A NONCOMMUTATIVE THEORY OF BADE FUNCTIONALS

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1. Prelminaries. Since the pioneering work of W. G. Bade [**3, 4**] a great deal of work has been done on bounded Boolean algebras of projections on a Banach space ([**11**, XVII.3.XVIII.3], [**21**, V.3], [**16**], [**6**], [**12**], [**13**], [**14**], [**17**], [**18**], [**23**], [**24**]). Via the Stone representation space of the Boolean algebra, the theory can be studied through Banach modules over $C(K)$, where K is a compact Hausdorff space. One of the key concepts in the theory is the notion of Bade functionals. If X is a Banach $C(K)$-module and $x \in X$, then a *Bade functional* of x with respect to $C(K)$ is a continuous linear functional α on X such that, for each a in $C(K)$ with $a \geq 0$, we have

(i) $\alpha(ax) \geq 0$,

(ii) if $\alpha(ax) = 0$, then $ax = 0$.

It is clear that the definition of Bade functionals makes sense when $C(K)$ is replaced by an arbitrary C^*-algebra. In this paper we show, using elementary C^*-algebraic techniques, that most of the known results on Bade functionals for $C(K)$ carry over to the C^*-algebra setting. Moreover, we prove an existence theorem in the C^*-algebraic case that is new even in the $C(K)$ setting. This result shows that Bade functionals always exist in many important cases, e.g. when the C^*-algebra is separable or when the Banach space is either separable or the dual of a separable space. In [**14**], T. A. Gillespie showed that, for a Banach $C(K)$-module X, Bade functionals always exist if X does not contain a copy of c_0, and he asked if the converse is true. However Bade functionals always exist when $X = c_0$, since c_0 is separable.

Although the C^*-algebraic point of view clarifies many of the results in [**11**, XVII.3, XVIII.3], this paper is written for a readership that is not assumed to be expert in C^*-algebras. We therefore provide a brief account of the properties of C^*-algebras and of Arens extensions that we need.

Throughout, $\mathscr{A}$ will denote a C^*-algebra with 1. The Gelfand–Naimark theorem says that we can assume that $\mathscr{A}$ is a C^*-subalgebra of the algebra $L(H)$ of all (bounded linear) operators on a Hilbert space H. We will choose H to have the additional properties listed in the following lemma. The interested reader can consult [**22**] for details. If X is a Banach space, we follow the notation of [**15**] and let $X^\#$ denote the normed dual of X. We will use $\mathscr{A}'$ to denote the commutant of $\mathscr{A}$, and we use $*$ to denote the involution in a C^*-algebra. However, we still use w^* to denote the weak-star topology.

LEMMA 1. *Every C^*-algebra is $*$-isomorphic to a C^*-algebra $\mathscr{A}$ of operators on a Hilbert space H with the following properties.*

(1) *For each φ in $\mathscr{A}^\#$, there are vectors e, f in H with $\|e\|^2 = \|f\|^2 = \|\varphi\|$ such that $\varphi(A) = (Ae, f)$ for all A in $\mathscr{A}$.*

(2) *The second dual $\mathscr{A}^{\#\#}$ is isometrically isomorphic to the weak operator closure $\mathscr{A}''$ of $\mathscr{A}$, with each T in $\mathscr{A}''$ acting on a φ in $\mathscr{A}^\#$ (defined as in (1)) by $T(\varphi) = (Te, f)$.*

(3) *Under the identification of $\mathscr{A}^{\#\#}$ with $\mathscr{A}''$ in (2), the w^*-topology on $\mathscr{A}^{\#\#}$ corresponds to the weak operator topology on $\mathscr{A}''$.*

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(4) *Every w^*-closed left (resp. right) ideal in $\mathscr{A}^{\#\#}$ has the form $\mathscr{A}^{\#\#}P$ (resp. $P\mathscr{A}^{\#\#}$) for some selfadjoint projection P in $\mathscr{A}^{\#\#}$.*

(5) *If $\mathscr{A}$ is separable, then H is separable.*

Throughout, X will denote a (left) Banach $\mathscr{A}$-module, i.e., X is an $\mathscr{A}$-module and a Banach space, $1x = x$ for all x in X, and the map $(a, x) \to ax$ is continuous from $A \times X$ into X. Equivalently, there is a bounded homomorphism $m: \mathscr{A} \to L(X)$ such that $m(1) = 1$ (i.e., $m(a)(x) = ax$). If we define a new norm $\| \|'$ on X by $\|x\|' = \sup\{\|m(a)x\| : \|a\| \le 1\}$, then $\|x\| \le \|x\|' \le \|m\| \, \|x\|$, so $\| \|'$ is an equivalent norm with respect to which m is a contraction. By replacing $\mathscr{A}$ with the C^*-algebra $\mathscr{A}/\ker(m)$, we can always assume that m is $1-1$. The following lemma shows that we can also assume that m is an isometry.

