THE ASTROPHYSICAL JOURNAL SUPPLEMENT SERIES, 127 : 509–512, 2000 April

THE EFFECT OF BOUNDARY CONDITIONS ON RAYLEIGH-TAYLOR INSTABILITY

DEFNE UCER
Physics Department, Middle East Tech. University, Ankara, 06531, Turkey; ducer@rorqual.cc.metu.edu.tr
AND
VLADIMIR V. MIRNOV
Physics Department, Middle East Tech. University, Ankara, 06531, Turkey; vladimir@rorqual.cc.metu.edu.tr
Received 1999 January 19; accepted 1999 December 13

ABSTRACT

Laser experiments are widely used to investigate excitation of Rayleigh-Taylor modes, which are of great importance for astrophysical applications. Measured growth rates are normally compared with either the sharp interface or the smooth gradient model. In the present paper an analytical solution is obtained that is valid for arbitrary density gradient scale L. It is a further development of the Mikaelian & Lindl model. New explicit presentation $\omega(k)$ is found which describes all discrete modes at all transverse wavenumbers k with one parametric expression. A critical value of kL is shown to exist when two independent solutions for the fastest growing main mode become degenerate, in this case the growth rate is calculated exactly. The focus is on astrophysical applications when boundary conditions are at infinity. The case of rigid walls is also considered to study the interrelation with the Chandrasekhar model. Results are supposed to be used for nonlinear RT treatment to analyze mixing in supernovae and other RT-driven objects.

Subject headings: hydrodynamics — instabilities — methods: analytical — supernovae: general — X-ray

1. INTRODUCTION

The problem of the Rayleigh-Taylor (RT) instability is of great importance for many applications. It has been modeled and investigated in a number of laboratory experiments. Most recently, the Nova laser facilities (Remington et al. 1995) have been used for excitation and treatment of RT dynamics in imploded targets. Further laser-driven RT experiments are supposed to be carried out for modeling of the mixing and other processes important for astrophysical applications.

The measured growth rates are normally compared with the analytical results and hydrodynamic simulations. Two well-known expressions, one for sharp interface, $\omega_{\rm sh}^2 = -kg(\rho_2 - \rho_1)/(\rho_2 + \rho_1)$, and one for smooth density gradient, ${\omega_{\rm sm}}^2 = -g/L$, are used, where $\rho(z)$ is the vertical profile of unperturbed mass density, L, is the characteristic scale of mass density gradient, and k is the transverse wavenumber of the perturbations. These two cases are often combined with one extrapolation (Munro 1988): $\omega^2 = -kg/(1 + kL)$, which gives correct zeroth order terms but wrong asymptotical behavior.

In this paper incompressible Rayleigh-Taylor modes are treated for an exponential density profile $\rho(z)$. The analytical approach is presented, which is the further development of the model suggested by Mikaelian & Lindl (1984). A new explicit form of the dispersion relation $\omega(k)$ is found, which allows us to describe all discrete modes at all transverse wavelengths with one parametric expression. It is then used to follow the transition between sharp and smooth limiting cases.

Vertical eigenmodes are shown to be discrete not only in the fluid, which is bounded with the rigid walls, but in the case of infinite space as well. Short and long transverse wavelength limits are investigated. Furthermore, an intermediate value of $k_{\rm cr} L$ is found at which two fundamental solutions describing the fastest-growing main mode become degenerate and for this case the growth rate is calculated exactly. In the model by Chandrasekhar (1968), the growth of perturbations inside a single layer with a similar exponential density profile was analyzed. The inter-relation between this model and our approach is discussed as well.

Results obtained are valid for arbitrary scale L. They are planned to be used for nonlinear treatment of RT to study the process of mixing. In particular, mixing causes an effective smoothing of the density gradient that can be taken into account as time-dependent $L(t)$. Another purpose of this consideration is to create an accurate quantitative theory for comparison with the experiments investigating the early phase of supernova explosions by means of laser-irradiated targets.

