TÜBİTAK

Mikro-Plakların Modelleme ve Analizi İçin Yeni Yöntemler

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Önsöz

Mikro-plakların modelleme ve analizi başlıklı ve 213M606 numaralı proje Türkiye Bilimsel ve Teknolojik Araştırma Kurumu (TÜBİTAK) tarafından desteklenmiştir. Proje kapsamında fonksiyonel derecelendirilmiş mikro-plaklar için gerinim gradyanı elastisite teorisi bazlı yeni analiz ve modelleme yöntemleri geliştirilmiştir. Yürütülen faaliyetler analitik türetim, sayısal çözüm yöntemlerinin geliştirilmesi, sayısal çözüm uygulaması, ve parametrik analizler gibi ana kısımlardan oluşmaktadır. Proje 2014 yılı Nisan ayında başlamış ve 2016 yılı Nisan ayında tamamlanmıştır. Proje yürütücülüğü Prof. Dr. Serkan Dağ tarafından yapılmıştır. Projede Doç. Dr. Ender Ciğeroğlu araştırmacı, Ata Alipour Ghassabi bursiyer olarak görev almıştır. Çalışmalar Orta Doğu Teknik Üniversitesi Makine Mühendisliği Bölümü'nde yürütülmüştür.

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Öz

Bu araştırma projesinin temel amacı mekanik veya termal yükleme altındaki mikro-plakların analizi için yeni yöntemler ortaya koymaktır. Malzemelerin makro-ölçekte mekanik analizini yapmakta kullanılan teoriler mikro-ölçekte geçerli değildir. Bunun nedeni uzunluk ölçeği küçüldükçe etkisi artış gösteren boyut etkisidir. Mikro-ölçekli yapıların analizi için gerinim gradayanı elastisite teorisi ve modifiye edilmiş kuvvet çifti gerilmesi teorisi gibi yüksek dereceden sürekli ortam teorileri geliştirilmiştir. Teknik literatürde, mikro-plakların yüksek dereceden sürekli ortam teorileri ile modellenmesi üzerine çeşitli çalışmalar bulunmaktadır. Bu araştırmalarda hem fonksiyonel derecelendirilmiş malzemelerden (FDM) yapılmış mikro-plaklar hem de homojen mikro-plaklar analiz edilmiştir. Ancak, yapısal mekanik ile ilgili bazı önemli problemler bu makalelerde incelenmemiştir. İlgili çalışmalar sadece mekanik yükleme altındaki mikro-plaklar için yapılmış; ve çevresel ve elektrik etkiler gibi nedenlerle oluşabilecek termal yüklemeler ele alınmamıştır. Ayrıca, fonksiyonel derecelendirilmiş mikro-plaklar üzerine yürütülen çalışmalarda, hacim oranlarındaki değişimler nedeniyle uzaysal koordinatların fonksiyonları olması gereken uzunluk ölçeği parametreleri sabit olarak kabul edilmiştir. Bu araştırma projesinde, termal etkiler ve FDM'lerin uzunluk ölçeği parametrelerindeki değişimler göz önüne alınarak yeni analiz yöntemleri geliştirilmiştir.

Yeni yöntemler geliştirilirken, öncelikle termal yükleme altındaki uzunluk ölçeği parametreleri değişken fonksiyonel derecelendirilmiş mikro-plaklar için bağlaşık kısmi diferansiyel denklemler ve sınır koşulları türetilmiştir. Bu formülasyonda yüksek dereceden sürekli ortam teorisi olarak gerinim gradyanı elastisite teorisi kullanılmıştır. Kirchhoff, Mindlin, ve üçüncü dereceden plak teorileri olarak belirlenen üç farklı plak teorisi için sonuç üretebilmek amacıyla, genel bir formülasyon yaklaşımı ortaya konulmuştur. Matematiksel olarak, modifiye edilmiş kuvvet çifti gerilmesi teorisi, gerinim gradyanı elastisite teorisinin özel bir halidir; dolayısıyla basitleştirme yoluyla modifiye edilmiş kuvvet çifti gerilmesi teorisi için geçerli sonuçlar da bulunabilmektedir. Benzer şekilde, homojen mikro-plaklar için geçerli olan sonuçlar, FDM mikro-plaklar için türetilen formülasyon kullanılarak bulunabilmektedir. Sonuç itibariyle, geliştirilen formülasyon olabilecek en genel formda yapılandırılmış ve eğilme, burkulma, ve serbest titreşim gibi yapısal problemlerin çözümünde kullanılmıştır. Bağlaşık denklemleri sayısal olarak çözebilmek için, diferansiyel kare yapma metodunu baz alan sayısal algoritmalar hazırlanmıştır. Bu algoritmalar MATLAB adlı matematik yazılımına entegre edilmiştir. Formülasyonun ve sayısal çözüm tekniklerinin geçerliliklerini gösterebilmek amacıyla özel durumlar için geçerli olan ve literatürde bulunan sayısal sonuçlarla karşılaştırmalar yapılmıştır. Yürütülen detaylı sayısal analizler aracılığıyla, sıcaklık farkı, uzunluk ölçeği parametrelerindeki uzaysal değişimler, heterojenlik sabitleri, ve

geometrik parametrelerin, mikro-plakların statik deformasyonları, burkulma yükleri, ve serbest titreşim doğal frekansları üzerlerindeki etkileri belirlenmiştir.

Proje önerisinde tanımlanan bu çalışmalara ek olarak modifiye edilmiş kuvvet çifti teorisi kullanılarak halka şeklinde ve dairesel FDM mikro-plaklar için ve lokal olmayan elastisite teorisi aracılığı ile dikdörtgen FDM nano-plaklar için formülasyon ve sayısal çözüm çalışmaları yapılmıştır. Bu çalışmalarla statik eğilme ve serbest titreşim davranışları ile ilgili ek sonuçlar üretilmiştir.

Anahtar Kelimeler: Mikro-plaklar, fonksiyonel derecelendirilmiş malzemeler, eğilme, serbest titreşimler, burkulma.

Abstract

The main objective of this research project is to put forward new methods for the analysis of micro-plates that are under the effect of mechanical or thermal loading. The theories applicable for mechanical analysis of materials at the macro-scale are known not to be valid for micro-scale structures. This fact stems from the so-called size effect, which plays an increasingly important role on the mechanical behavior as the length scale gets smaller. Higher order continuum theories are developed for the analysis of micro-scale structures, among which we can mention strain gradient elasticity, and modified couple stress theory. There are a number of studies in the technical literature on the analysis of micro-plates by means of higher order continuum theories. Both micro-plates made of functionally graded materials (FGMs) and homogeneous micro-plates are examined in these investigations. However, certain important aspects of structural problems are not considered in these articles. In all relevant studies in the literature, only micro-plates subjected to mechanical loads are analyzed and the influence of thermal loading, which may arise as a result of environmental or electrical effects, is ignored. Furthermore, in the analysis of functionally graded micro-plates, the length scale parameters are assumed to be constants, which in reality are functions of spatial coordinates due to smooth spatial variations of volume fractions in functionally graded materials. Presented research is undertaken to be able to develop new analysis methods that can take into account thermal effects and the spatial variations in the length scale parameters of FGMs.

In the development of these new methods, first governing partial differential equations and boundary conditions are derived for thermally-loaded functionally graded micro-plates, that possess variable length scale parameters. As the higher order continuum theory, strain gradient elasticity is employed in the derivations. A unified formulation is constructed to be able to produce results for three different plate theories which are Kirchhoff, Mindlin, and third-order plate theories. Mathematically, modified couple stress theory is a special case of strain gradient elasticity thus results for modified couple stress theory can be computed through simplification. Moreover, results pertaining to homogeneous micro-plates can be found using the formulation derived for FGM micro-plates. Consequently, developed formulation is in the most general form, and applicable in the solutions of structural problems including bending, buckling, and free vibrations. Solution algorithms based on the differential quadrature method (DQM) are prepared so as to numerically solve the governing equations. These algorithms are integrated into the math software MATLAB. The formulation and numerical solution techniques are verified by making comparisons to results available in the literature for special cases. Systematic numerical analyses are carried out in order to determine the influences of temperature difference, variations in length scale parameters, inhomogeneity constants, and geometric parameters upon static deflections, buckling loads, and free vibration frequencies of micro-plates.

In addition to this work defined in the project proposal, formulation and numerical solution techniques are developed for annular and circular FGM micro-plates considering modified couple stress theory and for rectangular FGM nano-plates considering nonlocal elasticity theory. Additional new results regarding static bending and free vibrations are produced through these applications.

Key words: Micro-plates, functionally graded materials, bending, free vibrations, buckling.

1. GİRİŞ

Mikro-elektro-mekanik sistemler (MEMS) günümüzde birçok teknolojik uygulamada özellikle mikro-sensörler, mikro-aktüatörler, ve mikro-rezonatörler olarak kullanılmaktadır. Uygulama alanlarının sürekli olarak çeşitlilik ve yaygınlık kazanmasıyla beraber, bu sistemler üzerine yapılan yatırım ve araştırmalar da artmıştır. Mikro-kiriş ve mikro-plak gibi yapılar birçok MEMS cihazında kullanılmaktadır; ve bu yapıların mekanik, termal, ve elektrik yüklemeler altındaki davranışlarının doğru olarak belirlenmesi tasarım süreci içerisinde büyük önem kazanmıştır. Bu araştırma projesinin temel amacı mikro-plakların mekanik analizi için, literatürde bulunan yöntemlere göre, daha genel ve daha doğru sonuç veren yöntemler geliştirmektir. Proje kapsamında çalışmalar ile termal yükleme altındaki homojen ve fonksiyonel derecelendirilmiş malzemeden (FDM) yapılmış mikro-plaklar için analiz yöntemleri geliştirilmiş; ayrıca FDM mikro-plaklar için yapılan formülasyon ve analizlerde uzunluk ölçeği parametrelerindeki uzaysal koordinatlara bağlı değişimler göz önüne alınmıştır.

