

Destruction of the family of steady states in the planar problem of Darcy convection

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Abstract

The natural convection of incompressible fluid in a porous medium causes for some boundary conditions a strong non-uniqueness in the form of a continuous family of steady states. We are interested in the situation when these boundary conditions are violated. The resulting destruction of the family of steady states is studied via computer experiments based on a mimetic finite-difference approach. Convection in a rectangular enclosure is considered under different perturbations of boundary conditions (heat sources, infiltration). Two scenarios of the family of equilibria are found: the transformation to a limit cycle and the formation of isolated convective patterns.

Key words: Darcy convection, porous medium, families of equilibria, cosymmetry, finite-difference method

1 Introduction

The Darcy model of two-dimensional incompressible fluid convection in a porous medium [13,3,15,16] demonstrates the nontrivial effect of the appearance of a continuous family of steady states (equilibria) with a variable spectrum of stability. It was explained in [15] that the cosymmetry can lead to this phenomenon. Cosymmetry differs from symmetry due to variability of spectrum where all members of the family have the identical spectrum of stability. The effects of cosymmetry in Darcy convection was analyzed theoretically in

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[15,16,17]. The computation of the families of steady states and their evolution was carried out in [4,7,5,6,8,10]. Partial instability of the family (the occurrence of unstable steady states) was studied in [4,5,6]. The effects of collision and reorganization of the families were analyzed in [5,6,8].

A related interesting phenomenon is the destruction of a family of equilibria under some perturbations [2,17,12]. In [2] weak seeping of the fluid through the boundaries was considered. The breakdown of the family of stationary solutions was analyzed using the normal forms technique. V. Yudovich [17] studied the violation of the cosymmetry under low-intense heat sources or weak fluid infiltration and found that the continuous family of equilibria disintegrates or even disappears. It was predicted that in the case of heat sources a family of steady states may disintegrate to produce several separate stationary regimes. The case of weak infiltration may lead to the appearance of a long-period limit cycle instead of the family of equilibria [2,17].

In this paper we study the problem of the destruction of the family of steady states in the planar problem of Darcy convection in a rectangle. The finite-difference approximation of the Darcy equation allows us to consider the perturbation listed in [17] but our numerical studies are not restricted to the cases of small disturbances of heat source and low velocity of infiltration. We carried out several computer experiments with the harmonic perturbations of the temperature on the bottom (the case of a finite number of heat sources located on the boundary). The disintegration of the family of continuous equilibria is demonstrated numerically by showing the appearance of limit cycles and the formation of isolated convective patterns.

The paper is organized as follows. In Section 2 we give the Darcy equation and the cosymmetry property briefly. In Section 3 the finite-difference discretization is outlined. In Section 4 we present the numerical results for different perturbations of the boundary conditions.

2 Darcy model and cosymmetry

We consider an incompressible fluid which saturates a porous medium in a rectangular enclosure $\mathcal{D} = [0, a] \times [0, b]$ which is heated below. The dimensionless equations of the fluid flow in a porous medium according to Darcy law can be written as [16]:

$$\theta_t = \Delta\theta + \lambda\psi_x + J(\theta, \psi) \equiv F \tag{1}$$

$$0 = \Delta\psi - \theta_x \equiv G, \tag{2}$$

where Δ is the Laplacian and $J(\theta, \psi)$ denotes the Jacobian operator over (x, y) $J(\theta, \psi) = \theta_x \psi_y - \theta_y \psi_x$. The unknown functions $\psi(x, y, t)$ and $\theta(x, y, t)$ denote, respectively, the stream function and the perturbation of temperature from the basic state of rest with a linear conductive profile, λ is the Rayleigh number.

The boundary conditions are:

$$\begin{aligned} \theta(x, 0, t) &= \eta f(x); & \theta(x, y, t) &= 0, & (x, y) &\in \partial\mathcal{D} \setminus \{y = 0\}; \\ \psi(x, y, t) &= \mu y, & (x, y) &\in \partial\mathcal{D}, \end{aligned} \quad (3)$$

here the function $f(x)$ gives the nonhomogeneous distribution of the temperature at the bottom with magnitude η , and μ is the velocity of infiltration through the lateral sides of an enclosure.

The problem is supplied by the initial conditions

$$\theta(x, y, 0) = \theta_0(x, y), \quad (4)$$

where θ_0 denotes the initial perturbation of the temperature from the linear profile. For a given θ_0 , the stream function ψ can be obtained from (1)–(3) as the solution of the Dirichlet problem via Green's operator.

