graphs in general, but that it is a function of the order of the graph.

References

1. R. E. L. Aldred, Paths through *m* vertices in 3-connected cubic graphs, *Ars Combin.* **17** (1984), 85–92.
2. R. E. L. Aldred, S. Bau, D. A. Holton and B. McKay, Cycles through 23 vertices in 3-connected cubic planar graphs, *Graphs Combin.* **15** (1999), 373–376.
3. R. E. L. Aldred and C. Thomassen, On the number of cycles in 3-connected cubic graphs, *J. Combin. Theory (B)* **71** (1997), 79–84.
4. M. Aung, Circumference of a regular gğraph, *J. Graph Theory* **13** (1989), 149–155.
5. C. A. Barefoot, L. Clark and R. Entringer, Cubic graphs with the minimum number of cycles, *Congress. Numer.* **53** (1986), 49–62.
6. D. W. Barnette, Conjecture 5, *Recent Progress in Combinatorics* (ed. W. T. Tutte), Academic Press (1969), 343.
7. S. Bau and D. H. Holton, Cycles in regular graphs, *Proc. Twelfth British Combinatorial Conference, Ars Combin.* **29** (1990), 175–183.
8. D. Bauer, H. Broersma and E. Schmeichel, Toughness in graphs – a survey, *Graphs Combin.* **22** (2006), 1–35.
9. D. Bauer, H. Broersma and H. J. Veldman, Not every 2-tough graph is hamiltonian, *Discrete Appl. Math.* **99** (2000), 317–321.
10. D. Bauer, N. Kahl, L. McGuire and E. Schmeichel, Long cycles in 2-connected triangle-free graphs, *Ars Combin.* **86** (2008), 295–304.
11. T. Böhme, H. J. Broersma and H. Veldman, Toughness and longest cycles in 2-connected graphs, *J. Graph Theory* **239** (1996), 257–263.
12. T. Böhme, T. Gerlach and M. Stiebitz, Ordered and linked chordal graphs, *Discuss. Math. Graph Theory* **28** (2008), 367–373.
13. B. Bollobás and A. Thomason, Highly linked graphs, *Combinatorica* **16** (1996), 313–320.
14. J. A. Bondy, Pancyclic graphs: recent results, *Infinite and Finite Sets, Colloq. Math. Soc. János Bolyai* **10** (1975), 181–187.
15. J. A. Bondy, Hamiltonian cycles in graphs and digraphs, *Congress. Numer.* **21** (1978), 3–28.

16. J. A. Bondy, Basic graph theory: paths and cycles, *Handbook of Combinatorics* (ed. R. Graham, M. Grötschel and L. Lovász), MIT Press and North Holland (1995), 3–110.
17. J. A. Bondy and V. Chvátal, A method in graph theory, *Discrete Math.* **15** (1976), 111–135.
18. J. A. Bondy and S. C. Locke, Relative lengths of paths and cycles in 3-connected graphs, *Discrete Math.* **33** (1981), 111–122.
19. S. Brandt, O. Favaron and Z. Ryjáček, Closure and stable properties in claw-free graphs, *J. Graph Theory* **34** (2000), 30–41.
20. H. J. Broersma, Z. Ryjáček and I. Schiermeyer, Closure concepts: a survey, *Graphs Combin.* **16** (2000), 17–48.
21. H. J. Broersma, J. Van den Heuvel, H. A. Jung and H. Veldman, Long paths and cycles in tough graphs, *Graphs Combin.* **9** (1993), 3–17.
22. G. Chartrand, R. J. Gould and A. D. Polimeni, A note on locally connected and hamiltonian-connected graphs, *Israel J. Math.* **33** (1979), 5–8.
23. G. Chen, R. Faudree and R. Gould, Intersections of longest cycles in k-connected graphs, *J. Combin. Theory (B)* **72** (1998), 143–149.
24. G. Chen and X. Yu, Long cycles in 3-connected graphs, *J. Combin. Theory (B)* **86** (2002), 80–99.
25. S. A. Choudum and M. S. Paulraj, Regular factors in $K_{1,3}$-free graphs, *J. Graph Theory* **15** (1991), 259–265.
26. V. Chvátal, Tough graphs and hamiltonian circuits, *Discrete Math.* **5** (1973), 215–228.
27. V. Chvátal and P. Erdős, A note on Hamiltonian circuits, *Discrete Math.* **2** (1972), 111–113.
28. L. H. Clark and R. C. Entringer, The minimum number of cycles in graphs with given cycle rank and small connectivity, *J. Combin. Math. Combin. Comput.* **3** (1988), 169–181.
