Scalar Waves in a Wormhole Topology

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Global monochromatic solutions of the scalar wave equation are obtained in flat wormholes of dimensions 2+1 and 3+1. The solutions are in the form of infinite series involving cylindirical and spherical wave functions and they are elucidated by the multiple scattering method. Explicit solutions for some limiting cases are illustrated as well. The results presented in this work constitute instances of solutions of the scalar wave equation in a spacetime admitting closed timelike curves.

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I. INTRODUCTION

In this paper global solutions of the scalar wave equation in 2+1 and 3+1 dimensional flat wormholes are obtained. As a significant consequence of their non-trivial topology, wormholes admit closed timelike curves (CTC's). As such they constitute a suitable framework for the study of the solutions of the scalar wave equation in a spacetime admitting closed timelike curves. Again due to the topology of a wormhole, no single coordinate chart is sufficient to express the global geometry of the whole wormhole spacetime and it becomes necessary to develop techniques to handle global issues on the one hand and to investigate the propagation of scalar waves near closed timelike curves. It should be mentioned that there are works that study the scalar waves that are valid locally in a certain regions (such as may be termed the "throat") of the wormhole [1].

Wormholes are widely studied and discussed, especially after the paper of Morris and Thorne, in the context of time travel [2],[3]. The wave equation attracts attention in this sense that whether the anomalies of causality violations due to CTC's has corresponding footprints in the solutions of wave equation.

The metric of the spacetime, as well as the topology, maybe the origin of the closed timelike curves, as in the case of Gödel's universe [4]. Recently Bachelot has studied the properties of wave equation on a class of spacetimes of this type [5]. It may be an interesting question, whether the origin of the CTC's (being topology induced or metric induced) effects the global properties of the solutions. Nevertheless, as proposed in the concluding remarks, the presence of CTC's due to topology does not seem to have a significant effect on the solutions of wave equation.

Cauchy problem of the scalar wave equation in the flat wormhole considered here is studied throughout by Friedman and Morris with a variety of other spacetimes admitting closed timelike curves [6],[7]. They also proved that there exist a unique solution of Cauchy problem for a class of spacetimes, including our case, with initial data given at past null infinity [8].

Due to the wormhole structure, the boundary conditions imposed in solving the Helmholtz

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equation depends on the frequency. Therefore spectral theorem is not applicable in a straight forward manner to express the solution of the wave equation as a superposition of monochromatic wave solutions found in this work. However, in [8], it is proved using limiting absorption method that, the superpositions of the monochromatic wave solutions of the problem converge to the solution of wave equation.

The problem can be handled as a Cauchy problem with given initial data at past null infinity or alternatively as a scattering problem, i.e. finding scattered waves from wormhole handle given incident wave.

Our approach is similar to that used in scattering from infinite parallel cylinders [9]. Ψ_1 and Ψ_2 represents outgoing cylindrical (or for 3+1 dimensions spherical) waves emerging from the first and the second wormhole mouth respectively. In order to be able to apply the boundary conditions conveniently which arise from the peculiar topology of the wormhole in our case, it is necessary to express Ψ_1 in terms of cylindrical (spherical) waves centered at second mouth and vice versa. Addition theorems for cylindrical and spherical wave functions are employed for this purpose.

The equations for the scattering coefficients of Ψ_1 and Ψ_2 that result from the boundary conditions in question are in general not amenable to direct algebraic manipulation . The multiple scattering method is applied to obtain an infinite series solution. On the other hand for some important limiting cases the equations solved explicitly. The solutions by these both methods are consistent with one other.

The outline of the paper is as follows: In section II, the spacetime is described and the general formulation of the problem is presented. In section III, 2+1 dimensional case is studied. The equations are presented, explicit solutions for two limiting cases are obtained, and finally the multiple scattering solution is applied. In Section IV the same scheme as section III is followed for 3+1 dimensional case. In section V numerical verifications of the results obtained in section III are presented. Section VI contains some concluding remarks.

II. FLAT WORMHOLE

Given a Riemannian manifold M, a solution $F : M \times \mathbb{R} \to \mathbb{C}$ of the scalar wave equation

$$\Delta F = \frac{\partial^2 F}{\partial t^2}$$

is said to be a *monochromatic solution* with angular frequency $\omega \in \mathbb{R} - \{0\}$ if it is of the form $F(m,t) = \Psi(m)e^{i\omega t}$ for some $\Psi : M \to \mathbb{C}$. Clearly Ψ is a solution of the Helmholtz equation

$$\Delta\Psi + \omega^2\Psi = 0. \tag{1}$$

On a general Lorentzian spacetime the concept of monochromatic solution makes sense provided the spacetime has an almost product structure that singles out the time direction locally.

A simple example of a wormhole topology is the flat wormhole described in [8]. This 3+1 dimensional flat wormhole spacetime is constructed as follows: Let $a, d, \tau \in \mathbb{R}$ with $d > 2a > 0$. Consider

$$N = \mathbb{R}^3 - (\Delta_+ \cup \Delta_-),$$
$$\hat{M} = N \times \mathbb{R},$$

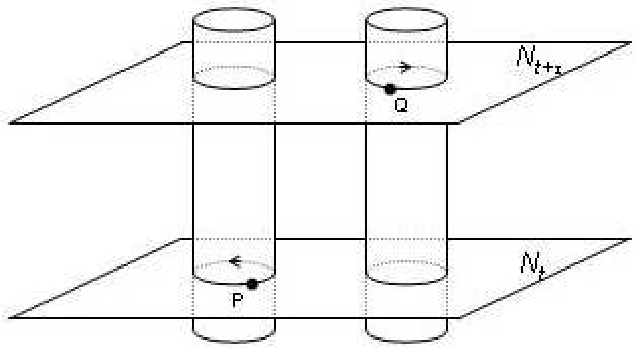

FIG. 1: 2+1 dimensional flat wormhole. P is identified with Q. Arrows indicate the direction of the identification.

where Δ_+, Δ_- are open balls of radius $a > 0$ and respective centers $(0, 0, d/2), (0, 0, -d/2)$ in $\mathbb{R}^3$. The boundaries of Δ_+ and Δ_- are designated as Σ_+ and Σ_- respectively. The *wormhole spacetime* M of width d, radius a, and lag τ is the semiriemannian manifold obtained as the quotient space of $\hat{M}$ by identifying events P, Q on $\Sigma_+ \times \mathbb{R}, \Sigma_- \times \mathbb{R}$ respectively if P is the reflection of Q in the xyt-plane after a translation by τ along the t-axis, the semiriemannian metric being naturally inherited from the ordinary Minkowski metric on $\mathbb{R}^4$. M is clearly a flat Lorentzian spacetime. To be precise:

$$\Sigma_+ = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + (z - d/2)^2 = a^2\},$$
$$\Sigma_- = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + (z + d/2)^2 = a^2\}.$$

For $(x, y, z) \in \Sigma_+$, P and Q are identified where

$$P = (x, y, z, t),$$
$$Q = (x, y, -z, t + \tau).$$

In 2+1 dimensions the manifold is defined in the same way except that:

$$N = \left(\mathbb{R}^2 - (\Delta_+ \cup \Delta_-) \right),$$

Δ_+, Δ_- are open disks of radius $a > 0$ with respective centers $(d/2, 0), (-d/2, 0)$ in $\mathbb{R}^2$ and P is the reflection of Q in the yt-plane after a translation by τ along the t-axis.

The geometry for 2+1 dimensions is shown in fig. 1.

Two wormhole conditions arise from this identification map defining the topology. These conditions will function as boundary conditions imposed on the general solution of Helmholtz equation in a flat spacetime.

The two wormhole conditions will be denoted as C-1 and C-2. C-1 is

$$F(P) = F(Q),$$

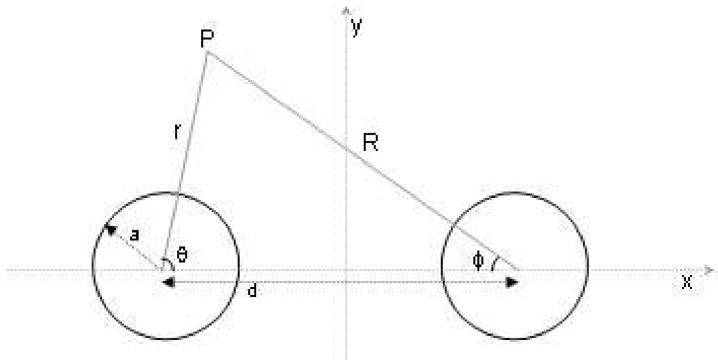

FIG. 2: Coordinates used for 2+1 dimensions.

and C-2 is

$$\hat{n}_P \cdot \nabla F(P) = -\hat{n}_Q \cdot \nabla F(Q).$$

where $\hat{n}_Q$ is the unit outward normal to Σ_- at Q and $\hat{n}_P$ is the unit outward normal to Σ_+ at P. In terms of Ψ, C-1 and C-2 are:

$$\Psi(\omega, p) = e^{i\omega\tau}\Psi(\omega, q),$$

$$\hat{n}_P \cdot \nabla\Psi(\omega, p) = -e^{i\omega\tau}\hat{n}_Q \cdot \nabla\Psi(\omega, q),$$

where p and q are the projections of P and Q on N respectively.

