INTERDEPENDENT DEMAND IN THE TWO-PERIOD NEWSVENDOR PROBLEM

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Abstract. The newsvendor problem is a classical task in inventory management. The present paper considers a two-period newsvendor problem where demand of different periods is interdependent (not independent), and seeks to follow this approach to develop a two-period newsvendor problem with unsatisfied demand or unsold quantity. Concerning the complexity of solution of multiple integrals, the problem is assessed for only two periods. In the course of a numerical solution, the probability distribution function of demand pertaining to each period is assumed to be given (in the form of a bivariate normal distribution). The optimal solution is presented in the form of the initial inventory level that maximizes the expected profit. Finally, all model parameters are subjected to a sensitivity analysis. This model can be used in a number of applications, such as procurement of raw materials in projects (e.g., construction, bridge-building and molding) where demand of different periods is interdependent. Proposed model takes into account interdependent demand oughts to provide a better solution than a model based on independent demand.

1. **Introduction.** The newsvendor problem is a classical problem in Operational Research and inventory management [25]. Over time, this problem has undergone a significant development through the addition of new assumptions and constraints. The classical form of this problem seeks to determine the order quantity for a single product for a single period to maximize the expected profit under a series of assumptions, based on which demands are probabilistic and products are subject to discount or deterioration [11].

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The appeal and importance of the newsvendor problem have encouraged a number of researchers to develop comprehensive literature reviews describing the wide-ranging extensions and modifications applied to the basic form of this problem. Most notable among these works are those by [25] Qin et al. and Khouja [16]. According to the classification proposed by Khouja, extensions have been developed through incorporation of factors, such as different objectives and utility functions, different supplier pricing policies, different newsvendor pricing policies [7], different levels of information about demand, constrained multi-products, multi-echelon systems, multi-location models, and multi-period models.

2. **Literature review.** The classical newsvendor problem cannot investigate scenarios where products remain unsold or demand remains unsatisfied over several periods.

2.1. **Multi-period newsvendor problem.** Matsuyama [17] developed a model for a multi-period newsvendor problem to deal with these issues. He assumed that some portion of products that remain unsold at the end of a period can be stored for the next period. He stated that, to determine the ordering quantity, one must first determine the amounts of products that remain stored or demand that remains unsatisfied from the last period. In his model, distribution of demand in different periods was assumed to be independent. Mileff and Nehéz [18] assessed an effective management of supplier corporations in a multi-product multi-period newsvendor problem. They proposed a heuristic-based optimal solution to satisfy the global capacity constraint policy. Altintas et al. [2] studied the problem of designing an optimal all-unit quantity discount with a single price break from the perspective of a distributor supplying a multi-period newsvendor buyer. The aim of this model was to maximize the supplier's profit, i.e., the income gained from the retailers minus the distribution costs. Behret and Kahraman [4] developed a fuzzy model for a multi-period newsvendor problem with a preseason extension and an emphasis on innovative products.This model optimizes the order period and order quantity by minimizing the sum of all fuzzy expected costs. Zhang [32] studied a multi-product newsvendor problem with both supplier quantity discounts and budget constraints. Given the presence of price discounts, the problem was formulated as a mixed-integer nonlinear programming model, it was solved by a Lagrangian relaxation approach. Zhang and Du [30] studied a multi-product newsvendor system with production capacity, where production can also be outsourced with zero or nonzero lead time. They developed the structural properties and solution procedures for profit maximization. Zhang and Hua [31] investigated a portfolio approach to a multi-product newsvendor problem with budget constraints, too, where the procurement strategy for each newsvendor's product is designed as a contract portfolio. They then proposed an efficient solution procedure and compared three models with different procurement contracts: a fixed-price contract, an option contract, and a contract portfolio. In some situations, the procurement strategy used for purchase can include a budget constraint such as a fixed price or an option or portfolio contract. This scenario has been modeled by Huang et al. [13], who demonstrated the advantage of a portfolio model. Ding and Gao [10] developed a methodology for determining the optimal (σ, S) policy for multi-product newsvendor problems that feature elements of uncertainty. They used uncertainty theory to deal with human uncertainty, which was given by the demand distribution. They designed their model such that ordering products triggered not only a linear ordering cost, but

also a fixed setup cost, which depended on whether a joint or individual order was placed. Hanasusanto et al. [12] developed a risk-averse multi-dimensional newsvendor model for multiple products with strongly correlated demand. This problem was tailored for demand with strong dependence on mostly unknown future fashion trends whose distribution (called multimodal distribution) is in the form of spatially separated clusters of probability mass. This work, based on the assumption that distributional ambiguity will be addressed by minimization of the worst-case risks of order portfolios for all distributions compatible with the assumed modality. The NP-hard complexity of this problem was proven. An efficient, accurate, and conservative numerical solution was developed with quadratic decision rules. It was also shown that a solution that disregards ambiguity or multimodality may be unstable and fail to exhibit adequate quality and robustness under stress tests. Alwan et al. [3] formulated a newsvendor problem with correlated demand. They compared the performance of a traditional approach with a dynamic forecast-based approach. The authors showed that a minimum mean squared error (MSE) forecast model a better cost savings performance than the traditional approach. The performance of this MSE-optimal approach and the traditional approach was also compared with the performance of widely used alternative forecasting methods, such as the moving average and exponential smoothing methods. This article reported that when using alternative forecasting methods, sometimes the traditional approach to the newsvendor problem superior better results, and it is better to disregard correlations and forecasting. Orders for different periods show some interperiod dependency, which has been addressed in the recent literature ([3, 12]).

Table 1 summarizes the literature dedicated to the inventory control and newsvendor problems. The approach most suitable for order planning is the multi-period newsvendor problem formulated by Matsuyama [17], in which interdependence of demand for different periods is disregarded.

The present study's innovation focuses on an incorporation of interdependence into the two-period newsvendor problem approach. The resulting formulations are applicable to large-scale projects where a majority of the raw materials is of the same nature.

To the best of the authors' knowledge, and as shown in Table 1, all multi-period and multi-product models available in the literature have assumed independent demand between periods. In many projects or manufacturing processes, materials and resources need to be ordered for two or more periods; so demand seems to be correlated. Therefore, to achieve a higher level of conformity with real-world applications, the present paper omits the assumption of independent demand and proposes a general solution for a two-period newsvendor problem with interdependent demand.

Section 3 presents the notations and assumptions used for formulation of the model. Section 4 describes the proposed model and proves the optimal solution. Section 5 presents the numerical results obtained with known variables. Section 6 discloses the results of our sensitivity analysis performed on all parameters. Some proofs are given in the Appendix.

3. **Problem description.** The significant distinction between the present paper and the one by Matsuyama [17] is the assumption that demand of two periods is correlated. When demand of two periods is interdependent, an increase in the demand of one period leads to a decrease in the demand of the other. Therefore, the following notations and assumptions are defined to develop the problem model.

TABLE 1. Classification of the literature.

Reference	Fuzzy	Single-period	Multi-period	Multi-product	Risk	Demand	Product	Market	Discount
Bouakiz and Sobel [5]			1		1	Independent			
Perakis and Sood [24]			1			Independent	Perishable	Competitive	
Matsuyama [17]			1			Independent			
Mileff and Nehz [18]			1	1		Independent			
Burnetas et al. [6]		1				Independent			Incremental
Altintas et al. [2]			1			Independent			All-Unit
Wang and Webster [29]		1			1	Independent			
Behret and Kahraman [4]	1		1			Independent			
Chen and Ho [8]	1	1				Independent			
Zhang [32]				1		Independent			All-Units
Zhang and Du [30]				1		Independent			
Zhang and Hua [31]				1		Independent			
Huang et al. [13]				1		Independent			
Sana [27]		1				Independent			
Chen and Ho [9]	1	1				Independent			
Ray and Jenamani [26]		1			1	Independent			
Ding and Gao [10]				1		Independent			
Kamburowski [14]		1				Independent			
Kamburowski [14]		1				Independent			
Pal and Sana [22]		1				Independent			
Hanasusanto et al. [12]				1	1	Interdependent product			
Alwan et al. [3]		1				Interdependent			
Summary	3	10	6	7	4	20 Independent 2 Interdependent	1 Perishable	1 Competitive	1 Incremental 2 All-Unit
The present study			1			Interdependent Demand			

3.1. **Assumptions.** This problem is modeled based on the following assumptions:

- The assumptions of the classic newsvendor model.
- There is a single product.
- In each project, the demand for a period depends on the demand for the other period.
- Demand for independent periods is an independent random variable.
- Surplus supplies of a period will be used in the next period.
- Out-of-stock supplies of a period will be procured in the next period or will be lost.
- The only objective is to maximize the mean profit from the planned periods.

3.2. **Definitions.**

Indices:

J: Index of periods ($J = \{1\}$),

S: Index of states ($S = \{1, 2, 3, 4\}$);

Parameters:

q_j:	Selling price (per unit) of commodity at period j,
p_j:	Buying price (per unit) of commodity at period j,
s_j:	Holding cost (per unit) of commodity during period j,
π:	Penalty cost (per unit) of commodity when the demand is not met,
C_j:	Setup cost at period j,
x_j:	Total amount of demand during period j,
$f(x_j, x_{j+1})$:	Joint probability density function for demand, $j = 1$,
α:	Ratio of the amount stocked to the amount unsold, $0 \leq \alpha \leq 1$,
β:	Ratio of the amount sold at the beginning of the next period to the amount of unsatisfied demand in the present period, $0 \leq \beta \leq 1$,
δ:	Ratio of selling price in each period, $0 \leq \delta \leq 1$,
$\delta q_j + (1-\delta)q_{j+1}$:	Selling price for the portion of demand that is not satisfied during the period j but is sold at the beginning of period $j + 1$, $j = 1$,

L:	Minimum demand in each period,
N:	Maximum demand in each period,
k_j, k'_j, A, B:	Helping variable in differentiation of h_{js};

Decision variables:

l_j: Inventory level at the beginning of period j;

Objectives:

h_{js} Profit at period j in state s,

H_j: Expected profit at period j,

H: Total expected profit.

This model is based on the assumptions of multi-period newsvendor problem, but here the main distinctive assumption is that $f(x_j, x_{j+1}) \neq \prod_{i=1}^{n} f_j(x_i)$, which means that the demand distributions are interdependent. Another assumption of the model is the presence of two periods, which simplifies the model.