29. K. Corradi and A. Hajnal, On the maximal number of independent circuits of a graph, *Acta Math. Acad. Sci. Hungar.* **14** (1963), 423–443.
30. Q. Cui, Y. Hu and J. Wang, Long cycles in 4-connected planar graphs, *Discrete Math.* **309** (2009), 1051–1059.
31. G. A. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.* **27** (1952), 69–81.
32. G. A. Dirac, In abstrakten Graphen vorhandene vollständige 4-Graphen und ihre Unterteilungen, *Math. Nachr.* **22** (1960), 61–85.
33. Y. Egawa, Vertex disjoint cycles of the same length, *J. Combin. Theory (B)* **66** (1996), 168–200.
34. Y. Egawa, R. Glas and S. C. Locke, Cycles and paths through specified vertices in k-connected graphs, *J. Combin. Theory (B)* **52** (1991), 20–91.
35. M. N. Ellingham, D. A. Holton and C. H. C. Little, Cycles through ten vertices in 3-connected cubic graphs, *Combinatorica* **4** (1984), 265–273.
36. H. Enomoto, B. Jackson, P. Katerinis and A. Saito, Toughness and the existence of k-factors, *J. Graph Theory* **9** (1985), 87–95.
37. P. Erdős, R. J. Faudree, C. C. Rousseau and R. H. Schelp, The number of cycle lengths in graphs of given minimum degree and girth, *Discrete Math.* **200** (1999), 55–60.
38. P. Erdős and E. Győri, Any four independent edges of a 4-connected graph are contained in a circuit, *Acta Math. Hungar.* **46** (1985), 311–313.
39. G. Fan, Longest cycles in regular graphs, *J. Combin. Theory (B)* **39** (1985), 325–345.

40. R. J. Faudree, Connectivity and cycles in graphs, *Congr. Numer.* **187** (2007), 97–131.
41. R. J. Faudree, E. Flandrin and Z. Ryjáček, Claw-free graphs – A survey, *Discrete Math.* **164** (1997), 87–147.
42. R. J. Faudree, R. J. Gould, T. Lindquester and R. H. Schelp, On k-linked graphs, *Combinatorics, Graph Theory and Algorithms*, Vols. I, II, New Issues Press (1999), 387–400.
43. E. Flandrin, H. Li, A. Marczyk, I. Schiermeyer and M. Woźniak, Chvátal–Erdős condition and pancyclism, *Discuss. Math. Graph Theory* **26** (2006), 335–342.
44. W. Goddard, 4-connected maximal planar graphs are 4-ordered, *Discrete Math.* **257** (2002), 405–410.
45. P. R. Goodey, Hamiltonian circuits in polytopes with even sides, *Israel J. Math.* **22** (1975), 52–56.
46. R. Gould, Advances on the Hamiltonian problem – A survey, *Graphs Combin.* **19** (2003), 7–52.
47. M. Grötschel, On intersections of longest cycles, *Graph Theory and Combinatorics* (ed. B. Bollobás), Academic Press (1984), 171–189.
48. E. Győri and M. Plummer, A nine vertex theorem for 3-connected claw-free graphs, *Studia Sci. Math. Hungar.* **38** (2001), 233–244.
49. R. Häggkvist and C. Thomassen, Circuits through specified edges, *Discrete Math.* **41** (1982), 29–34.
50. F. Harary and C. St.J. A. Nash-Williams, On Eulerian and Hamiltonian graphs and line graphs, *Canad. Math. Bull.* **8** (1965), 701–709.
51. G. R. T. Hendry, Extending cycles in graphs. *Discrete Math.* **85** (1990), 59–72.
52. D. A. Holton, Cycles through specified vertices in k-connected regular graphs, *Ars Combin.* **13** (1982), 129–143.
53. D. A. Holton, Cycles in 3-connected cubic planar graphs, *Cycles in Graphs, Math. Stud.* **115** (1985), 219–226.
54. D. A. Holton, B. Manvel and B. D. McKay, Hamiltonian cycles in cubic 3-connected bipartite planar graphs, *J. Combin. Theory (B)* **38** (1985), 279–297.
55. D. A. Holton and B. D. McKay, The smallest non-Hamiltonian 3-connected cubic planar graphs have 38 vertices, *J. Combin. Theory (B)* **45** (1988), 305–319.