LEMMA 2. *Suppose $m: \mathscr{A} \to L(X)$ is a unital contractive homomorphism. Then*
(1) *if $a, b \in \mathscr{A}$ and $a^*a \le b^*b$, then $\|m(a)x\| \le \|m(b)x\|$ for every x in X;*
(2) *if m is $1-1$, then m is an isometry.*

Proof. (1) For each positive integer n, let $f_n: [0, \infty) \to [0, \infty)$ be defined by

$$f_n(t) = \begin{cases} 1/t & \text{if } t \ge 1/n \\ n & 0 \le t \le 1/n \end{cases}.$$

Then $tf_n(t) \le 1$ and $t(1 - tf_n(t))^2 \le 1/n$ for all $t \ge 0$. Also $a^*a \le b^*b$ implies that $\|d^*a^*ad\| \le \|d^*b^*bd\|$ for all d in $\mathscr{A}$. Let $c_n = af_n(b^*b)b^*$ for each $n \ge 1$. Then

$$\begin{aligned} \|c_n\|^2 = \|c_n c_n^*\| &= \|af_n(b^*b)(b^*b)^{1/2}\|^2 \\ &= \|(b^*b)^{1/2}f_n(b^*b)(a^*a)f_n(b^*b)(b^*b)^{1/2}\| \\ &\le \|(b^*b)^{1/2}f_n(b^*b)(b^*b)f_n(b^*b)(b^*b)^{1/2}\| = \|f_n(b^*b)(b^*b)\|^2 \le 1. \end{aligned}$$

Also

$$\begin{aligned} \|a - c_n b\|^2 &= \|a(1 - (b^*b)f_n(b^*b))\|^2 \\ &= (1 - (b^*b)f_n(b^*b))(a^*a)(1 - (b^*b)f_n(b^*b))\| \\ &\le \|(1 - (b^*b)f_n(b^*b))(b^*b)(1 - (b^*b)f_n(b^*b))\| \le 1/n. \end{aligned}$$

Since $\|c_n\| \le 1$ for every n and $\|a - c_n b\| \to 0$, it follows that $\|m(a)x\| = \lim \|m(c_n b)x\| \le \|m(b)x\|$ for every x in X.

(2). It follows from (1) that $\|m(a)x\| = \|m((a^*a)^{1/2})x\|$ for every x in X and every a in $\mathscr{A}$. To show that m is an isometry we need only show that $\|m(a)\| = \|a\|$ for $0 \le a \le 1$ and $\|a\| = 1$. For each positive integer n let $g_n(t) = \min(t, 1 - 1/n)$ and $h_n(t) = \max(0, 1 - n(1 - t))$. Then $\|a - g_n(a)\| \to 0$ and $h_n(a) \ne 0$ and $g_n(a)h_n(a) = (1 - 1/n)h_n(a)$ for each n. Since m is $1-1$, we see that $1 - 1/n$ is in the spectrum of $m(g_n(a))$, and since $m(g_n(a)) \to m(a)$ and the set of noninvertible elements in $L(X)$ is closed, we conclude that 1 is in the spectrum of $m(a)$. Thus $\|m(a)\| \ge 1$.

Associated with the module multiplication

$$\mathscr{A} \times X \to X : (a, x) \to ax \tag{1}$$

we define three other bilinear maps:

$$X \times X^{\#} \to \mathscr{A}^{\#} : (x, \alpha) \to \mu_{x,\alpha};\ \mu_{x,\alpha}(a) = \alpha(ax), \tag{2}$$

$$X^{\#} \times \mathscr{A}^{\#\#} \to X^{\#} : (\alpha, a) \to \alpha a;\ (\alpha a)(x) = a(\mu_{x,\alpha}), \tag{3}$$

$$\mathscr{A}^{\#\#} \times X^{\#\#} \to X^{\#\#} : (\alpha, \beta) \to \alpha\beta;\ (\alpha\beta)(\alpha) = \beta(\alpha a). \tag{4}$$

When (1) is taken as the product on $\mathscr{A}$ (i.e., $X = \mathscr{A}$), then (4) becomes the *Arens product* on $\mathscr{A}^{\#\#}$ **[5]** (which coincides with the operator product when $\mathscr{A}$ is as in Lemma 1). Furthermore, (3) defines a right Banach $\mathscr{A}^{\#\#}$-module structure on $X^{\#}$ that gives an antihomomorphism $m^{\#} : \mathscr{A}^{\#\#} \to L(X^{\#})$ defined by $m^{\#}(a)(\alpha) = \alpha a$. (The only nontrivial thing to check is the associativity, which is straightforward using Lemma 1.) The map (4) defines a Banach $\mathscr{A}^{\#\#}$-module structure on $X^{\#\#}$. Since the $\mathscr{A}^{\#\#}$-module structures on $X^{\#}$ and $X^{\#\#}$ extend the canonical induced $\mathscr{A}$-module structures on these spaces, we call (3) and (4) the *Arens extensions* of the module multiplication on X. For information on Arens extensions the reader can consult **[2]** and **[10]**.