When $\eta = \mu = 0$ the system (1)–(3) imposes the equilibrium $\theta = \psi = 0$. This state of rest remains stable while $\lambda < \lambda_{11}$. The critical values λ_{ij} are given by $\lambda_{ij} = \pi^2(i^2/a^2 + j^2/b^2)$, $i, j \in \mathbb{Z}$. It was shown in [16] that the first critical value λ_{11} is two-times for any domain \mathcal{D} and larger values of λ implies an onset of the family of steady states. This result is a consequence of the cosymmetry that exists for the system (1)–(3) when $\eta = \mu = 0$ [15]. Indeed, the vector-function $(\theta, -\psi)$ is orthogonal to the right-hand side of (1), (2) in L_2 , i.e. $\int_{\mathcal{D}} (F\psi - G\theta) dx dy = 0$. In the case of nonzero η and μ we obtain using integration by parts and Green's formulae

$$\begin{aligned} \int_{\mathcal{D}} (F\psi - G\theta) dx dy &= -\eta \int_0^a \psi_y(x, 0, t) f(x) dx - \mu b \int_0^a \theta_y(x, b, t) dx \\ &\quad - \mu \int_0^b y [\theta_x(a, y, t) - \theta_x(0, y, t)] dy. \end{aligned}$$

Thus, the cosymmetry condition fails whenever $\mu \neq 0$ or $\eta \neq 0$.

3 Finite difference discretization

We use here the discretization and numerical scheme derived in [9]. Let us introduce the uniform arrangement of the nodes on the interval for spatial coordinates:

$$\begin{aligned} x_n &= nh_x, & n &= 0, \dots, N+1, & h_x &= a/(N+1), \\ y_m &= mh_y, & m &= 0, \dots, M+1, & h_y &= b/(M+1). \end{aligned}$$

The temperature θ and stream function ψ are defined at the nodes (x_n, y_m) . It is useful to introduce the auxiliary (staggered) nodes on each of the coordinates $x_{n+1/2} = \frac{1}{2}(x_n + x_{n+1})$, $n = 0, \dots, N$, and $y_{m+1/2} = \frac{1}{2}(y_m + y_{m+1})$, $m = 0, \dots, M$.

To approximate equations (1)–(2) we introduce difference and averaging operators based on the two-node stencils

$$(\delta_x f)_{n+1/2, m} = \frac{f_{n+1, m} - f_{n, m}}{h_x}, \quad (\delta_y f)_{n, m+1/2} = \frac{f_{n, m+1} - f_{n, m}}{h_y}, \quad (5)$$

$$(\delta_{0x} f)_{n+1/2, m} = \frac{f_{n+1, m} + f_{n, m}}{2}, \quad (\delta_{0y} f)_{n, m+1/2} = \frac{f_{n, m+1} + f_{n, m}}{2}. \quad (6)$$

These formulas are valid for integer and half-integer values of n and m . Using operators (5)–(6) we derive the operators on three-node stencils

$$D_x = \delta_x \delta_{0x} \equiv \delta_{0x} \delta_x, \quad D_y f = \delta_y \delta_{0y} \equiv \delta_{0y} \delta_y, \quad (7)$$

the averaging and difference operators on a four-point stencil

$$d_0 = \delta_{0x} \delta_{0y} \equiv \delta_{0y} \delta_{0x}, \quad d_x = \delta_x \delta_{0y} \equiv \delta_{0y} \delta_x, \quad d_y = \delta_y \delta_{0x} \equiv \delta_{0x} \delta_y, \quad (8)$$

and the discrete analog of Laplacian

$$(\Delta_h f)_{n, m} = (\delta_x \delta_x f + \delta_y \delta_y f)_{n, m}. \quad (9)$$

To approximate the Jacobian we use the Arakawa scheme [1]

$$\begin{aligned} J_{n, m}(\theta, \psi) &= \frac{1}{3} [D_x (\theta D_y \psi) - D_y (\theta D_x \psi)]_{n, m} \\ &+ \frac{2}{3} [d_x (d_0 \theta d_y \psi) - d_y (d_0 \theta d_x \psi)]_{n, m}. \end{aligned} \quad (10)$$