Note that the contribution of the present paper is the provision of a general solution based on the assumptions above, which is not unlikely in real-world applications. In the next section, the profit function of the model is formulated according to the assumption below:

$$L \leq x_j \leq N, \tag{1}$$

$$f_j(x_j) = 0 \ \ \text{if} \ \ x_j \geq N \ \text{or} \ \ x_j \leq L. \tag{2}$$

In columns 2 and 3 of Table 2, $L \leq x_j \leq N$ means that demand and initial inventory of each period is usually limited to lower and upper bounds (L, N). Equations (1)-(2) are defined accordingly and are considered to be one of the model's assumptions. Therefore, the following assumption is proved:

$j = 1$:

$$L \leq x_j \leq N \ \ \text{and} \ L \leq l_j \leq N \Rightarrow \quad 0 \leq l_j - x_j \leq N - L \tag{3}$$

$$L \leq l_{j+1} \leq N \quad \Rightarrow l_{j+1} - \alpha(l_j - x_j) \geq L - (N - L) = 2L - N \geq 0, \ \text{finally}; \tag{4}$$

$$N \leq 2L. \tag{5}$$

When demand in each period has lower and upper bounds (L, N), initial inventory in period l_j and the next period will have lower and upper bounds as shown in Equations (3)-(4), respectively; finally, it is proved in Equation (5) that $N \leq 2L$.

Depending on the status of demand, multiple conditions can be assumed for the model formulation; these conditions are shown in Table 2.

The second column in Table 2 describes a state where demand x_j is less than initial inventory l_j, sale income equals $q_j x_j$, buying cost equals $p_j l_j$, quantity of unsold products equals $(l_j - x_j)$, quantity of stocked products equals $\alpha(l_j - x_j)$, holding cost of unsold products equals $s_j \alpha(l_j - x_j)$, penalty for unsatisfied demand is zero, and order of next period equals $l_{j+1} - \alpha(l_j - x_j)$.

The third column in Table 2 describes a state where demand x_j is greater than inventory level l_j, sale income equals $q_j l_j$, buying cost equals $p_j l_j$, quantities of unsold and stocked products and holding cost of unsold products are all zero, unsatisfied demand equals $\beta\left(x_j - l_j\right)$, penalty for unsatisfied demand equals $\pi(x_j - l_j)$, and order of next period equals $l_{j+1} + \beta(x_j - l_j)$.

In columns 2 and 3 of Table 2, $L \leq x_j \leq N$, meaning that the demand of each period is usually limited to lower and upper bounds (L, N), which is one of the model's assumptions. Figure 1 shows the step-by-step process of implementation of the conceptual model.

TABLE 2. Conceptual Model.

Description	Period j	
Status of demand	$L \leq x_j \leq l_j \leq N$	$L \leq l_j \leq x_j \leq N$
Sale income	$q_j x_j$	$q_j l_j$
Buying cost	$p_j l_j$	$p_j l_j$
Unsold	$(l_j - x_j)$	0
Stocked amount	$\alpha(l_j - x_j)$	0
Holding cost of amount unsold	$s_j \alpha(l_j - x_j)$	0
Unsatisfied demand	0	$\beta\left(x_j - l_j\right)$
Penalty for unsatisfied demand	0	$\pi(x_j - l_j)$
Order of period j+1	$l_{j+1} - \alpha(l_j - x_j)$	$l_{j+1} + \beta(x_j - l_j)$

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│  Defining the known parameters and multivariate distribution function │
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                              ⤵
┌─────────────────────────────────────────────────────────────────┐
│  Computing the optimal initial inventory between periods (Section 4, Appendix 1) │
│       through numerical computation (Section 4, Appendex 2)       │
└─────────────────────────────────────────────────────────────────┘
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│                    Sensitivity analysis (Section 6)              │
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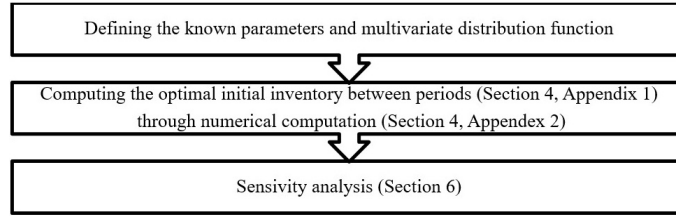
FIGURE 1. Process of implementation of the conceptual model.

4. **Mathematical model for interdependent demand in a Two-Period newsvendor problem.** The main objective of the formulation of this newsvendor problem is to construct the objective function with respect to the range of demand and initial inventory level. In the formulation provided by Matsuyama [17] for a two-period problem with uncorrelated demands, $h_{js}(\alpha, \beta, \delta, \pi, l_1, l_2, x_1, x_2)$ is assumed to be the profit function for period j and $\alpha, \beta, \delta, \pi$ are assumed to be as follows.
In the first period, only two states are possible. In one state, demand is lower than the initial inventory ($x_1 \leq l_1$) and profit equals sales (determined by demand) minus inventory purchase costs (Equation (6)). In the second state, demand is greater than the initial inventory ($x_1 > l_1$) and profit equals sales minus purchase costs, lost sales and setup costs (Equation (7)) [17]:

$$j = 1 \text{ and } s = 1:$$
$$x_1 \leq l_1 \Rightarrow h_{11}(\alpha, \beta, \delta, \pi; l_1; x_1) = q_1 x_1 - p_1 l_1 - C_1, \tag{6}$$
$$j = 1 \text{ and } s = 2:$$
$$x_1 > l_1 \Rightarrow h_{12}(\alpha, \beta, \delta, \pi; l_1; x_1) = q_1 l_1 - p_1 l_1 - \pi(x_1 - l_1) - C_1$$
$$= (q_1 - p_1 + \pi) l_1 - \pi x_1 - C_1. \tag{7}$$

In the second period, four states are possible. In the first state, demand of the first period is lower than the second initial inventory ($x_1 \leq l_1$) and demand of the second period is also lower than the inventory level ($x_2 \leq l_2$). As shown by Equation (8), here, overall profit equals profit of the first period minus holding cost plus sales of second period (with respect to its demand) minus initial purchase cost minus cost of the percentage of products that are moved to this period, and also

minus setup costs [17]:

$$
\begin{aligned}
j = 2 \text{ and } s = 1 : \\
x_1 \leq l_1 \ , \ x_2 \leq l_2 \ &\Rightarrow h_{21}\left(\alpha, \beta, \delta, \pi; l_1, l_2; x_1, x_2\right) \\
&= h_{11}\left(\alpha, \beta, \delta, \pi; l_1; x_1\right) - s_1 \alpha\left(l_1 - x_1\right) + q_2 x_2 \\
&\quad - p_2\left(l_2 - \alpha\left(l_1 - x_1\right)\right) - C_2.
\end{aligned}
\tag{8}
$$

In the second state, demand of the first period is less than the initial inventory $(x_1 \leq l_1)$, but demand of the second period is greater than the second inventory level $(x_2 > l_2)$. As shown by Equation (9), here, overall profit equals profit of first period minus holding cost plus sales of initial inventory minus inventory purchase cost minus cost of the percentage of products that are moved to this period minus shortage cost of the second period and setup costs [17]:

$$
\begin{aligned}
j = 2 \text{ and } s = 2 : \\
x_1 \leq l_1 \ , \ x_2 > l_2 \ &\Rightarrow h_{22}\left(\alpha, \beta, \delta, \pi; l_1, l_2; x_1, x_2\right) \\
&= h_{11}\left(\alpha, \beta, \delta, \pi; l_1; x_1\right) - s_1 \alpha\left(l_1 - x_1\right) + q_2 l_2 \\
&\quad - p_2\left(l_2 - \alpha\left(l_1 - x_1\right)\right) - \pi\left(x_2 - l_2\right) - C_2.
\end{aligned}
\tag{9}
$$

In the third state, demand of the first period is greater than initial inventory $(x_1 > l_1)$ and demand of the second period is lower than the inventory of the second period $(x_2 \leq l_2)$. As shown by Equation (10), here, overall profit equals profit of first period plus sales of leftovers of the first period (with mixed price) plus sales of the second period minus costs of purchase of the second inventory and costs of purchase of the percentage of products that are moved to this period minus setup costs [17]:

$$
\begin{aligned}
j = 2 \text{ and } s = 3 : \\
x_1 > l_1 \ , \ x_2 \leq l_2 \ &\Rightarrow h_{23}\left(\alpha, \beta, \delta, \pi; l_1, l_2; x_1, x_2\right) \\
&= h_{12}\left(\alpha, \beta, \delta, \pi; l_1; x_1\right) + \left(\delta q_1 + (1 - \delta) q_2\right) \beta\left(x_1 - l_1\right) \\
&\quad + q_2 x_2 - p_2\left(l_2 + \beta\left(x_1 - l_1\right)\right) - C_2.
\end{aligned}
\tag{10}
$$

In the fourth state, demand of the first period is greater than the initial inventory $(x_1 > l_1)$ and demand of the second period is greater than the initial inventory $(x_2 > l_2)$. As shown by Equation (11), here, overall profit equals profit of the first period plus sales of leftovers of the first period (with mixed price) plus sales of the initial inventory minus cost of purchase of the initial inventory plus cost of purchase of the percentage of products that are moved to this period minus shortage cost of the second period and setup costs [17]:

$$
\begin{aligned}
j = 2 \text{ and } s = 4 : \\
x_1 > l_1 \ , \ x_2 > l_2 \ &\Rightarrow h_{24}\left(\alpha, \beta, \delta, \pi; l_1, l_2; x_1, x_2\right) \\
&= h_{12}\left(\alpha, \beta, \delta, \pi; l_1; x_1\right) + \left(\delta q_1 + (1-\delta) q_2\right) \beta\left(x_1 - l_1\right) + q_2 l_2 \\
&\quad - p_2\left(l_2 + \beta\left(x_1 - l_1\right)\right) - \pi\left(x_2 - l_2\right) - C_2.
\end{aligned}
\tag{11}
$$

Demands of periods are assumed to be interdependent, so in the next step, the expected profit function has a bivariate demand probability distribution function of $f\left(x_j, x_{j+1}\right) \neq \prod_{i=1}^{n} f_j\left(x_i\right)$:

$$H\left(l_1, l_2; x_1, x_2\right) = \int_{-\infty}^{l_2} \int_{-\infty}^{l_1} h_{21}\left(l_1, l_2; x_1, x_2\right) f\left(x_1, x_2\right) dx_1 dx_2$$

$$+ \int_{l_2}^{\infty} \int_{-\infty}^{l_1} h_{22}\left(l_1, l_2; x_1, x_2\right) f\left(x_1, x_2\right) dx_1 dx_2$$

$$+ \int_{-\infty}^{l_2} \int_{l_1}^{\infty} h_{23}\left(l_1, l_2; x_1, x_2\right) f\left(x_1, x_2\right) dx_1 dx_2$$

$$+ \int_{l_2}^{\infty} \int_{l_1}^{\infty} h_{24}\left(l_1, l_2; x_1, x_2\right) f\left(x_1, x_2\right) dx_1 dx_2. \tag{12}$$

In Equation (12), $H\left(l_1, l_2; x_1, x_2\right)$ is a function expressing the expected profit of the second period. This function is a dual integral on initial inventory of the first and second periods and can be obtained by multiplication of demand distribution function by profit functions of the four above-mentioned states. In the first part of Equation (12), the profit function Equation (8) bounded by $x_1 \leq l_1,\ x_2 \leq l_2$ has been multiplied by $f\left(x_1, x_2\right)$. In the second part of Equation (12), the profit function Equation (9) bounded by $x_1 \leq l_1,\ x_2 > l_2$ has been multiplied by $f\left(x_1, x_2\right)$. In the third part of Equation (12), the profit function Equation (10), bounded by $x_1 > l_1,\ x_2 \leq l_2$, has been multiplied by $f\left(x_1, x_2\right)$. Finally, in the fourth part of Equation (12), the profit function Equation (11) bounded by $x_1 > l_1,\ x_2 > l_2$ has been multiplied by $f\left(x_1, x_2\right)$.

