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NON-LINEAR SIGMA MODEL ON THE FUZZY SUPERSPHERE

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Abstract

In this note we develop fuzzy versions of the supersymmetric non-linear sigma model on the supersphere $S^{(2,2)}$. In [1] Bott projectors have been used to obtain the fuzzy $\mathbb{C}P^1$ model. Our approach utilizes the use of supersymmetric extensions of these projectors. Here we obtain these (super)-projectors and quantize them in a fashion similar to the one given in [1]. We discuss the interpretation of the resulting model as a finite dimensional matrix model.

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1 Introduction

In past few years studies of field theories on non-commutative manifolds have been very fruitful. To construct such theories one usually starts with a continuum theory on a manifold $\mathcal{M}$ and replaces the commutative algebra $\mathcal{A}$ of functions on $\mathcal{M}$ by a non-commutative algebra A which preserves most of the symmetries of the continuum theory and which approximates the commutative algebra $\mathcal{A}$ and hence the continuum theory in the commutative limit. It is possible to realize a large class of such non-commutative field theories as finite dimensional matrix models. Field theories on the non-commutative (fuzzy) sphere S_F^2 and the fuzzy supersphere $S_F^{(2,2)}$ are two such examples. As non-commutative manifolds the former is based on the irreducible representations of the $su(2)$ Lie algebra, whereas the latter is described by the irreducible representations of the Lie superalgebra $osp(2,1)$. To date many studies on different and novel aspects of field theories on S_F^2 have been carried out [1, 2, 3, 4, 5, 6].

Recently, $\mathbb{C}P^1$ model on the fuzzy sphere S_F^2 have been studied from several different points of view [1, 7, 8]. In [1] the commutative theory have been reformulated by replacing the non-linear fields with a certain class of projectors called "Bott Projectors". A discrete (fuzzy) version of these projectors are easily obtained and they have permitted the construction of a fuzzy $\mathbb{C}P^1$ model in a rather straightforward way.

In this paper we address the question of constructing a fuzzy supersymmetric non-linear sigma model on $S^{(2,2)}$. For this purpose we obtain the supersymmetric extensions of the Bott projectors and quantize them in a similar manner as discussed in [1]. Using the quantized (super)-projectors and the already known description of $S^{(2,2)}$ in terms of the Lie superalgebras $osp(2,1)$ and $osp(2,2)$ and their associated Lie supergroups we construct the fuzzy supersymmetric non-linear sigma model on $S^{(2,2)}$. We interpret the resulting theory as a finite dimensional matrix model and comment on its various physical properties.

2 $\mathbb{C}P^1$ Sigma Model and Bott Projectors

Non-linear sigma models are customarily defined in terms of a field that maps the world-sheet to the target manifold. In the case of the $\mathbb{C}P^1$ models both world-sheet and the target manifolds are 2-spheres (S^2) and the field $\vec{n}$ maps the point x of the world-sheet

$$S^2 = \left\langle x = (x_1, x_2, x_3) \in \mathbb{R}^3 \middle| \, x_i x_i = 1 \right\rangle \tag{1}$$

to a point on the target manifold

$$S^2 = \left\langle \vec{n}(x) = (n_1(x), n_2(x), n_3(x)) \in \mathbb{R}^3 \middle| \, n_a(x) n_a(x) = 1 \right\rangle . \tag{2}$$

As is well known these maps are classified in terms of an integer κ called the winding number since the second homotopy class $\pi_2(S^2) = \mathbb{Z}$.

An alternative formulation of $\mathbb{C}P^1$ model which happens to be more convenient for passage to fuzzy $\mathbb{C}P^1$ model have been considered in [1]. This formulation uses certain class of projectors, known as Bott projectors instead of the non-linear fields. At the topological sector $\kappa = 1$ the Bott projector can be expressed in terms of $\vec{n}$ as

$$P(x) = \frac{1 + \vec{\tau} \cdot \vec{n}(x)}{2} \tag{3}$$

where $\vec{\tau}$ are the Pauli matrices. $P(x)$ is a projector since $P^2(x) = P(x)$ and $P^\dagger(x) = P(x)$. At the topological sector κ, Bott projector can be expressed by introducing the partial isometries[1] $\vartheta_\kappa^\dagger$ (for $\kappa > 0$) [9]

$$\vartheta_\kappa^\dagger(z) = \begin{pmatrix} \bar{z}_1^\kappa & \bar{z}_2^\kappa \end{pmatrix} \frac{1}{\sqrt{Z_\kappa}}, \quad \vartheta_\kappa(z) = \begin{pmatrix} z_1^\kappa \\ z_2^\kappa \end{pmatrix} \frac{1}{\sqrt{Z_\kappa}}, \qquad Z_\kappa = |z_1|^{2\kappa} + |z_2|^{2\kappa} \tag{4}$$