We list some properties of $m^{\#}$ for easy reference.

LEMMA 3. *The map $m^{\#}$ has the following properties.*

(1) *For each a in $\mathscr{A}$, $m^{\#}(a)$ is the adjoint in $L(X^{\#})$ of the operator $m(a)$ in $L(X)$.*

(2) *$m^{\#}$ is $(w^*, w^*$-operator)-continuous.*

(3) *For each α in $X^{\#}$ the linear map from $\mathscr{A}^{\#\#}$ into $X^{\#}$ that sends a to αa is (w^*, w^*)-continuous.*

(4) *For each α in $X^{\#}$, there is a selfadjoint projection e_α in $\mathscr{A}^{\#\#}$ such that $\{a \in \mathscr{A}^{\#\#} : \alpha a = 0\} = (1 - e_\alpha)\mathscr{A}^{\#\#}$.*

The proof of the above lemma is straightforward if one recalls that the w^*-operator topology on $L(X^{\#})$ is generated by the seminorms $T \to |\Sigma T(\alpha_i)(x_i)|$ for finite subsets $\alpha_i \in X^{\#}$ and $x_i \in X$, $i = 1, 2, \ldots, n$. By taking adjoints we embed $L(X)$ into $L(X^{\#})$, and on $L(X)$ the w^*-operator topology coincides with the weak operator topology.

Note that the projection in part (4) of the above lemma is called the *carrier projection* of α in $\mathscr{A}^{\#\#}$.

Considering the Arens $\mathscr{A}^{\#\#}$-module structure on $X^{\#\#}$ and the natural embedding of X in $X^{\#\#}$, we let $\langle X \rangle$ denote the norm closed $\mathscr{A}^{\#\#}$-submodule generated by X, i.e., $\langle X \rangle = \mathrm{sp}\{ax : a \in \mathscr{A}^{\#\#}, x \in X\}$. On $\langle X \rangle$ the $\mathscr{A}^{\#\#}$-module structure behaves nicely. Let $\sigma = \sigma(\langle X \rangle, X^{\#})$ denote the relative w^*-topology on $\langle X \rangle \subset X^{\#\#}$.

LEMMA 4. *The following are true.*

(1) *For each z in $\langle X \rangle$, the linear map from $\mathscr{A}^{\#\#}$ to $\langle X \rangle$ that sends a to az is (w^*, σ)-continuous.*

(2) *Each z in $\langle X \rangle$ has a carrier projection e_z in $\mathscr{A}^{\#\#}$, i.e., $\{a \in \mathscr{A}^{\#\#} : az = 0\} = \mathscr{A}^{\#\#}(1 - e_z)$.*

(3) *The following are equivalent:*

(*a*) *$\langle X \rangle = X$,*

(*b*) *For each x in X, the map from $\mathscr{A}$ to X that sends a to ax is (norm, weak)-compact.*

(c) *The map $m : \mathscr{A} \to L(X)$ has an (unique) extension $\hat{m} : \mathscr{A}^{\#\#} \to L(X)$ that is $(w^*$, weak-operator)-continuous.*

(4) *If the conditions in* (3) *hold and* $\{p_\lambda\}$ *is an increasingly directed net of selfadjoint projections in* $\mathcal{A}^{\#\#}$, *and* $p_\lambda \to p(w^*)$, *then* $\hat{m}(p_\lambda) \to \hat{m}(p)$ *in the strong operator topology.*