Next, to obtain the optimal solutions for initial inventory of the first (l_1^*) and second (l_2^*) periods, it is necessary to differentiate $H\left(l_1, l_2; x_1, x_2\right)$ with respect to $l_1,\ l_2$ and equate this then to zero:

$$\frac{\partial}{\partial l_1} H\left(l_1, l_2; x_1, x_2\right) = 0, \tag{13}$$

$$\frac{\partial}{\partial l_2} H\left(l_1, l_2; x_1, x_2\right) = 0. \tag{14}$$

To obtain optimal solution l_1, the derivative of H is found:

$$H\left(l_1, l_2; x_1, x_2\right) = \sum_{j=1}^{4} \int_{q_1(l_1)}^{q_2(l_1)} \int_{p_1(l_2)}^{p_2(l_2)} h_{2j}\left(l_1, l_2; x_1, x_2\right) f(x_1, x_2) dx_1 dx_2, \tag{15}$$

$$H_j\left(l_1, l_2; x_1, x_2\right) = h_{2j}\left(l_1, l_2; x_1, x_2\right) f\left(x_1, x_2\right), \tag{16}$$

$$\frac{\partial}{\partial l_1} H_j\left(l_1, l_2; x_1, x_2\right) = \frac{\partial}{\partial l_1} h_{2j}\left(l_1, l_2; x_1, x_2\right) f\left(x_1, x_2\right)$$

$$= f\left(x_1, x_2\right) \frac{\partial}{\partial l_1} h_{2j}\left(l_1, l_2; x_1, x_2\right). \tag{17}$$

Let be $k_j = \frac{\partial}{\partial l_1} h_{2j}\left(l_1, l_2; x_1, x_2\right);$ then, \hfill (18)

$$\frac{\partial}{\partial l_1} H_j\left(l_1, l_2; x_1, x_2\right) = k_j f\left(x_1, x_2\right), \tag{19}$$

$$\frac{\partial}{\partial l_1} H\left(l_1, l_2; x_1, x_2\right) = \frac{\partial}{\partial l_1} \sum_{j=1}^{4} \int_{p_1(l_2)}^{p_2(l_2)} \int_{q_1(l_1)}^{q_2(l_1)} H_j\left(l_1, l_2; x_1, x_2\right) dx_1 dx_2$$

$$= \sum_{j=1}^{4} \int_{p_1(l_2)}^{p_2(l_2)} \frac{\partial}{\partial l_1} \int_{q_1(l_1)}^{q_2(l_1)} H_j\left(l_1, l_2; x_1, x_2\right) dx_1 dx_2$$

$$= \sum_{j=1}^{4} \int_{p_1(l_2)}^{p_2(l_2)} [\int_{q_1(l_1)}^{q_2(l_1)} \frac{\partial}{\partial l_1} H_j \left(l_1, l_2; x_1, x_2\right) dx_1$$

$$+ \frac{\partial}{\partial l_1} q_2 \left(l_1\right) H_j \left(l_1, l_2, q_2 \left(l_1\right), x_2\right)$$

$$- \frac{\partial}{\partial l_1} q_1(l_1) H_j \left(l_1, l_2, q_1 \left(l_1\right), x_2\right)] dx_2. \tag{20}$$

Equation (12) has been changed into parametric form in Equation (15), so $H_j \left(l_1, l_2; x_1, x_2\right)$ has been replaced with its short form $h_{2j} \left(l_1, l_2; x_1, x_2\right) f \left(x_1, x_2\right)$ (Equation (16)). In Equation (17), with respect to l_1, it has been differentiated, and then a change in variable $\frac{\partial}{\partial l_1} h_{2j} \left(l_1, l_2; x_1, x_2\right)$ with k_j shown in Equation (18) results in Equation (19). This change of variable allows the dual integral of l_1 to be easily differentiated (Equation (20)):

$$\frac{\partial}{\partial l_1} H \left(l_1, l_2; x_1, x_2\right) = \sum_{j=1}^{4} \int_{p_1(l_2)}^{p_2(l_2)} [\int_{q_1(l_1)}^{q_2(l_1)} k_j f \left(x_1, x_2\right) dx_1$$

$$+ \frac{\partial}{\partial l_1} q_2 \left(l_1\right) H_j \left(l_1, l_2, q_2 \left(l_1\right), x_2\right)$$

$$- \frac{\partial}{\partial l_1} q_1(l_1) H_j \left(l_1, l_2, q_1 \left(l_1\right), x_2\right)] dx_2$$

$$= \int_{-\infty}^{l_2} \int_{-\infty}^{l_1} k_1 f \left(x_1, x_2\right) dx_1 dx_2 + \int_{l_2}^{\infty} \int_{-\infty}^{l_1} k_2 f \left(x_1, x_2\right) dx_1 dx_2$$

$$+ \int_{-\infty}^{l_2} \int_{l_1}^{\infty} k_3 f \left(x_1, x_2\right) dx_1 dx_2 + \int_{l_2}^{\infty} \int_{l_1}^{\infty} k_4 f \left(x_1, x_2\right) dx_1 dx_2$$

$$+ \int_{-\infty}^{l_2} [H_1 \left(l_1, l_2, l_1, x_2\right) - H_3 \left(l_1, l_2, l_1, x_2\right)] dx_2$$

$$+ \int_{l_2}^{\infty} [H_2 \left(l_1, l_2, l_1, x_2\right) - H_4 \left(l_1, l_2, l_1, x_2\right)] dx_2. \tag{21}$$

Considering the presence of $\frac{\partial}{\partial l_1} H_j \left(l_1, l_2; x_1, x_2\right)$ in Equation (20), the dual integral has been differentiated. Then, in Equation (21), it has been replaced with $k_j f \left(x_1, x_2\right)$. Although this equation contains the terms $H_1 \left(l_1, l_2; l_1, x_2\right) - H_3 \left(l_1, l_2; l_1, x_2\right)$ and $H_2 \left(l_1, l_2; l_1, x_2\right) - H_4 \left(l_1, l_2; l_1, x_2\right)$, it will be proven that these terms are zero.

Corollary 1.

$$H_1 \left(l_1, l_2, q_2 \left(l_1\right), x_2\right) = h_{21} \left(l_1, l_2, l_1, x_2\right) f \left(l_1, x_2\right)$$

$$= [q_1 l_1 - p_1 l_1 - C_1 + q_2 x_2 - C_2] f \left(l_1, x_2\right), \tag{22}$$

$$H_3 \left(l_1, l_2, q_2 \left(l_1\right), x_2\right) = h_{23} \left(l_1, l_2, l_1, x_2\right) f \left(l_1, x_2\right)$$

$$= [q_1 l_1 - p_1 l_1 - C_1 + q_2 x_2 - C_2] f \left(l_1, x_2\right), \tag{23}$$

$$H_2 \left(l_1, l_2, q_2 \left(l_1\right), x_2\right) = h_{22} \left(l_1, l_2, l_1, x_2\right) f \left(l_1, x_2\right)$$

$$= [q_1 l_1 - p_1 l_1 - C_1 + q_2 l_2 - \pi \left(x_2 - l_2\right) - C_2] f \left(l_1, x_2\right), \tag{24}$$

$$H_4 \left(l_1, l_2, q_2 \left(l_1\right), x_2\right) = h_{24} \left(l_1, l_2, l_1, x_2\right) f \left(l_1, x_2\right)$$

$$= [q_1 l_1 - p_1 l_1 - C_1 + q_2 l_2 - \pi \left(x_2 - l_2\right) - C_2] f \left(l_1, x_2\right), \tag{25}$$

$$H_1(l_1, l_2; l_1, x_2) - H_3(l_1, l_2; l_1, x_2) = 0, \tag{26}$$

$$H_2(l_1, l_2; l_1, x_2) - H_4(l_1, l_2; l_1, x_2) = 0. \tag{27}$$

Equation (21) contains the terms $H_1(l_1, l_2, l_1, x_2)$, $H_3(l_1, l_2, l_1, x_2)$, $H_2(l_1, l_2, l_1, x_2)$, $H_4(l_1, l_2, l_1, x_2)$, but Equations (22)-(27) clearly show that these terms cancel each other, and finally, it has been proven that $H_1(l_1, l_2; l_1, x_2) - H_3(l_1, l_2; l_1, x_2)$ and $H_2(l_1, l_2; l_1, x_2) - H_4(l_1, l_2; l_1, x_2)$ are both equal to zero.