2. BASIC EQUATIONS

Rayleigh-Taylor gravitational modes are analyzed by using linearized incompressible fluid equations. The uniform static gravitational field g is assumed to be along the z-axis, such that all parameters of the equilibrium state are z-dependent and uniform in the xy-plane. Linear perturbations are taken in the form $f(t, y, z) = f(z) \exp(-i\omega t + iky)$. The eigenfunction equation for z component of fluid velocity $v(z)$ can be obtained as follows:

$$[\rho(z)v']' - k^2[(g/\omega^2)\rho'(z) + \rho(z)]v = 0 . \qquad (1)$$

This equation describes the "vertical" structure of the incompressible Rayleigh-Taylor modes. Normally, either the rigid wall ($v = 0$) or free interface boundary condition $[v' = (gk^2/\omega^2)v]$ are imposed on the upper and lower boundaries at $z = \pm h$. However, regardless of what particular case is considered, some general properties of eigenvalues ω^2 can be derived. The function $v(z)$ is described by a uniform linear differential equation with uniform linear boundary conditions. This implies that before initial conditions are specified, the profile $v(z)$ is defined with the accu-

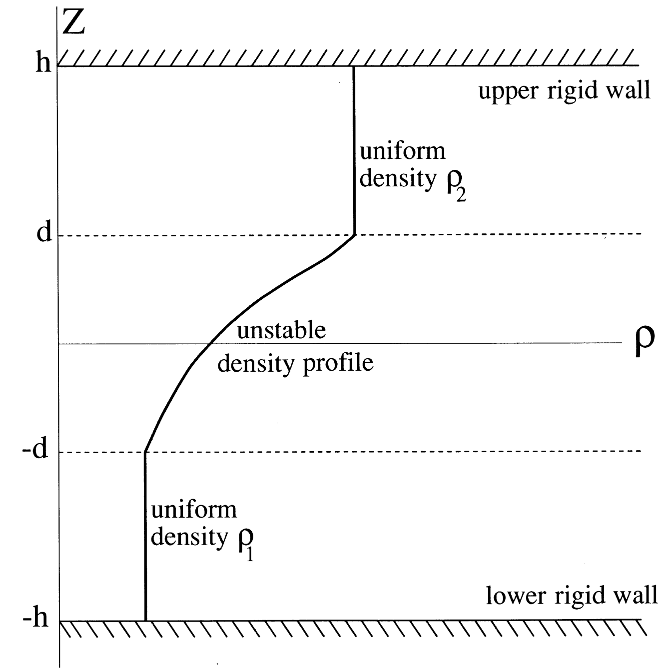

FIG. 1.—Unperturbed density profile, $\rho(z)$. The two uniform fluids of densities ρ_1, ρ_2 are bounded with rigid walls at $z = \pm h$, and separated by the layer, $-d < z < d$, with the exponential density profile.

racy to an arbitrary nonzero constant. This constant can even be a complex number (of course, complete solution $v[t, y, z]$ satisfying physical initial conditions turns out to be pure real). This freedom allows us to impose any integral constraint suitable for calculation. Let us multiply equation (1) with complex conjugate $v^*(z)$ and integrate over the fluid volume. If the integral constraint is chosen in the form, $\int \rho(z)\,|v(z)|^2\,dz = 1$, equation (1) takes the following form:

$$\omega^2 = -g\,\frac{\int \rho'(z)\,|v(z)|^2\,dz + \rho(-h)\,|v(-h)|^2 - \rho(h)\,|v(h)|^2}{1 + k^{-2}\int \rho(z)\,|v'(z)|^2\,dz}\,. \qquad (2)$$

In the case of rigid wall boundary conditions, the second and the third terms in the numerator are zero. In all cases the eigenvalues of ω^2 turn out to be pure real. Moreover, if the case of rigid walls is considered and density gradient is positive, that is, $\rho'(z) \geq 0$, then all ω^2 are negative, indicating that all eigenmodes are unstable.