Mikro-yapıların analiz ve modellemesinde standard sürekli ortam mekaniği yaklaşımları boyut etkisi nedeniyle geçerliliğini yitirmektedir. Elastisite teorisi gibi yaygın kullanılan teoriler mikro-yapıların mekanik davranışını açıklamakta yetersiz kaldığından, bu yapılar için geçerli olan modifiye edilmiş kuvvet çifti gerilmesi teorisi (Yang vd., 2002) ve gerinim gradyanı elastisite teorisi (Lam vd., 2003) gibi teoriler geliştirilmiştir. Bu teoriler kullanılarak, Literatür Özeti bölümünde anlatıldığı gibi, mikro-plakların mekanik davranışlarının belirlenmesini sağlayacak çeşitli çözümler geliştirilmiştir. Ancak, tüm bu çalışmalar incelendiğinde, çevresel ve elektriksel etkiler sonucunda oluşabilecek termal yüklemelerin göz önüne alınmadığı; ayrıca FDM mikro-plaklar üzerine yapılan çalışmalarda uzaysal koordinatlara bağlı fonksiyonlarla temsil edilmesi gereken uzunluk ölçeği parametrelerinin sabit olarak alındığı görülmüştür. Bu araştırma projesi kapsamında mikro-plakların mekanik modellemesinde termal yüklemeler göz önüne alınmış ve FDM mikro-plaklar ile ilgili yapılan geliştirmelerde, uzunluk ölçeği parametreleri sürekli fonksiyonlarla temsil edilerek daha gerçekçi sonuç veren yöntemler ortaya konulmuştur.

Bu hedeflere ulaşabilmek için öncelikle termal yükleme altındaki FDM mikro-plaklar için genel bir formülasyon geliştirilerek bağlaşık kısmi diferansiyel denklemler ve sınır koşulları türetilmiştir. Homojen mikro-plak problemi matematiksel olarak FDM mikro-plak probleminin özel bir hali olduğundan, bu denklemlerden homojen mikro-plaklar için geçerli olan denklemleri elde etmek mümkündür. Formülasyonun oluşturulmasında gerinim gradyanı elastisite teorisi kullanılmıştır. Modifiye edilmiş kuvvet çifti gerilmesi teorisi de, gerinim gradyanı elastisite teorisinin özel bir halidir. Bu nedenle geliştirilen formülasyon her iki teori için de geçerlidir. Ayrıca, formülasyon üç farklı plak deformasyon teorisi için sonuç verecek şekilde oluşturulmuştur. Göz önüne alınan deformasyon teorileri Kirchhoff, Mindlin, ve üçüncü dereceden plak teorileridir. Türetilen denklemlerin sayısal çözümü diferansiyel kare yapma yöntemi (Shu, 2000) aracılığıyla gerçekleştirilmiştir.

Yürütülen araştırma sonucunda, termal veya mekanik yüklemeye tabi tutulmuş mikro-plakların eğilme, burkulma, ve serbest titreşim analizlerinin yapılmasını sağlayacak araçlar ortaya konulmuştur. Bu araçlar ile malzeme özelliklerinin, geometrik parametrelerin, ve FDM mikro-plaklarda uzunluk ölçeği parametrelerindeki değişimlerin, statik deformasyon, kritik burkulma yükü, ve serbest titreşim doğal frekansları üzerlerindeki etkilerini açığa çıkarmak mümkün olmaktadır. Ortaya konulan yeni yöntemler, MEMS cihazları tasarımı konusunda çalışan bilim adamları ve mühendisler tarafından kullanılabilecek; ve mikro-kabuklar ile eğik mikro-paneller gibi yapıların incelenmesi üzerine yeni araştırma projelerinin tasarlanmasını sağlayacaktır.

2. LİTERATÜR ÖZETİ

Genel olarak mikro-fabrikasyon teknikleri ile üretilen minyatürize edilmiş mekanik ve elektrik bileşenlerden oluşan sistemler olarak tanımlanan mikro-elektro-mekanik sistemler (MEMS) ile ilgili günümüzde yoğun araştırma ve uygulama çalışmaları yürütülmektedir. Teknolojik uygulamalarda en sık kullanılan MEMS cihazları arasında mikro-sensörler, mikro-aktüatörler, ve mikro-rezonatörler gösterilebilir. Mikro-sensörler genellikle sıcaklık, basınç, ivme, ve magnetik alan gibi fiziksel nicelikleri ölçmek amacıyla kullanılmaktadır. Yoğunlukla kullanılan mikro-aktüatör çeşitleri arasında ise mikro-konumlama cihazlarını, mikro-pompaları, mikro-flapları, ve mikro-tutucuları gösterebiliriz. Mikro-rezonatör uygulamaları özellikle otomotiv, telekomunikasyon, ve biyomedikal sektörlerinde yaygınlık kazanmaktadır. Mikro-fabrikasyona dayalı üretim yöntemlerinde ulaşılan gelişmelerle üretim maliyetlerinin düşürülmesi gibi etkenler de MEMS cihazlarının uygulama alanlarını hızla genişletmekte ve bu cihazların uluslararası ekonomi içerisindeki önemini arttırmaktadır.

Araştırma projemiz kapsamında incelenen mikro-plaklar çeşitli türde MEMS cihazlarının önemli bileşenlerinden birisidir. Elektriksel olarak aktive edilen mikro-plaklar, mikro-sensörler ve mikro-aktüatörlerde bulunan havalı kondensatörlerde yer almaktadır (Zhao, 2004). Aynı şekilde dikdörtgen ve dairesel mikro-plaklar, mikro-rezonatörlerde yaygın olarak kullanılmaktadır (Li vd., 2012). Bu nedenlerle mikro-plakların dış etkiler altındaki mekanik davranışlarının doğru olarak belirlenmesi, MEMS tasarımı sürecinde büyük önem taşımaktadır. Mikro-plağın kullanıldığı uygulamaya bağlı olarak eğilme, burkulma ya da serbest titreşim analizinin yapılması, mekanik davranışın dış etkiler altında nasıl olacağı sorusunun yanıtlanabilmesi için gereklidir. Yürütülen projenin başlıca amacı bu incelemelerin yapılabilmesi için yeni modelleme ve analiz yöntemleri geliştirmektir.

Standard analiz yöntem ve yaklaşımları boyut etkisi nedeniyle mikro-düzeyde geçerliliğini kaybetmekte; ve bu yöntemler mikro-yapılar için yeterli doğrulukta sonuç verememektedir. Örneğin ankastre bir makro-kirişte uç noktasındaki yerdeğiştirme, kirişin uzunluk-kalınlık oranının sabit olması halinde, kalınlık değiştikçe değişmemektedir. Ancak, aynı gözlem ankastre mikro-kirişler için yapılamamaktadır. Ankastre mikro-kirişlerde uç noktasındaki yerdeğiştirme, uzunluk-kalınlık oranının sabit tutulduğu durumda dahi kalınlıkla değişmektedir (Lam vd., 2003). Boyut etkisi olarak adlandırılan bu durum mikro-kiriş, mikro-plak ve mikro-kabuk gibi diğer yapısal elemanlar için de geçerlidir. Boyut etkisini göz önüne alabilmek için yüksek dereceden sürekli ortam teorileri olarak adlandırılan teorileri kullanmak gereklidir. Yürüttüğümüz proje çalışmasında da mikro-plakların modelleme ve analizini yapabilmek için yüksek dereceden sürekli ortam teorileri kullanılmıştır.

Son yıllarda mikro-yapıların modellenmesinde özellikle iki farklı yüksek dereceden teori temel alınmıştır. Bunlardan birincisi Yang vd. (2002) tarafından geliştirilen modifiye edilmiş kuvvet çifti gerilmesi teorisidir. Bu teoride bilinen malzeme özelliklerine ek olarak, formülasyonda bir de uzunluk ölçeği parametresi kullanılmaktadır. İkinci teori ise gerinim gradyanı elastisite teorisi olarak adlandırılmaktadır; ve bu teoride üç farklı uzunluk ölçeği parametresine ihtiyaç vardır (Lam vd., 2003). Modifiye edilmiş kuvvet çifti gerilmesi teorisini, gerinim gradyanı elastisite teorisinin özel bir hali olarak da tanımlamak mümkündür. Teknik literatürde, mikro-plakların mekaniği konusunda bu teorileri baz alarak yapılmış çeşitli çalışmalar bulunmaktadır. Bu çalışmalar incelendiğinde, bir kısmının homojen mikro-plaklar üzerine bir kısmının da fonksiyonel derecelendirilmiş malzemelerden (FDM) yapılmış mikro-plaklar ile ilgili olduğu görülecektir. Fonksiyonel derecelendirilmiş malzemeler birden çok faz içeren ve fazların hacim oranlarının uzaysal koordinatlara bağlı olarak değiştiği malzemelerdir (Suresh ve Mortensen, 1998). Bu değişimler kullanılarak termal bariyer kaplamaları, sürtünme ve aşınmaya dayanıklı yüzeyler, biomedikal malzemeler, ve yakıt hücreleri gibi teknolojik uygulamalarda kullanılabilecek yeni malzeme ve yapıların tasarlanması mümkün olmaktadır. Fonksiyonel derecelendirilmiş malzemelerin mikro-elektro-mekanik sistemlerde kullanımı konusunda da araştırmalar devam etmektedir (Fu vd., 2003; Witrouw ve Mehta, 2005).