56. D. A. Holton, B. D. McKay, M. D. Plummer and C. Thomassen, A nine point theorem for 3-connected graphs, *Combinatorica* **2** (1982), 53–62.
57. D. A. Holton and M. D. Plummer, On the cyclability of k-connected $(k + 1)$-regular graphs, *Ars Combin.* **23** (1987), 37–56.
58. B. Jackson, Hamiltonian cycles in regular 2-connected graphs, *J. Combin. Theory (B)* **29** (1980), 27–46.
59. B. Jackson, Long cycles in bipartite graphs, *J. Combin. Theory (B)* **38** (1985), 118–131.
60. B. Jackson and T. Parsons, A shortness exponent for r-regular r-connected graphs, *J. Graph Theory* **6** (1982), 169–176.
61. S. Jendrol and Z. Skupień, Exact numbers of longest cycles with empty intersection, *Europ. J. Combin.* **18** (1997), 575–578.
62. B. Jackson and N. Wormald, Long cycles and 3-connected spanning subgraphs of bounded degree in 3-connected $K_{1,d}$-free graphs, *J. Combin. Theory (B)* **63** (1995), 163–169.
63. B. Jackson and K. Yoshimoto, Even subgraphs of bridgeless graphs and 2-factors of line graphs, *Discrete Math.* **307** (2007), 2775–2785.
64. B. Jackson and K. Yoshimoto, Spanning even subgraphs of 3-edge-connected graphs, *J. Graph Theory* **62** (2009), 37–47.

65. H. A. Jung, Eine Verallgemeinerung des n-fachen Zusammenhangs für Graphen, *Math. Ann.* **187** (1970), 95–103.
66. T. Kaiser and P. Vrána, Hamiltonian cycles in 5-connected line graphs, preprint.
67. A. Kaneko and A. Saito, Cycles intersecting a prescribed vertex set, *J. Graph Theory* **15** (1991), 655–664.
68. A. Kaneko and K. Yoshimoto, A 2-factor with two components of a graph satisfying the Chvátal–Erdős condition, *J. Graph Theory* **43** (2003), 269–279.
69. S. V. Kanetkar and P. R. Rao, Connected, locally 2-connected $K_{1,3}$-free graphs are panconnected, *J. Graph Theory* **8** (1984), 347–353.
70. K. Kawarabayashi, One or two disjoint circuits cover independent edges, Lovász-Woodall conjecture. *J. Combin. Theory (B)* **84** (2002), 1–44.
71. K. Kawarabayashi, Cycles through a prescribed vertex set in n-connected graphs, *J. Combin. Theory (B)* **90** (2004), 315–323.
72. K. Kawarabayashi, k-linked graphs with girth condition, *J. Graph Theory* **45** (2004), 48–50.
73. A. K. Kelmans and M. V. Lomonosov, When m vertices in a k-connected graph cannot be walked round along a simple cycle, *Discrete Math.* **38** (1982), 317–322.
74. M. Knor, On the number of cycles in k-connected graphs, *Acta Math. Univ. Comenianae* **63** (1994), 315–321.
75. D. G. Larman and P. Mani, On the existence of certain configurations within graphs and the 1-skeletons of polytopes, *Proc. London Math. Soc.* **20** (1970), 144–160.
76. M. C. Li, Longest cycles in regular 2-connected claw-free graphs, *Discrete Math.* **137** (1995), 277–295.
77. H. Liu, M. Lu and F. Tian, Relative length of longest paths and cycles in graphs, *Graphs Combin.* **23** (2007), 433–443.
78. S. C. Locke, Relative lengths of paths and cycles in k-connected graphs, *J. Combin. Theory (B)* **32** (1982), 206–222.
79. L. Lovász, Problem 5, *Period. Math. Hungar.* **4** (1974), 82.
80. L. Lovász, Problem 6.67, *Combinatorial Problems and Exercises*, North-Holland (1979), 46.
81. W. Mader, Homomorphieeigenschaften und mittlere Kantendichte von Graphen, *Math. Annalen* **174** (1967), 265–268.
82. W. Mader, Topological subgraphs in graphs of large girth, *Combinatorica* **18** (1998), 405–412.
83. J. Malkevitch, On the lengths of cycles in planar graphs, *Proceedings of the Conference on Graph Theory and Combinatorics at St. John's University, Jamaica, NY, Lecture Notes in Mathematics* **186**, Springer (1970), 191–195.