where $z = (z_1, z_2)$ is a point on $S^3 = \langle z = (z_1, z_2) \in \mathbb{C}^2 \big| \, |z|^2 := |z_1|^2 + |z_2|^2 = 1 \rangle$ and "bar" stands for complex conjugation. Using the Hopf fibration $U(1) \to S^3 \to S^2$, points x on the world-sheet S^2 is expressed in terms of z as

$$x_i = z^\dagger \tau_i z \,. \tag{5}$$

By definition $\vartheta_\kappa^\dagger$ is a partial isometry if and only if $\vartheta_\kappa(z)\vartheta_\kappa^\dagger(z)$ is a projection. It is straightforward to check that $P_\kappa(x)$ in the topological sector κ given as

$$P_\kappa(x) = \vartheta_\kappa(z)\vartheta_\kappa^\dagger(z) = \frac{1}{Z_\kappa} \begin{pmatrix} |z_1|^{2\kappa} & z_1^\kappa \bar{z}_2^\kappa \\ z_2^\kappa \bar{z}_1^\kappa & |z_2|^{2\kappa} \end{pmatrix} \tag{6}$$

is a projector: $P_\kappa(x)^2 = P_\kappa(x) \, , P_\kappa(x)^\dagger = P_\kappa(x)$.

The field $n_a^\kappa(x)$ is associated to $P_\kappa(x)$ by the formulas

$$n_a^\kappa(x) = Tr\, \tau_a P_\kappa(x) = \vartheta_\kappa^\dagger(z) \tau_a \vartheta_\kappa(z) \, , \qquad P_\kappa(x) = \frac{1 + \vec{\tau} \cdot \vec{n}^\kappa(x)}{2} \,. \tag{7}$$

A phase change $z \to z e^{i\theta}$ induces the change $\vartheta_\kappa(z) \to \vartheta_\kappa(z) e^{i\kappa\theta}$. Nevertheless, this phase cancels in $\vartheta_\kappa(z)\vartheta_\kappa^\dagger(z)$ and $P_\kappa(x)$ is a function of x only.

In [1] an intuitive argument as well as an explicit calculation is given to show that κ appearing in equations (4) through (7) is indeed the winding number. Here we recollect the former. For $\kappa > 0$ consider the κ points (up to an overall phase of z which cancels out on x) of S^2 labeled by ℓ:

$$z_\ell = (z_1 e^{i \frac{2\Pi}{\kappa} \ell}, z_2) \qquad \ell \in (0 \, , \kappa - 1) \,. \tag{8}$$

All z_ℓ map to the same point on the target manifold S^2 or equivalently, they all have the same projection via $P_\kappa(x)$, giving winding number κ.

It must be noticed that the form of $P_\kappa(x)$ is very particular. Nevertheless, the most general projector $\mathcal{P}_\kappa(x)$ can be obtained from

$$\mathcal{P}_\kappa(x) = U(x) P_\kappa(x) U(x)^\dagger \tag{9}$$

where $U(x) \in U(2)$ is a 2×2 unitary matrix. The field associated to $\mathcal{P}_\kappa(x)$ is nothing but

$$n_a^{\kappa\prime}(x) = Tr \tau_a \mathcal{P}_\kappa(x) \tag{10}$$

where $n_a^{\kappa\prime}(x) = R_{ab} n_b^\kappa(x)$, $U^\dagger \tau_a U = R_{ab} \tau_b$ and $R \in O(3)$. The unitary transformation do not affect the the winding number since $\pi_2(U(2)) = \{e\}$.

[1]To be more precise the partial isometry $\vartheta_\kappa^\dagger$ in the algebra $\mathcal{A} = C^\infty(S^3) \otimes Mat_{2\times 2}\mathbb{C}$ is the matrix $\begin{pmatrix} \bar{z}_1^\kappa & \bar{z}_2^\kappa \\ 0 & 0 \end{pmatrix}$. But for all practical calculations it is perfectly safe to call (4) as the partial isometry, thus we do so from now on.

3 On the Actions

A Euclidean action in the κ-th topological sector is given in terms of the fields $n_a^\kappa(x)$ [2] by

$$S_\kappa = -\frac{1}{8\pi}\int_{S^2} d\Omega(\mathcal{L}_i n_a^\kappa)(\mathcal{L}_i n_a^\kappa) \tag{11}$$

where $\mathcal{L}_i = -i(x \wedge \nabla)_i$ is the angular momentum operator and $d\Omega = dcos\theta d\psi$. In terms of the projectors, S_κ can be expressed as

$$S_\kappa = -\frac{1}{4\pi}\int_{S^2} d\Omega\, Tr\big(\mathcal{L}_i\mathcal{P}_\kappa\big)\,\big(\mathcal{L}_i\mathcal{P}_\kappa\big)\,. \tag{12}$$

The well known formulae for the winding number and BPS bound of this model can also be rewritten in terms of the projectors $\mathcal{P}_\kappa$. The actions given in (11) and (12) both do have discrete versions when the $\mathbb{C}P^1$ model is formulated on the fuzzy sphere S_F^2. However, it seems that the latter is better adapted for formulation of fuzzy $\mathbb{C}P^1$ sigma models; as will be discussed in section 6 it is possible to quantize the projectors in a straightforward manner. For a detailed discussion on the fuzzy $\mathbb{C}P^1$ model the reader is refered to [1].