Homojen mikro-plaklar ile ilgili bilimsel araştırmalarda farklı mekanik problemleri ele alınmıştır. Bu araştırmaların bir kısmında modifiye edilmiş kuvvet çifti gerilmesi teorisi baz alınmış ve mikro-plakların statik eğilme, burkulma, ve serbest titreşim analizleri yapılmıştır (Akgöz ve Civelek, 2013; Asghari, 2012; Jomehzadeh vd., 2011; Ke vd., 2012a; Ma vd., 2011; Tsiatas, 2009; Wang vd., 2013; Yin vd., 2010). Bu incelemelerde formülasyon genellikle en temel plak teorisi olan Kirchhoff plak teorisine göre oluşturulmuş olmakla birlikte, Ke vd. (2012a) ve Ma vd. (2011) tarafından yürütülen çalışmalarda Mindlin plak teorisinin geçerli olduğu varsayılmıştır. Modifiye edilmiş kuvvet çifti gerilmesi teorisi dışında gerinim gradyanı elastisite teorisi de homojen mikro-plak analizinde kullanılmıştır (Lazopoulos, 2009; Ramezani, 2012; Ramezani, 2013; Wang vd., 2011). Yüksek dereceden sürekli ortam teorisi temel alınarak, fonksiyonel derecelendirilmiş mikro-plakların davranışları üzerine yapılmış bilimsel araştırmalar da bulunmaktadır. Bu araştırmaların çoğunda formülasyon modifiye edilmiş kuvvet çifti gerilmesi teorisine göre geliştirilmiştir. Bu teorinin kullanıldığı çalışmalara örnek olarak Ke vd. (2012b), Kim ve Reddy (2013), Reddy ve Kim (2012), Thai ve Choi (2013), Thai ve Kim (2013), ve Thai ve Vo (2013) tarafından yazılmış makaleler gösterilebilir. Sahmani ve Ansari (2013) tarafından FDM mikro-plaklar ile ilgili yapılan çalışmada ise gerinim gradyanı elastisite teorisi esas alınmıştır. FDM mikro-plaklar konusunda yapılan tüm bu incelemelerde Kirchhoff, Mindlin, ve üçüncü dereceden plak teorilerinden biri kullanılmıştır. Verilen sonuçlar uzunluk ölçeği parametrelerinin eğilme, burkulma, ve serbest titreşim davranışları üzerindeki etkilerini ortaya koymaktadır.

Homojen ve FDM mikro-plaklar üzerine yapılmış ve yukarıdaki paragrafta özetlenen çalışmalar incelendiğinde, bazı önemli noktaların göz önüne alınmadığı görülmektedir. Bunlardan birincisi termal yüklemedir. Termal yükleme gerek mikro-plakların bulunduğu ortamdaki sıcaklık değişiklikleri nedeniyle, gerekse de elektriksel etkiler sonucunda ortaya çıkabilir. Bu yükleme ve termal gerilmeler bir mikro-plağın mekanik davranışını önemli ölçüde değiştirmektedir. Termal yüklemenin mikro-kirişlerin hem statik eğilme hem de serbest titreşim davranışını göz ardı edilemeyecek bir biçimde değiştirdiği gösterilmiştir (Aghazadeh, 2013). Aynı etkinin mikro-plaklar için de geçerli olması beklenmelidir. Bu nedenle proje kapsamında, mikro-plakların incelenmesinde termal yüklemeler de ele alınmış; ve oluşabilecek sıcaklık farklarının statik deformasyon, gerilme dağılımı, burkulma yükü, ve doğal frekanslar üzerlerindeki etkileri araştırılmıştır.

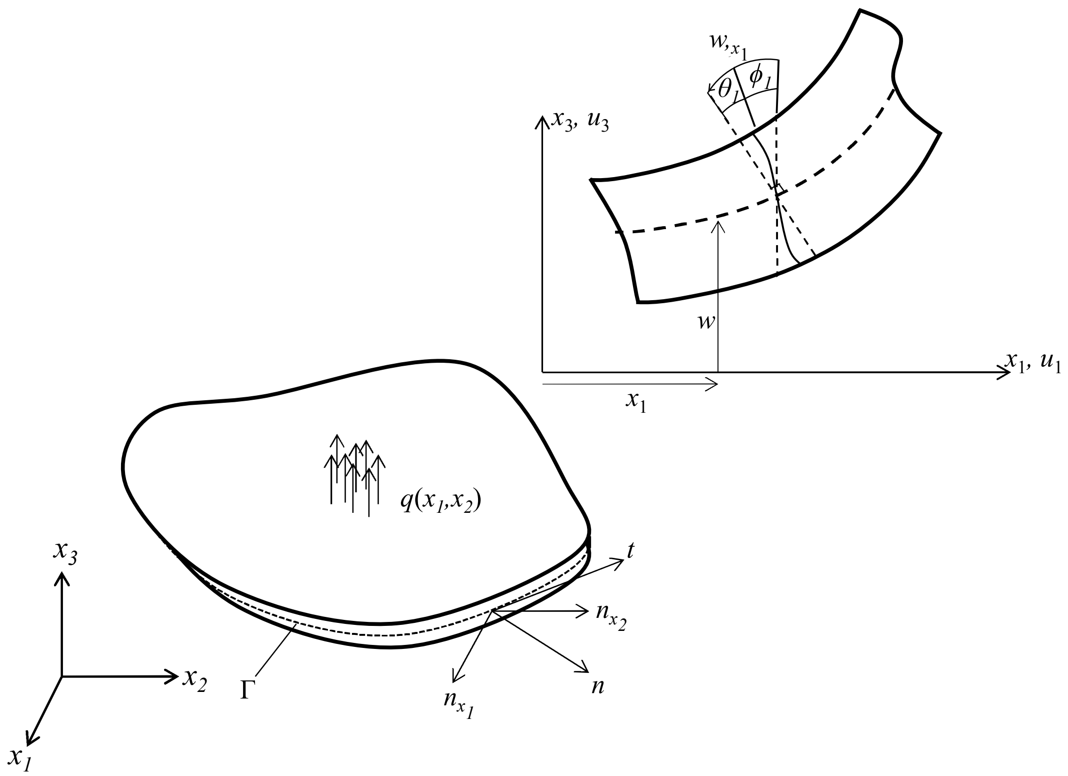
Literatürde bulunan çalışmalarda göz önüne alınmayan ikinci bir önemli nokta ise fonksiyonel derecelendirilmiş mikro-plaklarda uzunluk ölçeği parametrelerindeki uzaysal koordinatlara bağlı değişimlerdir. Literatür incelemesinde değinilen FDM mikro-plaklar ile ilgili makalelerin tamamında, uzunluk ölçeği parametrelerinin sabit oldukları varsayılmıştır. Ancak, derecelendirilmiş bir mikro-plak için bu doğru bir yaklaşım değildir. Uzunluk ölçeği parametresi de mekanik bir malzeme özelliği olduğundan, FDM'lerde diğer mekanik özellikler gibi uzaysal koordinatlara bağlı olarak değişmektedir. Araştırma projesi kapsamında, FDM mikro-plaklar için yapılan formülasyonda, uzunluk ölçeği parametrelerinin kalınlığa bağlı olarak değiştiği varsayılmış, ve mikro-plakların mekanik davranışı ile ilgili literatürde verilen sonuçlara göre daha gerçekçi sonuçlar elde edilmiştir.

Yürüttüğümüz çalışmalarla fonksiyonel derecelendirilmiş mikro-plaklar için hem termal etkilerin hem de uzunluk ölçeği parametrelerindeki değişimlerin göz önüne alındığı genel bir formülasyon geliştirilmiş, ve bağlaşık kısmi diferansiyel denklemler ile sınır koşulları türetilmiştir. Denklemler bulunurken gerinim gradyanı elastisite teorisi kullanılmıştır. Modifiye edilmiş kuvvet çifti gerilmesi teorisi, gerinim gradyanı elastisite teorisinin özel bir hali olduğundan elde edilen denklem sistemi çeşitli sadeleştirmelerle modifiye edilmiş kuvvet çifti teorisi için de uygulanabilmektedir. Denklemlerin sayısal çözümü için diferansiyel kare yapma metoduna (Shu, 2000) bağlı sayısal teknikler geliştirilmiştir. Statik eğilme, serbest titreşim, ve burkulma problemleri için geliştirdiğimiz sayısal çözüm yöntemleri MATLAB adlı matematik yazılımı içersine entegre edilmiştir. Ortaya koyduğumuz yöntemleri doğrulamak için, literatürde bulunan özel durumlar için geçerli hesaplamalı veriler kullanılmıştır. Yapılan karşılaştırmalar, proje çalışmaları ile oluşturduğumuz çözüm prosedürlerinin yüksek doğruluk derecesinde sonuç verdiğini kanıtlamıştır. Parametrik analizler ile termal yükleme, uzunluk ölçeği parametrelerindeki değişimler, malzeme özellikleri, ve geometrik özellikler gibi faktörlerin statik deformasyon, gerilme dağılımı, burkulma yükü, ve serbest titreşim doğal frekansları üzerlerindeki etkileri araştırılmıştır.

3. DİFERANSİYEL DENKLEMLER VE SINIR KOŞULLARININ BULUNMASI

3.1 Yerdeğiştirme Alanı ve Bünye Denklemleri

Şekil 1'de genel bir fonksiyonel derecelendirilmiş malzemeden (FDM) yapılmış mikro-plak konfigürasyonu gösterilmektedir. FDM mikro-plağın özellikleri x_3 yönünde değişmektedir. x_1 ve x_2 ise düzlem içi koordinatlardır.



Şekil 1. FDM mikro-plak konfigürasyonu.