The solution will be expressed in three components: An everywhere regular part of the wave, Ψ_0, which may be considered as originating from the sources at past null infinity (or alternatively as the incident wave if the problem is considered as a scattering problem), and two outgoing waves originating from each wormhole mouth (or scattered waves from each mouth), Ψ_1 and Ψ_2. Obviously $\Psi = \Psi_0 + \Psi_1 + \Psi_2$

There are two wormhole conditions that enable one to determine two of Ψ_0, Ψ_1 and Ψ_2. The problem will be handled like a scattering problem and the scattered waves Ψ_1 and Ψ_2. will be solved given the incident wave Ψ_0.

III. 2+1 DIMENSIONS:

In 2+1 dimensions, solution of Helmholtz equation in cylindrical coordinates yields Bessel (or Hankel) functions. Being everywhere regular, Ψ_0 is expressed in terms of $J_n(r)$,while $\Psi_1$and Ψ_2 represent outgoing waves radiating from the wormhole mouths Δ_- and Δ_+, respectively. Outgoing waves are expressed by Hankel functions of the first kind, $H_n^{(1)}(r)$. Referring to fig. 2, Ψ_1 has its natural coordinates (r, θ) centered at $(-d/2, 0)$, and Ψ_2 has its natural coordinates (R, ϕ) centered at $(d/2, 0)$. The coordinate variables, θ and ϕ are chosen in this way to make use of the mirror symmetry of the geometry of the wormhole with respect to y axis. Since Ψ_0, $\Psi_1$and Ψ_2 are valid in exterior domain, they are expressed in terms of integer order Bessel (Hankel) functions only. Therefore the expansion of Ψ_0, $\Psi_1$and Ψ_2 in terms of Bessel (Hankel) functions are:

$$\Psi_0 = \sum_{n=-\infty}^{\infty} A_n J_n(\omega r) e^{in\theta},$$
$$\Psi_1 = \sum_{n=-\infty}^{\infty} B_n H_n^{(1)}(\omega r) \cdot e^{in\theta},$$
$$\Psi_2 = \sum_{n=-\infty}^{\infty} C_n H_n^{(1)}(\omega R) \cdot e^{in\phi}.$$

B_n and C_n will be found given the coefficients of the incident wave A_n. The two wormhole conditions supply the two equations to determine the unknown coefficients B_n and C_n.

The wormhole conditions C-1 and C-2 are:

$$\Psi|_{R=a,\phi=\theta} = e^{i\omega\tau} \Psi|_{r=a,\theta} \qquad -\pi < \theta \leq \pi,$$
$$\frac{\partial}{\partial R}\Psi|_{R=a,\phi=\theta} = -e^{i\omega\tau} \frac{\partial}{\partial r}\Psi|_{r=a,\theta} \qquad -\pi < \theta \leq \pi.$$

To compute Ψ at $R = a$ and $r = a$ it is necessary to write down Ψ_0, Ψ_1 in (R, ϕ) coordinates and Ψ_2 in (r, θ) coordinates. The addition theorem for cylindrical harmonics is used for expressing a cylindrical wave in terms of cylindrical waves of a translated origin [10]. It should be noted that, unlike the everywhere regular Bessel functions $J_n(\omega r)$, there are two different versions of the addition theorems of Hankel functions. For $\vec{r} = \vec{d} + \vec{R}$, addition theorems yield

$$H_n^{(1)}(\omega R) e^{in(\pi-\phi)} = \begin{cases} \sum_{k=-\infty}^{\infty} J_k(\omega d) H_{n+k}^{(1)}(\omega r) e^{i(n+k)\theta} & \text{if } r > d \\ \sum_{k=-\infty}^{\infty} H_k^{(1)}(\omega d) J_{n+k}(\omega r) e^{i(n+k)\theta} & \text{if } r < d \end{cases}, \tag{2}$$

$$H_n^{(1)}(\omega r) e^{in(\theta-\pi)} = \begin{cases} \sum_{k=-\infty}^{\infty} J_k(\omega d) H_{n+k}^{(1)}(\omega R) e^{-i(n+k)\phi} & \text{if } R > d \\ \sum_{k=-\infty}^{\infty} H_k^{(1)}(\omega d) J_{n+k}(\omega R) e^{-i(n+k)\phi} & \text{if } R < d \end{cases}, \tag{3}$$

$$J_n(\omega r) e^{in(\theta-\pi)} = \sum_{k=-\infty}^{\infty} J_k(\omega d) J_{n+k}(\omega R) e^{i(n+k)\phi}. \tag{4}$$

Wormhole conditions require the expression at $r = a$ and $R = a$. Since $a < d$, $r < d$ versions of (2) and (3) should be used.

Accordingly, the wave functions are expressed as a sum of Bessel functions at translated origin as

$$\sum_{n=-\infty}^{\infty} A_n J_n(\omega r) e^{in\theta} = \sum_{n=-\infty}^{\infty} \bar{A}_n \cdot J_n(\omega R) e^{in\phi},$$
$$\sum_{n=-\infty}^{\infty} B_n H_n^{(1)}(\omega r) e^{in\theta} = \sum_{n=-\infty}^{\infty} \bar{B}_n \cdot J_n(\omega R) \cdot e^{in\phi},$$
$$\sum_{n=-\infty}^{\infty} C_n H_n^{(1)}(\omega R) e^{in\phi} = \sum_{n=-\infty}^{\infty} \bar{C}_n \cdot J_n(\omega r) \cdot e^{in\theta}.$$

The expressions for $\bar{A}_n$, $\bar{B}_n$ and $\bar{C}_n$ are found using (2), (3) and (4) as:

$$\bar{A}_n = \sum_{k=-\infty}^{\infty} A_{k-n} J_k(\omega d), \tag{5}$$
$$\bar{B}_n = \sum_{k=-\infty}^{\infty} B_{k-n} H_k^{(1)}(\omega d), \tag{6}$$
$$\bar{C}_n = \sum_{k=-\infty}^{\infty} C_{k-n} H_k^{(1)}(\omega d). \tag{7}$$

Having obtained the expression of the wave in the coordinates centered at each mouth, application of wormhole conditions give necessary equations for the unknown coefficients B_n and C_n.

C-1 leads to

$$\sum_{n=-\infty}^{\infty} (A_n \cdot J_n(\omega a) + B_n \cdot H_n^{(1)}(\omega a) + \bar{C}_n J_n(\omega a)) e^{in\theta}$$
$$= e^{i\omega\tau} \sum_{n=-\infty}^{\infty} (\bar{A}_n J_n(\omega a) + \bar{B}_n J_n(\omega a) + C \cdot H_n^{(1)}(\omega a)) e^{in\theta},$$

$$B_n - e^{i\omega\tau} C_n = -\frac{J_n(\omega a)}{H_n^{(1)}(\omega a)} (A_n - e^{i\omega\tau} \bar{A}_n + \bar{C}_n - e^{i\omega\tau} \bar{B}_n), \tag{8}$$

and C-2 leads to

$$\sum_{n=-\infty}^{\infty} \left(A_n \cdot \frac{\partial}{\partial r} J_n(\omega r)|_{r=a} + B_n \cdot \frac{\partial}{\partial r} H_n^{(1)}(\omega r)|_{r=a} + \bar{C}_n \frac{\partial}{\partial r} J_n(\omega r)|_{r=a} \right) e^{in\theta}$$
$$= -e^{i\omega\tau} \sum_{n=-\infty}^{\infty} \left(\bar{A}_n \frac{\partial}{\partial r} J_n(\omega r)|_{r=a} + \bar{B}_n \frac{\partial}{\partial r} J_n(\omega r)|_{r=a} + C_n \frac{\partial}{\partial r} H_n^{(1)}(\omega r)|_{r=a} \right) e^{in\theta},$$

$$B_n + e^{i\omega\tau} C_n = -\frac{\dfrac{\partial}{\partial r} J_n(\omega r)|_{r=a}}{\dfrac{\partial}{\partial r} H_n^{(1)}(\omega r)|_{r=a}} (A_n + e^{i\omega\tau} \bar{A}_n + \bar{C}_n + e^{i\omega\tau} \bar{B}_n). \tag{9}$$

Solving (8) and (9) for B_n and C_n, one finds

$$B_n = -\gamma_n^+(\omega a)\bar{C}_n + e^{i\omega\tau}\gamma_n^-(\omega a)\bar{B}_n - \gamma_n^+(\omega a)A_n + e^{i\omega\tau}\gamma_n^-(\omega a)\bar{A}_n, \tag{10}$$

$$C_n = -\gamma_n^+(\omega a)\bar{B}_n + e^{-i\omega\tau}\gamma_n^-(\omega a)\bar{C}_n - \gamma_n^+(\omega a)\bar{A}_n + e^{-i\omega\tau}\gamma_n^-(\omega a)A_n, \tag{11}$$

where

$$\gamma_n^+(\omega a) \triangleq \frac{1}{2}\left(\frac{J_n(\omega a)}{H_n^{(1)}(\omega a)} + \frac{\dfrac{\partial}{\partial r} J_n(\omega r)|_{r=a}}{\dfrac{\partial}{\partial r} H_n^{(1)}(\omega r)|_{r=a}} \right),$$

$$\gamma_n^-(\omega a) \triangleq \frac{1}{2}\left(\frac{J_n(\omega a)}{H_n^{(1)}(\omega a)} - \frac{\dfrac{\partial}{\partial r} J_n(\omega r)|_{r=a}}{\dfrac{\partial}{\partial r} H_n^{(1)}(\omega r)|_{r=a}} \right).$$

For the sake of simplicity the known parts of (10) and (11) will be denoted by E_n and F_n respectively.

$$E_n = -\gamma_n^+(\omega a)A_n + e^{i\omega\tau}\gamma_n^-(\omega a)\bar{A}_n, \tag{12}$$

$$F_n = -\gamma_n^+(\omega a)\bar{A}_n + e^{-i\omega\tau}\gamma_n^-(\omega a)A_n. \tag{13}$$

This pair of equations (10) and (11) are not solvable explicitly; however it is possible to solve them for the limiting cases $a \ll d$ and $a \ll 1$.