In section 5 we develop the supersymmetric extension of the projectors $\mathcal{P}_\kappa(x)$ and apply this result to the description of non-linear sigma model first on the supersphere and then on the fuzzy supersphere. The latter will require the supersymmetric extension of quantized projectors.

4 The Commutative and Non-Commutative (Fuzzy) Superspheres

4.1 The Supersphere $S^{(2,2)}$

In this section we would like to collect some preliminary differential geometric and group theoretical formulae that is used to characterize the supersphere $S^{2,2}$ and its non-commutative (fuzzy) version $S_F^{(2,2)}$. The details of the very brief discussion below can be found in [10, 11].

The structure underlying the supersphere $S^{(2,2)}$ comes from the Lie superalgebras $osp(2,1)$ and $osp(2,2)$ and their associated Lie supergroups $OSP(2,1)$ and $OSP(2,2)$. $osp(2,1)$ is build up of the Lie algebra $su(2)$ (even part) with generators $L_i\,, (i = 1,2,3)$ and $su(2)$ spinors $V_\alpha(\alpha = +,-)$ (odd part). $osp(2,2)$ Lie superalgebra is constructed by augmenting $osp(2,1)$ generators with an additional pair of spinors $D_\alpha(\alpha = +,-)$ and an additional even generator Γ. The graded commutation relations of $osp(2,2)$ generators read

$$\begin{aligned}
[L_i, L_j] &= i\epsilon_{ijk}L_k\,, \quad [L_i, V_\alpha] = \frac{1}{2}(\sigma_i)_{\beta\alpha}V_\beta\,, \quad \{V_\alpha, V_\beta\} = \frac{1}{2}(C\sigma_i)_{\alpha\beta}L_i\,,\\
[L_i, \Gamma] &= 0\,, \quad [\Gamma, V_\alpha] = D_\alpha\,, \quad [\Gamma, D_\alpha] = V_\alpha\,, \quad [L_i, D_\alpha] = \frac{1}{2}(\sigma_i)_{\beta\alpha}D_\beta\,,\\
\{D_\alpha, D_\beta\} &= -\frac{1}{2}(C\sigma_i)_{\alpha\beta}L_i\,, \quad \{D_\alpha, V_\beta\} = \frac{1}{4}C_{\alpha\beta}\Gamma\,.
\end{aligned} \tag{13}$$

where $i,j = 1,2,3,\ \alpha,\beta = \pm$ and $C = i\sigma_2$. The graded commutation relations for the $osp(2,1)$ generators is given by the first line of (13).

[2]For brevity we drop the "prime" on the fields $n_a(x)$.

In the corresponding enveloping algebras there are central polynomials - the Casimir operators in representations given by the formulas:

$$\begin{aligned} K_2^{osp(2,1)} &= L_i L_i + C_{\alpha\beta} V_\alpha V_\beta \,, \\ K_2^{osp(2,2)} &= L_i L_i + C_{\alpha\beta} V_\alpha V_\beta - \Big(C_{\alpha\beta} D_\alpha D_\beta + \frac{1}{4}\Gamma^2 \Big) \,. \end{aligned} \tag{14}$$

These Lie superalgebras are endowed with a grade dagger operation ‡ replacing the usual adjoint operation on the Lie algebras. Generators of $osp(2,2)$ fulfill the following reality conditions implemented by ‡:

$$L_i^\ddagger = L_i^\dagger = L_i, \qquad V_\alpha^\ddagger = C_{\alpha\beta} V_\beta \,, \qquad D_\alpha^\ddagger = -C_{\alpha\beta} D_\beta \,, \qquad \Gamma^\ddagger = \Gamma^\dagger = \Gamma \,. \tag{15}$$

The reality conditions fulfilled by $osp(2,1)$ is obtained by restricting to the relations fulfilled by L_i and V_α. The graded conjugation is extended to homogeneous elements A and B in enveloping algebras by

$$(AB)^\ddagger = (-1)^{|A||B|} B^\ddagger A^\ddagger \,. \tag{16}$$

Here $|A|$ and $|B|$ denote the degrees of A and B, respectively. By linearity the conjugation is extended to the whole enveloping algebra. The Casimir elements, given above, are real.