Proof. (1) Fix α in $X^\#$, and for each z in $X^{\#\#}$ we define a map $\varphi_{z,\alpha}$ in $\mathcal{A}^{\#\#\#}$ by $\varphi_{z,\alpha}(a) = (az)(\alpha)$. Since $\mathcal{A}^\#$ is a norm closed subspace of $\mathcal{A}^{\#\#\#}$, and since the linear map from $X^{\#\#}$ into $\mathcal{A}^{\#\#\#}$ that sends z to $\varphi_{z,\alpha}$ is norm continuous it follows that $M = \{z \in X^{\#\#} : \varphi_{z,\alpha} \in \mathcal{A}^\#\}$ is a norm closed subspace of $X^{\#\#}$. Since $\varphi_{x,\alpha} = \mu_{x,\alpha}$ for every x in $X (\subset X^{\#\#})$, we know that $X \subset M$. By Lemma 1, M is clearly closed under multiplication by elements in $\mathcal{A}^{\#\#}$; thus $\langle X \rangle \subset M$. If $\{a_n\}$ is a net in $\mathcal{A}^{\#\#}$ and $a_n \to 0(w^*)$, and if $\alpha \in X^\#$, and if $z \in \langle X \rangle$, then $(a_n z)(\alpha) = \varphi_{z,\alpha}(a_n) \to 0$. This proves (1).

(2) This follows from (1) and part (4) of Lemma 1.

(3) The implication (a) implies (c) follows from (1), since σ is the weak topology on X.

To prove (c) implies (b), suppose $x \in X$. Then, by (c), $\hat{m}(\text{ball}(\mathcal{A}^{\#\#}))$ is compact in the weak operator topology and contains $m(\text{ball}(\mathcal{A}))$ as a weak-operator dense subset. Thus $w\text{-cl}[(\text{ball}(\mathcal{A}))x] = \hat{m}(\text{ball}(\mathcal{A}^{\#\#}))x$ is weakly compact.

To prove (b) implies (a), suppose $a \in \mathcal{A}^{\#\#}$ and $x \in X$. We wish to show that $ax \in X$. Choose a bounded net $\{a_n\}$ in $\mathcal{A}$ so that $a_n \to a(w^*)$. By (b), we can assume that $a_n x \to y(w)$ for some y in X. However, by (1), $a_n x \to ax(w^*)$ in $X^{\#\#}$. Thus $ax = y \in X$. Since $\mathcal{A}^{\#\#} X \subset X$, we have $X = \langle X \rangle$.

(4) It follows from (3) that $p_\lambda x \to px(w)$ for every x in X. Fix x in X and suppose $\varepsilon > 0$. It follows from the Hahn Banach theorem that there are indices $\lambda_1, \lambda_2, \ldots, \lambda_n$ and positive numbers $t_1, t_2, \ldots, t_n$ such that $\sum t_i = 1$ and $\|(p - \sum t_i p_{\lambda_i})x\| < \varepsilon$. If $\lambda \geq \lambda_i$ for $1 \leq i \leq n$, then p_λ commutes with the p_{λ_i}'s, and since $p - p_\lambda \leq p - \sum t_i p_{\lambda_i}$, it follows from Lemma 2 that $\|(p - p_\lambda)x\| \leq \|(p - \sum t_i p_{\lambda_i})x\| < \varepsilon$. Thus $\lim_\lambda \|(p - p_\lambda)x\| = 0$.

The following lemma, which is based on the polar decomposition of linear functionals on a C^*-algebra (see **[22]**), is our main tool in proving the existence of Bade functionals.

LEMMA 5. *If* $x \in X$ *and* $\alpha \in X^\#$, *then there is a partial isometry* B *in* $\mathcal{A}^{\#\#}$ *such that, for each* A *in* $\mathcal{A}$ *with* $A \geq 0$,

(1) $(\alpha B)(Ax) \geq 0$,

(2) *if* $(\alpha B)(Ax) = 0$, *then* $\alpha(CAx) = 0$ *for every* C *in* $\mathcal{A}$,

(3) $\mu_{x,\alpha} BB^* = \mu_{x,\alpha}$.

Proof. Assume without loss of generality that $\|\mu_{x,\alpha}\| = 1$. Since the closed unit ball of $\mathcal{A}^{\#\#}$ is w^*-compact, we can choose a net $\{B_n\}$ in ball$(\mathcal{A})$ and a B in ball$(\mathcal{A}^{\#\#})$ so that $\mu_{x,\alpha}(B_n) \to 1 = \|\mu_{x,\alpha}\|$ and $B_n \to B(w^*)$. Assume that $\mathcal{A} \subset L(H)$ with the properties of Lemma 1. Then there are unit vectors e, f in H such that $\mu_{x,\alpha}(A) = (Ae, f)$ for all A in $\mathcal{A}$. Since $\mu_{x,\alpha}(B_n) \to (Be, f) = (e, B^*f) = 1$, we have equality in the Cauchy–Schwarz inequality; thus $Be = f$ and $B^*f = e$. Since $B^*Be = e$, we know that $(B^*B)^{1/2}e = e$; thus if $B = V(B^*B)^{1/2}$ is the polar decomposition of B, then $Ve = f$ and $V^*f = e$. Hence we can assume that B is a partial isometry. Note that $(\alpha B)(Ax) = (BAe, f) = (Ae, B^*f) = (Ae, e)$. Statements (1) and (2) are now obvious and (3) follows from the fact that $BB^*f = f$.