After proving Corollary 1, Equations (26)-(27), the results need to be inserted into Equation (21), revealing the (l_1, l_2) in $\int_{-\infty}^{l_2} \int_{-\infty}^{l_1} f(x_1, x_2) \, dx_1 dx_2$, which is a cumulative demand distribution:

$$
\begin{aligned}
\frac{\partial}{\partial l_1} H(l_1, l_2; x_1, x_2) &= \int_{-\infty}^{l_2} \int_{-\infty}^{l_1} k_1 f(x_1, x_2) \, dx_1 dx_2 + \int_{l_2}^{\infty} \int_{-\infty}^{l_1} k_2 f(x_1, x_2) \, dx_1 dx_2 \\
&+ \int_{-\infty}^{l_2} \int_{l_1}^{\infty} k_3 f(x_1, x_2) \, dx_1 dx_2 + \int_{l_2}^{\infty} \int_{l_1}^{\infty} k_4 f(x_1, x_2) \, dx_1 dx_2 \\
&= k_1 \int_{-\infty}^{l_2} \int_{-\infty}^{l_1} f(x_1, x_2) \, dx_1 dx_2 + k_2 \int_{l_2}^{\infty} \int_{-\infty}^{l_1} f(x_1, x_2) \, dx_1 dx_2 \\
&+ k_3 \int_{-\infty}^{l_2} \int_{l_1}^{\infty} f(x_1, x_2) \, dx_1 dx_2 + k_4 \int_{l_2}^{\infty} \int_{l_1}^{\infty} f(x_1, x_2) \, dx_1 dx_2 \\
&= k_1 F(l_1, l_2) + k_2 (F(l_1) - F(l_1, l_2)) + k_3 (F(l_2) - F(l_1, l_2)) \\
&+ k_4 (1 - F(l_1) - F(l_2) + F(l_1, l_2)) \\
&= (k_1 - k_2 - k_3 + k_4) F(l_1, l_2) + (k_2 - k_4) F(l_1) \\
&+ (k_3 - k_4) F(l_2) + k_4. \tag{28}
\end{aligned}
$$

The process of differentiation continues until $\frac{\partial}{\partial l_1} H(l_1, l_2; x_1, x_2)$ becomes $(k_1 - k_2 - k_3 + k_4) F(l_1, l_2) + (k_2 - k_4) F(l_1) + (k_3 - k_4) F(l_2) + k_4$ (Equation (28)):

$$k_1 = \frac{\partial}{\partial l_1} h_{21}(l_1, l_2, x_1, x_2) = -p_1 - s_1 \alpha + \alpha p_2, \tag{29}$$

$$k_2 = \frac{\partial}{\partial l_1} h_{22}(l_1, l_2, x_1, x_2) = -p_1 - s_1 \alpha + \alpha p_2, \tag{30}$$

$$k_3 = \frac{\partial}{\partial l_1} h_{23}(l_1, l_2, x_1, x_2) = q_1 - p_1 + \pi - \beta(\delta q_1 + (1 - \delta) q_2) + p_2 \beta, \tag{31}$$

$$k_4 = \frac{\partial}{\partial l_1} h_{24}(l_1, l_2, x_1, x_2) = q_1 - p_1 + \pi - \beta(\delta q_1 + (1 - \delta) q_2) + p_2 \beta, \tag{32}$$

$$k_1 = k_2, k_3 = k_4. \tag{33}$$

In this step, given as above k_1, k_2, k_3, k_4, we differentiate h_{2j} with respect to l_1 (Equations (29)-(32)). As a result, $k_1 = k_2$ and $k_3 = k_4$; then these parameters are inserted into Equation (28):

$$\frac{\partial}{\partial l_1} H(l_1, l_2, x_1, x_2) = (k_2 - k_4) F(l_1) + k_4 = 0, \tag{34}$$

$$F(l_1^*) = \frac{k_4}{k_4 - k_2} = \frac{q_1 - p_1 + \pi - \beta(\delta q_1 + (1 - \delta) q_2) + p_2 \beta}{q_1 + \pi - \beta(\delta q_1 + (1 - \delta) q_2) + p_2 \beta + \alpha(s_1 - p_2)}. \tag{35}$$

In the following, to obtain the optimal solutions for the initial inventory of the first and second periods l_1^*, in Equation (35), we need to make $\frac{\partial}{\partial l_1} H(l_1, l_2, x_1, x_2)$

equal to zero. after moving the known variables to the right side, as Equation (35) shows, $F(l_1^*)$ equals $k_4/(k_4 - k_2)$:

$$F(l_1^*) = \int_{-\infty}^{\infty} \int_{-\infty}^{l_1^*} f(x_1, x_2)\, dx_1 dx_2$$
$$= \frac{q_1 - p_1 + \pi - \beta(\delta q_1 + (1-\delta)q_2) + p_2\beta}{q_1 + \pi - \beta(\delta q_1 + (1-\delta)q_2) + p_2\beta + \alpha(s_1 - p_2)}. \tag{36}$$

Finally, it has been proved that the cumulative demand function (CDF) of l_1^* equals the constant parameter of Equation (36), and to determine l_1^* in Equation (36), F should be inverted. However, the infinity term in the above equation and the approximation of l_1^* are approximately in the following through numerical computation:

$$F(l_2^*) = \frac{k_4'}{k_4' - k_3'} = \frac{q_2 - p_2 + \pi}{q_2 - p_2 + \pi - (-p_2)} = \frac{q_2 - p_2 + \pi}{q_2 + \pi}, \tag{37}$$

$$F(l_2^*) = \int_{-\infty}^{\infty} \int_{-\infty}^{l_2^*} f(x_1, x_2)\, dx_2 dx_1 = \frac{q_2 - p_2 + \pi}{q_2 + \pi}. \tag{38}$$

Similarly, $F(l_2^*)$ can be obtained from Equations (47)-(74) in Appendix 1, and the final results are shown in Equations (38)-(39). Although the values of l_1^*, l_2^* have been computed, to determine whether l_1^* and l_2^* are extrema or optimal solutions, a Hessian matrix needs to be constructed with respect to l_1^* and l_2^* ([1, 23]):

$$\nabla^2 H = \begin{pmatrix} \frac{\partial^2}{\partial l_1^2} H & \frac{\partial^2}{\partial l_1 \partial l_2} H \\ \frac{\partial^2}{\partial l_2 \partial l_1} H & \frac{\partial^2}{\partial l_2^2} H \end{pmatrix}$$
$$= \begin{pmatrix} (k_2 - k_4) \frac{\partial}{\partial l_1} F(l_1) \Big|_{\substack{l_1 = l_1^* \\ l_2 = l_2^*}} & 0 \\ 0 & (k_3' - k_4') \frac{\partial}{\partial l_2} F(l_2) \Big|_{\substack{l_1 = l_1^* \\ l_2 = l_2^*}} \end{pmatrix}, \tag{39}$$

$$\det\left(\nabla^2 H\right) = (k_2 - k_4)(k_3' - k_4') \frac{\partial}{\partial l_1} F(l_1) \Big|_{\substack{l_1 = l_1^* \\ l_2 = l_2^*}} \frac{\partial}{\partial l_2} F(l_2) \Big|_{\substack{l_1 = l_1^* \\ l_2 = l_2^*}}, \tag{40}$$

$$\Rightarrow \frac{\partial}{\partial l_1} F(l_1) = \int_{-\infty}^{\infty} f(l_1, x_2)\, dx_2 > 0, \quad \frac{\partial}{\partial l_2} F(l_2) = \int_{-\infty}^{\infty} f(x_1, l_2)\, dx_1 > 0, \tag{41}$$

$$(k_3' - k_4') = -p_2 - (q_2 - p_2 + \pi) = -q_2 - \pi < 0. \tag{42}$$

After computing the determinant of the Hessian matrix, since the determinant becomes positive and the values of $(k_3' - k_4')$ and $(k_2 - k_4)$ become less than zero, l_1^* and l_2^* develop into optimal solutions.

According to $L \leq x_1 \leq N$ and $L \leq x_2 \leq N$ from Equations (1)-(2), l_1^* and l_2^* need to be determined by a truncated distribution with correlated demands, negative and positive infinity are changed to L and N [28]:

$$f(x_1, x_2 | \ L \leq x_1 \leq N, L \leq x_2 \leq N) = \frac{f(x_1, x_2)}{F(N,N) - F(L,N) - F(N,L) + F(L,L)}. \tag{43}$$

In Equation (43), the function $f(x_1, x_2 | \ L \leq x_1 \leq N, L \leq x_2 \leq N)$ has been replaced by $f(x_1, x_2)$ with upper and lower bounds of $L \leq x_1 \leq l_1^*$ and $L \leq x_1 \leq N$,

$L \le x_2 \le N$ and $L \le x_2 \le l_2^*$:

$$F\left(l^*{}_1, N\right) = \int_L^N \int_L^{l^*{}_1} f\left(x_1, x_2 \mid L \le x_1 \le N, L \le x_2 \le N\right) \, dx_1 dx_2$$

$$= \frac{1}{\left[F\left(N, N\right) - F\left(L, N\right) - F\left(N, L\right) + F\left(L, L\right)\right]} \int_L^N \int_L^{l_1^*} f\left(x_1, x_2\right) dx_1 dx_2$$

$$= \frac{\left[F\left(l^*{}_1, N\right) - F\left(L, N\right) - F\left(l^*{}_1, L\right) + F\left(L, L\right)\right]}{\left[F\left(N, N\right) - F\left(L, N\right) - F\left(N, L\right) + F\left(L, L\right)\right]}$$

$$= \frac{q_1 - p_1 + \pi - \beta\left(\delta q_1 + \left(1 - \delta\right) q_2\right) + p_2 \beta}{q_1 + \pi - \beta\left(\delta q_1 + \left(1 - \delta\right) q_2\right) + p_2 \beta + \alpha(s_1 - p_2)},$$

(44)

$$F\left(N, l_2^*\right) = \int_L^N \int_L^{l_2^*} f\left(x_1, x_2 \mid L \le x_1 \le N, L \le x_2 \le N\right) \, dx_2 dx_1$$

$$= \frac{1}{\left[F\left(N, N\right) - F\left(L, N\right) - F\left(N, L\right) + F\left(L, L\right)\right]} \int_L^N \int_L^{l_2^*} f\left(x_1, x_2\right) dx_2 dx_1$$

$$= \frac{\left[F\left(N, l_2^*\right) - F\left(N, L\right) - F\left(L, l_2^*\right) + F\left(L, L\right)\right]}{\left[F\left(N, N\right) - F\left(L, N\right) - F\left(N, L\right) + F\left(L, L\right)\right]}$$

$$= \frac{q_2 - p_2 + \pi}{q_2 + \pi}.$$

(45)

After truncating the distribution and replacing $f\left(x_1, x_2 \mid L \le x_1 \le N, L \le x_2 \le N\right)$ by $f\left(x_1, x_2\right)$ in Equations (44)-(45), l_1^* and l_2^* have been approximated through numerical computation. The method by which numerical computation has been utilized is shown in Appendix 3.

5. **Numerical results.** In this example, demand is assumed to have a bivariate normal distribution and demand distribution of the two periods is assumed to be bivariate normal $N(243, \ 190, \ 134.49, \ 63.5, -50\%)$. Subsequently, we see the model parameters by which l_1^* and l_2^* can be computed:

Parameters:
$\beta = 100\%$, $\alpha = 100\%$, $\delta = 60\%$, $L = 144.33$, $N = 288.67$, $q_1, q_2 = 10$, $P_1, P_2 = 3$, $S_1, S_2 = 1$, $\pi = 1$, $\epsilon = 0.001$, $f\left(x_1, x_2\right) \sim N\left(243, 190, 134.49, 63.5, -50\%\right)$.

