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Category oriented Luce rule*

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Abstract

We propose a new random choice model that extends the Luce rule such that the agents first categorize, and then choose by using category contingent Luce weights. In our analysis, we show that the category-oriented Luce rules are the only random choice functions (RCFs) that satisfy *weak regularity* and *weak odds supermodularity*.

Key words: Random choice, Luce rule, supermodularity, zero probability choices.

JEL codes: D01, D07, D09.

1. Introduction

Luce rule asserts that each alternative has a fixed positive weight and from each choice set the probability that an alternative is chosen equals to its relative weight in the choice set.[1] Since the weight that is assigned to each alternative is positive, each alternative is chosen from each choice set with a positive probability.[2]

In the empirical literature, problems emanating from ignoring 'zero probability choices' have recently received attention. The common method to deal with zero probability choices (market shares of products) empirically is to drop such

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1 Luce rule corresponds to the logit model ((McFadden, 1978), the most widely used random choice model in economics.

2 This is called the positivity axiom. In the original formulation by Luce (1959), it is allowed that some alternatives are never chosen from any choice set. However, it is not allowed that an alternative may be chosen in a choice set but not in another, which is our focus in here.

sample points, or replace them with small numbers. However, both Hortacsu & Joo (2016) and Gandhi et al. (2013) argue that when samples with zero market shares are dropped or replaced by small shares, price coefficient estimates are either biased upward or the direction of the bias becomes unpredictable. Hortacsu and Joo demonstrate that ignoring zero choice probabilities can even result in upward-sloping demand curves. Motivated with these observations, we propose a choice model that extends the Luce rule by retaining its simplicity, and allow zero probability choices.

We propose and analyze the *category-oriented Luce rule*, which extends the Luce rule by taking into account the possibility that an agent can group the choice problems (sets) into different *categories*, and choose from different categories with different Luce weights. In psychology, categorization is taken as a central concept for human decision making and analyzed extensively (for a comprehensive reference see Cohen & Lefebvre, 2005). Our category-oriented Luce rule offers a plausible choice rule that leaves room for agents to form reasonable categories.

To illustrate our category-oriented Luce rule, let us modify Luce and Raiffa's dinner example ((Luce & Raiffa, 1957) in which they choose *chicken* when the menu consists of *steak* and *chicken* only, yet go for the *steak* when the menu consists of *steak*, *chicken*, and *frog's legs*. More generally, presence of a specific alternative (*frog's leg*) signals about the quality of the other items in the menu. Consequently, the decision maker categorizes the menus into two: those that contain *frog's leg*, and those that do not. Suppose in the former, he assigns a zero weight to *chicken;* equal and positive weights to *steak* and *frog's leg*, thus he chooses evenly between the two. In the latter, suppose he only assigns *chicken* a positive weight, thus chooses *chicken* deterministically when compared to *steak* alone.[3]

In our analysis, we show that a new choice axiom, *weak odds supermodularity*, which weakens the *odds supermodularity* introduced by Doğan & Yıldız (2019) is key for relating these models to the observables. In our Proposition 1, we show that the category-oriented Luce rules are the only *random choice functions* (RCFs) that satisfy *weak regularity* and *weak odds supermodularity*.

1.1. Relation to the literature

Echenique & Saito (2019) and Ahumada & Ülkü (2018) analyze a general Luce model that accommodates zero probability choices. In these models, an agent first forms his consideration sets in any arbitrary way, then applies the Luce rule.

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[3] Note that this rule is not a Luce rule, since there is no single weight assignment that is consistent with choosing chicken deterministically when compared to steak, and choosing chicken with zero probability when frog's leg is added to the menu. Thus, it fails to satisy Luce's axioms.

They show that the *(expansive) cyclical independence* characterizes this general model. A category-oriented Luce rule can be thought as a general Luce rule with specific consideration set structures. Another key difference between our work and that of Echenique & Saito (2019) and Ahumada & Ülkü (2018) is the axiomatic characterizations. In that, *weak odds supermodularity* plays the main role in our results, whereas *cyclical independence* plays the main role in theirs. We compare and contrast these axioms after presenting our result.