Detailed analysis of vertical eigenmodes is based on exact solution of equation (1). To treat the problem analytically, we choose the density profile suggested in Mikaelian & Lindl (1984) and the rigid wall boundary conditions $v(\pm h) = 0$ $(h \geq d)$ (see Fig. 1)

$$\rho(z) = \begin{cases} \rho_2\,, & z \geq d\,, \\ \sqrt{\rho_1 \rho_2}\,\exp(z/L)\,, & |z| \leq d\,, \\ \rho_1\,, & z \leq -d, \end{cases} \qquad (3)$$

where $d = (L/2)\ln(\rho_2/\rho_1)$. Substituting the density profile (eq. [3]) to equation (1) and making use of the normalization condition above, the following useful inequality is obtained:

$$-g/L < \omega^2 < 0\,. \qquad (4)$$

3. STRUCTURE OF THE EIGENMODES

Discrete structure of eigenvalues and their dependence on k is found by analyzing three different parts of the profile (eq. [3]) separately. Solutions in upper and lower uniform fluids are $v_{2,1}(z) = C_{2,1}\sinh[k(z \mp h)]$. The velocity profile, $v(z)$, in the intermediate nonuniform region is given by

$$v(z) = C_+ \exp \lambda_+ z + C_- \exp \lambda_- z\,, \qquad (5)$$

where

$$\lambda_\pm = (-\ln(\rho_2/\rho_1) \pm \sqrt{\psi^2})/4d\,,$$
$$\psi^2 = \ln^2(\rho_2/\rho_1) + [2kL\ln(\rho_2/\rho_1)]^2 \times (1 + g/\omega^2 L)\,. \qquad (6)$$

In accordance with definition (eq. [6]), the function ψ^2 is pure real and has either a positive or a negative sign. In fact, the first term in equation (6) is positive while the second term is always negative due to equation (4).

The frequency spectrum of the problem consists of an infinite number of discrete "vertical" eigenmodes. For the fastest growing main mode that has the smallest vertical wave number, $n = 1$, ψ^2 is positive if the transverse wave vector k is small enough. The ψ^2 changes sign and becomes negative at some critical value of $k(k_{\rm cr})$. However, for all other modes with $n > 1$, ψ^2 is negative at all values of kL and therefore ψ is pure imaginary. Following this picture, we will analyze the structure of the modes considering the situations where k is smaller, equal, or greater than $k_{\rm cr}$.

3.1. *The Main Mode, in Long Transverse Wave Limit* $k < k_{\rm cr}$ $(\psi^2 > 0)$

With the help of equation (6) eigenfrequencies of all modes can be expressed in terms of ψ and k as follows:

$$\omega^2 = -\frac{g}{L}\left\{1 + \frac{1}{4k^2L^2}\left[1 - \frac{\psi^2(k)}{\ln^2(\rho_2/\rho_1)}\right]\right\}^{-1}. \qquad (7)$$

The dependence $\psi^2(k)$ is derived by matching $v(z)$ and $v'(z)$ at points $z = \pm d$. At this point, it is suitable to introduce the variable k', which is related to k as $k' = k\coth[k(h - d)]$. The dependence of ψ on k', obtained under the assumption that $\psi^2 > 0$ is given by the equation

$$\exp\psi = \frac{(\psi + a_-)(\psi - a_+)}{(\psi - a_-)(\psi + a_+)}\,,$$
$$a_\pm = \ln(\rho_2/\rho_1)(1 \pm 2k'L)\,. \qquad (8)$$

When rigid walls are moved to infinity, $k' = k$. First, we will analyze this basic case. Then by substituting k' instead of k, the effect of boundary conditions will be taken into account.