In the following, the parameters have been inserted into the formulas. Additionally, according to Appendix 2, the optimal value obtained through two steps of numerical computation is $l_1^* = 220.99$ and $l_2^* = 240.88$.

After computing the optimal solution, l_1^* and l_2^* should be inserted into Equation (46), where the value of H can be obtained as the expected value:

$$H(l_1^* = 220.99, \ l_2^* = 240.88, x_1, x_2)$$

$$= \int_{-\infty}^{l_2^*} \int_{-\infty}^{l_1^*} h_2\left(l_1^*, l_2^*, x_1, x_2\right) f\left(x_1, x_2\right) dx_1 dx_2$$

$$+ \int_{l_2^*}^{\infty} \int_{-\infty}^{l_1^*} h_2\left(l_1^*, l_2^*, x_1, x_2\right) f\left(x_1, x_2\right) dx_1 dx_2$$

$$+ \int_{-\infty}^{l_2^*} \int_{l_1^*}^{\infty} h_2\left(l_1^*, l_2^*, x_1, x_2\right) f\left(x_1, x_2\right) dx_1 dx_2$$

$$+ \int_{l_2^*}^{\infty} \int_{l_1^*}^{\infty} h_2 \left(l_1^*, l_2^*, x_1, x_2\right) f\left(x_1, x_2\right) dx_1 dx_2 = 866.59. \qquad (46)$$

Table 3 and Figure 2 show that proposed model leads to a better profit (H) than Matsuyama [17] our instance problems. As a result, a model that takes into account interdependent demand oughts to provide a better solution than a model based on independent demand.

TABLE 3. Differences between the proposed model and [17].

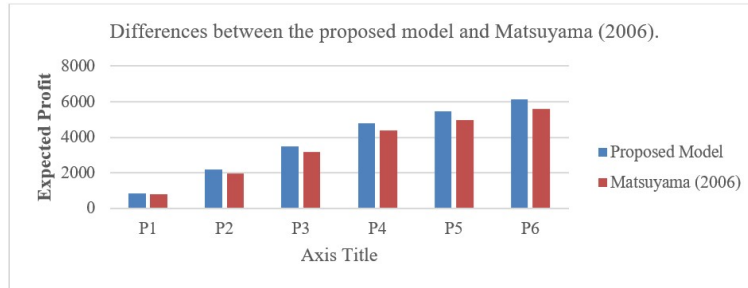| Problem	Expected Profit (H^*) of Proposed Model Corrolation = -0.5	Expected Profit of Matsuyama [17] Correlation = 0	Gap
P1	866.59	790.38	8.79%
P2	2173.1	1983	8.75%
P3	3486.3	3181.7	8.74%
P4	4801.6	4382.3	8.73%
P5	5459.7	4983	8.73%
P6	6118	5583.9	8.73%
		Mean(Gap)	8.75%
		Variance(Gap)	0.0000061%

FIGURE 2. Chart of Differences between the proposed model and Matsuyama [17].

However, we would like to recall that our results in Table 3 are based a bivariate normal distribution and numerical calculation; these results may change when distribution and parameters are becoming varied. The subject of future research will be about the question why the results obtained by our this model are better than the outcomes of previous methods, from a rigorous mathematical perspective.

We underline that this model can be used in a number of further applications, such as procurement of raw materials in projects (e.g., construction, bridge-building and molding) where demand of different periods is interdependent ([19, 21]).

6. **Sensitivity analysis.** The formulation developed for the proposed problem is based on a number of assumptions that involve several parameters; therefore, the sensitivity of objective functions to variations in the parameters needs to be analyzed. Table 4 shows the results of sensitivity analysis obtained by determining the expected profit after making slight changes in seven parameters. The most important results are presented in Table 4, and proofs of all formulations are presented in Appendix 3.

TABLE 4. Sensitivity analysis of the proposed model.

Parameter	Result of differentiation	Proof
α	$\frac{\partial}{\partial \alpha} H\left(l_1, l_2, x_1, x_2\right) = \frac{p_2 - s_1}{\|p_2 - s_1\|}$	Appendix 3 Proof 2
β	$\frac{\partial}{\partial \beta} H\left(l_1, l_2, x_1, x_2\right) = \frac{(\delta q_1 + (1-\delta)q_2) - p_2}{\|(\delta q_1 + (1-\delta)q_2) - p_2\|}$	Appendix 3 Proof 2
δ	$\frac{\partial}{\partial \delta} H\left(l_1, l_2, x_1, x_2\right) = \frac{q_1 - q_2}{\|q_1 - q_2\|}$	Appendix 3 Proof 2
π	$\frac{\partial}{\partial \pi} H\left(l_1, l_2, x_1, x_2\right) < 0, \qquad \forall \pi$	Appendix 3 Proof 2
q_1	$\frac{\partial}{\partial q_1} H\left(l_1, l_2, x_1, x_2\right) > 0 \qquad \forall\, q_1$	Appendix 3 Proof 3
q_2	$\frac{\partial}{\partial q_2} H\left(l_1, l_2, x_1, x_2\right) > 0 \qquad \forall\, q_2$	Appendix 3 Proof 3
p_1	$\frac{\partial}{\partial p_1} H\left(l_1, l_2, x_1, x_2\right) < 0, \qquad \forall\, p_1$	Appendix 3 Proof 4
p_2	$\frac{\partial}{\partial p_2} H\left(l_1, l_2, x_1, x_2\right) = -l_2 + \alpha\left(l_1 - \mu_1\right)$ $+ (\alpha - \beta) \int_{-\infty}^{\infty} \int_{l_1}^{\infty} (x_1 - l_1) f\left(x_1, x_2\right) dx_1 dx_2$	Appendix 3 Proof 4

Table 4 shows that for all q_1, q_2 the value of $H\left(l_1, l_2, x_1, x_2\right)$ is ascending and for π, p_1 the value of $H\left(l_1, l_2, x_1, x_2\right)$ is descending. The value of other parameters such as β and δ depend on $p_2 - s_1$, $(\delta q_1 + (1 - \delta)\, q_2) - p_2$ and $q_1 - q_2$. Moreover, $H\left(l_1, l_2, x_1, x_2\right)$ is ascending whenever it is positive, and is descending otherwise. However, the parameter p_2 depends on other parameters and its status is unknown. Proofs of these sensitivity analyses are presented in Appendix 3.

Tables 5-7 and Figures 3-5 show a sensitivity analysis on expected value (H) of the ratio α, β, δ that have proved based on differentiation in Table 4. Parameters vary between 20% to 100%, both increasing and decreasing of the parameters is considerd in Tables 5-7.

TABLE 5. Sensitivity analysis on expected profit (H) of the ratio $(0 \leq \alpha \leq 1)$.

α	H^*	l_1^*	l_2^*
20%	861.47	190.73	240.88
40%	862.27	195.58	240.88
60%	863.3	201.73	240.88
80%	864.65	209.82	240.88
100%	866.59	220.99	240.88

Table 5 and Figure 3 reveal that expected value (H) increases when α increases; this has been proved based on differentiation in Table 4.
Table 6 and Figure 4 show that the expected value (H) increases when β increases; this was demonstrated through differentiation in Table 4.
By Table 7 and Figure 5, we learn that the expected value (H) increases when δ rises, this was proved via differentiation in Table 4. All the above examples disclose that our model is verified and meets the requirements a good matematical model.
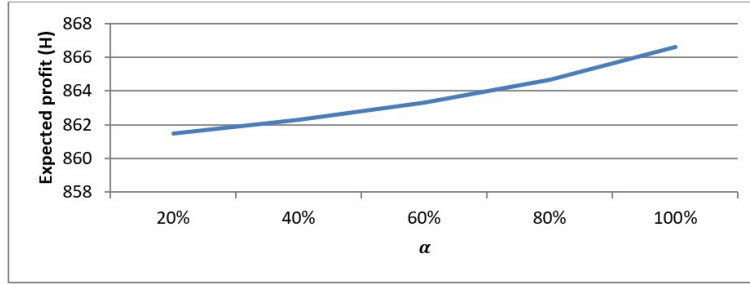
FIGURE 3. Chart of the ratio (α).

TABLE 6. Sensitivity analysis on expected profit (H) of the ratio $(0 \leq \beta \leq 1)$.

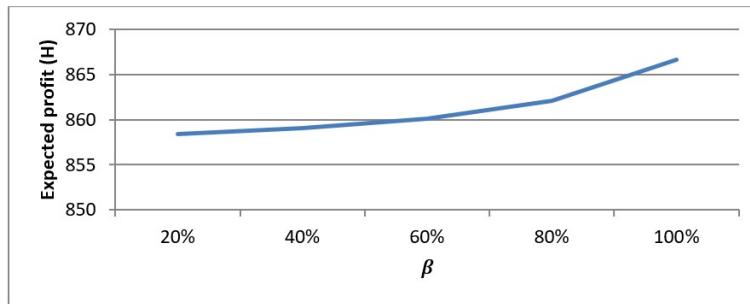| β	H^*	l_1^*	l_2^*
20%	858.4	270.2	240.88
40%	859.07	266.14	240.88
60%	860.13	259.76	240.88
80%	862.04	248.29	240.88
100%	866.59	220.99	240.88

FIGURE 4. Chart of the ratio (β).

TABLE 7. Sensitivity analysis on expected profit (H) of the ratio $(0 \leq \delta \leq 1)$.

δ	H^*	l_1^*	l_2^*
0%	1532.1	271.1	240.88
20%	1532.5	274.55	240.88
40%	1533.1	271.1	240.88
60%	1534	265.4	240.88
80%	1535.9	254.17	240.88
100%	1541.3	220.98	240.88

7. **Conclusion and outlook.** The present paper sought to develop the formulation provided by Matsuyama [17] to study the two-period newsvendor problem with unsatisfied demand or unsold quantity and interdependent demand as a new
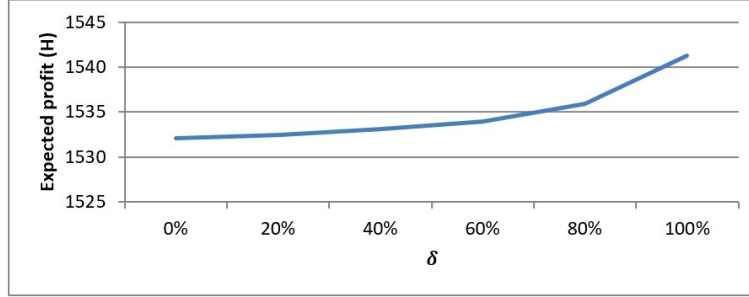
FIGURE 5. Chart of the ratio (δ).

approach to this problem. Due to the complexity of the solution of multiple integrals, the problem was assessed for only two periods. The aim was to calculate the optimal value of the initial inventory at each period; because of the emergence of double integrals in the distribution function, the initial inventory was approximated by numerical computation. In numerical results, bivariate normal distribution was used to obtain the optimal solution, defined as the initial inventory at each period. Formulation of the proposed problem was based on a series of assumptions involving multiple parameters, so it was necessary to determine the sensitivity of the objective functions with respect to variations in the parameters. Therefore, all the parameters underwent a subjected to sensitivity analysis. The proposed model takes into account interdependent demand oughts to provide a better solution than a model based on independent demand. it can be used in a number of applications, including procurement of raw materials and products required by projects in the demands of periods are interdependent.