84. M. Matthews and D. Sumner, Hamiltonian results in $K_{1,3}$-free graphs, *J. Graph Theory* **9** (1984), 139–146.
85. W. D. McCuaig and M. Rosenfeld, Cyclability of r-regular r-connected graphs, *Bull. Austral. Math. Soc.* **29** (1984), 1–11.
86. K. Menger, Zur allgemeinen Kurventheorie, *Fund. Math.* **10** (1927), 95–115.
87. G. H. J. Meredith, Regular n-valent n-connected non-Hamiltonian non-n-edge-colorable graphs, *J. Combin. Theory (B)* **14** (1973), 55–60.
88. K. Mészáros, On low degree k-ordered graphs, *Discrete Math.* **308** (2008), 2418–2426.
89. J. W. Moon and L. Moser, Simple paths on polyhedra, *Pacific J. Math.* **13** (1963), 629–631.
90. J. W. Moon and L. Moser, On hamiltonian bipartite graphs, *Israel J. Math.* **1** (1963), 163–165.

91. L. Ng and M. Schultz, k-ordered hamiltonian graphs, *J. Graph Theory* **24** (1997), 45–57.
92. D. J. Oberly and D. P. Sumner, Every connected, locally connected nontrivial graph with no induced claw is hamiltonian, *J. Graph Theory* **3** (1979), 351–356.
93. J. Petersen, Die Theorie der regulären Graphs, *Acta Math.* **15** (1891), 193–220.
94. M. Rosenfeld, The number of cycles in 2-factors of cubic graphs, *Discrete Math.* **84** (1990), 285–294.
95. Z. Ryjáček, On a closure concept in claw-free graphs, *J. Combin. Theory (B)* **70** (1997), 217–224.
96. A. Saito, Long cycles through specified vertices in a graph, *J. Combin. Theory (B)* **47** (1989), 220–230.
97. D. P. Sanders, On circuits through five edges, *Discrete Math.* **159** (1996), 199–215.
98. D. P. Sanders, On paths in planar graphs, *J. Graph Theory* **24** (1997), 341–345.
99. B. Sudakov and J. Verstraëte, Cycle lengths in sparse graphs, *Combinatorica* **28** (2008), 357–372.
100. R. Thomas and P. Wollan, An improved linear edge bound for graph linkages, *Europ. J. Combin.* **26** (2005), 309–324.
101. C. Thomassen, A theorem on paths in planar graphs, *J. Graph Theory* **7** (1983), 169–176.
102. C. Thomassen, Reflections on graph theory, *J. Graph Theory* **10** (1986), 309–324.
103. M. Trenkler, On 4-connected, planar 4-almost pancyclic graphs, *Math. Slovaca* **39** (1989), 13–20.
104. W. T. Tutte, A theorem on planar graphs, *Trans. Amer. Math. Soc.* **82** (1956), 99–116.
105. H. Wang, On long cycles in a bipartite graph, *Graphs Combin.* **12** (1996), 373–384.
106. H. Wang, Maximal total length of k disjoint cycles in bipartite graphs, *Combinatorica* **25** (2005), 367–377.
107. M. E. Watkins and D. M. Mesner, Cycles and connectivity in graphs, *Canad. J. Math.* **19** (1967), 1319–1328.
108. B. Wei, On the circumference of regular 2-connected graphs, *J. Combin. Theory (B)* **75** (1999), 88–99.
109. B. Wei and Y. Zhu, The Chvátal–Erdős condition for panconnectivity of triangle-free graphs, *Discrete Math.* **252** (2002), 203–214.
110. D. R. Woodall, Sufficient conditions for circuits in graphs, *Proc. London Math. Soc.* **24** (1972), 739–755.
111. K. Yoshimoto, On the number of components in 2-factors of claw-free graphs, *Discrete Math.* **307** (2007), 2808–2819.
112. S. Zhan, On hamiltonian line graphs and connectivity, *Discrete Math.* **89** (1991), 89–95.
113. Y. Zhu, Z. Liu and Z. Yu, An improvement of Jackson's result on Hamilton cycles in 2-connected regular graphs, *Cycles in Graphs* (eds. B. R. Alspach and C. D. Godsil), *Ann. Discrete Math.* **27** (1985), 237–247.
114. Y. Zhu, Z. Liu and Z. Yu, 2-connected k-regular graphs on at most $3k + 3$ vertices to be hamiltonian, *J. Systems Sci. Math. Sci.* **6** (1986), 36–49.