There is a nontrivial solution to equation (8) when k takes values between $0 \leq k \leq k_{\rm cr}$. Let us focus on the limit of small kL within this interval. Expanding equation (8) in powers of kL, an asymptotical expression for the growth rate of the main mode is obtained:

$$\omega^2 = -kg\,\frac{\rho_2 - \rho_1/\rho_2 + \rho_1}{1 + kLf(\rho_2/\rho_1)}\,, \qquad (9)$$

where

$$f(\rho_2/\rho_1) = \frac{4\rho_2\rho_1}{(\rho_2-\rho_1)^2}\left(\ln\frac{\rho_2}{\rho_1} - 2\frac{\rho_2-\rho_1}{\rho_2+\rho_1}\right).$$

The function $f(\rho_2/\rho_1)$ has a maximum at $\ln(\rho_2/\rho_1) = 1.8$, with $f_{max} = 0.35$. This result indicates that corrections to the sharp interface model, due to finite values of $kL \ll 1$, become important starting from relatively large values of $kL \simeq 3$. This conclusion will be confirmed below by the exact calculation of the growth rate at $k_{cr}L$.

3.2. *The Main Mode, at Intermediate Value of $k = k_{cr}$ (Degenerate Case, $\psi^2 = 0$)*

The transition of the main mode eigenfunction from real to complex appears at some critical value of k when two fundamental solutions of equation (5) become degenerate, $\lambda_+ = \lambda_-$, $\psi^2 = 0$. The growth rate in this case can be calculated exactly as a function of density ratio ρ_2/ρ_1. Instead of exponential functions in equation (5) the degenerate main mode is described by the following solution:

$$v(z) = (C_+ + C_- z)\exp(-z/2L)\,. \tag{10}$$

Matching $v(z)$ and $v_{1,2}$ at points $z = \pm d$ yields a critical value for k when degeneration takes place:

$$\frac{k_{cr}L}{\tanh[k_{cr}(h-d)]} = R(\rho_2/\rho_1) = \frac{\ln(\rho_2/\rho_1)}{4(1+\sqrt{1+\ln^2(\rho_2/\rho_1)/4})}\,. \tag{11}$$

Here the effect of rigid boundaries is taken into account explicitly.

The growth rate of the perturbations with the critical wavelength is as follows:

$$\omega^2 = -4(g/L)k_{cr}^2L^2/(1+4k_{cr}^2L^2)\,. \tag{12}$$

The critical k exists if boundaries are placed far enough from the unstable region, that is, $L/(h-d) < R(\rho_2/\rho_1)$. If rigid walls are infinitively far and don't affect instability, the critical values always exist, $k_{cr}L = R(\rho_2/\rho_1)$, yielding the growth rate $\omega^2 = -g/2L$ at $\ln(\rho_2/\rho_1) \gg 1$ and $\omega^2 = -(g/16L)(\ln \rho_2/\rho_1)^2$ in the opposite limiting case. Comparing these results with the sharp interface growth rate ω_{sh}^2, (see § 1), one can conclude that expression for ω_{sh}^2 is valid not only asymptotically, when $kL \to 0$, but gives correct values at finite $kL \simeq 1$ as well.

3.3. *The Range of the Short Wavelength $k_{cr} < k$, ($\psi^2 < 0$)*

In the short wave range $k > k_{cr}$, there are no more solutions to equations (7) and (8); therefore negative values of $\psi^2(k)$ have to be considered. This can be formally achieved by substitution of $i\psi$ instead of ψ in the equations (7)–(8). Because the left-hand side of the resulting equation (8) is a periodic function of ψ, it has infinitely many discrete solutions $\psi_n(k)$ for all values $0 \le k < \infty$. One of these solutions, $0 \le \psi_1 < 2\pi$ appears only if $k > k_{cr}$ and corresponds to the continuation of the main mode. Other solutions with larger ψ exist for all values of k. When ψ varies between $2\pi(n-1) < \psi < 2\pi n$, k increases from zero to ∞ and it is repeated periodically with ψ. All modes can be labeled by integer n ($n = 1, 2, 3, \ldots$) in accordance with the interval of $2\pi(n-1) < \psi < 2\pi n$, which corresponds to the given mode. As a conclusion, at large values of $kL \gg 1$ the asymptotical expression for the growth rates of all modes is as follows:

$$\omega_n^2 = -\frac{g}{L}\left\{1 + \frac{1}{4k^2L^2}\left[1 + \frac{(2\pi n)^2}{\ln^2(\rho_2/\rho_1)}\right]\right\}^{-1}\,. \tag{13}$$