The supersphere $S^{(2,2)}$ is the adjoint orbit of the Lie supergroup $OSP(2,1)$. It can be obtained through a super generalization of the Hopf fibration for the 2-sphere. In the supersymmetric case this becomes $U(1) \to S^{(3,2)} \to S^{(2,2)}$ where $S^{(3,2)} \equiv OSP(2,1)$ and

$$S^{(2,2)} = S^{(3,2)} \big/ U(1) \,. \tag{17}$$

The superspace $\mathbb{R}^{(3,2)}$ is defined as the algebra of polynomials in generators x_i and θ_α satisfying reality conditions

$$x_i^\ddagger = x_i \,, \qquad \theta_\alpha^\ddagger = C_{\alpha\beta} \theta_\beta \,. \tag{18}$$

These conditions are extended as in (16) to all polynomials. The equation characterizing the adjoint orbit $S^{(2,2)}$ of $osp(2,1)$ is

$$S^{(2,2)} = \Big\langle (x_i \,, \theta_\alpha) \in \mathbb{R}^{(3,2)} \;\big|\; x_i^2 + C_{\alpha\beta}\, \theta_\alpha \theta_\beta = \frac{1}{4} \Big\rangle \,. \tag{19}$$

The action of $osp(2,1)$ on $S^{(2,2)}$ is the adjoint action and is given in terms of the differential operators

$$\begin{aligned} \ell_i &= -i\varepsilon_{ijk} x_j \partial_k - \frac{1}{2} (\sigma_i)_{\beta\alpha} \theta_\beta \partial_{\theta^\alpha} \,, \\ v_\alpha &= -\frac{1}{2} (\sigma_i)_{\beta\alpha} \theta_\beta \partial_i + \frac{1}{2} (C\sigma_i)_{\alpha\beta} x_i \partial_{\theta^\beta} \,. \end{aligned} \tag{20}$$

corresponding to the $osp(2,1)$ generators L_i and V_α, respectively. It can be extended to an $osp(2,2)$ action which is not an adjoint action but it is closely related to it. (for details see [10] [11]). The additional differential operators have the form

$$\begin{aligned} d_\alpha &= -r\Big(1 + \frac{2}{r^2}\Big) C_{\alpha\beta} \partial_{\theta^\beta} + \frac{1}{2r} (\sigma_i)_{\beta\alpha} \theta_\beta \mathcal{L}_i - \frac{\theta_\alpha}{2r} x_i \partial_i \,, \\ \gamma &= \Big(\frac{\theta_+ x_3}{r} + \frac{\theta_- x_+}{r} \Big) \partial_+ + \Big(\frac{\theta_+ x_-}{r} - \frac{\theta_- x_3}{r} \Big) \equiv 2(\theta_- v_+ - \theta_+ v_-) \,. \end{aligned} \tag{21}$$

corresponding to the generators D_α and Γ of $osp(2,2)$.

4.2 The Fuzzy Supersphere $S_F^{(2,2)}$

The fuzzy supersphere $S_F^{(2,2)}$ is obtained replacing $(x_i\,,\theta_\alpha) \in \mathbb{R}^{(3,2)}$ by suitable rescaled $osp(2,1)$ generators $X_i = \lambda L_i$ and $\Theta = \lambda V_\alpha$ with λ determined by the value of $osp(2,1)$ Casimir operator:

$$\frac{1}{4\lambda^2} = K_2^{osp(2,1)}\,. \tag{22}$$

The fuzzy parameters then satisfy the supersphere's defining relation

$$X_i X_i + C_{\alpha\beta}\Theta_\alpha\Theta_\beta = \frac{1}{4}\,. \tag{23}$$

The non-commutativity of the supersphere follows from the graded commutation relations of X_i and Θ_α. For details we refer the reader to [10], [11].

5 Non-Linear Sigma Model on $S^{(2,2)}$

5.1 Preliminaries

The superfield Φ on $S^{(2,2)}$ is a function of the variables $(x_i\,,\theta_\alpha)$; it is real provided that $\Phi^\ddagger = \Phi$. For a free real superfield multiplet the action is related to the $osp(2,1)$ invariant given as the difference of the quadratic Casimir operators:

$$K_2^{osp(2,1)} - K_2^{osp(2,2)} = C_{\alpha\beta}D_\alpha D_\beta + \frac{1}{4}\Gamma^2\,. \tag{24}$$

The action takes the form

$$S^{SUSY} = \frac{1}{4\pi}\int d\mu\Big(d_\alpha\Phi d_\alpha\Phi + \frac{1}{2}\gamma\Phi\frac{1}{2}\gamma\Phi\Big) \tag{25}$$

where $d\mu = d^3x^i d\theta^+ d\theta^- \delta(x_i^2 + C_{\alpha\beta}\theta^\alpha\theta^\beta - \frac{1}{4})$, and d_α and γ are the differential operators given in (21).