Future research could use this approach with multi-period (n–period) models, which can be predicted to be in the form of multiple integrals. Another approach would be to use a fuzzy demand distribution function, another multivariate distribution to achieve more conformity with real-world applications or considering budget constraints [20]. Another plausible approach would be to develop the association of competitive environments with cooperative and non-cooperative game theory in multi-newsvendor environments. A further suggestion is to develop this model by adding quantity discounts and nonzero lead times to motivate customers to buy perishable goods in single- or multi-product problems.

8. Appendices.
8.1. Appendix 1.
Proof 1: Differentiation of expected profit value with respect to l_2

To obtain the optimal solution l_2^* with a process similar to that followed in Equations (6)-(36), $H(l_1, l_2, x_1, x_2)$ should be differentiated with respect to l_2^*:

$$H(l_1, l_2, x_1, x_2) = \sum_{j=1}^{4} \int_{q_1(l_1)}^{q_2(l_1)} \int_{p_1(l_2)}^{p_2(l_2)} H_j(l_1, l_2, x_1, x_2)\, dx_2 dx_1, \qquad (47)$$

$$\frac{\partial}{\partial l_2} H(l_1, l_2, x_1, x_2) = 0, \qquad (48)$$

$$H_j(l_1, l_2, x_1, x_2) = h_{2j}(l_1, l_2, x_1, x_2)\, f(x_1, x_2), \qquad (49)$$

$$\frac{\partial}{\partial l_2} H_j\left(l_1, l_2, x_1, x_2\right) = \frac{\partial}{\partial l_2} h_{2j}\left(l_1, l_2, x_1, x_2\right) f\left(x_1, x_2\right)$$

$$= f\left(x_1, x_2\right) \frac{\partial}{\partial l_2} h_{2j}\left(l_1, l_2, x_1, x_2\right). \tag{50}$$

In Equations (48)-(49), $H\left(l_1, l_2, x_1, x_2\right)$ is differentiated, and $H_j\left(l_1, l_2, x_1, x_2\right)$ is changed to $h_{2j}\left(l_1, l_2, x_1, x_2\right) f\left(x_1, x_2\right)$ in Equation (49). $\frac{\partial}{\partial l_2} H_j\left(l_1, l_2, x_1, x_2\right)$ is differentiated to obtain $f\left(x_1, x_2\right) \frac{\partial}{\partial l_2} h_{2j}\left(l_1, l_2, x_1, x_2\right)$ in Equation (50).

$$Let bek_j' = \frac{\partial}{\partial l_2} h_{2j}\left(l_1, l_2, x_1, x_2\right); \text{ then,} \tag{51}$$

$$\frac{\partial}{\partial l_2} H_j\left(l_1, l_2, x_1, x_2\right) = k_j' f\left(x_1, x_2\right), \tag{52}$$

$$\frac{\partial}{\partial l_2} H\left(l_1, l_2, x_1, x_2\right) = \frac{\partial}{\partial l_2} \sum_{j=1}^{4} \int_{q_1(l_1)}^{q_2(l_1)} \int_{p_1(l_2)}^{p_2(l_2)} H_j\left(l_1, l_2, x_1, x_2\right) dx_2 dx_1$$

$$= \sum_{j=1}^{4} \int_{q_1(l_1)}^{q_2(l_1)} \frac{\partial}{\partial l_2} \int_{p_1(l_2)}^{p_2(l_2)} H_j\left(l_1, l_2, x_1, x_2\right) dx_2 dx_1$$

$$= \sum_{j=1}^{4} \int_{q_1(l_1)}^{q_2(l_1)} [\int_{p_1(l_2)}^{p_2(l_2)} \frac{\partial}{\partial l_2} H_j\left(l_1, l_2, x_1, x_2\right) dx_2$$

$$+ \frac{\partial}{\partial l_2} p_2\left(l_2\right) H_j\left(l_1, l_2, x_1, p_2\left(l_2\right)\right)$$

$$- \frac{\partial}{\partial l_2} p_1(l_2) H_j\left(l_1, l_2, x_1, p_2\left(l_2\right)\right)] dx_1$$

$$= \sum_{j=1}^{4} \int_{q_1(l_1)}^{q_2(l_1)} [\int_{p_1(l_2)}^{p_2(l_2)} k_j' f\left(x_1, x_2\right) dx_2$$

$$+ \frac{\partial}{\partial l_2} p_2(l_2) H_j\left(l_1, l_2, x_1, p_2\left(l_2\right)\right)$$

$$- \frac{\partial}{\partial l_2} p_1(l_2) H_j\left(l_1, l_2, x_1, p_2\left(l_2\right)\right)] dx_2, \tag{53}$$

$$\frac{\partial}{\partial l_2} H(l_1, l_2, x_1, x_2)$$

$$= \int_{-\infty}^{l_1} \int_{-\infty}^{l_2} k_1' f\left(x_1, x_2\right) dx_2 dx_1 + \int_{-\infty}^{l_1} \int_{l_2}^{\infty} k_2' f\left(x_1, x_2\right) dx_2 dx_1$$

$$+ \int_{l_1}^{\infty} \int_{-\infty}^{l_2} k_3' f\left(x_1, x_2\right) dx_2 dx_1 + \int_{l_1}^{\infty} \int_{l_2}^{\infty} k_4' f\left(x_1, x_2\right) dx_2 dx_1$$

$$+ \int_{-\infty}^{l_1} [H_1\left(l_1, l_2, x_1, l_2\right) - H_2\left(l_1, l_2, x_1, l_2\right)] dx_1$$

$$+ \int_{l_2}^{\infty} [H_3\left(l_1, l_2, x_1, l_2\right) - H_4\left(l_1, l_2, x_1, l_2\right)] dx_1. \tag{54}$$

According to $\frac{\partial}{\partial l_2} H\left(l_1, l_2, x_1, x_2\right)$ in Equations (52)-(53), the differentials of the dual integral need to be obtained and then the resulting term needs to be inserted into Equation (54); in the following, the terms $H_1\left(l_1, l_2; l_1, x_2\right) - H_3\left(l_1, l_2; l_1, x_2\right)$ and $H_2\left(l_1, l_2; l_1, x_2\right) - H_4\left(l_1, l_2; l_1, x_2\right)$ in Equation (55) are calculated.

Corollary 2.

Proof.

$$H_1\left(l_1, l_2, x_1, l_2\right) = h_{21}\left(l_1, l_2, x_1, l_2\right) f\left(x_1, l_2\right), \tag{55}$$

$$H_2\left(l_1, l_2, x_1, l_2\right) = h_{22}\left(l_1, l_2, x_1, l_2\right) f\left(x_1, l_2\right), \tag{56}$$

$$H_3\left(l_1, l_2, x_1, l_2\right) = h_{23}\left(l_1, l_2, x_1, l_2\right) f\left(x_1, l_2\right), \tag{57}$$

$$H_4\left(l_1, l_2, x_1, l_2\right) = h_{24}\left(l_1, l_2, x_1, l_2\right) f\left(x_1, l_2\right), \tag{58}$$

$$H_1\left(l_1, l_2, x_1, l_2\right) - H_2\left(l_1, l_2, x_1, l_2\right) = \left(h_{21}\left(l_1, l_2, x_1, l_2\right)\right.$$
$$\left. - h_{22}\left(l_1, l_2, x_1, l_2\right)\right) f\left(x_1, l_2\right), \tag{59}$$

$$h_{21}\left(l_1, l_2, x_1, l_2\right) - h_{22}\left(l_1, l_2, x_1, l_2\right) = 0, \tag{60}$$

$$H_1\left(l_1, l_2, x_1, l_2\right) - H_2\left(l_1, l_2, x_1, l_2\right) = 0, \tag{61}$$

$$H_3\left(l_1, l_2, x_1, l_2\right) - H_4\left(l_1, l_2, x_1, l_2\right) = \left(h_{23}\left(l_1, l_2, x_1, l_2\right)\right.$$
$$\left. - h_{24}\left(l_1, l_2, x_1, l_2\right)\right) f\left(x_1, l_2\right), \tag{62}$$

$$h_{23}\left(l_1, l_2, x_1, l_2\right) - h_{24}\left(l_1, l_2, x_1, l_2\right) = 0, \tag{63}$$

$$H_3\left(l_1, l_2, x_1, l_2\right) - H_4\left(l_1, l_2, x_1, l_2\right) = 0. \tag{64}$$

$$\square$$

According to $\frac{\partial}{\partial l_2} H\left(l_1, l_2, x_1, x_2\right)$, $H_1\left(l_1, l_2, l_1, x_2\right)$, $H_3\left(l_1, l_2, l_1, x_2\right)$, $H_2\left(l_1, l_2, l_1, x_2\right)$, $H_4\left(l_1, l_2, l_1, x_2\right)$ in Equation (54), it can be easily shown that the terms computed in Equations (55)-(64) cancel each other. Finally, it has been proved that $H_1\left(l_1, l_2; l_1, x_2\right) - H_3\left(l_1, l_2; l_1, x_2\right)$ and $H_2\left(l_1, l_2; l_1, x_2\right) - H_4\left(l_1, l_2; l_1, x_2\right)$ are equal to zero in Equations (61)-(64):

$$\frac{\partial}{\partial l_2} H(l_1, l_2, x_1, x_2)$$

$$= \int_{-\infty}^{l_1} \int_{-\infty}^{l_2} k_1' f\left(x_1, x_2\right) dx_2 dx_1 + \int_{-\infty}^{l_1} \int_{l_2}^{\infty} k_2' f\left(x_1, x_2\right) dx_2 dx_1$$

$$+ \int_{l_1}^{\infty} \int_{-\infty}^{l_2} k_3' f\left(x_1, x_2\right) dx_2 dx_1 + \int_{l_1}^{\infty} \int_{l_2}^{\infty} k_4' f\left(x_1, x_2\right) dx_2 dx_1$$

$$= k_1' \int_{-\infty}^{l_1} \int_{-\infty}^{l_2} f\left(x_1, x_2\right) dx_2 dx_1 + k_2' \int_{-\infty}^{l_1} \int_{l_2}^{\infty} f\left(x_1, x_2\right) dx_2 dx_1$$

$$+ k_3' \int_{l_1}^{\infty} \int_{-\infty}^{l_2} f\left(x_1, x_2\right) dx_2 dx_1 + k_4' \int_{l_1}^{\infty} \int_{l_2}^{\infty} f\left(x_1, x_2\right) dx_2 dx_1$$

$$= k_1' F\left(l_1, l_2\right) + k_2' \left(F\left(l_1\right) - F\left(l_1, l_2\right)\right) + k_3' \left(F\left(l_2\right) - F\left(l_1, l_2\right)\right)$$

$$+ k_4' \left(1 - F\left(l_1\right) - F\left(l_2\right) + F\left(l_1, l_2\right)\right)$$

$$= \left(k_1' - k_2' - k_3' + k_4'\right) F\left(l_1, l_2\right) + \left(k_2' - k_4'\right) F\left(l_1\right) + \left(k_3' - k_4'\right) F\left(l_2\right) + k_4'. \tag{65}$$