REMARKS. 1. So far we have considered only left Banach $\mathcal{A}$-modules. However, the Arens extensions defined previously have obvious analogues for right modules and the analogue of the preceding lemma is also valid for right Banach $\mathcal{A}$-modules.

2. Notice that most of the proof of the preceding lemma takes place in the Hilbert space H and not in X. All we require is that $X^{\#}$ be an $\mathcal{A}^{\#\#}$-module, not necessarily a Banach $\mathcal{A}^{\#\#}$-module. If X is a locally convex space and an $\mathcal{A}$-module such that the multiplication $\mathcal{A} \times X$ into X is (jointly) continuous, then the Arens extension make sense to the extent that $X^{\#}$ is a right $\mathcal{A}^{\#\#}$-module. In this setting, the preceding lemma is valid.

2. Existence of Bade Functionals. The following theorem is new even in the case where $\mathcal{A} = C(K)$ for some compact Hausdorff space K. We say that a (not necessarily closed) linear subspace M of X is w^*-countably *$\mathcal{A}$-separated* if there is a sequence $\{\alpha_n\}$ in $X^{\#}$ such that $\text{sp}\{\alpha_n a : n \geq 1, a \in \mathcal{A}\}$ separates the points of M. Suppose that X is a Banach $\mathcal{A}$-module and α is a Bade functional for x. Then $\alpha\mathcal{A}$ separates the points of $\mathcal{A}x$. (Proof: $\alpha\mathcal{A}(ax) = 0 \Rightarrow \alpha(a^*ax) = 0 \Rightarrow a^*ax = 0 \Rightarrow ax = 0$.) Hence, if x has a Bade functional, then $\mathcal{A}x$ is w^*-singly $\mathcal{A}$-separated. The following theorem is a strong converse of this fact.

THEOREM 6. *Suppose that X is a Banach $\mathcal{A}$-module and $x \in X$. If $\mathcal{A}x$ is w^*-countably $\mathcal{A}$-separated, then x has a Bade functional with respect to $\mathcal{A}$.*

Proof. Choose a sequence $\{\alpha_n\}$ of unit vectors in $X^{\#}$ so that $\alpha_1\mathcal{A} + \alpha_2\mathcal{A} + \dots$ separates the points of $\mathcal{A}x$. By Lemma 5, we can choose a sequence $\{B_n\}$ in the unit ball of $\mathcal{A}^{\#\#}$ such that, for each positive A in $\mathcal{A}$ and each n, $(\alpha_n B_n)(Ax) \geq 0$, and $(\alpha_n B_n)(Ax) = 0$ implies that $\alpha_n(CAx) = 0$ for every C in $\mathcal{A}$, i.e., each element of $\alpha_n\mathcal{A}$ annihilates Ax. Thus if we let $\alpha = \sum_n (\alpha_n B_n)/2^n$, then α is a Bade functional for x.

REMARKS. 1. The analogue of the preceding theorem for right Banach $\mathcal{A}$-modules is true (see the remarks following Lemma 5).

2. The preceding theorem has an analogue for locally convex $\mathcal{A}$-modules. In the proof, the existence of the B_n's still holds (see the remarks following Lemma 5). The only problem is in being able to find a sequence $\{t_n\}$ of positive numbers such that $\sum_n t_n\alpha_n B_n$ converges to a functional in $X^{\#}$. One way to insure this is by requiring $X^{\#}$ to be sequentially complete in the w^*-topology (i.e., the $\sigma(X^{\#}, X)$-topology) and the sequence $\{\alpha_n\}$ to be strongly bounded.

COROLLARY 7. *If $X^{\#}$ contains a sequence separating the points of X, then every vector in X has a Bade functional with respect to $\mathcal{A}$. In particular, if X is separable or the dual of a separable Banach space, then every vector in X has a Bade functional with respect to $\mathcal{A}$.*

Proof. First suppose that X is separable and $\{x_n\}$ is dense in X. Choose a sequence $\{\alpha_n\}$ in the unit ball of $X^{\#}$ so that $\alpha_n(x_n) = \|x_n\|$ for $n \geq 1$. Then $\{\alpha_1, \alpha_2, \dots\}$ separates the points of X, so $\text{sp}\{\alpha_1, \alpha_2, \dots\}$ is w^*-dense in $X^{\#}$. Thus X is w^*-countably $\mathcal{A}$-separated for every C^*-algebra $\mathcal{A}$ acting on X.