The closest study is Doğan & Yıldız (2019), who provide a new characterization of the Luce rule in terms of *positivity* and a new choice axiom called *odds supermodularity* that strengthens the *regularity* axiom. Doğan & Yıldız (2019) also analyze the *preference-oriented Luce rule*, in which from each choice set, an agent first shortlists the best alternatives according to a preference relation, then chooses each shortlisted alternative with a probability that equals the alternative's relative weight in the shortlist.

2. Category-oriented Luce rule

Given an alternative set X, any nonempty subset S is called a **choice set**. Let Ω denote the collection of all choice sets. A **random choice function** (RCF) ρ assigns each choice set $S \in \Omega$ a probability measure over S. We denote by $\rho_x(S)$ the probability that alternative x is chosen from choice set S. For each choice set S, $\rho^+(S)$ is the set of alternatives that are chosen with positive probability from S, i.e. $\rho^+(S) = \{x \in S \ : \ \rho_x(S) > 0\}$.

For a given RCF ρ, a *categorization* $\mathcal{P}$ is a collection of *categories* $\{\Omega_i\}_{i \in I}$ such that (1) each choice set is contained by at least one category in this collection, (2) each category is *maximal*, that is, the set of alternatives chosen with positive probability from some choice set in a category is not contained in that of another, (3) each category is *comprehensive*, that is, if a set of alternatives is chosen with positive probability from a category, then so is each subset of it.

Definition 1 *A* ***categorization*** $\mathcal{P} = \{\Omega_i\}_{i \in I}$ ***for*** *an RCF ρ is a collection of families of choice sets such that*

(1) $\bigcup_{i \in I} \Omega_i = \Omega$,

(2) each Ω_i is maximal, i.e. for each distinct $i, j \in I$, $\bigcup_{S_i \in \Omega_i} \rho^+(S_i) \not\subset \bigcup_{S_j \in \Omega_j} \rho^+(S_j)$, and

(3) each Ω_i is comprehensive, i.e. if $S \subset \rho^+(S_i)$ for some $S_i \in \Omega_i$, then $S \in \Omega_i$.

An RCF ρ is a *category-oriented Luce rule* if there exist a categorization $\{\Omega_i\}_{i\in I}$ for ρ, and a collection of Luce rules $\{v_i\}_{i\in I}$, where each $v_i: X \to [0,1]$, such that for each choice set S, if S belongs to a category Ω_i, then $\rho(S)$ is obtained by applying the Luce rule v_i to S.

Definition 2 *An RCF ρ is a* ***category-oriented Luce rule*** *(COLR) if there exist a categorization $\mathcal{P} = \{\Omega_i\}_{i\in I}$ for ρ, and a collection of Luce rules $\{v_i\}_{i\in I}$ such that for each $S \in \Omega$ and $i \in I$, if $S \in \Omega_i$, then for each $x \in S$,*

$$\rho_x(S) = \frac{v_i(x)}{\sum_{y\in S} v_i(y)}$$

Note that a choice set S might appear under different categories. If an RCF is a category-oriented Luce rule, then each Luce rule for each category that contains S should yield the same probability distribution on S.

2.1. Result

We introduce two choice axioms and show that the category-oriented Luce rules are the only RCFs that satisfy these two axioms. Our first axiom, *weak regularity*, is a weakening of Luce's *regularity* axiom. To understand *weak regularity*, let S be a set of alternatives that are chosen with positive probability from a choice set. *Weak regularity* requires each alternative be chosen with positive probability from S. Put differently, *weak regularity* asks for *regularity* for each choice set S such that $S \subset \rho^+(T)$ for some choice set T. In contrast to *regularity*, *weak regularity* allows that removal of an alternative may decrease a remaining alternative's choice probability. Thus, in contrast to *regularity*, *weak regularity* does not rule out one of the most well documented choice behavior, called *attraction effect*.[4]

Weak regularity: For each $S, T \in X$, if $S \subset \rho^+(T)$, then $\rho^+(S) = S$.