After proving Equations (55)-(64), the result should be inserted into Equation (54) and the process of differentiation continues until $\frac{\partial}{\partial l_2} H\left(l_1, l_2; x_1, x_2\right)$ becomes $\left(k_1' - k_2' - k_3' + k_4'\right) F\left(l_1, l_2\right) + \left(k_2' - k_4'\right) F\left(l_1\right) + \left(k_3' - k_4'\right) F\left(l_2\right) + k_4'$ in Equation (65).

$$\text{Let be } k_j' = \frac{\partial}{\partial l_2} h_{2j}\left(l_1, l_2, x_1, x_2\right); \text{ then,} \tag{66}$$

$$k_1' = \frac{\partial}{\partial l_2} h_{21}\left(l_1, l_2, x_1, x_2\right) = -p_2, \tag{67}$$

$$k_2' = \frac{\partial}{\partial l_2} h_{22}\left(l_1, l_2, x_1, x_2\right) = q_2 - p_2 + \pi, \tag{68}$$

$$k_3' = \frac{\partial}{\partial l_2} h_{23}\left(l_1, l_2, x_1, x_2\right) = -p_2, \tag{69}$$

$$k_4' = \frac{\partial}{\partial l_2} h_{24}\left(l_1, l_2, x_1, x_2\right) = q_2 - p_2 + \pi, \tag{70}$$

$$k_1' = k_3', k_2' = k_4'. \tag{71}$$

In this step, h_{2j} has been differentiated with respect to l_2, and the results are k_1', k_2', k_3', k_4' in Equations (66)-(71). As a result, $k_1' = k_3'$ and $k_2' = k_4'$ in Equation (71), then these parameters have been inserted into Equation (65):

$$\frac{\partial}{\partial l_2} H\left(l_1, l_2, x_1, x_2\right) = \left(k_1' - k_2' - k_3' + k_4'\right) F\left(l_1, l_2\right) + \left(k_2' - k_4'\right) F\left(l_1\right)$$
$$+ \left(k_3' - k_4'\right) F\left(l_2\right) + k_4'$$
$$= \left(k_3' - k_4'\right) F\left(l_2\right) + k_4' = 0, \tag{72}$$

$$F\left(l_2\right) = \frac{k_4'}{k_4' - k_3'} = \frac{q_2 - p_2 + \pi}{q_2 - p_2 + \pi - \left(-p_2\right)} = \frac{q_2 - p_2 + \pi}{q_2 + \pi}, \tag{73}$$

$$F\left(l_2^*\right) = \int_{-\infty}^{\infty} \int_{-\infty}^{l_2^*} f\left(x_1, x_2\right) dx_2 dx_1 = \frac{q_2 - p_2 + \pi}{q_2 + \pi}. \tag{74}$$

Finally, it has been proved that the cumulative demand function of l_2^* equals a constant parameter in Equation (74). For determining l_2^* in Equation (74), F should be inverted. However, the infinity term in the above equation and approximation of l_1^* have been computed through the numerical computations provided in Appendix 2.

8.2. Appendix 2: Numerical computation.
After truncation of the normal distribution, l_1^* needs to be obtained through numerical computation:

$$F\left(l_1^*, N\right) = \frac{F\left(l_1^*, N\right) - F\left(L, N\right) - F\left(l_1^*, L\right) + F\left(L, L\right)}{F\left(N, N\right) - F\left(L, N\right) - F\left(N, L\right) + F\left(L, L\right)}$$
$$= \frac{q_1 - p_1 + \pi - \beta\left(\delta q_1 + \left(1 - \delta\right) q_2\right) + p_2\beta}{q_1 + \pi - \beta\left(\delta q_1 + \left(1 - \delta\right) q_2\right) + p_2\beta + \alpha(s_1 - p_2)}. \tag{75}$$

Let be $A = \frac{q_1 - p_1 + \pi - \beta(\delta q_1 + (1-\delta)q_2) + p_2\beta}{q_1 + \pi - \beta(\delta q_1 + (1-\delta)q_2) + p_2\beta + \alpha(s_1 - p_2)}$; then,

$$F\left(l_1^*, N\right) = A\left[F\left(N, N\right) - F\left(L, N\right) - F\left(N, L\right) + F\left(L, L\right)\right] + F\left(L, N\right)$$
$$+ F\left(l_1^*, L\right) - F\left(L, L\right), \tag{76}$$

$$F\left(l_1^{*'}, N\right) \cong A\left[F\left(N, N\right) - F\left(L, N\right) - F\left(N, L\right) + F\left(L, L\right)\right] + F\left(L, N\right)$$
$$+ F\left(N, L\right) - F\left(L, L\right), \tag{77}$$

$$F\left(l_1^*, N\right) \cong \left[F\left(N, N\right) - F\left(L, N\right) - F\left(N, L\right) + F\left(L, L\right)\right] + F\left(L, N\right)$$
$$+ F\left(l_1^{*'}, L\right) - F\left(L, L\right), \tag{78}$$

$$\left| F\left(l_1^*, N\right) - F\left(l_1^{*'}, N\right)\right| < \epsilon. \tag{79}$$

Here, l_1^* has been received through almost two steps of numerical computation, until $\left| F\left(l_1^*, N\right) - F\left(l_1^{*'}, N\right) \right| < \epsilon$ occurs in Equations (75)-(79).

Moreover, after truncation of the normal distribution, l_2^* also needs to be obtained through numerical computation:

$$F\left(N, l_2^*\right) = \frac{F\left(N, l_2^*\right) - F\left(N, L\right) - F\left(L, l_2^*\right) + F\left(L, L\right)}{F\left(N, N\right) - F\left(L, N\right) - F\left(N, L\right) + F\left(L, L\right)} = \frac{q_2 - p_2 + \pi}{q_2 + \pi}, \quad (80)$$

Let be $B = \frac{F\left(N, l_2^*\right) - F\left(N, L\right) - F\left(L, l_2^*\right) + F\left(L, L\right)}{F\left(N, N\right) - F\left(L, N\right) - F\left(N, L\right) + F\left(L, L\right)} = \frac{q_2 - p_2 + \pi}{q_2 + \pi}$; then,

$$F\left(N, l_2^*\right) - F\left(N, L\right) - F\left(L, l_2^*\right) + F\left(L, L\right) = B[F\left(N, N\right) - F\left(L, N\right)$$
$$- F\left(N, L\right) + F\left(L, L\right)], \quad (81)$$

$$F\left(N, l_2^*\right) = B\left[F\left(N, N\right) - F\left(L, N\right) - F\left(N, L\right) + F\left(L, L\right)\right] + F\left(N, L\right)$$
$$+ F\left(L, l_2^*\right) - F\left(L, L\right), \quad (82)$$

$$F\left(N, l_2^{*'}\right) \cong B\left[F\left(N, N\right) - F\left(L, N\right) - F\left(N, L\right) + F\left(L, L\right)\right] + F\left(N, L\right)$$
$$+ F\left(L, N\right) - F\left(L, L\right), \quad (83)$$

$$F\left(N, l_2^*\right) \cong B\left[F\left(N, N\right) - F\left(L, N\right) - F\left(N, L\right) + F\left(L, L\right)\right] + F\left(N, L\right)$$
$$+ F\left(L, l_2^{*'}\right) - F\left(L, L\right), \quad (84)$$

$$\left| F\left(N, l_2^*\right) - F\left(N, l_2^{*'}\right) \right| < \epsilon. \quad (85)$$

Now l_2^* has also been got through almost two steps of numerical computation, until $\left| F\left(N, l_2^*\right) - F\left(N, l_2^{*'}\right) \right| < \epsilon$ occurs in Equations (80)-(85).

8.3. Appendix 3: Sensitivity analysis.
Proof 2: Differentiation of expected profit value with respect to α, β, δ, π

The process of differentiation of the expected profit value $H\left(l_1, l_2, x_1, x_2\right)$ with respect to α is similar to the one with respect to l_j:

$$\frac{\partial}{\partial \alpha} H_j\left(l_1, l_2, x_1, x_2\right) = \frac{\partial}{\partial \alpha} h_{2j}\left(l_1, l_2, x_1, x_2\right) f\left(x_1, x_2\right)$$

$$= f\left(x_1, x_2\right) \frac{\partial}{\partial \alpha} h_{2j}\left(l_1, l_2, x_1, x_2\right), \quad (86)$$

$$\frac{\partial}{\partial \alpha} h_{21}\left(l_1, l_2, x_1, x_2\right) = \frac{\partial}{\partial \alpha} h_{22}\left(l_1, l_2, x_1, x_2\right) = -s_1\left(l_1 - x_1\right) + p_2\left(\left(l_1 - x_1\right)\right)$$

$$= \left(p_2 - s_1\right)\left(l_1 - x_1\right), \quad (87)$$

$$\frac{\partial}{\partial \alpha} h_{23}\left(l_1, l_2, x_1, x_2\right) = \frac{\partial}{\partial \alpha} h_{24}\left(l_1, l_2, x_1, x_2\right) = 0, \quad (88)$$

$$[b]\frac{\partial}{\partial \alpha} H\left(l_1, l_2, x_1, x_2\right) = \frac{\partial}{\partial \alpha} \sum_{j=1}^{4} \int_{q_1(l_1)}^{q_2(l_1)} \int_{p_1(l_2)}^{p_2(l_2)} H_j\left(l_1, l_2, x_1, x_2\right) dx_2 dx_1$$

$$= \sum_{j=1}^{4} \int_{q_1(l_1)}^{q_2(l_1)} \int_{p_1(l_2)}^{p_2(l_2)} \frac{\partial}{\partial \alpha} H_j\left(l_1, l_2, x_1, x_2\right) dx_2 dx_1$$

$$= \int_{-\infty}^{l_2} \int_{-\infty}^{l_1} \left(p_2 - s_1\right)\left(l_1 - x_1\right) f\left(x_1, x_2\right) dx_1 dx_2$$

$$+ \int_{l_2}^{\infty} \int_{-\infty}^{l_1} (p_2 - s_1) (l_1 - x_1) f(x_1, x_2) dx_1 dx_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{l_1} (p_2 - s_1) (l_1 - x_1) f(x_1, x_2) dx_1 dx_2$$

$$= (p_2 - s_1) \int_{-\infty}^{\infty} \int_{-\infty}^{l_1} (l_1 - x_1) f(x_1, x_2) dx_1 dx_2, \qquad (89)$$

$$\frac{\partial}{\partial \alpha} H(l_1, l_2, x_1, x_2) = \frac{p_2 - s_1}{|p_2 - s_1|}. \qquad (90)$$