At small values of $kL \ll 1$ the square of the growth rates of all modes except $n = 1$ tends to zero as $(kL)^2$:

$$\omega_n^2 = -(2kL)^2\frac{g}{L}\left[1 + \frac{4\pi^2(n-1)^2}{\ln^2(\rho_2/\rho_1)}\right]^{-1}\,, \tag{14}$$

while for the main mode, $n = 1$, it tends to zero linearly.

Using the information collected so far, an explicit dependence of the growth rate on transverse wave number can be obtained by the help of parametric expressions, where ψ is used as a parameter. Making use of equation (8), kL can be expressed as a function of ψ as following:

$$kL = -\frac{\psi\coth(\psi/2)}{2\ln(\rho_2/\rho_1)} + \sqrt{\frac{\psi^2\,\mathrm{csch}^2(\psi/2)}{4\ln^2(\rho_2/\rho_1)} + \frac{1}{4}}\,, \tag{15}$$

while ω^2 is expressed as a function of ψ with the equation (7). In order to get a parametric representation for the main mode, parameter ψ has to be ranged as follows: first it is considered as real, starts from $\ln(\rho_2/\rho_1)$ and varies between $\ln(\rho_2/\rho_1) > \psi > 0$. After it becomes zero, it takes imaginary values $i\psi$ where ψ increases from zero up to 2π. Other modes with $n > 1$ correspond to the range $2\pi(n-1) < \psi < 2\pi n$. These parametric expressions allow us to observe the explicit form of dispersion for all wavelengths as shown in Figure 2.

3.4. *The Effect of the Rigid Wall Boundaries*

The effect of the rigid walls on the dispersion relation in equations (7) and (8) is presented by the dependence $k' = k\coth[k(h-d)]$. It shows that boundaries are important for the long transverse waves. In fact, if $k \to 0$ then

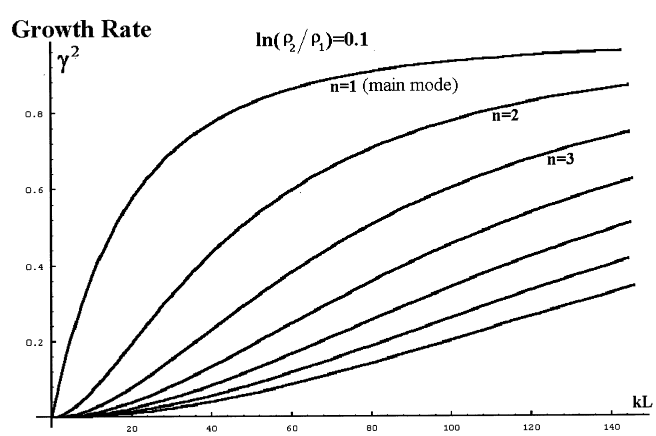

FIG. 2.—Explicit dependence of the growth rate on transverse wave number kL ($\rho_2/\rho_1 = 1.105$).

$k' \rightarrow 1/(h - d)$. Suppose the boundaries are placed far away so that $(h - d)^{-1} \ll k_{cr} = R/L$, then expanding ψ in the vicinity of $k = 0$ and substituting $\psi(k')$ into equation (7) we have an asymptotical expression for the growth rate of the main mode, $n = 1$, in the long wave limit:

$$\omega^2 = -kg \tanh [k(h - d)](\rho_2 - \rho_1/\rho_2 + \rho_1) . \quad (16)$$

The modes with the higher vertical wave numbers $n > 1$ are less sensitive to the distant rigid walls, and therefore equation (14) is still a good approximation in this case.