Plağın herhangi bir noktasının t zamanında x_1, x_2 ve x_3 doğrultularında yerdeğiştirmeleri, sırasıyla u_1, u_2 ve u_3 ile gösterilmiştir. Yerdeğiştirme alanı aşağıdaki gibi ifade edilmiştir:

$$u_1\left(x_1, x_2, x_3, t\right) = u\left(x_1, x_2, t\right) - x_3 w_{,x_1} + f\left(x_3\right)\theta_1\left(x_1, x_2, t\right) \tag{1}$$

$$u_2\left(x_1, x_2, x_3, t\right) = v\left(x_1, x_2, t\right) - x_3 w_{,x_2} + f\left(x_3\right)\theta_2\left(x_1, x_2, t\right) \tag{2}$$

$$u_3\left(x_1, x_2, x_3, t\right) = w\left(x_1, x_2, t\right) \tag{3}$$

Burada virgül türevi temsil etmektedir; u, v ve w orta düzlemin sırasıyla x_1, x_2 ve x_3 yönlerinde yerdeğiştirmeleridir. θ_1 ve θ_2, orta düzlem üzerindeki (x_1, x_2) noktasının sırasıyla $x_1 - x_3$ ve $x_2 - x_3$ düzlemlerindeki kayma gerinimleridir. θ_1 ve θ_2 ile dönmeleri temsil eden ϕ_1 ve ϕ_2 arasındaki ilişkiler şu şekildedir:

$$\theta_1(x_1, x_2, t) = w_{,x_1}(x_1, x_2, t) + \phi_1(x_1, x_2, t), \qquad (4)$$

$$\theta_2(x_1, x_2, t) = w_{,x_2}(x_1, x_2, t) + \phi_2(x_1, x_2, t). \qquad (5)$$

Denklem (1) ve (2)'de f şekil fonksiyonudur ve plağın kalınlığı boyunca kayma geriniminin dağılımını belirlemektedir. Klasik plak teorisi (Kirchhoff teorisi) enine kayma gerinim etkisini içermez. Birinci derece kayma deformasyon teorisi (Mindlin teorisi) kayma geriniminin plağın kalınlığı boyunca sabit olduğunu varsaymaktadır. Yüksek derece kayma deformasyonu plak teorilerinde kayma gerinimi lineer olmayan bir şekilde x_3 koordinatına bağlıdır. Üçüncü derece kayma deformasyon teorisi en yaygın yüksek derece plak teorilerinden biridir. Kirchhoff, Mindlin ve üçüncü derece plak teorisi (Reddy plak teorisi) için f fonksiyonları aşağıdaki ifadeyle verilmektedir:

$$f(x_3) = \begin{cases} 0, & \text{Kirchhof plak teorisi için,} \\ x_3, & \text{Mindlin plak teorisi için,} \\ x_3\left(1 - \dfrac{4x_3^2}{3h^2}\right), & \text{üçüncü derece plak teorisi için.} \end{cases} \qquad (6)$$

Termal etki altında olan bir mikro-plak için gerinim gradyanı elastisite teorisine göre şekil değiştirme enerjisi

$$U = \frac{1}{2}\int_{\Omega}\left(\sigma_{ij}\left(\varepsilon_{ij} - \alpha \Delta T \delta_{ij}\right) + p_i \gamma_i + \tau_{ijk}^{(1)} \eta_{ijk}^{(1)} + m_{ij}^s \chi_{ij}^s\right) dV, \qquad (7)$$

formunda ifade edilir. Bu tanımda V plağın hacmini; δ_{ij} Kronecker deltayı; α ısıl genleşme katsayısını temsil eder. $\Delta T = T - T_o$ olarak tanımlıdır; burada T sıcaklık, T_o ise referans sıcaklığıdır. ε_{ij}, γ_i, $\eta_{ijk}^{(1)}$ ve χ_{ij}^{s} sırasıyla gerinim, dilatasyon gradyan, deviatorik streç gradyan ve dönme gradyanının simetrik tensörleridir. Bu değişkenler şu şekilde tanımlanırlar:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{8}$$

$$\gamma_i = \varepsilon_{mm,i}, \tag{9}$$

$$\eta_{ijk}^{(1)} = \frac{1}{3}\left(\varepsilon_{jk,i} + \varepsilon_{ki,j} + \varepsilon_{ij,k}\right) - \frac{1}{15}\delta_{ij}\left(\varepsilon_{mm,k} + 2\varepsilon_{mk,m}\right) - \frac{1}{15}\left\{\delta_{jk}\left(\varepsilon_{mm,i} + 2\varepsilon_{mi,m}\right) + \delta_{ki}\left(\varepsilon_{mm,j} + 2\varepsilon_{mj,m}\right)\right\}, \tag{10}$$

$$\chi_{ij}^{s} = \frac{1}{2}\left(e_{ipq}\varepsilon_{qj,p} + e_{jpq}\varepsilon_{qi,p}\right). \tag{11}$$

Bu denklemlerdeki e_{ijk} permutasyon sembolüdür. (7) numaralı eşitlikte, σ_{ij} klasik Cauchy gerilme tensörü ve p_i, $\tau_{ijk}^{(1)}$ ve m_{ij}^{s} yüksek derece gerilimlerdir. Gerinim gradyanı elastisite teorisine göre temel bünye denklemleri aşağıdaki gibi ifade edilir:

$$\sigma_{ij} = \lambda tr\left(\varepsilon\right)\delta_{ij} + 2\mu\varepsilon_{ij} - \alpha\left(3\lambda + 2\mu\right)\Delta T\delta_{ij}, \tag{12}$$

$$p_i = 2\mu l_0^2 \gamma_i, \tag{13}$$

$$\tau_{ijk}^{(1)} = 2\mu l_1^2 \eta_{ijk}^{(1)}, \tag{14}$$

$$m_{ij}^{s} = 2\mu l_2^2 \chi_{ij}^{s}. \tag{15}$$

(13)-(15) numaralı denklemlerdeki l_0, l_1, ve l_2 uzunluk ölçeği parametreleridir. Literatürdeki çalışmalarda uzunluk ölçeği parametrelerinin sabit oldukları varsayılmıştır. Ancak, bu parametreler malzeme özelliği olduğundan, FDM bir mikro-plak için diğer malzeme özellikleri gibi kalınlık koordinatı x_3 boyunca değişim göstermeleri gereklidir. Bu nedenle bu çalışmada l_0, l_1, ve l_2'nin x_3'e bağlı olarak değiştikleri kabul edilmiştir. Modifiye edilmiş kuvvet çifti teorisinde ise l_0=l_1=0 olarak alınır. Dolayısıyla gerinim gradyanı elastisite teorisi için geliştirilmiş sayısal çözüm algoritmasında l_0 ve l_1 sıfıra eşitlenerek, modifiye edilmiş kuvvet çifti teorisi ile ilgili sayısal sonuçları elde etmek mümkün olmaktadır.

(12) numaralı eşitlikteki λ ve μ Lame katsayılarıdır. Bu parametreler Young modülü E ve Poisson oranı υ ile şu şekilde bağıntılıdır:

$$\lambda = \frac{E\upsilon}{(1+\upsilon)(1-2\upsilon)}, \tag{16}$$

$$\mu = \frac{E}{2(1+\upsilon)}. \tag{17}$$

(1)-(3) ve (8)-(15) numaralı denklemler kullanılarak gerinim ve gerilmeler şu şekilde ifade edilmiştir:

$$\begin{aligned}
\varepsilon_{11} &= \frac{\partial u}{\partial x_1} - x_3 \frac{\partial^2 w}{\partial x_1^2} + f \frac{\partial \theta_1}{\partial x_1}, \\
\varepsilon_{22} &= \frac{\partial v}{\partial x_2} - x_3 \frac{\partial^2 w}{\partial x_2^2} + f \frac{\partial \theta_2}{\partial x_2}, \\
\varepsilon_{12} = \varepsilon_{21} &= \frac{1}{2}\frac{\partial u}{\partial x_2} + \frac{1}{2}\frac{\partial v}{\partial x_1} - x_3 \frac{\partial^2 w}{\partial x_1 \partial x_2} + \frac{1}{2} f \frac{\partial \theta_1}{\partial x_2} + \frac{1}{2} f \frac{\partial \theta_2}{\partial x_1}, \\
\varepsilon_{13} = \varepsilon_{31} &= \frac{1}{2} f' \theta_1, \\
\varepsilon_{23} = \varepsilon_{32} &= \frac{1}{2} f' \theta_2,
\end{aligned} \tag{18}$$

$$\begin{aligned}
\gamma_1 &= \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_1 \partial x_2} - x_3 \frac{\partial^3 w}{\partial x_1^3} - x_3 \frac{\partial^3 w}{\partial x_1 \partial x_2^2} + f \frac{\partial^2 \theta_1}{\partial x_1^2} + f \frac{\partial^2 \theta_2}{\partial x_1 \partial x_2}, \\
\gamma_2 &= \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 v}{\partial x_2^2} - x_3 \frac{\partial^3 w}{\partial x_2^3} - x_3 \frac{\partial^3 w}{\partial x_1^2 \partial x_2} + f \frac{\partial^2 \theta_1}{\partial x_1 \partial x_2} + f \frac{\partial^2 \theta_2}{\partial x_2^2}, \\
\gamma_3 &= -\frac{\partial^2 w}{\partial x_1^2} - \frac{\partial^2 w}{\partial x_2^2} + f' \frac{\partial \theta_1}{\partial x_1} + f' \frac{\partial \theta_2}{\partial x_2},
\end{aligned} \tag{19}$$

$$\eta_{111}^{(1)} = \frac{2}{5}\left(\frac{\partial^2 u}{\partial x_1^2} - \frac{1}{2}\frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 v}{\partial x_1 \partial x_2} - x_3 \frac{\partial^3 w}{\partial x_1^3} + \frac{3}{2} x_3 \frac{\partial^3 w}{\partial x_1 \partial x_2^2} + f \frac{\partial^2 \theta_1}{\partial x_1^2} - \frac{1}{2} f \frac{\partial^2 \theta_1}{\partial x_2^2} - \frac{1}{2} f'' \theta_1 - f \frac{\partial^2 \theta_2}{\partial x_1 \partial x_2} \right)$$

