The Momentum 4-Vector Imparted by Gravitational Waves in Bianchi-type Metrics‡

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Abstract. Considering the Møller, Weinberg and Qadir-Sharif's definitions in general relativity, we find the momentum four-vector of the closed universe based on the Bianchi-type metrics. The momentum four-vector(due to matter plus fields) is found to be zero. This result supports the viewpoints of Albrow and Tryon and extends the previous works by Cooperstock-Israelit, Rosen, Johri *et al.*, Banerjee-Sen and Vargas who investigated the problem of the energy in Friedmann-Robertson-Walker universe and Saltı-Havare who studied the problem of the energy-momentum of the viscous Kasner-type space-times.

Keywords: Momentum 4-vector; Bianchi-types, general relativity.
PACs Numbers: 04.20.-q; 04.20.Jb; 04.50.+h

1. Introduction

The momentum four-vector and the angular momentum play a crucial role as they provide the first integrals of equations of motions, helping one to solve otherwise intractable problems[1]. Furthermore the energy content in a sphere of radius R in a given space-time gives a taste of the effective gravitational mass that a test particle situated at the same distance from the gravitating object experiences. A large number of researchers have devoted considerable attention to the problem of finding the energy as well as momentum and angular momentum associated with various space-times.

The problem of defining the energy and momentum distributions is considered not only for general relativity but also the tele-parallel theory of gravity. From the advents of these different gravitation theories various methods have been proposed to deduce the conservation laws that characterize the gravitational systems. The first

‡ Except Bianchi-type III metrics
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of such attempts was made by Einstein who proposed an expression for the energy-momentum distribution of the gravitational field. There have many attempts to resolve the energy-momentum problem[2, 3, 4, 5, 6, 7, 8, 9, 10]. There exists an opinion that the energy-momentum definitions are not useful to get finite and meaningful results in a given geometry. Virbhadra and his collaborators re-opened the problem of the energy and momentum by using the energy-momentum complexes. The Einstein energy-momentum complex, used for calculating the energy in general relativistic system, was followed by many complexes: e.g. Tolman, Papapetrou, Bergmann-Thomson, Møller, Landau-Liftshitz, Weinberg, Qadir-Sharif and the tele-parallel gravity analogs of the Einstein, Landau-Lifshitz, Bergmann-Thomson and Møller's. The energy-momentum complexes give meaningful results when we transform the line element in quasi-Cartesian coordinates. The energy and momentum complex of Møller gives the possibility to make the calculations in any coordinate system[11]. To this end Virbhadra and his collaborators have considered many space-time models and have shown that several energy-momentum complexes give the same and acceptable results for a given space-time[12, 13, 14, 15, 16, 17]. In Phys. Rev. D60-104041 (1999), Virbhadra, using the energy and momentum complexes of Einstein, Landau-Lifshitz, Papapetrou and Weinberg's for a general non-static spherically symmetric metric of the Kerr-Schild class, showed that all of these energy-momentum formulations give the same energy distribution as in Penrose energy-momentum formulation.

Albrow[18] and Tryon[19] suggested that in our universe, all conserved quantities have to vanish. Tryon's big bang model predicted a homogenous, isotropic and closed universe including of matter and anti-matter equally. They argue that any closed universe has zero energy. The subject of the energy-momentum distributions of the closed universes was opened by an interesting work of Cooperstock and Israelit[20]. They found the zero value energy for a closed homogenous isotropic universe described by a Friedmann-Robertson-Walker(FRW) metric in the context of general relativity. After this interesting result some general relativists studied the same problem, for instance: Rosen[21], Johri *et al.*[22], Banerjee-Sen[23], Vargas[10] and Saltı-Havare[24]. Johri *et al.* using the Landau-Liftshitz's energy-momentum complex, found that the total energy of a FRW spatially closed universe is zero at all times. Banerjee and Sen who investigated the problem of total energy of the Bianchi-I type space-times using the Einstein complex, obtained that the total energy is zero. This result agrees with the studies of Johri *et al.*. Because, the line element of the Bianchi-I type space-time reduces to the spatially flat FRW line element in a special case. Next, Vargas using the definitions of Einstein and Landau-Lifshitz in tele-parallel gravity, found that the total energy associated with the FRW space-time is zero. These results extend the works by Rosen and Johri *et al.*. After Vargas's work, Saltı and Havare considered the Bergmann-Thomson complex in both general relativity and tele-parallel gravity for the viscous Kasner-type metric and then they found the total energy and momentum are zero everywhere.

The layout of the paper is as follows. In Sec. II, we introduce the Bianchi-type space-times. Then, in Sec. III, we give the momentum 4-vector definitions of the Møller,

Table 1. List of Bianchi-type standard metrics. Here $0 \leq m \leq 1$.