A. Solutions for $a \ll d$ and $a \ll 1$:

The difficulty in solving (10) and (11) arises from the convolution sum present in the expressions of $\bar{B}_n$ and $\bar{C}_n$. However this term can be evaluated for special forms of $H_n^{(1)}(\omega d)$, namely when it is in complex exponential $e^{-in\alpha}$ form. When $|n| \ll \omega d$, asymptotically $H_n^{(1)}(\omega d)$ becomes $e^{-in\pi/2}$ as a function of n. $\gamma_n^+(\omega a)$ and $\gamma_n^-(\omega a)$ are almost zero for $|n| \gtrsim 2\omega a$, and so are B_n and C_n. Thus when $a \ll d$, the only terms that contribute to $\gamma_n^\pm(\omega a)\bar{B}_n$ ($\gamma_n^\pm(\omega a)\bar{C}_n$) are those satisfy $|n| \lesssim 2\omega a \ll \omega d$. The $a \ll d$ case is of practical importance in physics. In a wormhole universe, this corresponds to the case that the wormhole is connecting regions of the universe that are spatially far from each other compared to the radius of the wormhole.

This approximation is not valid for the high frequency limit in general.

When $a \ll 1$, $\gamma_n^\pm(\omega a)$ tends to zero unless $n \neq 0$, regardless of d. Accordingly, so are B_n and C_n.

These two cases in which approximate solutions are possible, $a \ll d$ and $a \ll 1$, are examined below.

1. $a \ll d$

For large ωd, asymptotic formula for $H_n^{(1)}(\omega d)$ is

$$H_n^{(1)}(\omega d) = z(\omega d)e^{-in\pi/2} \times (1 + i\frac{4n^2-1}{1!(8\omega d)} + i^2\frac{(4n^2-1)(4n^2-9)}{2!(8\omega d)^2} + i^3\frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3!(8\omega d)^3} + ...), \tag{14}$$

$$z(\omega d) \triangleq \sqrt{\frac{2}{\pi\omega d}}e^{i(\omega d-(\pi/4))}.$$

For $n^2 \ll \omega d$, the infinite sum inside the brackets can be approximated to 1 :

$$H_n^{(1)}(\omega d) \approx z(\omega d)e^{-in\pi/2}$$

This form of $H_n^{(1)}(\omega d)$ allows one to evaluate the sum $\bar{B}_n$:

$$\begin{aligned}\sum_{k=-\infty}^{\infty} B_{(k-n)}H_k^{(1)}(\omega d) = \sum_{m=-\infty}^{\infty} B_m H_{n+m}^{(1)}(\omega d) &\approx z(\omega d)(\sum_{m=-\infty}^{\infty} B_m e^{-im\pi/2})e^{-in\pi/2} \\ &= z(\omega d)\hat{B}(\pi/2)e^{-in\pi/2}\end{aligned}$$

where hat denotes the Fourier sum:

$$\hat{X}(\alpha) \triangleq \sum_{m=-\infty}^{\infty} X_m e^{-im\alpha}.$$

Substituting into (10) and (11)

$$B_n = z(\omega d)[-\hat{C}(\pi/2)\gamma_n^+ + e^{i\omega\tau}\hat{B}(\pi/2)\gamma_n^-]e^{-in\pi/2} + E_n, \tag{15}$$

$$C_n = z(\omega d)[-\hat{B}(\pi/2)\gamma_n^+ + e^{-i\omega\tau}\hat{C}(\pi/2)\gamma_n^-]e^{-in\pi/2} + F_n. \tag{16}$$

The right hand sides of (15) and (16) involves $\hat{C}(\pi/2)$ and $\tilde{B}(\pi/2)$ which are unknown yet. Multiplying each side by $e^{-in\pi/2}$ and sum over n's gives a pair of equations for $\hat{C}(\pi/2)$ and $\tilde{B}(\pi/2)$:

$$\begin{bmatrix}\hat{B}(\pi/2)\\ \hat{C}(\pi/2)\end{bmatrix} = \begin{bmatrix}1 - z(\omega d)e^{i\omega\tau}\tilde{\gamma}^-(\pi) & z(\omega d)\tilde{\gamma}^+(\pi) \\ z(\omega d)\tilde{\gamma}^+(\pi) & 1 - z(\omega d)e^{-i\omega\tau}\tilde{\gamma}^-(\pi)\end{bmatrix}^{-1}\begin{bmatrix}\hat{E}(\pi/2)\\ \hat{F}(\pi/2)\end{bmatrix} \tag{17}$$

The numerical results comparing the solutions obtained by these formulae and by the multiple scattering method is presented in the appendix.

To have a better approximation, the second term $i\frac{4n^2-1}{1!(8\omega d)}$ in the infinite sum of in (14) can be included:

$$H_n^{(1)}(\omega d) \approx \sqrt{\frac{2}{\pi\omega d}}e^{i(\omega d-(\pi/4))}e^{-in\pi/2}(1 + i\frac{4n^2-1}{1!(8z)})$$

In this case, the expression for $H_n^{(1)}(\omega d)$ involves $n^2 e^{-in\pi/2}$. This form of $H_n^{(1)}(\omega d)$ still enables one to evaluate $\bar{B}_n$ and $\bar{C}_n$, explicitly. This time there arise 6 unknowns and (17) is replaced by a 6x6 matrix equation. In this way it is possible to have better approximations by taking more terms into account in (14). The number of the linear equations is $4k-2$ when the first k term is taken into account in (14).

2. $a \ll 1$:

At $\omega a = 0$, the Bessel function $J_n(\omega a)$ is a discrete delta function with respect to variable n, and its derivative is zero for all n :

$$J_n(0) = \begin{cases} 1 \text{ if } n = 0 \\ 0 \text{ otherwise} \end{cases} ; \qquad \frac{\partial}{\partial r} J_n(\omega r)|_{r=0} = 0 \text{ for all } n. \tag{18}$$

Therefore, in the limit a goes to zero, $\gamma_n^+(\omega a)$ and $\gamma_n^-(\omega a)$ become discrete delta functions:

$$\gamma_n^+(\omega a) \approx \gamma_n^-(\omega a) \approx \frac{J_0(\omega a)}{H_0^{(1)}(\omega a)} \delta_n,$$

where [11],

$$\frac{J_0(\omega a)}{H_0^{(1)}(\omega a)} \triangleq \gamma_0 = \frac{1}{1 + \frac{2i}{\pi}(\ln(\frac{\omega a}{2}) + 0.5772)}. \tag{19}$$

The $\gamma_n^{\pm}(\omega a)$ factors standing in front of each term on the right hand sides of (10) and (11) make B_n and C_n delta functions as well.

$$B_n = B_0 \delta_n,$$
$$C_n = C_0 \delta_n;$$

so that,

$$\Psi_1 = B_0 \cdot H_0^{(1)}(\omega r),$$
$$\Psi_2 = C_0 \cdot H_0^{(1)}(\omega R).$$

B_0 and C_0 are found by substitution to (10) and (11):

$$\gamma_n^{\pm}(\omega a)\bar{B}_n \approx (\gamma_0 \delta_n)(B_0 H_n^{(1)}(\omega d)) = \gamma_0 B_0 H_0^{(1)}(\omega d)\delta_n,$$
$$\gamma_n^{\pm}(\omega a)\bar{C}_n \approx \gamma_0 C_0 H_0(\omega d)\delta_n,$$

$$\begin{bmatrix} B_0 \\ C_0 \end{bmatrix} = \begin{bmatrix} 1 - \gamma_0 H_0(\omega d) e^{i\omega\tau} & \gamma_0 H_0(\omega d) \\ \gamma_0 H_0(\omega d) & 1 - \gamma_0 H_0(\omega d) e^{-i\omega\tau} \end{bmatrix}^{-1} \begin{bmatrix} E_0 \\ F_0 \end{bmatrix},$$

where

$$E_0 = \gamma_0(-A_0 + \ e^{i\omega\tau} \sum_{m=-\infty}^{\infty} A_m J_m(\omega d)),$$

$$F_0 = \gamma_0(- \sum_{m=-\infty}^{\infty} A_m J_m(\omega d) + \ e^{-i\omega\tau} A_0).$$

B. Multiple scattering

An alternative approach is the use of multiple scattering [9],[12]. In the multiple scattering method, the scattered waves are decomposed into lower order scattered waves from each mouth. Initially each wormhole mouth is considered to be excited by only the incident wave and first order scattering coefficients are found by imposing wormhole conditions. Then each wormhole is considered to be excited by only the first order scattered wave from the other mouth and the second order scattering coefficient are found imposing wormhole conditions. k^{th} order scattering coefficients are found by continuing the same procedure. The scattered wave from each mouth is the sum of these k^{th} order scattering coefficients.