For a free triplet real superfield $\Phi^a = \Phi^a(x_i\,,\theta_\alpha)$, $(a = 1,2,3)$, we just replace in Φ by Φ^a (with the summation over repeated index a understood). Now we define the $O(3)$ sigma model [12] by putting on Φ^a the constraint

$$\Phi^a\Phi^a = 1 \qquad (a = 1,2,3)\,. \tag{26}$$

Then (25) and (26) defines the non-linear sigma model on the supersphere $S^{(2,2)}$ with the target manifold being S^2.

The superfield $\Phi^a(x_i\,,\theta_\alpha)$ can be expanded in powers of θ_α as

$$\Phi^a(x_i\,,\theta_\alpha) = n^a(x_i) + C_{\alpha\beta}\theta_\beta\psi_\alpha^a(x_i) + \frac{1}{2}F^a(x_i)C_{\alpha\beta}\theta_\alpha\theta_\beta \tag{27}$$

where $\psi^a(x_i)$ are two component Majorana spinors : $\psi_\alpha^{a\ddagger} = C_{\alpha\beta}\psi_\beta^a$, and $F^a(x_i)$ are auxiliary scalar fields. In terms of the component fields the constraint equation (26) splits to

$$n^a n^a = 1\,, \qquad n^a F^a = \frac{1}{2}\psi^{a\ddagger}\psi^a\,, \qquad n^a\psi_\alpha^a = 0\,. \tag{28}$$

5.2 Supersymmetric Extensions of Bott Projectors

A possible supersymmetric extension of the projector $\mathcal{P}_\kappa(x)$ can be obtained in the following way. Let $\mathcal{U}(x_i\,,\theta_\alpha)$ be a graded unitary operator with $\mathcal{U}\mathcal{U}^\ddagger = \mathcal{U}^\ddagger\mathcal{U} = 1$. $\mathcal{U}(x_i\,,\theta_\alpha)$ in general can be thought as a 2×2 supermatrix whose entries are functions on $S^{(2,2)}$. $\mathcal{U}(x_i\,,\theta_\alpha)$ acts on $\mathcal{P}_\kappa$ by conjugation and generates a set of supersymmetric extensions $\mathcal{Q}_\kappa(x_i\,,\theta_\alpha)$:

$$\mathcal{Q}_\kappa(x_i\,,\theta_\alpha) = \mathcal{U}^\ddagger\,\mathcal{P}_\kappa(x)\,\mathcal{U}\,. \tag{29}$$

It is easy to see that $\mathcal{Q}_\kappa(x_i\,,\theta_\alpha)$ satisfies $\mathcal{Q}_\kappa^2(x_i\,,\theta_\alpha) = Q_\kappa(x_i\,,\theta_\alpha)$ and $\mathcal{Q}_\kappa^\ddagger(x_i\,,\theta_\alpha) = \mathcal{Q}_\kappa(x_i\,,\theta_\alpha)$. Thus $\mathcal{Q}_\kappa(x_i\,,\theta_\alpha)$ is a (super)-projector. The real superfield on $S^{(2,2)}$ associated to $\mathcal{Q}_\kappa(x_i\,,\theta_\alpha)$ is given by

$$\Phi_a'(x_i\,,\theta_\alpha) = Tr\,\tau_a\mathcal{Q}_\kappa\,. \tag{30}$$

In order to perform a check that establishes that $\mathcal{Q}_\kappa(x_i\,,\theta_\alpha)$ are indeed the supersymmetric projectors that reproduces the superfields on $S^{(2,2)}$ subject to

$$\Phi_a'\Phi_a' = 1\,, \tag{31}$$

we proceed as follows. First we expand $\mathcal{U}(x_i\,,\theta_\alpha)$ in powers of the Grassmann variables as

$$\mathcal{U}(x_i\,,\theta_\alpha) = \mathcal{U}_0(x_i) + C_{\alpha\beta}\theta_\beta\mathcal{U}_\alpha(x_i) + \frac{1}{2}\mathcal{U}_2(x_i)C_{\alpha\beta}\theta_\alpha\theta_\beta \tag{32}$$

where $\mathcal{U}_0\,,\mathcal{U}_\alpha(\alpha=\pm)$ and $\mathcal{U}_2$ are all 2×2 graded unitary matrices. The requirement that $\mathcal{U}(x_i\,,\theta_\alpha)$ is graded unitary makes $\mathcal{U}_0(x_i)$ unitary, whereas $\mathcal{U}_\alpha(x_i)$ are uniquely determined by $\mathcal{U}_\alpha(x_i) = H_\alpha(x_i)\mathcal{U}_0(x_i)$ where H_α are 2×2 odd supermatrices with the reality condition $H_\alpha^\ddagger = -C_{\alpha\beta}H_\beta$. Moreover, with the ansatz that $\mathcal{U}_2 = A\mathcal{U}_0$ with A being an arbitrary 2×2 even supermatrix, graded unitarity of $\mathcal{U}(x_i\,,\theta_\alpha)$ further restricts the symmetric part of A as:

$$A + A^\dagger = -C_{\alpha\beta}H_\alpha H_\beta\,. \tag{33}$$

Using the expansion (32) in (29) and subsequently the resulting expression in (30) together with the properties listed above it is straightforward to extract the component fields of the superfield $\Phi_a'(x_i\,,\theta_\alpha)$. We find

$$n_a^{\kappa\prime} := Tr\,\tau_a U_0^\dagger\mathcal{P}_\kappa U_0\,, \tag{34}$$

$$\psi_\alpha^{a\prime} := -2i(\vec{n}^{\kappa\prime}\times\vec{H}_\alpha')^a = Tr\,\tau_a U_0^\dagger[H_\alpha\,,\mathcal{P}_\kappa]U_0\,, \tag{35}$$

and after using (33) that

$$\begin{aligned} F_a' &:= 4(\vec{H}_+'\cdot\vec{H}_-')n_a^{\kappa\prime} - 2\vec{H}_+^{a\prime}(\vec{n}^{\kappa\prime}\cdot\vec{H}_-') - (\vec{n}^{\kappa\prime}\cdot\vec{H}_+')2\vec{H}_-^{a\prime} + i(\vec{n}^{\kappa\prime}\times(\vec{A}'-\vec{A}^{\dagger\prime}))^a \\ &= Tr\,\tau_a U_0^\dagger(\mathcal{P}_\kappa A + A^\dagger\mathcal{P}_\kappa - C_{\alpha\beta}H_\beta\mathcal{P}_\kappa H_\alpha)U_0\,. \end{aligned} \tag{36}$$

where $\vec{H}_\alpha^{a\prime} = H_\alpha^{a\prime}\tau^a$ and $\vec{A}^{a\prime} = A^{a\prime}\tau^a$. By direct computation from above we find

$$n_a^{\kappa\prime}n_a^{\kappa\prime} = 1\,,\qquad n_a^{\kappa\prime}F_a' = \frac{1}{2}\psi_a^{\ddagger\prime}\psi_a'\,,\qquad n_a^{\kappa\prime}\psi_\pm^{a\prime} = 0\,. \tag{37}$$

Comparing (37) with (28) we observe that they are identical. Therefore we conclude that the superfield associated to the super-projector $\mathcal{Q}_\kappa$ is the same as the superfield in supersymetric non-linear sigma model of the previous subsection.

5.3 SUSY Action Revisited

We are now ready to give the formulation of non-linear sigma model on the supersphere using the (super)-projectors. In close analogy with the $\mathbb{C}P^1$ case the supersymmetric action in (25) with the constraint (26) translates to

$$S_\kappa^{SUSY} = \frac{1}{2\pi}\int d\mu\, Tr\Big[(d_\alpha \mathcal{Q}_\kappa)(d_\alpha \mathcal{Q}_\kappa) + \frac{1}{4}(\gamma\mathcal{Q}_\kappa)(\gamma\mathcal{Q}_\kappa)\Big]\,. \tag{38}$$

The even part of this action, as well as the one given in (25) is nothing but the action S_κ of the $\mathbb{C}P^1$ theory given in (12) and (11), respectively. In other words, the action S_κ^{SUSY} is the supersymmetric extension of S_κ on S^2 to $S^{(2,2)}$. Thus in the supersymmetric theory it is possible to interpret the index κ carried by the action as the winding number of the corresponding $\mathbb{C}P^1$ theory.

We recall that d_α and γ are both derivations in the superalgebra $Osp(2,2)$. Therefore they obey a graded Leibnitz rule and from $\mathcal{Q}_\kappa^2 = \mathcal{Q}_\kappa$ we find

$$\mathcal{Q}_\kappa d_\alpha \mathcal{Q}_\kappa = d_\alpha \mathcal{Q}_\kappa(\mathbf{1} - \mathcal{Q}_\kappa)\,. \tag{39}$$

This enables us to write

$$Tr d_\alpha \mathcal{Q}_\kappa(\mathbf{1}-\mathcal{Q}_\kappa)d_\alpha\mathcal{Q}_\kappa = Tr(\mathbf{1}-\mathcal{Q}_\kappa)(d_\alpha\mathcal{Q}_\kappa)^2 = \frac{1}{2}Tr(d_\alpha\mathcal{Q}_\kappa)^2\,. \tag{40}$$

Equations (39) and (40) continue to hold when d_α is replaced by γ as well. The action then takes the form

$$S_\kappa^{SUSY} = \frac{1}{\pi}\int d\mu\, Tr\Big[\mathcal{Q}_\kappa(d_\alpha \mathcal{Q}_\kappa)(d_\alpha \mathcal{Q}_\kappa) + \frac{1}{4}\mathcal{Q}_\kappa(\gamma\mathcal{Q}_\kappa)(\gamma\mathcal{Q}_\kappa)\Big]\,. \tag{41}$$

It is possible that this form of the action could play an important role for obtaining an supersymmetric generalization of the BPS equation since an analogues expression in the $\mathbb{C}P^1$ case [1] have been employed to achieve this result.