The discretized boundary conditions are formulated as follows

$$\begin{aligned}\psi_{n,0} &= 0, & \theta_{n,0} &= \eta f(x_n), & n &= 0, \dots, N+1, \\ \psi_{n,M+1} &= 0, & \theta_{n,M+1} &= 0, & n &= 0, \dots, N+1, \\ \theta_{0,m} &= \theta_{N+1,m} = 0, & \psi_{0,m} &= \psi_{N+1,m} = \mu y_m, & m &= 0, \dots, M+1.\end{aligned}\tag{11}$$

Using operators (7)–(10) we obtain the semi-discretized form of (1) and (2):

$$\dot{\theta}_{nm} = F_{nm} \equiv (\Delta_h \theta + \lambda D_x \psi)_{nm} + J_{nm}(\theta, \psi),\tag{12}$$

$$0 = G_{nm} \equiv (\Delta_h \psi - D_x \theta)_{nm},\tag{13}$$

here $n = 1, \dots, N$, $m = 1, \dots, M$, the dot denotes a differentiation on t .

After discretization the resulting system of ordinary differential equations (12)–(13) may be written in the following form

$$\dot{\Theta} = (A + \lambda B A^{-1} B) \Theta + F(\Theta) = \Phi(\Theta, \lambda, \mu, \eta).\tag{14}$$

Here Θ is vector of unknowns (temperature), matrices A and B are the approximation, respectively, of the Laplacian Δ and the first derivative on x , where $F(\Theta)$ represents the nonlinear Jacobian term.

To find nonstationary regimes or to converge to stable equilibrium we apply the Runge-Kutta method of fourth order. To compute a family of equilibria in the system (14), we apply the technique [7] based on the cosymmetric version of the implicit function theorem [16]. Different realization for ordinary differential equations *was also* developed in [4].

When λ is slightly larger than λ_{11} , then all points of the family are stable [16]. Starting from the vicinity of unstable zero equilibrium we integrate the ordinary differential equations (14) by *the* Runge-Kutta method up to a point Θ_0 close to some stable equilibrium on the family. Then we correct the point Θ_0 by the Newton method. To predict the next point on the family we determine the kernel of the linearization matrix (Jacobi matrix) at the point Θ_0 and use the Adams-Bashford method. We repeat this procedure to obtain the entire family.

4 Numerical results

Here we mainly consider convection for a wide enclosure with $a = 2$ and $b = 1$. Our previous computation of the families of steady states [7,8] showed that a

sufficiently good description of a number of the transitions can be obtained on a rather coarse grid. Several hundreds of internal nodes was enough to describe an appearance of a primary family of steady states [7] and follow a number transformations [8]. So we used the discretization with 16×8 nodes and compared with results on 32×16 nodes to study the destruction of a family of steady states under inhomogeneity of boundary conditions.

To present results we apply the special Nusselt coordinates [4]

$$Nu_h = \int_0^b \theta_x\left(\frac{a}{2}, y\right) dy, \quad Nu_v = \int_0^a \theta_y(x, 0) dx, \quad (15)$$

where Nu_h corresponds to the integral value of the heat flux from left to right defined for the centered vertical section. The value Nu_v may be considered as a cumulative heat flux through the bottom of the container.

4.1 Infiltration through the lateral sides

In [2,17] a scenario of the transformation of a family of steady states to a limit cycle of the large period was described. Fig. 1 illustrates this scenario

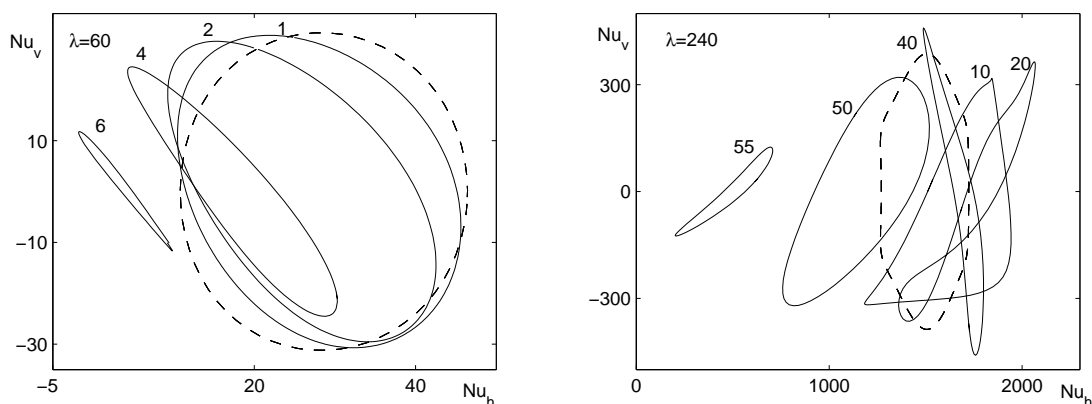


Fig. 1. Family of steady states (dashed) and limit cycles (solid) for several infiltration velocities μ : $\lambda = 60$ (left) $\lambda = 240$ (right)