Next suppose that $X = Y^{\#}$ for some separable Banach space Y, and let $\{y_n\}$ be a dense sequence in Y. Then $\text{sp}\{y_1, y_2, \dots\}$, considered as a subset of $X^{\#} = Y^{\#\#}$, is w^*-dense, and the desired conclusion follows as in the first case.

COROLLARY 8. *Every element of c_0, c, l^∞, $c_0 \times l^1$, $l^1 \times l^\infty$, $L(Y^{\#})$(Y a Banach space with $Y^{\#}$ separable), or $\mathcal{B}^{\#\#}$ ($\mathcal{B}$ a separable C^*-algebra) has a Bade functional with respect to any Banach $\mathcal{A}$-module structure on these spaces.*

COROLLARY 9. *If $\mathscr{A}$ is separable, and X is a Banach $\mathscr{A}$-module, then every vector in X has a Bade functional with respect to $\mathscr{A}$.*

Proof. The space $(\mathscr{A}x)^-$ is separable for every x in X.

The following theorem shows how Theorem 6 can be used to construct Bade functionals in nonseparable situations. The second corollary contains C^*-algebraic analogues of classical results of Bade [3, 4]. Suppose X is a Banach $\mathscr{A}^{\#\#}$-module, and $x \in X$. We say that $\mathscr{A}^{\#\#}$ *is countably decomposable* with respect to x if, whenever $\{p_\lambda\}$ is an orthogonal family of projections in $\mathscr{A}^{\#\#}$, then $p_\lambda x \neq 0$ for at most countably many values of λ.

REMARK. If x has a carrier projection in $\mathscr{A}^{\#\#}$ (i.e., $\{a \in \mathscr{A}^{\#\#} : ax = 0\}$ is w^*-closed), $\mathscr{A}^{\#\#}$ being countably decomposable with respect to x is equivalent to the stronger statement that, for every increasingly directed family $\{p_\lambda\}$ of selfadjoint projections in $\mathscr{A}^{\#\#}$ with w^*-limit p, there is an increasing sequence $\{p_{\lambda_n}\}$ with w^*-limit q such that $px = qx$. To see this let $\mathscr{P}$ denote the set of all w^*-limits of increasing sequences in $\{p_\lambda\}$. It is clear that $\mathscr{P}$ is a set of projections with the following properties.

(i) Any countable subset of $\mathscr{P}$ has an upper bound in $\mathscr{P}$.

(ii) If $b \in P$ and $(p-b)x \neq 0$, then there is a b' in $\mathscr{P}$ such that $b \leq b'$ and $(b'-b)x \neq 0$.

The first property is obvious. The second follows from the fact that $(p-b) = w^* - \lim_\lambda p_\lambda(p-b)$, and since $\{a \in \mathscr{A}^{\#\#} : ax = 0\}$ is w^*-closed, there must be a λ for which $p_\lambda(p-b)x \neq 0$. Choose b' in $\mathscr{P}$ so that $b, p_\lambda \leq b'$. Then $(b'-b)x \neq 0$, since $0 \neq p_\lambda(p-b)x = p_\lambda(b'-b)x$.

It follows from (i) and (ii) that, if the desired q does not exist, there is an increasing net $\{q_\alpha : \alpha < \Omega\}$, where Ω is the first uncountable ordinal, such that, for each $\alpha < \Omega$, $(q_{\alpha+1} - q_\alpha)x \neq 0$. This clearly violates the countable decomposability of $\mathscr{A}^{\#\#}$ with respect to x.

THEOREM 10. *Suppose that Y is a Banach $\mathscr{A}^{\#\#}$-module, $y \in Y$, $\mathscr{F} \subset Y^\#$, such that*

(1) $\mathscr{F}\mathscr{A}^{\#\#} = \{\alpha\mathscr{A}^{\#\#} : \alpha \in \mathscr{F}\}$ separates the points of $\mathscr{A}^{\#\#}y$,

(2) $\mu_{y,\alpha} \in \mathscr{A}^\#$ for every α in $\mathscr{F}$,

(3) $\mathscr{A}^{\#\#}$ is countably decomposable with respect to y.

Then y has a Bade functional with respect to $\mathscr{A}^{\#\#}$ in the norm closure of sp $\mathscr{F}\mathscr{A}^{\#\#}$.