$$\Psi_1 = \sum_{k=1}^{\infty} \Psi_1^k$$

where

$$\Psi_1^k = \sum_{n=-\infty}^{\infty} B_n^k H_n^{(1)}(\omega r) \cdot e^{in\theta} \text{ and } B_n = \sum_{k=1}^{\infty} B_n^k.$$

Similarly,

$$\Psi_2 = \sum_{k=1}^{\infty} \Psi_2^k,$$

$$\Psi_2^k = \sum_{n=-\infty}^{\infty} C_n^k H_n^{(1)}(\omega R) \cdot e^{in\phi}.$$

Wormhole conditions for the first order scattering coefficients are:

$$(\Psi_0 + \Psi_1^1)_{r=a} = e^{i\omega\tau}(\Psi_0 + \Psi_2^1)_{R=a,\phi=\theta}, \tag{20}$$

$$\frac{\partial}{\partial r}(\Psi_0 + \Psi_1^1)|_{r=a} = -e^{i\omega\tau}\frac{\partial}{\partial R}(\Psi_0 + \Psi_2^1)|_{R=a,\phi=\theta}. \tag{21}$$

Similarly for the $(k+1)^{th}$ order coefficients:

$$(\Psi_1^{k+1} + \Psi_2^k)_{r=a} = e^{i\omega\tau}(\Psi_1^k + \Psi_2^{k+1})|_{R=a,\phi=\theta},$$

$$\frac{\partial}{\partial r}(\Psi_1^{k+1} + \Psi_2^k)|_{r=a} = -e^{i\omega\tau}\frac{\partial}{\partial R}(\Psi_1^k + \Psi_2^{k+1})|_{R=a,\phi=\theta}.$$

It is easy to see that when 1^{st} and k^{th} order scattered waves satisfy wormhole conditions, total scattered wave satisfies as well.

$$\Psi|_{r=a} = (\Psi_0 + \Psi_1 + \Psi_2)|_{r=a} = (\Psi_0 + \Psi_1^1 + \sum_{k=1}^{\infty}(\Psi_1^{k+1} + \Psi_2^k))|_{r=a}$$
$$= e^{i\omega\tau}(\Psi_0 + \Psi_2^1)|_{R=a,\phi=\theta} + e^{i\omega\tau}\sum_{k=1}^{\infty}(\Psi_l^k + \Psi_2^{k+1})|_{R=a,\phi=\theta} = e^{i\omega\tau}(\Psi_0 + \Psi_1 + \Psi_2)|_{R=a,\phi=\theta}.$$

Imposing wormhole conditions to (20) and (21), first order scattering coefficients are obtained.

C-1 yields:

$$\sum_{n=-\infty}^{\infty} A_n \cdot J_n(\omega a)e^{in\theta} + B_n^1 \cdot H_n^{(1)}(\omega a) \cdot e^{in\theta} = e^{i\omega\tau}(\sum_{n=-\infty}^{\infty} \bar{A}_n J_n(\omega a)e^{in\theta} + \sum_{n=-\infty}^{\infty} C_n^1 H_n^{(1)}(\omega a) \cdot e^{in\theta}),$$

$$B_n^1 - e^{i\omega\tau}C_n^1 = -\frac{J_n(\omega a)}{H_n^{(1)}(\omega a)}(A_n - e^{i\omega\tau}\bar{A}_n).$$

C-2 yields:

$$B_n^1 + e^{i\omega\tau}C_n^1 = -\frac{\frac{\partial}{\partial r}J_n(\omega r)|_{r=a}}{\frac{\partial}{\partial r}H_n^{(1)}(\omega r)|_{r=a}}(A_n + e^{i\omega\tau}\bar{A}_n).$$

Solving for B_n^1 and C_n^1 :

$$B_n^1 = -\gamma_n^+(\omega a)A_n + e^{i\omega\tau}\gamma_n^-(\omega a)\bar{A}_n, \tag{22}$$
$$C_n^1 = -\gamma_n^+(\omega a)\bar{A}_n + e^{-i\omega\tau}\gamma_n^-(\omega a)A_n. \tag{23}$$

Note that B_n^1 and C_n^1 are equal to the known parts of (10) and (11), E_n and F_n, respectively.

k^{th}order scattering coefficients are obtained similarly as:

$$B_n^{k+1} = -\gamma_n^+(\omega a)\bar{C}_n^k + e^{i\omega\tau}\gamma_n^-(\omega a)\bar{B}_n^k, \tag{24}$$
$$C_n^{k+1} = -\gamma_n^+(\omega a)\bar{B}_n^k + e^{-i\omega\tau}\gamma_n^-(\omega a)\bar{C}_n^k. \tag{25}$$

IV. 3+1 DIMENSIONS

In 3+1 dimensions, the solutions of wave equation in spherical coordinates, i.e. spherical wave functions, involve spherical Bessel functions and spherical harmonics [13]. In agreement

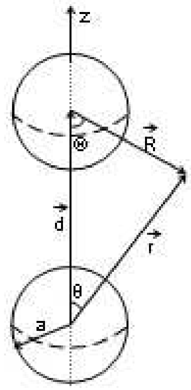

FIG. 3: Coordinates used for 3+1 dimensions.

with the 2+1 dimensional case, Ψ_0 is expressed in terms of usual spherical Bessel functions, while Ψ_1 and Ψ_2 are expressed in terms of spherical Hankel functions. Referring to fig.3,

$$\Psi_0 = \sum_{l=-\infty}^{\infty} \sum_{m=-l}^{l} A_{lm} \cdot j_l(\omega r) Y_{lm}(\theta, \varphi), \tag{26}$$

$$\Psi_1 = \sum_{l=-\infty}^{\infty} \sum_{k=-l}^{l} B_{lm} \cdot h_l^{(1)}(\omega r) Y_{lm}(\theta, \varphi), \tag{27}$$

$$\Psi_2 = \sum_{l=-\infty}^{\infty} \sum_{m=-l}^{l} C_{lm} \cdot h_l^{(1)}(\omega R) Y_{lm}(\Theta, \varphi), \tag{28}$$

and the wormhole conditions are,

$$\Psi|_{r=a,\theta,\varphi} = e^{i\omega\tau} \Psi|_{R=a,\Theta=\theta,\varphi}; \qquad 0 \leq \theta \leq \pi,\ -\pi < \varphi \leq \pi,$$
$$\frac{\partial}{\partial r} \Psi|_{r=a,\theta,\varphi} = -e^{i\omega\tau} \frac{\partial}{\partial R} \Psi|_{R=a,\Theta=\theta,\varphi}; \qquad 0 \leq \theta \leq \pi,\ -\pi < \varphi \leq \pi.$$

The addition theorems for the spherical wave functions, for $\vec{r} = \vec{d} + \vec{R}$ are [14],[15]:

$$j_l(\omega r) Y_{lm}(\theta, \varphi) = \sum_{l'm'} \alpha_{l'm'}^{lm+}(\vec{d}) j_{l'}(\omega R) Y_{l'm'}(\pi - \Theta, \varphi), \tag{29}$$

$$h_l^{(1)}(\omega r) Y_{lm}(\theta, \varphi) = \sum_{l'm'} \alpha_{l'm'}^{lm}(\vec{d}) j_{l'}(\omega R) Y_{l'm'}(\pi - \Theta, \varphi) \quad \text{for } R < d, \tag{30}$$

$$h_l^{(1)}(\omega R) Y_{lm}(\pi - \Theta, \varphi) = \sum_{l'm'} \alpha_{l'm'}^{lm}(-\vec{d}) j_{l'}(\omega r) Y_{l'm'}(\theta, \varphi) \quad \text{for } r < d, \tag{31}$$

where

$$\alpha_{l'm'}^{lm+}(\vec{x}) \triangleq \sum_{\lambda\mu} c(lm|l'm'|\lambda\mu) j_\lambda(\omega\,|x|) Y_{\lambda\mu}(\hat{x}), \tag{32}$$
$$\alpha_{l'm'}^{lm}(\vec{x}) \triangleq \sum_{\lambda\mu} c(lm|l'm'|\lambda\mu) h_\lambda^{(1)}(\omega\,|x|) Y_{\lambda\mu}(\hat{x}).$$

The coefficients $c(lm|l'm'|\lambda\mu)$ in terms of 3-j symbols are:

$$c(lm|l'm'|\lambda\mu) = i^{l+\lambda-1}(-1)^m[4\pi(2l+1)(2l'+1)(2\lambda+1)]^{1/2} \begin{pmatrix} l & l' & \lambda \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & \lambda \\ m & -m' & \mu \end{pmatrix} \tag{33}$$

The expansion (30) and (31) are valid for $R < d$ and $r < d$, respectively. and they cover region where the wormhole conditions are imposed: $R = a$ and $r = a$, $(a < d)$.