for two Rayleigh numbers in the case of small and large infiltration through the lateral sides. The case with $\lambda = 60$ corresponds to the initial stage of the evolution of the family of equilibria after its birth. The Rayleigh number $\lambda = 240$ shows the rather developed convective patterns. If infiltration velocity μ is small then the trajectory of a limit cycle lies in vicinity of the family of equilibria. When the velocity μ increases a limit cycle goes far away from the family. Moreover, large infiltration causes degradation of the nonstationary

convective regime. All fluid particles move horizontally without the formation of a convective pattern. With the increase of the Rayleigh parameter λ the convective regime becomes able to prevent a pure infiltration. Computation at $\lambda = 240$ shows that for small velocity μ a limit cycle occurs near a curve of the family of equilibria. Increasing of μ shifts *the limit cycle to the range of small Nusselt values Nu_h* . Afterwards we see a deformation of a limit cycle and rather large velocity of infiltration leads to suppression of the convection.

The period of a limit cycle T becomes smaller with increasing velocity μ and depends also on Rayleigh parameter λ . One can see in Fig. 2 that the dependence $T(\mu)$ is practically the same for different values of λ .

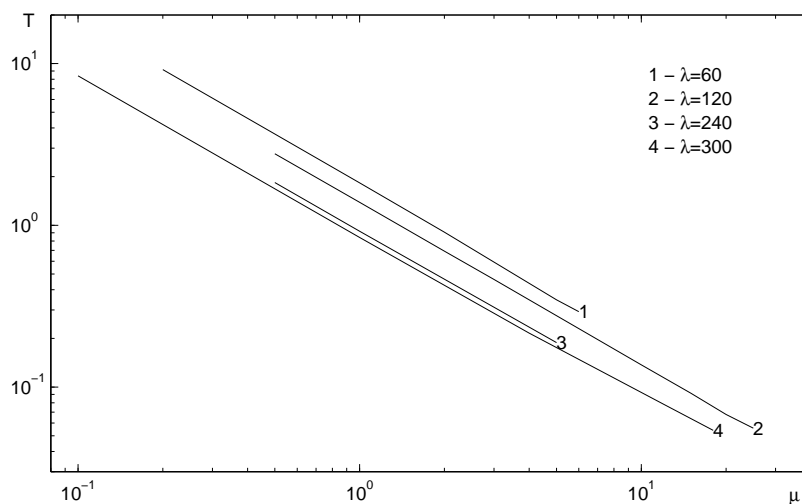


Fig. 2. Dependence of the period of a limit cycle from the velocity of infiltration μ for several λ

4.2 Inhomogeneous heating from below

For inhomogeneous heating and without infiltration ($\mu = 0$) we found two scenarios of destruction of the family of equilibria. We consider the case of the harmonic distribution of the bottom temperature $f(x) = \sin(\kappa\pi x/a)$. It allows us to consider the case with zero average deviation from uniform heating. Through the experiment we change the intensity of heating (Rayleigh number λ), the magnitude of inhomogeneity η and the amount of half-harmonics κ . Depending on the parameters, the family of steady states may transform to a limit cycle or to a number of isolated equilibria.

Computation with two harmonics in temperature distribution ($\kappa = 4$) shows that the family of steady states disintegrates onto two isolated equilibria, see