Proof. For each α in $\mathscr{F}$, let $\mathscr{I}_\alpha = \{a \in \mathscr{A}^{\#\#} : \mathscr{A}^{\#\#}ay \subset \ker(\alpha)\} = \{a \in \mathscr{A}^{\#\#} : \mathscr{A}^{\#\#}a \subset \ker \mu_{y,\alpha}\}$. Then, by (2), each $\mathscr{I}_\alpha$ is a w^*-closed left ideal in $\mathscr{A}^{\#\#}$. Hence, by Lemma 1, there is a projection q_α in $\mathscr{A}^{\#\#}$ such that $\mathscr{I}_\alpha = \mathscr{A}^{\#\#}(1-q_\alpha)$. It is clear from (1) that $a \in \bigcap_{\alpha \in \mathscr{F}} \mathscr{I}_\alpha$ if and only if $ay = 0$. Hence $ay = 0$ if and only if $aq_\alpha = 0$ for every α in $\mathscr{F}$.

We now think of $\mathscr{A}^{\#\#}$ as a von Neumann algebra on the Hilbert space H (Lemma 1). The condition $aq_\alpha = 0$ says that $a = 0$ on $q_\alpha(H)$. For each countable subset λ of $\mathscr{F}$, let p_λ be the projection onto $\overline{\text{sp}}\{q_\alpha(H) : \alpha \in \lambda\}$. Then each p_λ is in $\mathscr{A}^{\#\#}$ (since p_λ is the projection onto the closure of the range of $\sum_{\alpha \in \lambda} t_\alpha q_\alpha$, with each $t_\alpha > 0$, and $\sum_{\alpha \in \lambda} t_\alpha < \infty$). Moreover, the net $\{p_\lambda\}$ is increasingly directed and therefore converges (w^*) in $\mathscr{A}^{\#\#}$ to a projection p in $\mathscr{A}^{\#\#}$. It follows that $\mathscr{A}^{\#\#}(1-p) = \{a \in \mathscr{A}^{\#\#} : ay = 0\}$, since $ap = 0$ if and only if $aq_\alpha = 0$ for every α in $\mathscr{F}$. Thus p is a carrier projection in $\mathscr{A}^{\#\#}$ for y.

It follows from countable decomposability that there is an increasing sequence $\{p_{\lambda_i}\}$ with w^*-limit q in $\mathscr{A}^{\#\#}$ such that $qy = py$. Since $q \leq p$ and $(p-q)y = 0$, we conclude that $p - q = (p-q)(1-p) = 0$, i.e., $p = q$. Let $\lambda = \bigcup_i \lambda_i$. Clearly, $p = q = p_\lambda$. It follows that $ay = 0$ if and only if $aq_\alpha = 0$ for every α in λ. It follows from the definition of the q_α and the countability of λ that $\mathscr{A}^{\#\#}y$ is w^*-countably $\mathscr{A}^{\#\#}$-separated. Thus, by Theorem 6, y has a Bade functional with respect to $\mathscr{A}^{\#\#}$. The fact that the Bade functional is in sp $\mathscr{F}\mathscr{A}^{\#\#}$ is a consequence of the proof of Theorem 6.

REMARKS. 1. The analogue of the preceding result for right $\mathscr{A}^{\#\#}$-modules is also true (see the remarks following Theorem 6).

2. If Y is a locally convex $\mathscr{A}^{\#\#}$-module, then the above result holds if we assume that the set $\mathscr{F}$ is w^*-bounded in $Y^\#$ and that $Y^\#$ is w^*-sequentially complete.

3. All we really needed in the above theorem was that $\mathscr{A}^{\#\#}$ was a von Neumann algebra, not necessarily the second dual of a C^*-algebra. However, every von Neumann algebra is the dual of a unique Banach space and has a unique w^*-topology. When formulated for a von Neumann algebra, part (2) of the preceding theorem should be changed to say that each $\mu_{y,\alpha}$ is w^*-continuous.

We can now give a complete characterization of the existence of Bade functionals with respect to $\mathscr{A}^{\#\#}$ in terms of countable decomposability. This is somewhat related to a result of Gillespie [**14**, Theorem 2]. The following result arises by combining Theorem 10 with Lemma 4(1).

COROLLARY 11. *Suppose X is a Banach $\mathscr{A}$-module, $\alpha \in X^\#$, and $z \in \langle X \rangle$. Then*

(1) *z has a Bade functional in $X^\#$ with respect to $\mathscr{A}^{\#\#}$ if and only if $\mathscr{A}^{\#\#}$ is countably decomposable with respect to z;*

(2) *α has a Bade functional in $\langle X \rangle$ with respect to $\mathscr{A}^{\#\#}$ if and only if $\mathscr{A}^{\#\#}$ is countably decomposable with respect to α.*

If, for each x in X, the map from $\mathscr{A}$ to X that sends a to ax is weakly compact (see part (3) of Lemma 4), we say that $\mathscr{A}$ has *weakly compact action* on X. The notion of weakly compact action in the case $\mathscr{A} = C(K)$ was related to the Bade completeness of a bounded Boolean algebra of projections in [**18**]. It is clear from Lemma 4(3) that if $\mathscr{A}$ has weakly compact action on X, then X is a Banach $\mathscr{A}^{\#\#}$-module, and, by Lemma 4(4), $\mathscr{A}^{\#\#}$ is countably decomposable with respect to every vector x in X. In the presence of weakly compact action the preceding corollary reduces to the C^*-algebraic analogue of two results of Bade [**3, 4**] (see [**11**, XVII.3.1.12, XVIII.3.26]).