Using (29), (30) and (31), Ψ_0, Ψ_1 and Ψ_2 are expressed as a sum of wave functions at translated origin as:

$$\sum_{lm} A_{lm} \cdot j_l(\omega r) Y_{lm}(\theta, \varphi) = \sum_{lm} \bar{A}_{lm} j_l(\omega R) Y_{lm}(\Phi, \varphi),$$
$$\sum_{lm} B_{lm} \cdot h_l^{(1)}(\omega r) Y_{lm}(\theta, \varphi) = \sum_{lm} \bar{B}_{lm} j_l(\omega r) Y_{lm}(\Phi, \varphi),$$
$$\sum_{lm} C_{lm} \cdot h_l^{(1)}(\omega R) Y_{lm}(\Phi, \varphi) = \sum_{lm} \bar{C}_{lm} j_l(\omega r) Y_{lm}(\theta, \varphi).$$

where the analogues of the formulas (5), (6) and (7) are (see appendix A)

$$\bar{A}_{lm} = (-1)^{l+m} \sum_{l'} A_{l'm} \alpha_{lm}^{l'm+}(\vec{d}), \tag{34}$$
$$\bar{B}_{lm} = (-1)^{l+m} \sum_{l'} B_{l'm} \alpha_{lm}^{l'm}(\vec{d}), \tag{35}$$
$$\bar{C}_{lm} = (-1)^{l+m} \sum_{l'} C_{l'm} \alpha_{lm}^{l'm}(\vec{d}). \tag{36}$$

3-j symbols are zero unless $m - m' = \mu$ [16]. Furthermore, $\vec{d} = \hat{z}d$, and $Y_{\lambda\mu}(\hat{d}) = Y_{\lambda\mu}(0, \varphi)$ is nonzero only when $\mu = 0$. Thus $m' = m$ and that's why the summation over m' drops in (34), (35) and (36)

$$Y_{\lambda 0}(0, \varphi) = \sqrt{\frac{2\lambda+1}{4\pi}},$$
$$\alpha_{lm}^{l'm'}(\vec{d}) = \alpha_{lm}^{l'm}(\vec{d}) = \sum_{\lambda,\mu} c(l'm|lm|\lambda 0) h_\lambda^{(1)}(\omega d) \sqrt{\frac{2\lambda+1}{4\pi}},$$
$$\alpha_{lm}^{l'm'+}(\vec{d}) = \alpha_{lm}^{l'm+}(\vec{d}) = \sum_{\lambda,\mu} c(l'm|lm|\lambda 0) j_\lambda(\omega d) \sqrt{\frac{2\lambda+1}{4\pi}}.$$

Imposing the wormhole conditions and using the orthogonality of $Y_{lm}(\theta,\varphi)$ for different l,m, yields the 3+1 dimensional analogues of the equations found for 2+1 dimensions:

$$B_{lm} - e^{i\omega\tau}C_{lm} = -\frac{j_l(\omega a)}{h_l^{(1)}(\omega a)}(\bar{C}_{lm} - e^{i\omega\tau}\bar{B}_{lm} + A_{lm} - e^{i\omega\tau}\bar{A}_{lm}),$$
$$B_{lm} + e^{i\omega\tau}C_{lm} = -\frac{\frac{\partial}{\partial r}j_l(\omega r)|_{r=a}}{\frac{\partial}{\partial r}h_l^{(1)}(\omega r)|_{r=a}}(\bar{C}_{lm} - e^{i\omega\tau}\bar{B}_{lm} + A_{lm} - e^{i\omega\tau}\bar{A}_{lm}),$$

giving

$$B_{lm} = -\gamma_n^+(\omega a)\bar{C}_{lm} + e^{i\omega\tau}\gamma_n^-(\omega a)\bar{B}_{lm} + E_{lm}, \tag{37}$$
$$C_{lm} = -\gamma_n^+(\omega a)\bar{B}_{lm} + e^{-i\omega\tau}\gamma_n^-(\omega a)\bar{C}_{lm} + F_{lm}, \tag{38}$$

where E_{lm} and F_{lm} are known functions of A_{lm} :

$$E_{lm} = -\gamma_n^+(\omega a)A_{lm} + e^{i\omega\tau}\gamma_n^-(\omega a)\bar{A}_{lm},$$
$$F_{lm} = -\gamma_n^+(\omega a)\bar{A}_{lm} + e^{-i\omega\tau}\gamma_n^-(\omega a)A_{lm},$$

and $\gamma_l^{\pm}(\omega a)$ are defined similar to 2+1 dimensional case:

$$\gamma_l^+(\omega a) \triangleq \frac{1}{2}\left(\frac{j_l(\omega a)}{h_l^{(1)}(\omega a)} + \frac{\frac{\partial}{\partial r}j_l(\omega r)|_{r=a}}{\frac{\partial}{\partial r}h_l^{(1)}(\omega r)|_{r=a}}\right),$$
$$\gamma_l^-(\omega a) \triangleq \frac{1}{2}\left(\frac{j_l(\omega a)}{h_l^{(1)}(\omega a)} - \frac{\frac{\partial}{\partial r}j_l(\omega r)|_{r=a}}{\frac{\partial}{\partial r}h_l^{(1)}(\omega r)|_{r=a}}\right).$$

Similar to the 2+1 dimensional case, (37) and (38) can be solved for $a \ll 1$ and $a \ll d$ cases.

A. Solutions for $a \ll d$ and $a \ll 1$:

The asymptotic form of $h_l^{(1)}(\omega d)$ for $l \ll \omega d$ allows us to compute $\alpha_{lm}^{l'm'}(\vec{d})$. The similarity between 2+1 and 3+1 dimensional cases are remarkable. Indeed for 2+1 dimensional case, if we consider $\bar{X}_n = \sum_{k=-\infty}^{\infty} X_{k-n}H_k^{(1)}(\omega d)$ as an operator on $H_n^{(1)}(\omega d)$, the asymptotic form of $H_n^{(1)}(\omega d)$ for $n \ll \omega d$ is an eigenvalue of this operator. Similarly in the passage to the 3+1 dimensions, considering $\bar{X}_{lm} = (-1)^{l+m}\sum_{l'm'} X_{l'm'}\alpha_{lm}^{l'm'}(\vec{d})$ as an operator on $h_l^{(1)}(\omega d)$, asymptotic form of $h_l^{(1)}(\omega d)$ for $l \ll \omega d$ is an eigenfunction of $\bar{X}_{lm}$.

As in the 2+1 dimensional case, the presence of the $\gamma_n^{\pm}(\omega a)$ factor at each term of the right hand sides of (37) and (38), makes B_{lm} and C_{lm} vanish when $\omega a \ll l$. Thus when $a \ll d$ the asymptotic form of $h_\lambda^{(1)}(\omega d)$ for $l \ll \omega d$ can be used.

For $a \ll 1$, just like 2+1 dimensions, $h_l^{(1)}(\omega d)$ is zero unless $l = 0$ and (37) and (38) can be solved.

1. $a \ll d$:

$\gamma_l^+(\omega a)$ and $\gamma_l^-(\omega a)$ filter the terms with $l > 2\omega a$, thus when $a \ll d$ the only terms that contribute to $\bar{B}_{lm}$ and $\bar{C}_{lm}$are $l \ll \omega d$. In this case $h_\lambda^{(1)}(\omega d)$ has the asymptotic expression:

$$h_\lambda^{(1)}(\omega d) \approx i^{-(\lambda+1)} \frac{e^{i\omega d}}{\omega d}$$

Then,

$$\bar{B}_{lm} \approx \sum_{l'm'} \sum_{\lambda} B_{l'm'} c(l'm|lm|\lambda 0) i^{-(\lambda+1)} \frac{e^{i\omega d}}{\omega d} \sqrt{\frac{2\lambda+1}{4\pi}}$$

Substituting

$$c(l'm|lm|\lambda 0) = i^{l'+\lambda-1}(-1)^m [4\pi(2l+1)(2l'+1)(2\lambda+1)]^{1/2} \begin{pmatrix} l & l' & \lambda \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & \lambda \\ m & -m & 0 \end{pmatrix}$$

gives:

$$\bar{B}_{lm} \approx -\frac{e^{i\omega d}}{\omega d} \sum_{l'} B_{l'm} i^{l'} (-1)^m [(2l+1)(2l'+1)]^{1/2} \delta_{m0}$$

where in the last step the orthogonality property of the 3-j symbols is used [17]:

$$\sum_{\lambda\mu} (2\lambda+1) \begin{pmatrix} l & l' & \lambda \\ m_1 & m_2 & \mu \end{pmatrix} \begin{pmatrix} l & l' & \lambda \\ p_1 & p_2 & \mu \end{pmatrix} = \delta_{m1p_1} \delta_{m2p2}.$$

Thus,

$$\bar{B}_{lm} \approx -\frac{e^{i\omega d}}{\omega d} \sqrt{(2l+1)} \sum_{l'} B_{l'0} i^{l'} \sqrt{(2l'+1)} \delta_{m0} = -\frac{e^{i\omega d}}{\omega d} \sqrt{(2l+1)} T(B_{l0}) \delta_{m0},$$

where, for X_l being any function of l, the functional $T(X_l)$ is defined as:

$$T(X_l) \triangleq \sum_{l'} X_{l'} i^{l'} \sqrt{(2l'+1)}$$

If $m \neq 0$; $B_{lm} = E_{lm}, C_{lm} = F_{lm}$ and if $m = 0$:

$$B_{l0} = -e^{i\omega\tau} \frac{e^{i\omega d}}{\omega d} (-1)^l \sqrt{(2l+1)} \gamma_l^-(\omega a) T(B_{l0}) + \frac{e^{i\omega d}}{\omega d} (-1)^l \sqrt{(2l+1)} \gamma_l^+ T(C_{l0}) + E_{l0},$$
$$C_{l0} = -e^{-i\omega\tau} \frac{e^{i\omega d}}{\omega d} (-1)^l \sqrt{(2l+1)} \gamma_l^-(\omega a) T(C_{l0}) + \frac{e^{i\omega d}}{\omega d} (-1)^l \sqrt{(2l+1)} \gamma_l^+ T(B_{l0}) + F_{l0}.$$