6 Fuzzy Projectors and Sigma Models

6.1 Fuzzy $\mathbb{C}P^1$ Model

In [1] the $\mathbb{C}P^1$ model has been quantized as follows. Let $\xi = (\xi_1\,, \xi_2) \in \mathbb{C}^2\setminus\{0\}$. In terms of ξ we define

$$z = \frac{\xi}{|\xi|}\,, \qquad |\xi| = \sqrt{|\xi_1|^2 + |\xi_2|^2}\,, \qquad x_i = z^\dagger x_i z\,. \tag{42}$$

ξ_α and $\bar{\xi}_\alpha$ are quantized by replacing them with a pair of annihilation a_α and creation $a_\alpha^\dagger$ operators respectively. With this substitution $|\xi|$ becomes the square root of the number operator and we have

$$\begin{aligned}
\hat{N} &= \hat{N}_1 + \hat{N}_2\,, \quad \hat{N}_1 = a_1^\dagger a_1\,, \quad N_2 = a_2^\dagger a_2 \\
\hat{z}_\alpha^\dagger &= \frac{1}{\sqrt{\hat{N}}}a_\alpha^\dagger = a_\alpha^\dagger\frac{1}{\sqrt{\hat{N}+1}}\,, \quad \hat{z}_\alpha = \frac{1}{\sqrt{\hat{N}+1}}a_\alpha = a_\alpha\frac{1}{\sqrt{\hat{N}}}\,, \\
\hat{x}_i &= \frac{1}{\hat{N}}a^\dagger\tau_i a\,.
\end{aligned} \tag{43}$$

In the light of this conjecture it is easy to see that the quantized version of the partial isometry $\vartheta_\kappa^\dagger$ defined in (4) and its Hermitian conjugate reads

$$\begin{aligned} \hat{\vartheta}_\kappa^\dagger &= \frac{1}{\sqrt{\hat{Z}_\kappa}} \begin{pmatrix} a_1^{\dagger\kappa} & a_2^{\dagger\kappa} \end{pmatrix}, \qquad \hat{\vartheta}_\kappa = \begin{pmatrix} a_1^\kappa \\ a_2^\kappa \end{pmatrix} \frac{1}{\sqrt{\hat{Z}_\kappa}}, \qquad \hat{\vartheta}_\kappa^\dagger \hat{\vartheta}_\kappa = \mathbf{1}, \\ \hat{Z}_\kappa &= \hat{Z}_\kappa^{(1)} + \hat{Z}_\kappa^{(2)}, \quad \hat{Z}_\kappa^{(\alpha)} = \hat{N}_\alpha(\hat{N}_\alpha - 1)\dots(\hat{N}_\alpha - \kappa + 1). \end{aligned} \tag{44}$$

The fuzzy analogue of (6) can now be written as

$$\hat{P}_\kappa(x) = \hat{\vartheta}_\kappa \hat{\vartheta}_\kappa^\dagger = \begin{pmatrix} a_1^\kappa \frac{1}{\hat{Z}_\kappa} a_1^{\dagger\kappa} & a_1^\kappa \frac{1}{\hat{Z}_\kappa} a_2^{\dagger\kappa} \\ a_2^\kappa \frac{1}{\hat{Z}_\kappa} a_1^{\dagger\kappa} & a_2^\kappa \frac{1}{\hat{Z}_\kappa} a_2^{\dagger\kappa} \end{pmatrix} \tag{45}$$

where for example

$$a_1^\kappa \frac{1}{\hat{Z}_\kappa} = \frac{1}{(\hat{N}_1 + \kappa)\dots(\hat{N}_1 + 1) + \hat{Z}_\kappa^{(2)}} a_1^\kappa, \quad a_1^\kappa a_1^{\dagger\kappa} = (\hat{N}_1 + \kappa)\dots(\hat{N}_1 + 1). \tag{46}$$