Our second axiom is a natural weakening of *odds supermodularity* (OS) proposed by Doğan & Yıldız (2019). By using this axiom, Doğan & Yıldız (2019) provide a new characterization of the Luce rule. First let us introduce *odds supermodularity* of Doğan & Yıldız (2019) .

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4 Experimental evidence for the attraction effect is first presented by Payne & Puto (1982) and Huber & Puto (1983). Following their work, evidence for the attraction effect has been observed in a wide variety of settings. For a list of these results, consult Rieskamp et al. (2006).

Consider an alternative x that is chosen from a choice set S with positive probability. The *odds against x in S*, denoted by $\mathcal{O}(x,S)$, is the ratio of the probability that x is not chosen from S to the probability that x is chosen from S, that is

$$\mathcal{O}(x,S) = \frac{1-\rho_x(S)}{\rho_x(S)}.$$

If x is not chosen from a choice set S with positive probability, then $\mathcal{O}(x,S) = \infty$.

For each pair of choice sets S and T that contain only x in their intersection, *odds supermodularity* requires the sum of the odds against x in S and T be less than or equal to the odds against x in $S \cup T$, i.e. for each $S,T \in \Omega$ and $x \in X$ such that $S \cap T = \{x\}$, $\mathcal{O}(x,S) + \mathcal{O}(x,T) \leq \mathcal{O}(x,S \cup T)$.

As Doğan & Yıldız (2019) argue *odds supermodularity* can be thought as a strengthening of the *regularity* axiom, which requires that if new alternatives are added to a choice set, then the choice probability of an existing alternative x should not increase, put differently, the odds against x should not decrease. *Odds supermodularity* strengthens *regularity* by requiring that the odds against an existing alternative increases at least additively as new alternatives are added to the choice set.

The *weak odds supermodularity* we formulate here restricts *odds supermodularity* to alternatives that are chosen with positive probability from each choice set that appears in the formulation of the *odds supermodularity*. More precisely, consider a pair of choice sets S_1 and S_2 such that the set of alternatives that are chosen with positive probability from S_1 or S_2 is S. Now, if x is the only alternative that is chosen with positive probability from both S_1 and S_2, then the sum of odds against x in S_1 and S_2 should be less than or equal to odds against x in from S. Next, we first present this axiom formally, then present our characterization result and its proof.

Weak odds supermodularity (WOS): For each distinct $S, S_1, S_2 \in \Omega$ and $x \in X$, if $\rho^+(S) = \rho^+(S_1) \cup \rho^+(S_2)$ and $\{x\} = \rho^+(S_1) \cap \rho^+(S_2)$, then

$$\mathcal{O}(x,S_1) + \mathcal{O}(x,S_2) \leq \mathcal{O}(x,S).$$

Weak odds (super)modularity can be thought as a stochastic counterpart of Plott (1973)'s *path independence* axiom. This axiom requires that if the choice set is divided into smaller sets and the choices from the smaller sets are collected, then the choice made from this collection should be the same as the choice from the original choice set. Similarly, *weak odds supermodularity* requires the odds be independent of how the choice process is divided into parts.

Proposition 1 *An RCF ρ is a category-oriented Luce rule if and only if ρ satisfies weak regularity and weak odds supermodularity.*

Proof. *Only if part*: Let ρ be a category-oriented Luce rule represented by $\{\Omega_i, v_i\}_{i\in I}$. First, we show that ρ satisfies *weak regularity*. For each $S, T \in X$ such that $S \subset \rho^+(T)$, let $\rho^+(T) \in \Omega_i$. Since for each $i \in I$, Ω_i is comprehensive, it follows that $S \in \Omega_i$. Therefore, $\rho^+(S) = S$.