Multiplying each side of these equations by $i^l\sqrt{(2l+1)}$ and summing over l gives $T(B_{l0})$ and $T(C_{l0})$:

$$\begin{bmatrix} T(B_{l0}) \\ T(C_{l0}) \end{bmatrix} = \begin{bmatrix} 1 - e^{i\omega\tau}\frac{e^{i\omega d}}{\omega d}T((-i)^l(2l+1)\gamma_l^-(\omega a)) & \frac{e^{i\omega d}}{\omega d}T((-i)^l(2l+1)\gamma_l^+(\omega a)) \\ \frac{e^{i\omega d}}{\omega d}T((-i)^l(2l+1)\gamma_l^+(\omega a)) & 1 - e^{-i\omega\tau}\frac{e^{i\omega d}}{\omega d}T((-i)^l(2l+1)\gamma_l^-(\omega a)) \end{bmatrix}^{-1} \begin{bmatrix} T(E_{l0}) \\ T(F_{l0}) \end{bmatrix}$$

2. $a \ll 1$:

Similar to the 2+1 dimensional case, for $a \ll 1$, $\gamma_l^{\pm}(\omega a)$ becomes a discrete delta function, δ_l. Due to the factors of $\gamma_l^{\pm}(\omega a)$ in each term, B_{lm} and C_{lm} are nonzero for only $l = m = 0$. The problem reduces to finding the constants B_{00} and C_{00}.

$$\gamma_l^+(\omega a) \approx \gamma_l^-(\omega a) \approx \frac{\omega a}{i + \omega a}\delta_l,$$

$$\begin{aligned} B_{lm} &= B_{00}\delta_l\delta_m, \\ C_{lm} &= C_{00}\delta_l\delta_m. \end{aligned}$$

$l = 0$ implies $m = 0$ and $l' = \lambda$, so that

$$\begin{aligned} \bar{B}_{00} &= \sum_\lambda B_{00}\delta_\lambda c(\lambda 0|00|\lambda 0)h_\lambda^{(1)}(\omega d)\sqrt{\frac{2\lambda+1}{4\pi}} = B_{00}h_0^{(1)}(\omega d), \\ \bar{C}_{00} &= \sum_\lambda C_{00}\delta_\lambda c(\lambda 0|00|\lambda 0)h_\lambda^{(1)}(\omega d)(-1)^\lambda\sqrt{\frac{2\lambda+1}{4\pi}} = C_{00}h_0^{(1)}(\omega d). \end{aligned}$$

B_{00} and C_{00} are found as:

$$\begin{bmatrix} B_{00} \\ C_{00} \end{bmatrix} = \begin{bmatrix} 1 - e^{i\omega\tau}h_0^{(1)}(\omega d) & h_0^{(1)}(\omega d) \\ h_0^{(1)}(\omega d) & 1 - e^{-i\omega\tau}h_0^{(1)}(\omega d) \end{bmatrix}^{-1} \begin{bmatrix} E_{00} \\ F_{00} \end{bmatrix}.$$

B. Multiple scattering

Multiple scattering formulae for the 3+1 dimensions can be found. by the same steps followed as the 2+1 dimensional case The multiple scattering expansion of 3+1 dimensional wave functions

$$\begin{aligned} \Psi_1^k &= \sum_{lm} B_{lm}^k \cdot h_l^{(1)}(\omega r)Y_{lm}(\theta, \varphi), \\ \Psi_2^k &= \sum_{lm} C_{lm}^k \cdot h_l^{(1)}(\omega R)Y_{lm}(\Theta, \varphi), \end{aligned}$$

together with the wormhole conditions for the 1^{st} and the k^{th} order scattering coefficients

$$
\begin{aligned}
(\Psi_0 + \Psi_1^1)_{r=a} &= e^{i\omega\tau}(\Psi_0 + \Psi_2^1)_{R=a,\Theta=\theta}, \\
\frac{\partial}{\partial r}(\Psi_0 + \Psi_1^1)|_{r=a} &= -e^{i\omega\tau}\frac{\partial}{\partial R}(\Psi_0 + \Psi_2^1)|_{R=a,\Theta=\theta}, \\
(\Psi_1^{k+1} + \Psi_2^k)|_{r=a} &= e^{i\omega\tau}(\Psi_1^k + \Psi_2^{k+1})|_{R=a,\Theta=\theta}, \\
\frac{\partial}{\partial r}(\Psi_1^{k+1} + \Psi_2^k)|_{r=a} &= -e^{i\omega\tau}\frac{\partial}{\partial R}(\Psi_1^k + \Psi_2^{k+1})|_{R=a,\Theta=\theta},
\end{aligned}
$$

lead the formulas for the multiple scattering solution:

$$
\begin{aligned}
B_{lm}^1 &= -\gamma_l^+(\omega a)A_{lm} + e^{i\omega\tau}\gamma_l^-(\omega a)\bar{A}_{lm}, \\
C_{lm}^1 &= -\gamma_l^+(\omega a)\bar{A}_{lm} + e^{-i\omega\tau}\gamma_n^-(\omega a)A_{lm}, \\
B_{lm}^{k+1} &= -\gamma_l^+(\omega a)\bar{C}_{lm}^k + e^{i\omega\tau}\gamma_n^-(\omega a)\bar{B}_{lm}^k, \\
C_{lm}^{k+1} &= -\gamma_l^+(\omega a)\bar{B}_{lm}^k + e^{-i\omega\tau}\gamma_n^-(\omega a)\bar{C}_{lm}^k.
\end{aligned}
$$

V. NUMERICAL VERIFICATIONS:

In this section, the solutions for certain values of $a,\ d,\ \omega$ and τ are evaluated numerically for 2+1 dimensions and it is verified that they satisfy wormhole conditions. Numerical evaluation of solutions are done by using the multiple scattering results (22), (23), (24) and (25). Alternatively (10) and (11) are tested by an iteration method. For iteration, two initial test functions B_n^0 and C_n^0 are picked and substituted to right hand sides of (10) and (11) to obtain B_n^1 and C_n^1. Similarly B_n^1 and C_n^1 are substituted to (10) and (11) to obtain B_n^2 and C_n^2. Continuing this iteration, B_n^m and C_n^m are assumed to converge to the solution. No proof for the conditions of convergence is given, it is verified numerically that the solution found by iteration method converges to the multiple scattering solution for the parameter sets that are considered.

Moreover, to check the formulas found for $a \ll d$ the solutions found by this method is compared with the multiple scattering solution.

As the velocity of wave is taken as 1 in equation (1), $\omega d = 2\pi d/\lambda$ and $\omega a = 2\pi a/\lambda$ where λ is the wavelength of the wave. Practically if when a light wave in a wormhole universe is considered, these values supposed to be much larger (at least order of $\sim 10^{10}$) compared to what chosen in the below examples. However, numerical calculations with such large values were beyond the capacity of the PC used and there is no reason to think that the formulas will fail for large values.

The incident wave Ψ_0 is chosen as a plane wave and $A_n = e^{in\alpha}$, where α is the angle between direction of the incident wave and the y axis.

Referring to figure 2, the wormhole is located symmetrically with respect to the y axis. Consider the reflection operator R with respect to the y axis, i.e. $R\Psi(x,y) = \Psi(-x,y)$. According to the wormhole conditions C-1 and C-2

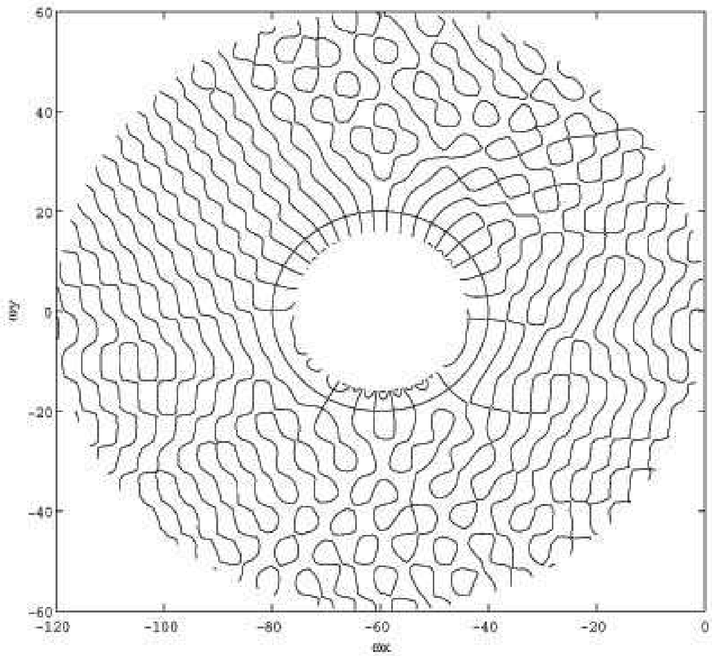

FIG. 4: The contour plot of $\mathrm{Re}(\Psi - e^{i\omega\tau}R\Psi)$ in the vicinity of left wormhole mouth Δ_-. The contour circle at $r = a$ shows that $(\Psi - e^{i\omega\tau}R\Psi)|_{r=a}$ is constant. ($\omega a = 20;\ \omega d = 120;\ \tau = 1;\ \alpha = \pi/3$)