Bianchi-types	Line elements (dl^2)
Type-I	$dx^2 + dy^2 + dz^2$
Type-II	$dx^2 + dy^2 + (1 + y^2)dz^2 - 2ydxdz$
Type-IV	$e^{2y}(1 + y^2)dx^2 + dy^2 + e^{2y}dz^2 - 2ye^{2y}dxdz$
Type-V	$e^{2y}dx^2 + dy^2 + e^{2y}dz^2$
Type-VI(m)	$e^{2(m-1)y}dx^2 + dy^2 + e^{(m+1)y}dz^2$
Type-VII(m)	$e^{2my}dx^2 + dy^2 + e^{2my}dz^2$
Type-VIII	$(1 + 2\sinh^2 y)dx^2 + dy^2 + dz^2 + 2\sinh y dxdz$
Type-IX	$dx^2 + dy^2 + dz^2 - 2\sin y dxdz$

Weinberg and Qadir-Sharif in the general theory of relativity and obtain the Bianchi-type solutions. Finally, Sec. IV is devoted to concluding remarks. Throughout this paper we choose units such that $G = 1$, $c = 1$ and follow the convention that indices take values from 0 to 3 otherwise stated.

2. The Bianchi-Type Metrics

The original work of Bianchi[25] has been reorganized into a contemporary formalism by theoretical cosmologists. Here, we shall generally adopt the scheme of MacCallum[26]. The Bianchi-type space-times generally defined by the following metric.

$$ds^2 = -dt^2 + dl^2 \tag{1}$$

where

$$dl^2 = g_{ab}dx^a dx^b \tag{2}$$

here dl^2 is the 3-dimensional line element and the Latin indices take value from 1 to 3. Now, we give list of Bianchi-type space-times[27] in table-1.

Using these metrics, we define the Bianchi-type space-times by the following form

$$ds^2 = -dt^2 + A^2(y)dx^2 + dy^2 + B^2(y)dz^2 - 2C(y)dxdz. \tag{3}$$

For the line element (3), $g_{\mu\nu}$ and $g^{\mu\nu}$ are written as

$$g_{\mu\nu} = -\delta^0_\mu\delta^0_\nu + A^2\delta^1_\mu\delta^1_\nu - C(\delta^1_\mu\delta^3_\nu + \delta^3_\mu\delta^1_\nu) + \delta^2_\mu\delta^2_\nu + B^2\delta^3_\mu\delta^3_\nu \tag{4}$$

$$\Delta^2 g^{\mu\nu} = -\Delta^2\delta^\mu_0\delta^\nu_0 + B^2\delta^\mu_1\delta^\nu_1 + C(\delta^\mu_1\delta^\nu_3 + \delta^\mu_3\delta^\nu_1) + \Delta^2\delta^\mu_2\delta^\nu_2 + A^2\delta^\mu_3\delta^\nu_3 \tag{5}$$

here $\Delta^2(y) = A^2B^2 - C^2$.

3. The Momentum 4-vector in general relativity

3.1. The Momentum Four-vector Definition of Mϕller

The energy-momentum complex of Mϕller[5] is given by

$$M_\mu^\nu = \frac{1}{8\pi}\chi_{\mu,\alpha}^{\nu\alpha} \tag{6}$$

satisfying the local conservation laws:

$$\frac{\partial M_\mu^\nu}{\partial x^\nu} = 0 \tag{7}$$

where the antisymmetric super-potential $\chi_\mu^{\nu\alpha}$ is

$$\chi_\mu^{\nu\alpha} = \sqrt{-g}[g_{\mu\beta,\gamma} - g_{\mu\gamma,\beta}]g^{\nu\gamma}g^{\alpha\beta}. \tag{8}$$

The locally conserved energy-momentum complex M_μ^ν contains contributions from the matter, non-gravitational and gravitational fields. M_0^0 is the energy density and M_a^0 are the momentum density components. The momentum four-vector of Mϕller is given by

$$P_\mu = \int\int\int M_\mu^0 dxdydz. \tag{9}$$

Using Gauss's theorem, this definition transforms into

$$P_\mu = \frac{1}{8\pi}\int\int \chi_\mu^{\nu\alpha}\mu_\alpha dS. \tag{10}$$

where μ_α is the outward unit normal vector over the infinitesimal surface element dS. P_i give momentum components P_1, P_2, P_3 and P_0 gives the energy. Using equation (8), we found the required components of $\chi_\mu^{\nu\alpha}$ to be zero. From this point of view, by using equation (10), we obtain

$$P_\mu = 0 \tag{11}$$

3.2. The Momentum Four-vector Definition of Weinberg

The energy-momentum complex of Weinberg[6] is given by:

$$W^{\mu\nu} = \frac{1}{16\pi}K^{\mu\nu\alpha}_{,\alpha} \tag{12}$$

where

$$K^{\mu\nu\alpha} = \frac{\partial h_\beta^\beta}{\partial x_\mu}\eta^{\nu\alpha} - \frac{\partial h_\beta^\beta}{\partial x_\nu}\eta^{\mu\alpha} - \frac{\partial h^{\beta\mu}}{\partial x^\beta}\eta^{\nu\alpha} + \frac{\partial h^{\beta\nu}}{\partial x^\beta}\eta^{\mu\alpha} + \frac{\partial h^{\mu\alpha}}{\partial x_\nu} - \frac{\partial h^{\nu\alpha}}{\partial x_\mu} \tag{13}$$

and

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}. \tag{14}$$

The indices on $h_{\mu\nu}$ or $\frac{\partial}{\partial x_\mu}$ are raised or lowered with the help of η's. It is clear that

$$K^{\mu\nu\alpha} = -K^{\nu\mu\alpha} \tag{15}$$

W^{00} and W^{a0} are the energy and momentum density components respectively. The Weinberg energy and momentum complex $W^{\mu\nu}$ contains contributions from the matter, non-gravitational and gravitational fields, and satisfies the local conservation laws

$$\frac{\partial W^{\mu\nu}}{\partial x^{\nu}} = 0 \tag{16}$$

The Momentum four-vector is given by

$$P^{\mu} = \int\int\int W^{\mu 0} dxdydz. \tag{17}$$

and the angular momentum is given by

$$J^{\mu\nu} = \int\int\int (x^{\mu} W^{0\nu} - x^{\nu} W^{0\mu}) dxdydz. \tag{18}$$

Further Gauss's theorem furnishes

$$P^{\mu} = \frac{1}{16\pi} \int\int K^{\mu\nu\alpha} \kappa_{\alpha} dS. \tag{19}$$

and the physically interesting components of the angular momentum are

$$J^{\mu\nu} = \frac{1}{16\pi} \int\int (x^{\mu} K^{\alpha 0\nu} - x^{\nu} K^{\alpha 0\mu} + \eta^{\alpha\mu} h^{0\nu} - \eta^{\alpha\nu} h^{0\mu}) \kappa_{\alpha} dS. \tag{20}$$

κ_{α} stands for the 3-components of unit vector over an infinitesimal surface element dS. P_i give momentum(energy current) components P_1, P_2, P_3 and P_0 gives the energy. We are interested in determining the momentum four-vector. Now using the line element (3) in equations (13) and (14) then we find all components of $K^{\mu\nu\alpha}$ vanish. Thus, from equation (19) we have

$$P^{\mu} = 0 \tag{21}$$

3.3. The Momentum Four-vector Definition of Qadir-Sharif

The Momentum four-vector of Qadir-Sharif[7] is given by:

$$P_{\mu} = \int F_{\mu} dxdydz. \tag{22}$$

where

$$F_0 = m\left[\left\{\ln\left(\frac{A}{\sqrt{g_{00}}}\right)\right\}_{,0} - \frac{g_{\mu\nu,0}\, g^{\mu\nu}_{,0}}{4A}\right], \qquad F_a = m\left(\ln\sqrt{g_{00}}\right)_{,\mu} \tag{23}$$

and

$$A = \left(\ln\sqrt{-g}\right)_{,0}. \tag{24}$$

This force definition depends on the choice of frame, which is not uniquely fixed. The quantity, whose proper time derivative is F_{μ}, is called the momentum four-vector for the test particle. Using the line element (3), equation (23) with (24) then we get

$$F_{\mu} = 0 \tag{25}$$

Using this result, we can easily see that the Qadir-Sharif momentum four-vector is obtained as

$$P_{\mu} = 0 \tag{26}$$

4. Concluding Remarks

The definitions of the energy and momentum in both the general theory of relativity and tele-parallel gravity have been very interesting; however, they have been associated with some debate. In literature, Albrow and Tryon suggested that in our universe, all conserved quantities have to vanish everywhere. Tryon's big bang model predicted a homogenous, isotropic and closed universe including of matter and anti-matter equally and any closed universe has zero energy.

In this paper, considering the context of the general relativity, we have calculated the momentum 4-vector associated with the Bianchi-type space-times(except type-III). In order to compute the gravitational part of the momentum 4-vector, we have considered the Møller, Weinberg and Qadir-Sharif energy and/or momentum definitions. Our first result is that the momentum 4-vector vanishes everywhere and the second one is that our study supports the viewpoints of Albrow and Tryon and extend the previous works by Cooperstock-Israelit, Rosen, Johri *et al.*, Banerjee-Sen, Vargas and Saltı-Havare.

Acknowledgment

MK and MS would like to thank Turkish Scientific Research Council (Tübitak)-Feza Gürsey Institute, Istanbul, for the hospitality my received in summer terms.

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