$$\eta_{222}^{(1)} = \frac{2}{5}\left(-\frac{\partial^2 u}{\partial x_1 \partial x_2} - \frac{1}{2}\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} - x_3 \frac{\partial^3 w}{\partial x_2^3} + \frac{3}{2} x_3 \frac{\partial^3 w}{\partial x_1^2 \partial x_2} - f \frac{\partial^2 \theta_1}{\partial x_1 \partial x_2} - \frac{1}{2} f \frac{\partial^2 \theta_2}{\partial x_1^2} + f \frac{\partial^2 \theta_2}{\partial x_2^2} - \frac{1}{2} f'' \theta_2 \right)$$

$$\eta_{333}^{(1)} = \frac{1}{5}\left(\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} - 2 f' \frac{\partial \theta_1}{\partial x_1} - 2 f' \frac{\partial \theta_2}{\partial x_2} \right)$$

$$\eta_{112}^{(1)} = \eta_{211}^{(1)} = \eta_{121}^{(1)}$$
$$= \frac{1}{15}\left(8 \frac{\partial^2 u}{\partial x_1 \partial x_2} + 4 \frac{\partial^2 v}{\partial x_1^2} - 3 \frac{\partial^2 v}{\partial x_2^2} + 3 x_3 \frac{\partial^3 w}{\partial x_2^3} - 12 x_3 \frac{\partial^3 w}{\partial x_1^2 \partial x_2} + 8 f \frac{\partial^2 \theta_1}{\partial x_1 \partial x_2} + 4 f \frac{\partial^2 \theta_2}{\partial x_1^2} - 3 f \frac{\partial^2 \theta_2}{\partial x_2^2} - f'' \theta_2 \right)$$

$$\eta_{113}^{(1)} = \eta_{311}^{(1)} = \eta_{131}^{(1)} = -\frac{1}{15}\left(4 \frac{\partial^2 w}{\partial x_1^2} - \frac{\partial^2 w}{\partial x_2^2} - 8 f' \frac{\partial \theta_1}{\partial x_1} + 2 f' \frac{\partial \theta_2}{\partial x_2} \right)$$

$$\eta_{221}^{(1)} = \eta_{122}^{(1)} = \eta_{212}^{(1)}$$
$$= \frac{1}{15}\left(-3 \frac{\partial^2 u}{\partial x_1^2} + 4 \frac{\partial^2 u}{\partial x_2^2} + 8 \frac{\partial^2 v}{\partial x_1 \partial x_2} + 3 x_3 \frac{\partial^3 w}{\partial x_1^3} - 12 x_3 \frac{\partial^3 w}{\partial x_1 \partial x_2^2} - 3 f \frac{\partial^2 \theta_1}{\partial x_1^2} + 4 f \frac{\partial^2 \theta_1}{\partial x_2^2} - f'' \theta_1 + 8 f \frac{\partial^2 \theta_2}{\partial x_1 \partial x_2} \right)$$

$$\eta_{223}^{(1)} = \eta_{322}^{(1)} = \eta_{232}^{(1)} = -\frac{1}{15}\left(-\frac{\partial^2 w}{\partial x_1^2} + 4 \frac{\partial^2 w}{\partial x_2^2} + 2 f' \frac{\partial \theta_1}{\partial x_1} - 8 f' \frac{\partial \theta_2}{\partial x_2} \right)$$

$$\eta_{331}^{(1)} = \eta_{133}^{(1)} = \eta_{313}^{(1)}$$
$$= -\frac{1}{15}\left(3 \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + 2 \frac{\partial^2 v}{\partial x_1 \partial x_2} - 3 x_3 \frac{\partial^3 w}{\partial x_1^3} - 3 x_3 \frac{\partial^3 w}{\partial x_1 \partial x_2^2} + 3 f \frac{\partial^2 \theta_1}{\partial x_1^2} + f \frac{\partial^2 \theta_1}{\partial x_2^2} - 4 f'' \theta_1 + 2 f \frac{\partial^2 \theta_2}{\partial x_1 \partial x_2} \right)$$

$$\eta_{332}^{(1)} = \eta_{233}^{(1)} = \eta_{323}^{(1)}$$
$$= -\frac{1}{15}\left(2 \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 v}{\partial x_1^2} + 3 \frac{\partial^2 v}{\partial x_2^2} - 3 x_3 \frac{\partial^3 w}{\partial x_2^3} - 3 x_3 \frac{\partial^3 w}{\partial x_1^2 \partial x_2} + 2 f \frac{\partial^2 \theta_1}{\partial x_1 \partial x_2} + f \frac{\partial^2 \theta_2}{\partial x_1^2} + 3 f \frac{\partial^2 \theta_2}{\partial x_2^2} - 4 f'' \theta_2 \right)$$

$$\eta_{123}^{(1)} = \eta_{312}^{(1)} = \eta_{231}^{(1)} = \eta_{132}^{(1)} = \eta_{213}^{(1)} = \eta_{321}^{(1)} = \frac{1}{3}\left(-\frac{\partial^2 w}{\partial x_1 \partial x_2} + f' \frac{\partial \theta_1}{\partial x_2} + f' \frac{\partial \theta_2}{\partial x_1} \right)$$

(20)

$$\chi_{11}^{s}=\frac{1}{2}\left(2\frac{\partial^2 w}{\partial x_1\partial x_2}-f'\frac{\partial\theta_2}{\partial x_1}\right),$$

$$\chi_{22}^{s}=-\frac{1}{2}\left(2\frac{\partial^2 w}{\partial x_1\partial x_2}-f'\frac{\partial\theta_1}{\partial x_2}\right),$$

$$\chi_{33}^{s}=-\frac{1}{2}\left(f'\frac{\partial\theta_1}{\partial x_2}-f'\frac{\partial\theta_2}{\partial x_1}\right),$$

$$\chi_{12}^{s}=\chi_{21}^{s}=\frac{1}{4}\left(-2\frac{\partial^2 w}{\partial x_1^2}+2\frac{\partial^2 w}{\partial x_2^2}+f'\frac{\partial\theta_1}{\partial x_1}-f'\frac{\partial\theta_2}{\partial x_2}\right), \tag{21}$$

$$\chi_{13}^{s}=\chi_{31}^{s}=\frac{1}{4}\left(-\frac{\partial^2 u}{\partial x_1\partial x_2}+\frac{\partial^2 v}{\partial x_1^2}-f\frac{\partial^2\theta_1}{\partial x_1\partial x_2}+f\frac{\partial^2\theta_2}{\partial x_1^2}-f''\theta_2\right),$$

$$\chi_{23}^{s}=\chi_{32}^{s}=\frac{1}{4}\left(-\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 v}{\partial x_1\partial x_2}-f\frac{\partial^2\theta_1}{\partial x_2^2}+f''\theta_1+f\frac{\partial^2\theta_2}{\partial x_1\partial x_2}\right),$$

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\\ \sigma_{13}\\ \sigma_{23}\end{Bmatrix}=\frac{E}{1-\upsilon^2}\begin{bmatrix}1 & \upsilon & 0 & 0 & 0\\ \upsilon & 1 & 0 & 0 & 0\\ 0 & 0 & 1-\upsilon & 0 & 0\\ 0 & 0 & 0 & k_s\left(1-\upsilon\right) & 0\\ 0 & 0 & 0 & 0 & k_s\left(1-\upsilon\right)\end{bmatrix}\begin{Bmatrix}\varepsilon_{11}-\alpha\Delta T\\ \varepsilon_{22}-\alpha\Delta T\\ \varepsilon_{12}\\ \varepsilon_{13}\\ \varepsilon_{23}\end{Bmatrix}, \tag{22}$$

$$p_1=2\mu l_0^2\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 v}{\partial x_1\partial x_2}-x_3\frac{\partial^3 w}{\partial x_1^3}-x_3\frac{\partial^3 w}{\partial x_1\partial x_2^2}+f\frac{\partial^2\theta_1}{\partial x_1^2}+f\frac{\partial^2\theta_2}{\partial x_1\partial x_2}\right),$$

$$p_2=2\mu l_0^2\left(\frac{\partial^2 u}{\partial x_1\partial x_2}+\frac{\partial^2 v}{\partial x_2^2}-x_3\frac{\partial^3 w}{\partial x_2^3}-x_3\frac{\partial^3 w}{\partial x_1^2\partial x_2}+f\frac{\partial^2\theta_1}{\partial x_1\partial x_2}+f\frac{\partial^2\theta_2}{\partial x_2^2}\right), \tag{23}$$

$$p_3=2\mu l_0^2\left(-\frac{\partial^2 w}{\partial x_1^2}-\frac{\partial^2 w}{\partial x_2^2}+f'\frac{\partial\theta_1}{\partial x_1}+f'\frac{\partial\theta_2}{\partial x_2}\right),$$

$$\tau_{111}^{(1)} = \frac{4}{5}\mu l_1^2 \left(\frac{\partial^2 u}{\partial x_1^2} - \frac{1}{2}\frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 v}{\partial x_1 \partial x_2} - x_3 \frac{\partial^3 w}{\partial x_1^3} + \frac{3}{2} x_3 \frac{\partial^3 w}{\partial x_1 \partial x_2^2} + f \frac{\partial^2 \theta_1}{\partial x_1^2} - \frac{1}{2} f \frac{\partial^2 \theta_1}{\partial x_2^2} - \frac{1}{2} f'' \theta_1 - f \frac{\partial^2 \theta_2}{\partial x_1 \partial x_2} \right),$$