Next, we show that ρ satisfies WOS. Let $S, S_1, S_2 \in \Omega$ and $x \in X$ such that $\rho^+(S) = \rho^+(S_1) \cup \rho^+(S_2)$ and $\{x\} = \rho^+(S_1) \cap \rho^+(S_2)$. Now, since ρ is a category-oriented Luce rule, we have

$$\rho_x(S_1) = \frac{v(x)}{v(x)+\sum_{y\in\rho^+(S_1)\setminus\{x\}} v(y)} \quad , \quad \rho_x(S_2) = \frac{v(x)}{v(x)+\sum_{y\in\rho^+(S_2)\setminus\{x\}} v(y)} \ , \ and$$

$$\rho_x(S) = \frac{v(x)}{v(x)+\sum_{y\in\rho^+(S)\setminus\{x\}} v(y)}.$$

It follows that

$$\mathcal{O}(x, S_1) = \frac{\sum_{y\in\rho^+(S_1)\setminus\{x\}} v(y)}{-1.5exv(x)} \quad , \quad \mathcal{O}(x, S_2) = \frac{\sum_{y\in\rho^+(S_2)\setminus\{x\}} v(y)}{-1.5exv(x)} \quad , \ and$$

$$\mathcal{O}(x, S) = \frac{\sum_{y\in\rho^+(S)\setminus\{x\}} v(y)}{-1.5exv(x)}$$

Since $\rho^+(S) = \rho^+(S_1) \cup \rho^+(S_2)$ and $\{x\} = \rho^+(S_1) \cap \rho^+(S_2)$, we obtain that $\mathcal{O}(x, S_1) + \mathcal{O}(x, S_2) = \mathcal{O}(x, S)$.

If part: Suppose ρ is an RCF that satisfies *weak regularity* and WOS. We construct a $\{\Omega_i, v_i\}_{i\in I}$ that recovers the choices of ρ.

First, let $\{M_1, \dots, M_k\} \in \rho^+(X)$ such that each M_i is maximal in $\rho^+(X)$ with respect to set-containment $\subseteq$. Let $I = \{1, \dots, k\}$, and for each $i \in I$, define $\Omega_i = \{S \in \Omega: \rho^+(S) \subseteq M_i\}$. Next, we argue that $\{\Omega_i\}_{i\in I}$ is a categorization for ρ. First, for each $S \in \Omega$, there exists $i \in I$ with $\rho^+(S) \subseteq M_i$. Therefore, $S \in \Omega_i$, which implies $\{\Omega_i\}_{i\in I}$ covers Ω. Second, since ρ satisfies *weak regularity*, for each $i \in I$, $\bigcup_{S_i\in\Omega_i} \rho^+(S_i) = M_i$. Since for each $i, j \in I$, $M_i \not\subseteq M_j$, Ω_i is maximal. Finally, to see that for each $i \in I$, Ω_i is comprehensive, let $S_i \in \Omega_i$ and $S \subset \rho^+(S_i)$. Since ρ satisfies *weak regularity*, $S \subset \rho^+(S_i)$ implies $\rho^+(S) = S$. Then, by the construction of Ω_i, we get $S \in \Omega_i$. It follows that Ω_i is comprehensive.

Next, for each $i \in I$, we construct the weight function $v_i: X \rightarrow [0,1]$. To do this we rely on the Luce characterization of Doğan & Yıldız (2019) (Corollary 1). They show that an RCF ρ is a Luce rule if and only if ρ satisfies *positivity*, which requires each alternative being chosen from each choice set with positive probability, and *odds supermodularity* that is defined above. Now for each $i \in I$, consider the RCF ρ_i defined on $2^{M_i} \backslash \emptyset$ such that for each $\emptyset \neq S \subseteq M_i$ and $x \in S$, $\rho_i(x, S) = \rho_x(S)$. Since ρ satisfies *weak regularity*, ρ_i satisfies positivity. Since ρ satisfies *WOS*, ρ_i satisfies OS. Then, it follows from Corollary 1 of Doğan & Yıldız (2019) that there exists $v_i: M_i \rightarrow [0,1]$ that renders a Luce representation for ρ_i. Now, for each $i \in I$, define $v'_i: X \rightarrow [0,1]$ such that for each $x \in X$, if $x \in M_i$ then $v'_i(x) = v_i(x)$, if not then $v'_i(x) = 0$. Then, it directly follows that for each $S \in \Omega$, if $S \in \Omega_i$ for some $i \in I$, then for each $x \in S$, $\rho_x(S) = \frac{v'_i(x)}{\sum_{y \in S} v'_i(y)}$ as desired.