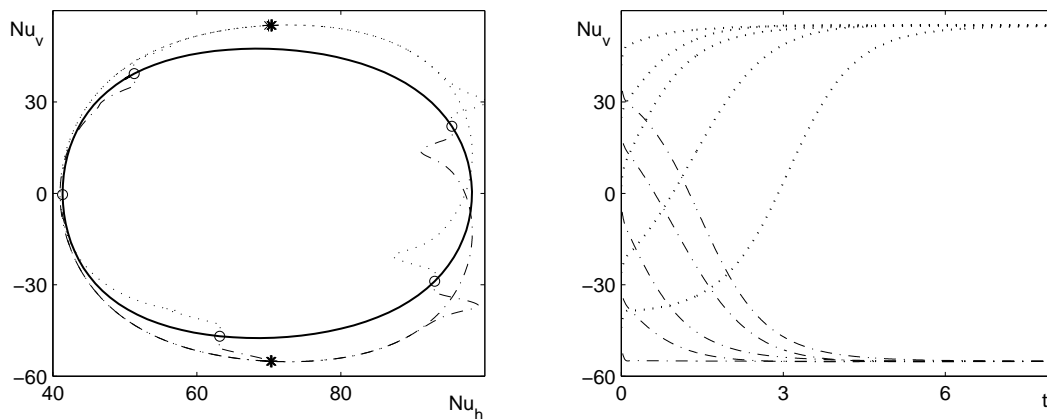


Fig. 3. Family of steady states and convergence from different initial points (circles) to the isolated stationary states (stars) under inhomogeneous heating below: $\eta = 4$ (dashed), $\eta = -4$ (dashed-point); $\lambda = 70$, $\kappa = 4$

Fig. 3. One of these equilibria is stable while the second is unstable, it depends on the sign of magnitude η .

The computer experiment was started from points taken from the curve of the family of equilibria. For small values of magnitude η the trajectories were very close to the curve of the family of equilibria. In this case we found very slow motion to the isolated stationary state.

In the case of one harmonics ($\kappa = 2$) the scenario of destruction of a family onto isolated equilibria was observed when $\lambda < 180$. The transformation of the family of steady states to a limit cycle was found for $\lambda \geq 180$. Fig. 4 gives relative location of the family of steady states and two limit cycles. For small magnitudes η the trajectory of the limit cycle lies close to the family.

The convective patterns and their number depend on the number of half-harmonics κ . Fig. 5 shows the results of disintegration of the families for two Rayleigh numbers: $\lambda = 120$ and $\lambda = 180$. We changed κ and fixed a magnitude of heating perturbation $\eta = 8$. For $\kappa = 2$ and $\kappa = 4$ only one isolated equilibrium was observed. The case $\kappa = 6$ produces two isolated stable equilibria instead of the family of steady states. The unstable equilibria also exist and may be obtained using computation with $\eta = -8$. Here we see a change of stability properties for isolated equilibria to which the family of steady states transforms. Stream functions for realized convective patterns are given in Fig. 6. It is clearly seen that one state for $\kappa = 6$ looks as if the equilibrium being formed under temperature distribution with two harmonics ($\kappa = 4$). The second possible state for $\kappa = 6$ is rather different. These experiments show that we can control the desirable state by the appropriate choosing of magnitude and amount of harmonics.

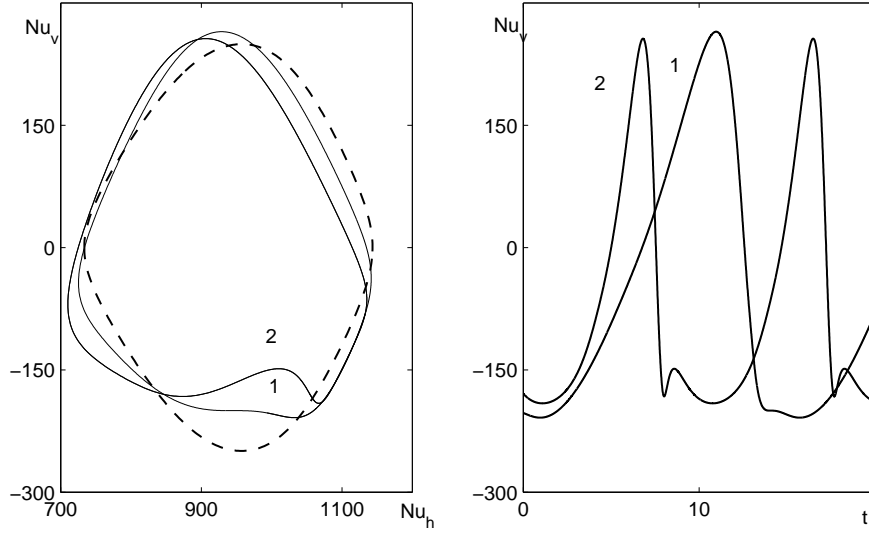


Fig. 4. The limit cycle (dashed) and the family of steady states (solid); $\lambda = 190$, $\eta = 8$ (curve 1), $\eta = 16$ (curve 2), $\kappa = 2$