$$\tau_{222}^{(1)} = \frac{4}{5}\mu l_1^2 \left(-\frac{\partial^2 u}{\partial x_1 \partial x_2} - \frac{1}{2}\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} - x_3 \frac{\partial^3 w}{\partial x_2^3} + \frac{3}{2} x_3 \frac{\partial^3 w}{\partial x_1^2 \partial x_2} - f \frac{\partial^2 \theta_1}{\partial x_1 \partial x_2} - \frac{1}{2} f \frac{\partial^2 \theta_2}{\partial x_1^2} + f \frac{\partial^2 \theta_2}{\partial x_2^2} - \frac{1}{2} f'' \theta_2 \right),$$

$$\tau_{333}^{(1)} = \frac{2}{5}\mu l_1^2 \left(\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} - 2 f' \frac{\partial \theta_1}{\partial x_1} - 2 f' \frac{\partial \theta_2}{\partial x_2} \right),$$

$$\tau_{112}^{(1)} = \tau_{211}^{(1)} = \tau_{121}^{(1)}$$

$$= \frac{2}{15}\mu l_1^2 \left(8\frac{\partial^2 u}{\partial x_1 \partial x_2} + 4\frac{\partial^2 v}{\partial x_1^2} - 3\frac{\partial^2 v}{\partial x_2^2} + 3x_3 \frac{\partial^3 w}{\partial x_2^3} - 12 x_3 \frac{\partial^3 w}{\partial x_1^2 \partial x_2} + 8 f \frac{\partial^2 \theta_1}{\partial x_1 \partial x_2} + 4 f \frac{\partial^2 \theta_2}{\partial x_1^2} - 3 f \frac{\partial^2 \theta_2}{\partial x_2^2} - f'' \theta_2 \right),$$

$$\tau_{113}^{(1)} = \tau_{311}^{(1)} = \tau_{131}^{(1)} = -\frac{2}{15}\mu l_1^2 \left(4\frac{\partial^2 w}{\partial x_1^2} - \frac{\partial^2 w}{\partial x_2^2} - 8 f' \frac{\partial \theta_1}{\partial x_1} + 2 f' \frac{\partial \theta_2}{\partial x_2} \right),$$

$$\tau_{221}^{(1)} = \tau_{122}^{(1)} = \tau_{212}^{(1)}$$

$$= \frac{2}{15}\mu l_1^2 \left(-3\frac{\partial^2 u}{\partial x_1^2} + 4\frac{\partial^2 u}{\partial x_2^2} + 8\frac{\partial^2 v}{\partial x_1 \partial x_2} + 3x_3 \frac{\partial^3 w}{\partial x_1^3} - 12 x_3 \frac{\partial^3 w}{\partial x_1 \partial x_2^2} - 3 f \frac{\partial^2 \theta_1}{\partial x_1^2} + 4 f \frac{\partial^2 \theta_1}{\partial x_2^2} - f'' \theta_1 + 8 f \frac{\partial^2 \theta_2}{\partial x_1 \partial x_2} \right),$$

$$\tau_{223}^{(1)} = \tau_{322}^{(1)} = \tau_{232}^{(1)} = -\frac{2}{15}\mu l_1^2 \left(-\frac{\partial^2 w}{\partial x_1^2} + 4\frac{\partial^2 w}{\partial x_2^2} + 2 f' \frac{\partial \theta_1}{\partial x_1} - 8 f' \frac{\partial \theta_2}{\partial x_2} \right),$$

$$\tau_{331}^{(1)} = \tau_{133}^{(1)} = \tau_{313}^{(1)}$$

$$= -\frac{2}{15}\mu l_1^2 \left(3\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + 2\frac{\partial^2 v}{\partial x_1 \partial x_2} - 3x_3 \frac{\partial^3 w}{\partial x_1^3} - 3 x_3 \frac{\partial^3 w}{\partial x_1 \partial x_2^2} + 3 f \frac{\partial^2 \theta_1}{\partial x_1^2} + f \frac{\partial^2 \theta_1}{\partial x_2^2} - 4 f'' \theta_1 + 2 f \frac{\partial^2 \theta_2}{\partial x_1 \partial x_2} \right),$$

$$\tau_{332}^{(1)} = \tau_{233}^{(1)} = \tau_{323}^{(1)}$$

$$= -\frac{2}{15}\mu l_1^2 \left(2\frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 v}{\partial x_1^2} + 3\frac{\partial^2 v}{\partial x_2^2} - 3x_3 \frac{\partial^3 w}{\partial x_2^3} - 3 x_3 \frac{\partial^3 w}{\partial x_1^2 \partial x_2} + 2 f \frac{\partial^2 \theta_1}{\partial x_1 \partial x_2} + f \frac{\partial^2 \theta_2}{\partial x_1^2} + 3 f \frac{\partial^2 \theta_2}{\partial x_2^2} - 4 f'' \theta_2 \right),$$

$$\tau_{123}^{(1)} = \tau_{312}^{(1)} = \tau_{231}^{(1)} = \tau_{132}^{(1)} = \tau_{213}^{(1)} = \tau_{321}^{(1)} = \frac{2}{3}\mu l_1^2 \left(-\frac{\partial^2 w}{\partial x_1 \partial x_2} + f' \frac{\partial \theta_1}{\partial x_2} + f' \frac{\partial \theta_2}{\partial x_1} \right), \tag{24}$$

$$
\begin{aligned}
m_{11}^{s} &= \mu l_2^2\left(2\frac{\partial^2 w}{\partial x_1 \partial x_2} - f'\frac{\partial \theta_2}{\partial x_1}\right),\\
m_{22}^{s} &= -\mu l_2^2\left(2\frac{\partial^2 w}{\partial x_1 \partial x_2} - f'\frac{\partial \theta_1}{\partial x_2}\right),\\
m_{33}^{s} &= -\mu l_2^2\left(f'\frac{\partial \theta_1}{\partial x_2} - f'\frac{\partial \theta_2}{\partial x_1}\right),\\
m_{12}^{s} = m_{21}^{s} &= \frac{1}{2}\mu l_2^2\left(-2\frac{\partial^2 w}{\partial x_1^2} + 2\frac{\partial^2 w}{\partial x_2^2} + f'\frac{\partial \theta_1}{\partial x_1} - f'\frac{\partial \theta_2}{\partial x_2}\right),\\
m_{13}^{s} = m_{31}^{s} &= \frac{1}{2}\mu l_2^2\left(-\frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 v}{\partial x_1^2} - f\frac{\partial^2 \theta_1}{\partial x_1 \partial x_2} + f\frac{\partial^2 \theta_2}{\partial x_1^2} - f''\theta_2\right),\\
m_{23}^{s} = m_{32}^{s} &= \frac{1}{2}\mu l_2^2\left(-\frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 v}{\partial x_1 \partial x_2} - f\frac{\partial^2 \theta_1}{\partial x_2^2} + f''\theta_1 + f\frac{\partial^2 \theta_2}{\partial x_1 \partial x_2}\right).
\end{aligned}
\tag{25}
$$

(22) numaralı denklemde k_s kayma gerilmesinin düzeltme faktörüdür ve Mindlin teorisinde dikdörtgen kesite sahip plaklar için genellikle 5/6 olarak alınmaktadır. Denklem (22)'de düzlem gerilme varsayımı kullanılmıştır. Düzlem gerilme varsayımında (16)'da verilen eşitlik yerine, şu tanım kullanılmaktadır:

$$
\lambda = \frac{E\upsilon}{\left(1-\upsilon^2\right)} \tag{26}
$$

3.2 Kısmi Diferansiyel Denklemler ve Sınır Koşulları

Kısmi diferansiyel denklemler ve sınır koşulları Hamilton prensibi kullanılarak türetilmiştir. Hamilton prensibi varyasyonel yönteme dayalıdır ve aşağıdaki gibi ifade edilir:

$$
\delta\int_{t_1}^{t_2}\left(K-\left(U-W\right)\right)dt = 0. \tag{27}
$$

Burada K kinetik enerji, U şekil değiştirme enerjisi ve W uygulanan kuvvetlerin yaptığı iştir. Şekil değiştirme enerjisinin varyasyonu, (18)-(25) numaralı denklemler kullanılarak aşağıdaki gibi yazılabilir:

$$
\delta U = \int_{A}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\sigma_{ij}\delta\varepsilon_{ij} + p_i\delta\gamma_i + \tau_{ijk}^{(1)}\delta\eta_{ijk}^{(1)} + m_{ij}^{s}\delta\chi_{ij}^{s}\right)dx_3 dA = \delta U_1 + \delta U_2 + \delta U_3 + \delta U_4, \tag{28}
$$

$$\delta U_1 = \iint\limits_{A}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ij}\delta\varepsilon_{ij}dx_3 dA = \iint\limits_{A}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\sigma_{11}\delta\varepsilon_{11} + \sigma_{22}\delta\varepsilon_{22} + 2\sigma_{12}\delta\varepsilon_{12} + 2\sigma_{13}\delta\varepsilon_{13} + 2\sigma_{23}\delta\varepsilon_{23}\right)dx_3 dA$$

$$= \iint\limits_{A}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{\sigma_{11}\delta\left(\frac{\partial u}{\partial x_1} - x_3\frac{\partial^2 w}{\partial x_1^2} + f\frac{\partial\theta_1}{\partial x_1}\right) + \sigma_{22}\delta\left(\frac{\partial v}{\partial x_2} - x_3\frac{\partial^2 w}{\partial x_2^2} + f\frac{\partial\theta_2}{\partial x_2}\right)\right.$$