Luce (1959) characterizes his model in terms of *positivity* and IIA, which says that the ratio of choice probabilities between two alternatives does not depend on other alternatives in the choice set, i.e. for each $S \in \Omega$ and $x, y \in S$, $\rho_x(\{x,y\})/\rho_y(\{x,y\}) = \rho_x(S)/\rho_y(S)$. As discussed in detail by Doğan & Yıldız (2019) , the behavioral message carried by *odds supermodularity* is substantially different from that of IIA. In that, *odds supermodularity* strengthens *regularity*, whereas IIA indicates that agents chooses as if taking conditional probabilities with respect to an underlying subjective probability measure.[5]

The *cyclical independence* in Echenique & Saito (2019) and Ahumada & Ülkü (2018) is a direct multiplicative extension of IIA, whereas *weak odds supermodularity* is obtained via a natural restriction of *odds supermodularity*. Therefore, the substantial behavioral difference between IIA and *odds supermodularity* remain present between *cyclical independence* and *weak odds supermodularity*.

2.2. Independence of the axioms

We show the independence of WOS and *weak regularity* via the following two examples.

Example 1 (violating WOS) *Let $X = \{x, y, z\}$. Consider the RCF ρ such that $\rho(x, X) = 3/4$, $\rho(y, X) = \rho(z, X) = 1/8$, and chooses each alternative evenly in a binary comparison. First note that ρ satisfies weak regularity, since ρ chooses each alternative from each choice set with positive probability. To see that ρ violates WOS, first note that $\rho^+(X) = X = \rho^+(\{x,y\}) \cup \rho^+(\{x,z\})$ and $\{x\} = \rho^+(\{x,y\}) \cap \rho^+(\{x,z\})$. However, $\mathcal{O}(x, X) = 1/3$ and $\mathcal{O}(x, \{x,y\}) = \mathcal{O}(x, \{x,z\}) = 1$.*

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[5] For the precise connection see Lemma 2 of Luce (1959).

Example 2 (violating regularity) *Let* $X = \{x, y, z\}$. *Consider the RCF* ρ *such that* $\rho(x, X) = \rho(y, X) = 1/2$, *and* $\rho(x, \{x, y\}) = \rho(z, \{x, z\}) = \rho(z, \{y, z\}) = 1$. *To see that* ρ *satisfies WOS, first note that, since for each pair of distinct binary choice sets* S_1 *and* S_2, *we have* $\rho^+(S_1) \cap \rho^+(S_2) = \emptyset$ *unless* $S_1 = \{x, z\}$ *and* $S_2 = \{y, z\}$. *Since* $\rho^+(X) = \{x\} \neq \{z\} = \rho^+(\{x, z\}) \cup \rho^+(\{y, z\})$, *the if part of the WOS requirement is never satisfied. Thus* ρ *satisfies WOS. To see that* ρ *violates weak regularity, note that* $\rho^+(X) = \{x, y\}$, *but* $\rho^+(\{x, y\}) = \{x\}$.

3. Conclusion

We analyzed a structured extension of the Luce rule that accommodates zero probability choices. We introduced a new choice axiom, *weak odds supermodularity* that weakens the *odds supermodularity* of Doğan & Yıldız (2019) . We show that the category-oriented Luce rules are the only *random choice functions* (RCFs) that satisfy *weak regularity* and *weak odds supermodularity*. We hope that relaxing *odds supermodularity* would be fruitful in analyzing other generalizations of the Luce rule.

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Özet

Kategori esaslı Luce kuralı

Karar vericilerin önce sınıflandırıp daha sonra sınıfa özel Luce ağırlıkları kullanarak seçim yaptığı, Luce kuralını genişleten bir olasılıksal seçim kuralı ileri sürüyoruz. Analizimizde, sınıf-yönlü Luce kuralının *zayıf düzenlilik* ve *zayıf ret-süpermodülerliği* sağlayan tek kural olduğunu gösteriyoruz.

Anahtar kelimeler: Olasılıksal seçim, Luce kuralı, süpermodülerlik.

JEL kodları: D01, D07, D09.