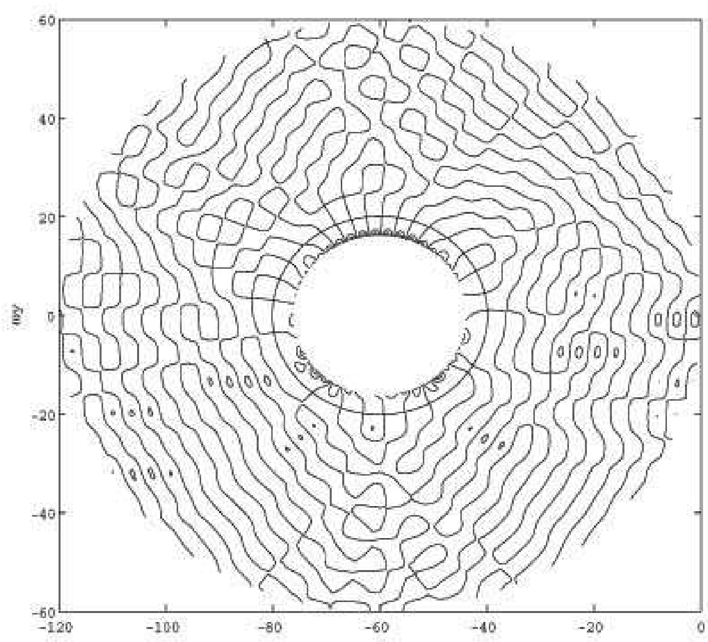

FIG. 5: The contour plot of $\mathrm{Re}(\dfrac{\partial}{\partial r}(\Psi + e^{i\omega\tau}R\Psi))$ in the vicinity of left wormhole mouth Δ_-. The same contour circle at $r = a$ is evident.($\omega a = 20;\ \omega d = 120;\ \tau = 1;\ \alpha = \pi/3$)

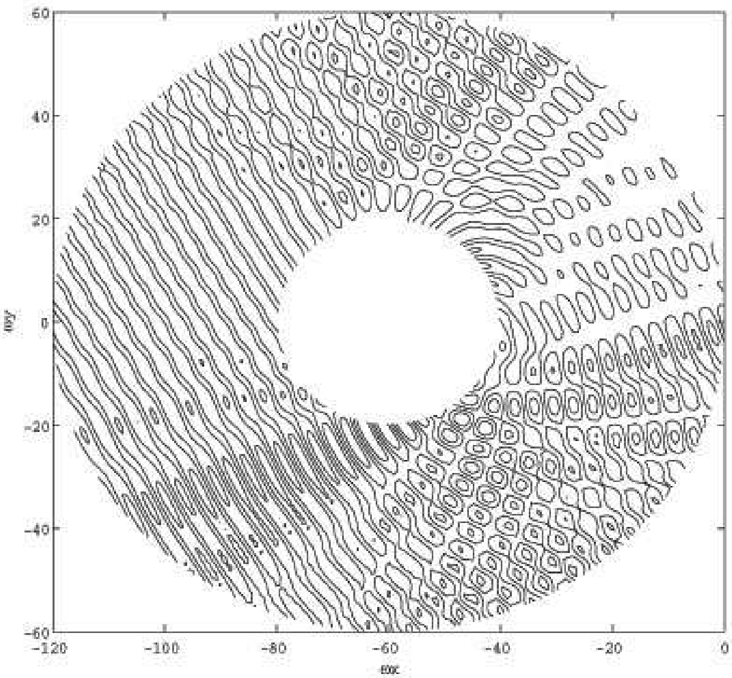

FIG. 6: The contour plot of Re(Ψ) in the vicinity of Δ_-. The incident wave is coming from the left with an angle $\pi/3$ and the shadow is on the opposite side. ($\omega a = 20;\ \omega d = 120;\ \tau = 1;\ \alpha = \pi/3$).

$$(\Psi - e^{i\omega\tau} R\Psi)|_{r=a} = 0 \tag{39}$$

$$\frac{\partial}{\partial r}(\Psi + e^{i\omega\tau} R\Psi)|_{r=a} = 0 \tag{40}$$

It is verified that the solution found satisfies (39)and (40) by plotting the contours at the vicinity of one of the wormhole mouths.

In fig.4, fig. 5, fig. 6 and fig. 7 the parameters are: $\omega a = 20$, $\omega d = 120$, $\alpha = \pi/3$, $\omega\tau = 1$.Figure 4 and Figure 5 show contour plots of the multiple scattering solution for real part of $\Psi(x, y) + e^{i\omega\tau} R(\Psi(x, y)$ and $\frac{\partial}{\partial r}(\Psi(x, y) - e^{i\omega\tau} R(\Psi(x, y))$, respectively. In both figures, the contour circles at $\omega r = \omega a = 20$ are clearly visible indicating that the values of each function are zero along $r = a$ circle. This shows that the wormhole conditions are satisfied. The contour plots of imaginary parts -which are not presented here- give the same contour circles at $r = a$. Although the contour is plotted for $0.8a < r < d$ to make the zero contour circle more visible, it should be remembered that the region $r < a$ is not a part of the spacetime. Figure 6 is a contour plot of the real part of the solution $\Psi(r, \theta)$ to give an example of a visual image of the solution. Figure 7 is a comparison of the multiple scattering solution and the iteration solution. The solid line with '+' markers show the $|B_n|$ that are found by multiple scattering and dashed line with 'x' markers are the difference of the absolute values of B_n found by the iteration method and the multiple scattering method. The difference is zero for all n; i.e. these two solutions are exactly the same. The results are obtained after 20 iterations. The test functions are chosen as constant, $B_n^0 = C_n^0 = 1$.

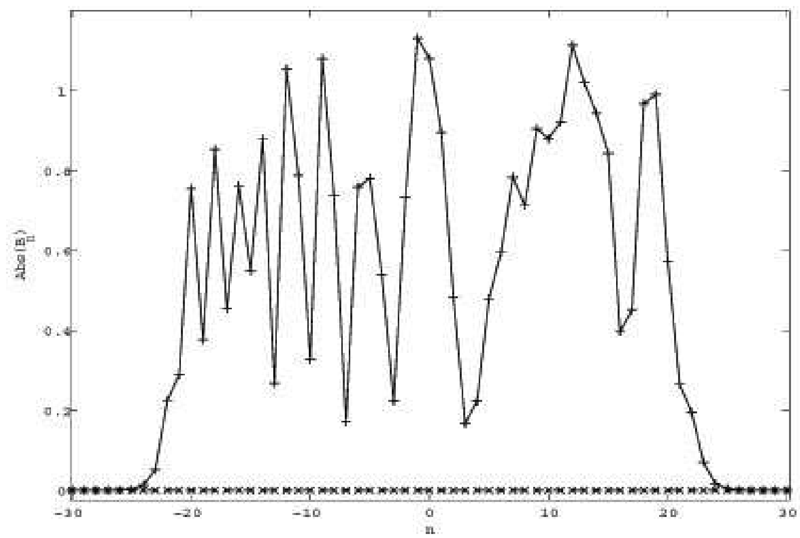

FIG. 7: Comparison of the multiple scattering and the iteration results. The difference of $|B_n|$ found by these two methods are points with marker 'x' which are zero for all n. ($\omega a = 20$; $\omega d = 120$; $\tau = 1$; $\alpha = \pi/3$)

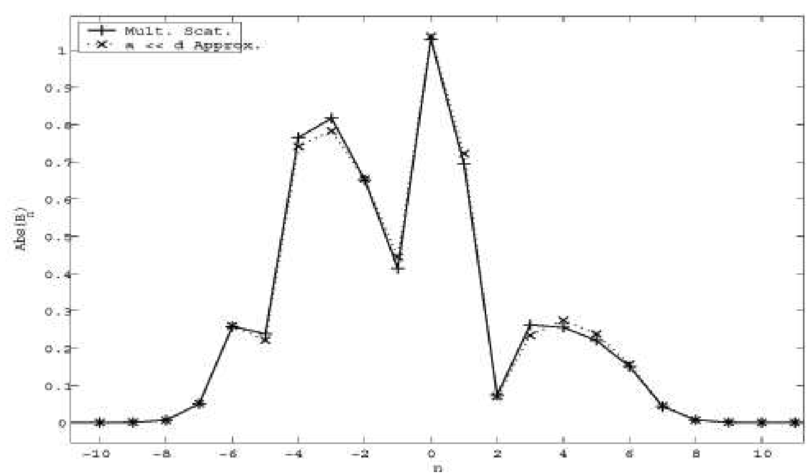

FIG. 8: Comparison of the multiple scattering and the $a \ll d$ approximation. ($\omega a = 5$; $\omega d = 1600$; $\tau = 1$; $\alpha = \pi/5$)

In fig. 8, the parameters are: $\omega a = 5,\ \omega d = 1600,\ \alpha = \pi/5,\ \omega\tau = 1$. This is an example for $a \ll d$ case. In the figure $|B_n|$ versus n is plotted. The solid line with '+' markers is the multiple scattering solution and the dashed line with 'x' markers is the $a \ll d$ approximation solutions given by (15) and (16).