$$\left. + \sigma_{12}\delta\left(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} - 2x_3\frac{\partial^2 w}{\partial x_1\partial x_2} + f\frac{\partial\theta_1}{\partial x_2} + f\frac{\partial\theta_2}{\partial x_1}\right) + \sigma_{13}\delta\left(f'\theta_1\right) + \sigma_{23}\delta\left(f'\theta_2\right)\right\}dx_3 dA, \tag{29}$$

$$\delta U_2 = \iint\limits_{A}\int_{-\frac{h}{2}}^{\frac{h}{2}} p_i\delta\gamma_i dx_3 dA = \iint\limits_{A}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(p_1\delta\gamma_1 + p_2\delta\gamma_2 + p_3\delta\gamma_3\right)dx_3 dA$$

$$= \iint\limits_{A}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left\{p_1\delta\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_1\partial x_2} - x_3\frac{\partial^3 w}{\partial x_1^3} - x_3\frac{\partial^3 w}{\partial x_1\partial x_2^2} + f\frac{\partial^2\theta_1}{\partial x_1^2} + f\frac{\partial^2\theta_2}{\partial x_1\partial x_2}\right)\right.$$

$$+ p_2\delta\left(\frac{\partial^2 u}{\partial x_1\partial x_2} + \frac{\partial^2 v}{\partial x_2^2} - x_3\frac{\partial^3 w}{\partial x_2^3} - x_3\frac{\partial^3 w}{\partial x_1^2\partial x_2} + f\frac{\partial^2\theta_1}{\partial x_1\partial x_2} + f\frac{\partial^2\theta_2}{\partial x_2^2}\right) \tag{30}$$

$$\left. + p_3\delta\left(-\frac{\partial^2 w}{\partial x_1^2} - \frac{\partial^2 w}{\partial x_2^2} + f'\frac{\partial\theta_1}{\partial x_1} + f'\frac{\partial\theta_2}{\partial x_2}\right)\right\}dx_3 dA,$$

$$\delta U_3=\iint_{A}\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{ijk}^{(1)}\delta\eta_{ijk}^{(1)}dx_3dA=\iint_{A}\int_{-\frac{h}{2}}^{\frac{h}{2}}\Big(\tau_{111}^{(1)}\delta\eta_{111}^{(1)}+\tau_{222}^{(1)}\delta\eta_{222}^{(1)}+\tau_{333}^{(1)}\delta\eta_{333}^{(1)}+3\tau_{112}^{(1)}\delta\eta_{112}^{(1)}+3\tau_{113}^{(1)}\delta\eta_{113}^{(1)}$$
$$+3\tau_{221}^{(1)}\delta\eta_{221}^{(1)}+3\tau_{223}^{(1)}\delta\eta_{223}^{(1)}++3\tau_{331}^{(1)}\delta\eta_{331}^{(1)}+3\tau_{332}^{(1)}\delta\eta_{332}^{(1)}+6\tau_{123}^{(1)}\delta\eta_{123}^{(1)}\Big)dx_3dA$$
$$=\iint_{A}\int_{-\frac{h}{2}}^{\frac{h}{2}}\Bigg\{\frac{2}{5}\tau_{111}^{(1)}\delta\left(\frac{\partial^2 u}{\partial x_1^2}-\frac{1}{2}\frac{\partial^2 u}{\partial x_2^2}-\frac{\partial^2 v}{\partial x_1\partial x_2}-x_3\frac{\partial^3 w}{\partial x_1^3}+\frac{3}{2}x_3\frac{\partial^3 w}{\partial x_1\partial x_2^2}+f\frac{\partial^2\theta_1}{\partial x_1^2}-\frac{1}{2}f\frac{\partial^2\theta_1}{\partial x_2^2}-\frac{1}{2}f''\theta_1-f\frac{\partial^2\theta_2}{\partial x_1\partial x_2}\right)$$
$$+\frac{2}{5}\tau_{222}^{(1)}\delta\left(-\frac{\partial^2 u}{\partial x_1\partial x_2}-\frac{1}{2}\frac{\partial^2 v}{\partial x_1^2}+\frac{\partial^2 v}{\partial x_2^2}-x_3\frac{\partial^3 w}{\partial x_2^3}+\frac{3}{2}x_3\frac{\partial^3 w}{\partial x_1^2\partial x_2}-f\frac{\partial^2\theta_1}{\partial x_1\partial x_2}-\frac{1}{2}f\frac{\partial^2\theta_2}{\partial x_1^2}+f\frac{\partial^2\theta_2}{\partial x_2^2}-\frac{1}{2}f''\theta_2\right)$$
$$+\frac{1}{5}\tau_{333}^{(1)}\delta\left(\frac{\partial^2 w}{\partial x_1^2}+\frac{\partial^2 w}{\partial x_2^2}-2f'\frac{\partial\theta_1}{\partial x_1}-2f'\frac{\partial\theta_2}{\partial x_2}\right)$$
$$+\frac{1}{5}\tau_{112}^{(1)}\delta\left(8\frac{\partial^2 u}{\partial x_1\partial x_2}+4\frac{\partial^2 v}{\partial x_1^2}-3\frac{\partial^2 v}{\partial x_2^2}+3x_3\frac{\partial^3 w}{\partial x_2^3}-12x_3\frac{\partial^3 w}{\partial x_1^2\partial x_2}+8f\frac{\partial^2\theta_1}{\partial x_1\partial x_2}+4f\frac{\partial^2\theta_2}{\partial x_1^2}-3f\frac{\partial^2\theta_2}{\partial x_2^2}-f''\theta_2\right)$$
$$-\frac{1}{5}\tau_{113}^{(1)}\delta\left(4\frac{\partial^2 w}{\partial x_1^2}-\frac{\partial^2 w}{\partial x_2^2}-8f'\frac{\partial\theta_1}{\partial x_1}+2f'\frac{\partial\theta_2}{\partial x_2}\right)$$
$$+\frac{1}{5}\tau_{221}^{(1)}\delta\left(-3\frac{\partial^2 u}{\partial x_1^2}+4\frac{\partial^2 u}{\partial x_2^2}+8\frac{\partial^2 v}{\partial x_1\partial x_2}+3x_3\frac{\partial^3 w}{\partial x_1^3}-12x_3\frac{\partial^3 w}{\partial x_1\partial x_2^2}-3f\frac{\partial^2\theta_1}{\partial x_1^2}+4f\frac{\partial^2\theta_1}{\partial x_2^2}-f''\theta_1+8f\frac{\partial^2\theta_2}{\partial x_1\partial x_2}\right)$$
$$-\frac{1}{5}\tau_{223}^{(1)}\delta\left(-\frac{\partial^2 w}{\partial x_1^2}+4\frac{\partial^2 w}{\partial x_2^2}+2f'\frac{\partial\theta_1}{\partial x_1}-8f'\frac{\partial\theta_2}{\partial x_2}\right)$$
$$-\frac{1}{5}\tau_{331}^{(1)}\delta\left(3\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+2\frac{\partial^2 v}{\partial x_1\partial x_2}-3x_3\frac{\partial^3 w}{\partial x_1^3}-3x_3\frac{\partial^3 w}{\partial x_1\partial x_2^2}+3f\frac{\partial^2\theta_1}{\partial x_1^2}+f\frac{\partial^2\theta_1}{\partial x_2^2}-4f''\theta_1+2f\frac{\partial^2\theta_2}{\partial x_1\partial x_2}\right)$$
$$-\frac{1}{5}\tau_{332}^{(1)}\delta\left(2\frac{\partial^2 u}{\partial x_1\partial x_2}+\frac{\partial^2 v}{\partial x_1^2}+3\frac{\partial^2 v}{\partial x_2^2}-3x_3\frac{\partial^3 w}{\partial x_2^3}-3x_3\frac{\partial^3 w}{\partial x_1^2\partial x_2}+2f\frac{\partial^2\theta_1}{\partial x_1\partial x_2}+f\frac{\partial^2\theta_2}{\partial x_1^2}+3f\frac{\partial^2\theta_2}{\partial x_2^2}-4f''\theta_2\right)$$
$$+2\tau_{123}^{(1)}\delta\left(-\frac{\partial^2 w}{\partial x_1\partial x_2}+f'\frac{\partial\theta_1}{\partial x_2}+f'\frac{\partial\theta_2}{\partial x_1}\right)\Bigg\}dx_3dA,\qquad (31)$$

$$\delta U_4=\iint_{A}\int_{-\frac{h}{2}}^{\frac{h}{2}}m_{ij}^{s}\delta\chi_{ij}^{s}dx_3dA$$
$$=\iint_{A}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(m_{11}^{s}\delta\chi_{11}^{s}+m_{22}^{s}\delta\chi_{22}^{s}+m_{33}^{s}\delta\chi_{33}^{s}+2m_{12}^{s}\delta\chi_{12}^{s}+2m_{13}^{s}\delta\chi_{13}^{s}+2m_{23}^{s}\delta\chi_{23}^{s}\right)dx_3dA$$
$$=\iint_{A}\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{1}{2}\Bigg\{m_{11}^{s}\delta\left(2\frac{\partial^2 w}{\partial x_1\partial x_2}-f'\frac{\partial\theta_2}{\partial x_1}\right)-m_{22}^{s}\delta\left(2\frac{\partial^2 w}{\partial x_1\partial x_2}-f'\frac{\partial\theta_1}{\partial x_2}\right)-m_{33}^{s}\delta\left(f'\frac{\partial\theta_1}{\partial x_2}-f'\frac{\partial\theta_2}{\partial x_1}\right)$$
$$+m_{12}^{s}\delta\left(-2\frac{\partial^2 w}{\partial x_1^2}+2\frac{\partial^2 w}{\partial x_2^2}+f'\frac{\partial\theta_1}{\partial x_1}-f'\frac{\partial\theta_2}{\partial x_2}\right)+m_{13}^{s}\delta\left(-\frac{\partial^2 u}{\partial x_1\partial x_2}+\frac{\partial^2 v}{\partial x_1^2}-f\frac{\partial^2\theta_1}{\partial x_1\partial x_2}+f\frac{\partial^2\theta_2}{\partial x_1^2}-f''\theta_2\right)$$
$$+m_{23}^{s}\delta\left(-\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 v}{\partial x_1\partial x_2}-f\frac{\partial^2\theta_1}{\partial x_2^2}+f''\theta_1+f\frac{\partial^2\theta_2}{\partial x_1\partial x_2}\right)\Bigg\}dx_3dA.\qquad (32)$$