In the opposite limiting case, when rigid walls are close to the unstable region, $(h - d)/L \ll \rho_2 - \rho_1/\rho_2 + \rho_1$, $k' \rightarrow \infty$ and equation (13) becomes valid for all modes and all k values. Note that this limit corresponds to the well-known solution of Chandrasekhar (1968) which is widely used in the text books for the illustration of Rayleigh-Taylor instability.

4. SUMMARY

Discrete structure of Rayleigh-Taylor incompressible gravitational modes is treated as a function of the transverse wave vector k in both cases of the infinite space and rigid wall boundaries. Explicit expression for the growth rate is constructed, which allows us to describe all modes for all transverse wavelengths as a function of one parameter ψ. The modes are labeled by integer n ($n = 1, 2, 3, \ldots$) in accordance with the intervals $2\pi(n - 1) < \psi < 2\pi n$.

1. There exists the main vertical mode, $n = 1$, with the highest growth rate. Its eigenfunction is pure real and the growth rate is close to the model with sharp interface ω_{sh} if $0 \le k \le k_{cr}$. Linear correction to ω_{sh}^2 at $kL \ll 1$ are found to be approximately 5 times smaller than in the paper by Munro (1988).

2. At some critical value $k_{cr} L$, the two fundamental solutions describing the main mode become degenerate and the growth rate is calculated exactly. The further increase of k results in main mode to become complex. Other modes with larger n ($n > 1$) are complex for all values of k.

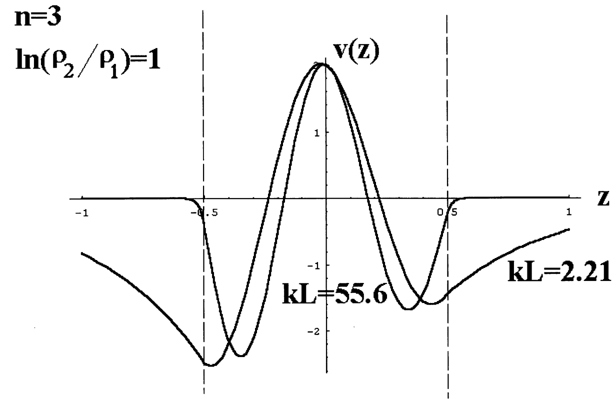

FIG. 3.—Profiles of eigenfunction $v(z)$ for mode $n = 3$ and different values of kL ($\rho_2/\rho_1 = 2.718$).

3. In the limit $kL \gg 1$, the vertical modes are localized inside the unstable region of the density profile while they spread out in the opposite limiting case. For the mode shown in Figure 3, in the case of large kL, the eigenfunctions are localized in the region of unstable density ($-d < z < d$); while for smaller kL, perturbations spread out through the region of uniform density.

If rigid walls are placed far away but at finite distance, ($\infty > h - d \gg L/R$), the main mode is modified in accordance with equation (16) while all other modes, $n > 1$, are not changed significantly in comparison with the case $h = \infty$. In the opposite limiting case the spectrum of all modes is described by equation (13).

The authors would like to acknowledge B. Remington who initiated this activity and visit of one of the authors (V. V. M.) to the 2d International Workshop on Laboratory Astrophysics with Intense Lasers (1998 March 19–21, Tucson, Arizona).

REFERENCES

Chandrasekhar, S. 1968, Hydrodynamic and Hydromagnetic Stability (London: Oxford Univ. Press)
Mikaelian, K. O., & Lindl, J. D. 1984, Phys. Rev. A, 29, 290
Munro, D. 1988, Phys. Rev. A, 38, 1433
Remington, B. A., Weber, S. V., Marinak, M. M., Haan, M. M., Kilkenny, J. D., Wallace, R. J., & Dimonte, G. 1995, Phys. Plasmas, 2, 241