The unitary matrix U introduced to generate all possible projectors $\mathcal{P}_\kappa$ from P_κ also have fuzzy analogue. It is a 2×2 unitary matrix $\hat{U}$, with matrix entries being polynomials in $a_\alpha^\dagger a_\beta$. Thus the most general fuzzy projectors are

$$\hat{\mathcal{P}}_\kappa = \hat{U} \hat{P}_\kappa \hat{U}^\dagger. \tag{47}$$

From (45) it is clear that $\hat{\mathcal{P}}_\kappa$ acts in general on $\mathcal{F}^2 := \mathcal{F} \otimes \mathbb{C}^2$ where $\mathcal{F}$ stands for the standard Fock space. It also follows from (45) that $\hat{P}_\kappa$ commutes with the number operator $\hat{N}$, as can be checked directly. Consequently, we can restrict ourselves to work on a finite dimensional subspace $\mathcal{F}_n$ of dimension $n+1$ of $\mathcal{F}$. Then $\hat{\mathcal{P}}_\kappa$ act on the finite dimensional Hilbert space $\mathcal{F}_n^2 := \mathcal{F}_n \otimes \mathbb{C}^2$ and this allows one to formulate a finite dimensional matrix model for projectors $\hat{\mathcal{P}}_\kappa$.

In [1] this has been done to write down the fuzzy $\mathbb{C}P^1$ model. They found that the fuzzy action corresponding to (12) is

$$S_{F,\kappa} = \frac{1}{4\pi} \frac{1}{2(n+1)} Tr_{\hat{N}=n} (\mathcal{L}_i \widehat{\mathcal{P}}_\kappa)(\mathcal{L}_i \widehat{\mathcal{P}}_\kappa) \tag{48}$$

where $\mathcal{L}_i \widehat{\mathcal{P}}_\kappa = [L_i\,, \widehat{\mathcal{P}}_\kappa]$ and the trace is over $\mathcal{F}_n^{(2)}$.

6.2 Fuzzy Supersymmetric Model

In much the same way the supersymmetric projectors $\mathcal{Q}_\kappa$ have been constructed from $\mathcal{P}_\kappa$ in section 5, we can construct supersymmetric extensions of $\widehat{\mathcal{P}}_\kappa$ by the graded unitary transformation

$$\widehat{\mathcal{Q}}_\kappa = \hat{\mathcal{U}}^\ddagger \widehat{\mathcal{P}}_\kappa \hat{\mathcal{U}} \tag{49}$$

where $\hat{\mathcal{U}}$ is a 2×2 supermatrix whose entries are polynomials in $a_\alpha^\dagger a_\beta$ and $b^\dagger b$ and where b and $b^\dagger$ are fermionic annihilation and creation operators with the anti-commutation relation $\{b\,, b^\dagger\} = 1$.

$\widehat{\mathcal{Q}}_\kappa$ defined (49) acts on the finite dimensional space $\tilde{\mathcal{F}}_n^2 = \tilde{\mathcal{F}}_n \otimes \mathbb{C}^2$. Here $\tilde{\mathcal{F}}$ is the $n+1$ dimensional subspace of the Hilbert space $\tilde{\mathcal{F}}$ spanned by the kets $|n_1, n_2, n_3\rangle$ where n_3 labels the fermionic part taking on the values 0 and 1 only. It is readily seen that $\widehat{\mathcal{Q}}_\kappa$ commutes with the supersymmetric number operator $\widehat{\mathcal{N}} = a^\dagger_\alpha a_\alpha + b^\dagger b$. In close analogy with the fuzzy $\mathbb{C}P^1$ model, it is now possible to write down a finite dimensional (super)matrix model for the (super)-projectors $\widehat{\mathcal{Q}}_\kappa$.

Making use of (24) once more the action for the fuzzy supersymmetric model is given as

$$S^{SUSY}_{F,\kappa} = \frac{1}{2\pi}\, Str_{\widehat{\mathcal{N}}=n}\left[(\mathcal{D}_\alpha\widehat{\mathcal{Q}}_\kappa)(\mathcal{D}_\alpha\widehat{\mathcal{Q}}_\kappa) + \frac{1}{4}(\Xi\widehat{\mathcal{Q}}_\kappa)(\Xi\widehat{\mathcal{Q}}_\kappa)\right], \tag{50}$$

where $\mathcal{D}_\alpha\widehat{\mathcal{Q}}_\kappa = \{D_\alpha\,,\widehat{\mathcal{Q}}_\kappa\}$ and $\Xi\widehat{\mathcal{Q}}_\kappa = [\Gamma\,,\widehat{\mathcal{Q}}_\kappa]$. "Str" in the above expression is the supertrace over the finite dimensional space $\tilde{\mathcal{F}}^2_n$. Obviously, in the large $\widehat{\mathcal{N}} = n$ limit (50) approximates the action given in (38).

7 Conclusions

In this paper we have obtained the fuzzy version of supersymmetric non-linear sigma model on $S^{(2,2)}$. Our approach has utilized the use of supersymmetric extensions of the Bott projectors and generalized results of $\mathbb{C}P^1$ model to supersymmetric theories. A natural question to be addressed is the supersymmetric generalization of the BPS equation. We hope to report any development on this issue elsewhere.

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