COROLLARY 12. *Suppose X is a Banach $\mathscr{A}$-module and $\mathscr{A}$ has weakly compact action on X. Then*

(1) *every vector in X has a Bade functional in $X^\#$ with respect to $\mathscr{A}^{\#\#}$;*

(2) *an α in $X^\#$ has a Bade functional in X with respect to $\mathscr{A}^{\#\#}$ if and only if $\mathscr{A}^{\#\#}$ is countably decomposable with respect to α.*

If $\mathscr{A}^{\#\#}$ is a von Neumann algebra acting on a separable Hilbert space, then every orthogonal family of nonzero projections must be countable. Thus $\mathscr{A}^{\#\#}$ is countably decomposable with respect to any vector.

COROLLARY 13. *If $\mathscr{A}$ is a separable C^*-algebra and X is an $\mathscr{A}$-module, then each vector in $X^\#$ has a Bade functional in $\langle X \rangle$ with respect to $\mathscr{A}^{\#\#}$.*

In **[13]** T. A. Gillespie proved that every bounded Boolean algebra of projections on a fixed Banach space X is contained in a Bade complete Boolean algebra of projections if and only if X does not contain a copy of c_0. Gillespie's result in one direction follows from the well-known theorem of Pelczynski **[19]**, which says that if a Banach space X does not contain a copy of c_0, then, for each compact Hausdorff space K, every operator from $C(K)$ into X is weakly compact. In the other direction, Gillespie **[13]** showed that if X does contain a copy of c_0, then there is a Banach c-module structure on X that does not have weakly compact action. Pelczynski's result was extended to C^*-algebras by Akemann, Dodds and Gamlen **[1]**, who proved that if X does not contain a copy of c_0, then, for every C^*-algebra $\mathscr{A}$ and every Banach $\mathscr{A}$-module structure on X, $\mathscr{A}$ has weakly compact action. Gillespie's result shows that the converse of the latter result is true.

COROLLARY 14. *If X is a Banach $\mathscr{A}$-module and X does not contain a copy of c_0, every element of X has a Bade functional.*

COROLLARY 15. *If X is a Banach $\mathscr{A}$-module and $X^\#$ does not contain a copy of c_0, then every vector in $X^\#$ has a Bade functional in $\langle X \rangle$ with respect to $\mathscr{A}^{\#\#}$.*

Gillespie **[14]** asked whether the existence of Bade functionals with respect to every bounded Boolean algebra of projections on a Banach space X implies that X does not contain a copy of c_0. As observed in Corollary 8, this is not true in general. However, Gillespie's methods **[13]** do imply the following proposition. If Γ is a nonempty set, we let $l^\infty(\Gamma)$ denote the C^*-algebra $\mathscr{A}$ of all bounded complex functions on Γ, we let $c_0(\Gamma)$ denote the norm closed subalgebra generated by the functions with finite support, and we let $c(\Gamma)$ denote the algebra generated by $c_0(\Gamma)$ and the identity. This proposition shows that the countable separation assumption in Theorem 6 cannot be increased to a higher cardinality. It also shows that Corollary 15 applies in cases in which not every vector in X has a Bade functional, since $c(\Gamma)^\#$ never contains a copy of c_0.

PROPOSITION 16. *Suppose X contains a copy of $c_0(\Gamma)$ for an uncountable index set Γ. Then on X there is a Banach $c(\Gamma)$-module structure with respect to which not every element of X has a Bade functional.*

3. Questions. (1) We have indicated in the remarks that some of our results can be extended to locally convex $\mathscr{A}$-modules. In view of the generalizations of Bade's work to locally convex spaces (**[7]**, **[8]**, **[9]**, **[20]**, **[11**, pp. 2107–2110**]**), it would be interesting to find exactly which of our results can be extended to this setting and the precise conditions that are needed for these extensions.

(2) Let us say that a Banach space X has the *$C(K)$-Bade functional property* if, for every compact Hausdorff space K and every Banach $C(K)$-module structure on X, every vector in X has a Bade functional. We define the *C^*-algebra Bade functional property* analogously. Are these two properties the same?

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