In Equations (86)-(90), the result of $\frac{\partial}{\partial \alpha} H(l_1, l_2, x_1, x_2)$ depends on α and $p_2 - s_1$; if they are positive, $H(l_1, l_2, x_1, x_2)$ is ascending, otherwise it is descending. This means that if $p_2 - s_1 \geq 0$ and α is increasing, then $H(l_1, l_2, x_1, x_2)$ is also increasing.

The process of differentiation of the expected profit value $H(l_1, l_2, x_1, x_2)$ with respect to β is similar to the one with respect to l_j:

$$\frac{\partial}{\partial \beta} H(l_1, l_2, x_1, x_2) = \frac{(\delta q_1 + (1 - \delta) q_2) - p_2}{|(\delta q_1 + (1 - \delta) q_2) - p_2|}. \qquad (91)$$

The result of $\frac{\partial}{\partial \beta} H(l_1, l_2, x_1, x_2)$ depends on β and $(\delta q_1 + (1-\delta) q_2) - p_2$; if they are positive, $H(l_1, l_2, x_1, x_2)$ is ascending, otherwise it is descending. This means that if $(\delta q_1 + (1 - \delta) q_2) - p_2 \geq 0$ and β are increasing, then $H(l_1, l_2, x_1, x_2)$ is also increasing.

The process of differentiation of the expected profit value $H(l_1, l_2, x_1, x_2)$ with respect to δ is similar to the one with respect to l_j:

$$\frac{\partial}{\partial \delta} H(l_1, l_2, x_1, x_2) = \frac{q_1 - q_2}{|q_1 - q_2|}. \qquad (92)$$

The result of $\frac{\partial}{\partial \delta} H(l_1, l_2, x_1, x_2)$ depends on δ and $(q_1 - q_2)$, if they are positive, $H(l_1, l_2, x_1, x_2)$ is ascending, otherwise it is descending. This means that if $(q_1 - q_2) \geq 0$ and δ is increasing, then $H(l_1, l_2, x_1, x_2)$ is also increasing.

The process of differentiation of the expected profit value $H(l_1, l_2, x_1, x_2)$ with respect to π is similar to the one with respect to l_j:

$$\frac{\partial}{\partial \pi} H(l_1, l_2, x_1, x_2) < 0. \qquad (93)$$

The result of $\frac{\partial}{\partial \pi} H(l_1, l_2, x_1, x_2)$ is descending. Hence, this means that if π is increasing, then $H(l_1, l_2, x_1, x_2)$ is also increasing.

Proof 3:

Differentiation of expected profit value with respect to q_1, q_2

The process of differentiation of the expected profit value $H(l_1, l_2, x_1, x_2)$ with respect to q_1 is similar to the one with respect to l_i:

$$\frac{\partial}{\partial q_1} H_j(l_1, l_2, x_1, x_2) = \frac{\partial}{\partial q_1} h_{2j}(l_1, l_2, x_1, x_2) f(x_1, x_2)$$

$$= f(x_1, x_2) \frac{\partial}{\partial q_1} h_{2j}(l_1, l_2, x_1, x_2), \qquad (94)$$

$$\frac{\partial}{\partial q_1} h_{21}(l_1, l_2, x_1, x_2) = \frac{\partial}{\partial q_1} h_{22}(l_1, l_2, x_1, x_2) = x_1, \qquad (95)$$

$$\frac{\partial}{\partial q_1} h_{23}(l_1, l_2, x_1, x_2) = \frac{\partial}{\partial q_1} h_{24}(l_1, l_2, x_1, x_2) = l_1 + \delta \beta (x_1 - l_1), \qquad (96)$$

$$\frac{\partial}{\partial q_1} H(l_1, l_2, x_1, x_2) = \frac{\partial}{\partial q_1} \sum_{j=1}^{4} \int_{q_1(l_1)}^{q_2(l_1)} \int_{p_1(l_2)}^{p_2(l_2)} H_j(l_1, l_2, x_1, x_2) \, dx_2 dx_1$$

$$= \sum_{j=1}^{4} \int_{q_1(l_1)}^{q_2(l_1)} \int_{p_1(l_2)}^{p_2(l_2)} \frac{\partial}{\partial q_1} H_j(l_1, l_2, x_1, x_2) \, dx_2 dx_1$$

$$= \int_{-\infty}^{l_2} \int_{-\infty}^{l_1} x_1 f(x_1, x_2) \, dx_1 dx_2$$

$$+ \int_{l_2}^{\infty} \int_{-\infty}^{l_1} x_1 f(x_1, x_2) \, dx_1 dx_2$$

$$+ \int_{-\infty}^{l_2} \int_{l_1}^{\infty} (l_1 + \delta\beta(x_1 - l_1)) f(x_1, x_2) \, dx_1 dx_2$$

$$+ \int_{l_2}^{\infty} \int_{l_1}^{\infty} (l_1 + \delta\beta(x_1 - l_1)) f(x_1, x_2) \, dx_1 dx_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{l_1} f(x_1, x_2) \, dx_1 dx_2$$

$$+ \int_{-\infty}^{\infty} \int_{l_1}^{\infty} (l_1 + \delta\beta(x_1 - l_1)) f(x_1, x_2) \, dx_1 dx_2$$

$$= \mu_1 + \int_{-\infty}^{\infty} \int_{l_1}^{\infty} (l_1 + \delta\beta(x_1 - l_1) - x_1) f(x_1, x_2) \, dx_1 dx_2$$

$$= \mu_1 + \int_{-\infty}^{\infty} \int_{l_1}^{\infty} (\delta\beta - 1)(x_1 - l_1) f(x_1, x_2) \, dx_1 dx_2$$

$$= \mu_1 + (\delta\beta - 1) \int_{-\infty}^{\infty} \int_{l_1}^{\infty} (x_1 - l_1) f(x_1, x_2) \, dx_1 dx_2, \tag{97}$$

$$\frac{\partial}{\partial q_1} H(l_1, l_2, x_1, x_2) > 0. \tag{98}$$

As Equations (94)-(98) show, $\frac{\partial}{\partial q_1} H(l_1, l_2, x_1, x_2)$ is ascending, which means that if q_1 is increasing, then $H(l_1, l_2, x_1, x_2)$ is also increasing.

$\frac{\partial}{\partial q_2} H(l_1, l_2, x_1, x_2)$ for parameter q_2 is ascending ($\frac{\partial}{\partial q_2} H(l_1, l_2, x_1, x_2) > 0$). This means that if q_2 is increasing, then $H(l_1, l_2, x_1, x_2)$ is also increasing.

Proof 4:

Differentiation of expected profit value with respect to p_1, p_2

The process of differentiation of the expected profit value $H(l_1, l_2, x_1, x_2)$ with respect to p_1 is similar to the one with respect to l_j:

$$\frac{\partial}{\partial p_1} H_j(l_1, l_2, x_1, x_2) = \frac{\partial}{\partial p_1} h_{2j}(l_1, l_2, x_1, x_2) f(x_1, x_2)$$

$$= f(x_1, x_2) \frac{\partial}{\partial p_1} h_{2j}(l_1, l_2, x_1, x_2), \tag{99}$$

$$\frac{\partial}{\partial p_1} h_{2j}(l_1, l_2, x_1, x_2) = -l_1, \tag{100}$$

$$\frac{\partial}{\partial p_1} H(l_1, l_2, x_1, x_2) = \frac{\partial}{\partial p_1} \sum_{j=1}^{4} \int_{q_1(l_1)}^{q_2(l_1)} \int_{p_1(l_2)}^{p_2(l_2)} H_j(l_1, l_2, x_1, x_2) \, dx_2 dx_1$$

$$= \sum_{j=1}^{4} \int_{q_1(l_1)}^{q_2(l_1)} \int_{p_1(l_2)}^{p_2(l_2)} \frac{\partial}{\partial p_1} H_j \left(l_1, l_2, x_1, x_2 \right) dx_2 dx_1$$

$$= \sum_{j=1}^{4} \int_{q_1(l_1)}^{q_2(l_1)} \int_{p_1(l_2)}^{p_2(l_2)} -l_1 f \left(x_1, x_2 \right) dx_2 dx_1$$

$$= -l_1 \sum_{j=1}^{4} \int_{q_1(l_1)}^{q_2(l_1)} \int_{p_1(l_2)}^{p_2(l_2)} f \left(x_1, x_2 \right) dx_2 dx_1 = -l_1, \qquad (101)$$

$$\frac{\partial}{\partial p_1} H \left(l_1, l_2, x_1, x_2 \right) = -l_1 < 0. \qquad (102)$$

As Equations (99)-(102) show, $\frac{\partial}{\partial \pi} H \left(l_1, l_2, x_1, x_2 \right)$ for parameter π is descending. This means that if p_1 is increasing, then $H \left(l_1, l_2, x_1, x_2 \right)$ is also increasing.

$\frac{\partial}{\partial p_2} H \left(l_1, l_2, x_1, x_2 \right)$ with regard to parameter p_2 depends of the other parameters, and its status is unknown:

$$\frac{\partial}{\partial p_2} H \left(l_1, l_2, x_1, x_2 \right)$$
$$= -l_2 + \alpha \left(l_1 - \mu_1 \right) + \left(\alpha - \beta \right) \int_{-\infty}^{\infty} \int_{l_1}^{\infty} \left(x_1 - l_1 \right) f \left(x_1, x_2 \right) dx_1 dx_2. \qquad (103)$$

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