VI. CONCLUDING REMARKS

The principal purpose of the present work was to investigate the scalar waves in wormhole topology. As the wormhole considered is flat, no complications due to curvature arise.

Although the spacetime considered in this work admits CTCs for sufficiently large values of time lag τ, their existence has no influence on monochromatic waves. The closed timelike curves emerge when time lag τ is greater than $d - 2a$. However, τ appears in the equations only as $\exp(i\omega\tau)$. Thus the solution remains the same for all integer k's where $\omega\tau = 2k\pi + \alpha$ and increasing τ does not change the nature of the solutions. This suggests that the presence of closed timelike curves does not have a dramatic effect on the scalar wave solutions.

This should not be surprising considering that, in a wave equation, what really matters is presence of closed null curves, rather than closed timelike curves. It is reasonable to think that the existence of closed timelike curves will not effect the nature of the solutions as long as closed null curves are not present. CTC's are present in the flat wormhole spacetime studied here, but still they don't have a significant effect on the solution. The reason is explained in [6]: In this kind of spacetimes, the closed null curves are a set of measure zero and due to the diverging lens property of the wormhole, the strength of the field is weakened by a factor $a/2d$ at each loop in the infinitely looping closed null geodesics.

The complications related to closed timelike curves are due to difficulty in specifying a Cauchy hypersurface when solving the Cauchy problem. Null geodesics are bicharacteristics of the wave equation and arbitrary initial data cannot be properly posed in a null direction [18]. A spacelike hypersurface never contains vectors in a null direction, thus are good candidates for specifying initial data. However there always exist a null direction on a timelike point of a hypersurface. In the light of these discussions it can be conjectured that no complications arise on the solution of wave equation due to CTC's. The complications are mainly due to the nature of Cauchy problem approach.

To make this point more apparent, let's consider a space-time with topology $\mathbb{R}^2 \times S^1$, where S^1 is the temporal direction. Clearly this spacetime admits closed timelike curves. In this spacetime, waves from only a discrete set of frequencies can exist due to periodicity conditions. But this restriction does not seem to be related to the existence of CTC's, because alternatively when a globally hyperbolic spacetime of topology $S^2 \times \mathbb{R}$ is considered, it is easy to see that it has the same property: Waves only from a discrete set of frequencies can exist in this spacetime, either. This observation also suggests that the restrictions on the solutions of wave equation are not due to existence of closed timelike curves. Compactness of the topology in either temporal or spatial directions have both similar consequences. Thus it is reasonable to expect that waves in a universe where time travel is possible, will have similar properties with globally hyperbolic ones.

If we consider the question in a purely mathematical point of view, the form of wave equation considered is almost symmetric with respect to time and space variables. For the 1+1 dimensions there is complete symmetry (remembering that the minus sign on the time derivative does not effect the symmetry since it is always possible to reverse the signs) and for higher dimensions the only difference is having more space variables. This suggest that there is no strong mathematical background for expecting disparate consequences of existence of CTC's compared to existence of closed curves along any space direction. On the other hand more space coordinates give rise to asymmetry between the spacelike hypersurfaces and the timelike hypersurfaces in Cauchy problem due to the shape of the null cone: Any timelike hypersurface passing through a spacetime point intersects the null cone of that point, while spacelike hypersurfaces does not.

There is a strong analogy between 2+1 and 3+1 cases, which suggests that the results can be extended to $n+1$ dimensions easily. In any dimensions, the solutions can be expressed in terms of spherical waves, $f(r)Y(\Omega)$, where r is the radial distance and Ω denotes the angular part [10]. In addition, to be able to apply the same method, an addition theorem similar to that of the 2+1 and 3+1 dimensions is needed for this higher dimension. The similarity of (10), (11) with (37), (38) suggests that the solution for higher dimensions are readily given by these equations where the expressions of $\bar{B}$ and $\bar{C}$ in terms of B and C will be found using addition theorems of those dimensions.

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APPENDIX A: CALCULATION OF $\bar{A}_{lm}$, $\bar{B}_{lm}$ AND $\bar{C}_{lm}$

Referring to the equations (26), (27) and (28)

$$\begin{aligned}
\sum_{lm} A_{lm} \cdot j_l(\omega r) Y_{lm}(\theta,\varphi) &= \sum_{lm} \bar{A}_{lm} j_l(\omega R) Y_{lm}(\Phi,\varphi), \\
\sum_{lm} B_{lm} \cdot h_l^{(1)}(\omega r) Y_{lm}(\theta,\varphi) &= \sum_{lm} \bar{B}_{lm} j_l(\omega r) Y_{lm}(\Phi,\varphi), \\
\sum_{lm} C_{lm} \cdot h_l^{(1)}(\omega R) Y_{lm}(\Phi,\varphi) &= \sum_{lm} \bar{C}_{lm} j_l(\omega r) Y_{lm}(\theta,\varphi),
\end{aligned}$$

(29), (30) and (31) can be employed to calculate $\bar{A}_{lm}$, $\bar{B}_{lm}$ and $\bar{C}_{lm}$. Considering Ψ_0,

$$\begin{aligned}
\Psi_0 &= \sum_{lm} A_{lm} \cdot j_l(\omega r) Y_{lm}(\theta,\varphi) = \sum_{lm} A_{lm} \sum_{l'm'} \alpha_{l'm'}^{lm+}(\vec{d}) j_{l'}(\omega R) Y_{l'm'}(\pi-\Theta,\varphi) \\
&= \sum_{lm} A_{lm} \sum_{l'm'} \alpha_{l'm'}^{lm+}(\vec{d}) j_{l'}(\omega R) (-1)^{l'+m'} Y_{l'm'}(\Theta,\varphi) \\
&= \sum_{lm} (-1)^{l+m} (\sum_{l'm'} A_{l'm'} \alpha_{lm}^{l'm'+}(\vec{d})) j_l(\omega R) Y_{lm}(\Theta,\varphi).
\end{aligned}$$

In the last step the order of summations and indices lm and $l'm'$ are interchanged. It is assumed that these series converge and changing the order of the summations is valid [15]. Thus

$$\bar{A} = (-1)^{l+m} (\sum_{l'm'} A_{l'm'} \alpha_{lm}^{l'm'+}(\vec{d})).$$

Calculation of the $\bar{B}_{lm}$ is identical except $\alpha_{lm}^{l'm'+}$ is replaced by $\alpha_{lm}^{l'm'}$. To find C_{lm}:

$$\begin{aligned}
\Psi_2 &= \sum_{lm} C_{lm} \cdot h_l^{(1)}(\omega R) Y_{lm}(\Theta,\varphi) = \sum_{lm} (-1)^{l+m} C_{lm} \cdot h_l^{(1)}(\omega R) Y_{lm}(\pi-\Theta,\varphi) \\
&= \sum_{lm} (-1)^{l+m} C_{lm} \sum_{l'm'} \alpha_{l'm'}^{lm}(-\vec{d}) j_{l'}(\omega r) Y_{l'm'}(\theta,\varphi) \\
&= \sum_{lm} (\sum_{l'm'} (-1)^{l'+m'} C_{l'm'} \alpha_{lm}^{l'm'}(-\vec{d})) j_l(\omega r) Y_{lm}(\theta,\varphi) \\
&= \sum_{lm} (\sum_{l'm'} (-1)^{l'+m'} C_{l'm'} \alpha_{lm}^{l'm'}(-\vec{d})) j_l(\omega r) Y_{lm}(\theta,\varphi). \qquad \text{(A1)}
\end{aligned}$$

Recalling (32):

$$\alpha_{lm}^{l'm'}(-\vec{d}) = \alpha_{lm}^{l'm}(-\vec{d}) \sum_{\lambda\mu} c(lm|l'm|\lambda 0) h_{\lambda}^{(1)}(\omega d) Y_{\lambda 0}(\pi, \varphi)$$
$$= \sum_{\lambda\mu} c(lm|l'm|\lambda 0) h_{\lambda}^{(1)}(\omega d)(-1)^{\lambda} \sqrt{\frac{2\lambda+1}{4\pi}}.$$

The $\begin{pmatrix} l & l' & \lambda \\ 0 & 0 & 0 \end{pmatrix}$ factor in $c(lm|l'm|\lambda 0)$ is zero when $l'+l+\lambda$ is odd. Thus $(-1)^{l+l'+\lambda} = 1$ and $(-1)^{\lambda} = (-1)^{l+l'}$, yielding

$$\alpha_{lm}^{l'm'}(-\vec{d}) = (-1)^{l+l'} \sum_{\lambda\mu} c(lm|l'm|\lambda 0) h_{\lambda}^{(1)}(\omega d) \sqrt{\frac{2\lambda+1}{4\pi}} = (-1)^{l+l'} \alpha_{lm}^{l'm}(\vec{d}).$$

Substituting in (A1)

$$\sum_{lm} C_{lm} \cdot h_l^{(1)}(\omega R) Y_{lm}(\Theta, \varphi) = \sum_{lm} ((-1)^{l+m} \sum_{l'} C_{l'm'} \alpha_{lm}^{l'm}(\vec{d})) j_l(\omega r) Y_{lm}(\theta, \varphi).$$

Hence

$$\bar{C}_{lm} = (-1)^{l+m} \sum_{l'} C_{l'm} \alpha_{lm}^{l'm}(\vec{d}).$$

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