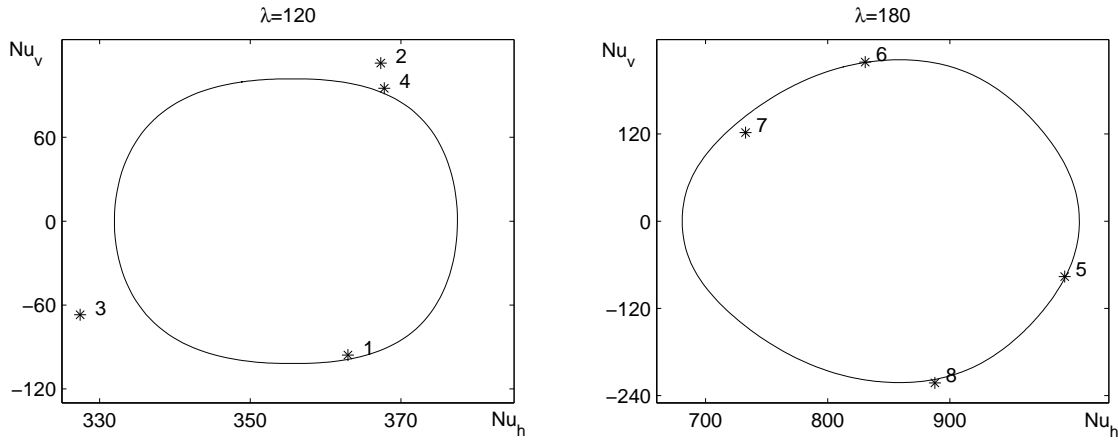


Fig. 5. Families of steady states (solid) and isolated states (stars) for heating with magnitude of perturbation $\eta = 8$ and different number of half-harmonics κ . Left: $\lambda = 120$, $\kappa = 2$ (1), $\kappa = 4$ (2), $\kappa = 6$ (3), $\kappa = 6$ (4); Right: $\lambda = 180$; $\kappa = 2$ (5), $\kappa = 4$ (6), $\kappa = 6$ (7), $\kappa = 6$ (8)

5 Conclusion

We have studied convection in the rectangular enclosure under the following perturbations of boundary conditions: heat sources at the bottom and infiltration through the lateral sides. Nonzero boundary values destroy the family of steady states which existed in the case of Dirichlet boundary conditions. Computer experiment allows to consider the case of large magnitudes of perturbations. Two scenario were studied: transformation of the family to a limit

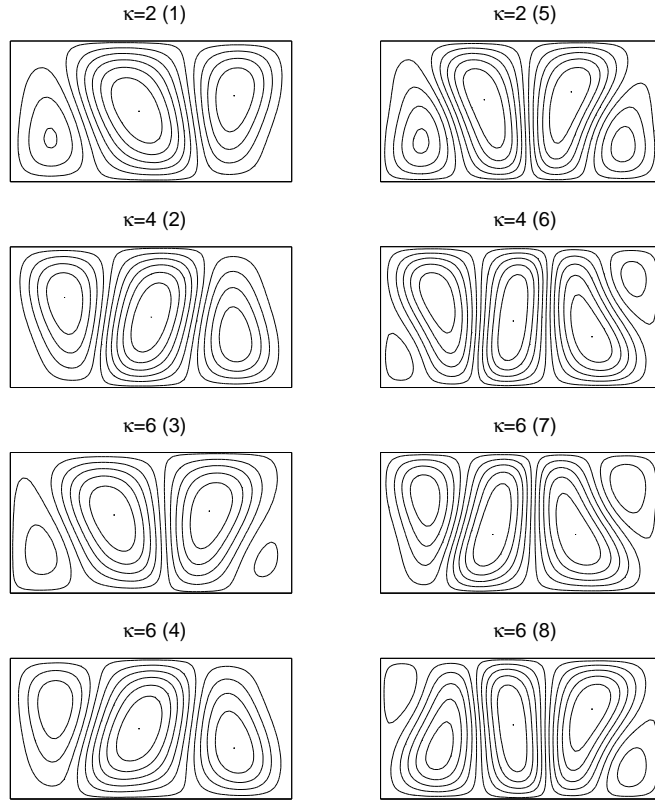


Fig. 6. Stream function of isolated steady states for different κ : $\lambda = 120$ (left), $\lambda = 180$ (right); $\eta = 8$

cycle and destruction of a family on a number of isolated convective patterns.

We found through computation that convergence to the isolated steady states may be very slow. It looks as a memory effect when the system remembers the location of the former family of equilibria. It should be noted that the dynamics on finite time span is very close to the results on selection of equilibria under bottom heat sources [14]. This may be used to control the preferable convective state via variation of boundary conditions.

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