Bu denklemlerde h mikro-plağın kalınlığını ve A ise orta düzlemin alanını temsil etmektedir. Gerilmelerin kalınlık boyunca integralleri alınarak aşağıda verilen kuvvet ve momentler tanımlanmıştır:

$$M_{11}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{11}\zeta_i dx_3,\ (i=0,\ 1,\ 2),\quad M_{22}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{22}\zeta_i dx_3,\ (i=0,\ 1,\ 2),$$
$$M_{12}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{12}\zeta_i dx_3,\ (i=0,\ 1,\ 2),\quad M_{13}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{13}\zeta_i dx_3,\ (i=3),\tag{33}$$
$$M_{23}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{23}\zeta_i dx_3,\ (i=3),$$

$$P_1^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}p_1\zeta_i dx_3,\ (i=0,\ 1,\ 2),\quad P_2^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}p_2\zeta_i dx_3,\ (i=0,\ 1,\ 2),\quad P_3^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}p_3\zeta_i dx_3,\ (i=0,\ 3),\tag{34}$$

$$T_{111}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{111}^{(1)}\zeta_i dx_3,\ (i=0,\ 1,\ 2,\ 4),\quad T_{222}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{222}^{(1)}\zeta_i dx_3,\ (i=0,\ 1,\ 2,\ 4),$$
$$T_{333}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{333}^{(1)}\zeta_i dx_3,\ (i=0,\ 3),\quad T_{112}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{112}^{(1)}\zeta_i dx_3,\ (i=0,\ 1,\ 2,\ 4),$$
$$T_{113}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{113}^{(1)}\zeta_i dx_3,\ (i=0,\ 3),\quad T_{221}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{221}^{(1)}\zeta_i dx_3,\ (i=0,\ 1,\ 2,\ 4),\tag{35}$$
$$T_{223}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{223}^{(1)}\zeta_i dx_3,\ (i=0,\ 3),\quad T_{331}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{331}^{(1)}\zeta_i dx_3,\ (i=0,\ 1,\ 2,\ 4),$$
$$T_{332}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{332}^{(1)}\zeta_i dx_3,\ (i=0,\ 1,\ 2,\ 4),\quad T_{123}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{123}^{(1)}\zeta_i dx_3,\ (i=0,\ 3),$$

$$Y_{11}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}m_{11}^{s}\zeta_i dx_3,\ (i=0,\ 3),\quad Y_{22}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}m_{22}^{s}\zeta_i dx_3,\ (i=0,\ 3),\quad Y_{33}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}m_{33}^{s}\zeta_i dx_3,\ (i=3),$$
$$Y_{12}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}m_{12}^{s}\zeta_i dx_3,\ (i=0,\ 3),\quad Y_{13}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}m_{13}^{s}\zeta_i dx_3,\ (i=0,\ 2,\ 4),\tag{36}$$
$$Y_{23}^{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}m_{23}^{s}\zeta_i dx_3,\ (i=0,\ 2,\ 4).$$

Burada $\zeta_0=1,\ \zeta_1=x_3,\ \zeta_2=f,\ \zeta_3=f',$ ve $\zeta_4=f''$ olarak tanımlıdır. Yukarıdaki ifadeler kullanılarak enerji varyasyonları aşağıdaki gibi ifade edilmiştir:

$$\delta U_1 = \int_A \left\{ \left(M_{11}^0 \delta \frac{\partial u}{\partial x_1} - M_{11}^1 \delta \frac{\partial^2 w}{\partial x_1^2} + M_{11}^2 \delta \frac{\partial \theta_1}{\partial x_1} \right) + \left(M_{22}^0 \delta \frac{\partial v}{\partial x_2} - M_{22}^1 \delta \frac{\partial^2 w}{\partial x_2^2} + M_{22}^2 \delta \frac{\partial \theta_2}{\partial x_2} \right) \right.$$

$$\left. + \left(M_{12}^0 \delta \frac{\partial u}{\partial x_2} + M_{12}^0 \delta \frac{\partial v}{\partial x_1} - 2M_{12}^1 \delta \frac{\partial^2 w}{\partial x_1 \partial x_2} + M_{12}^2 \delta \frac{\partial \theta_1}{\partial x_2} + M_{12}^2 \delta \frac{\partial \theta_2}{\partial x_1} \right) + M_{13}^3 \delta\theta_1 + M_{23}^3 \delta\theta_2 \right\} dA, \tag{37}$$

$$\delta U_2 = \int_A \left(P_1^0 \delta \frac{\partial^2 u}{\partial x_1^2} + P_1^0 \delta \frac{\partial^2 v}{\partial x_1 \partial x_2} - P_1^1 \delta \frac{\partial^3 w}{\partial x_1^3} - P_1^1 \delta \frac{\partial^3 w}{\partial x_1 \partial x_2^2} + P_1^2 \delta \frac{\partial^2 \theta_1}{\partial x_1^2} + P_1^2 \delta \frac{\partial^2 \theta_2}{\partial x_1 \partial x_2} \right)$$

$$+ \left(P_2^0 \delta \frac{\partial^2 u}{\partial x_1 \partial x_2} + P_2^0 \delta \frac{\partial^2 v}{\partial x_2^2} - P_2^1 \delta \frac{\partial^3 w}{\partial x_2^3} - P_2^1 \delta \frac{\partial^3 w}{\partial x_1^2 \partial x_2} + P_2^2 \delta \frac{\partial^2 \theta_1}{\partial x_1 \partial x_2} + P_2^2 \delta \frac{\partial^2 \theta_2}{\partial x_2^2} \right) \tag{38}$$

$$\left. + \left(-P_3^0 \delta \frac{\partial^2 w}{\partial x_1^2} - P_3^0 \delta \frac{\partial^2 w}{\partial x_2^2} + P_3^3 \delta \frac{\partial \theta_1}{\partial x_1} + P_3^3 \delta \frac{\partial \theta_2}{\partial x_2} \right) \right\} dA,$$

$$
\begin{aligned}
\delta U_3 = \int_A \Bigg\{ & \frac{2}{5}\left(T_{111}^0 \delta \frac{\partial^2 u}{\partial x_1^2} - \frac{1}{2} T_{111}^0 \delta \frac{\partial^2 u}{\partial x_2^2} - T_{111}^0 \delta \frac{\partial^2 v}{\partial x_1 \partial x_2} - T_{111}^1 \delta \frac{\partial^3 w}{\partial x_1^3} + \frac{3}{2} T_{111}^1 \delta \frac{\partial^3 w}{\partial x_1 \partial x_2^2} \right. \\
& \left. + T_{111}^2 \delta \frac{\partial^2 \theta_1}{\partial x_1^2} - \frac{1}{2} T_{111}^2 \delta \frac{\partial^2 \theta_1}{\partial x_2^2} - \frac{1}{2} T_{111}^4 \delta \theta_1 - T_{111}^2 \delta \frac{\partial^2 \theta_2}{\partial x_1 \partial x_2} \right) \\
& + \frac{2}{5}\left(-T_{222}^0 \delta \frac{\partial^2 u}{\partial x_1 \partial x_2} - \frac{1}{2} T_{222}^0 \delta \frac{\partial^2 v}{\partial x_1^2} + T_{222}^0 \delta \frac{\partial^2 v}{\partial x_2^2} - T_{222}^1 \delta \frac{\partial^3 w}{\partial x_2^3} + \frac{3}{2} T_{222}^1 \delta \frac{\partial^3 w}{\partial x_1^2 \partial x_2} \right. \\
& \left. - T_{222}^2 \delta \frac{\partial^2 \theta_1}{\partial x_1 \partial x_2} - \frac{1}{2} T_{222}^2 \delta \frac{\partial^2 \theta_2}{\partial x_1^2} + T_{222}^2 \delta \frac{\partial^2 \theta_2}{\partial x_2^2} - \frac{1}{2} T_{222}^4 \delta \theta_2 \right) \\
& + \frac{1}{5}\left(T_{333}^0 \delta \frac{\partial^2 w}{\partial x_1^2} + T_{333}^0 \delta \frac{\partial^2 w}{\partial x_2^2} - 2T_{333}^3 \delta \frac{\partial \theta_1}{\partial x_1} - 2T_{333}^3 \delta \frac{\partial \theta_2}{\partial x_2} \right) \\
& + \frac{1}{5}\left(8T_{112}^0 \delta \frac{\partial^2 u}{\partial x_1 \partial x_2} + 4T_{112}^0 \delta \frac{\partial^2 v}{\partial x_1^2} - 3T_{112}^0 \delta \frac{\partial^2 v}{\partial x_2^2} + 3T_{112}^1 \delta \frac{\partial^3 w}{\partial x_2^3} - 12T_{112}^1 \delta \frac{\partial^3 w}{\partial x_1^2 \partial x_2} \right. \\
& \left. + 8T_{112}^2 \delta \frac{\partial^2 \theta_1}{\partial x_1 \partial x_2} + 4T_{112}^2 \delta \frac{\partial^2 \theta_2}{\partial x_1^2} - 3T_{112}^2 \delta \frac{\partial^2 \theta_2}{\partial x_2^2} - T_{112}^4 \delta \theta_2 \right) \\
& - \frac{1}{5}\left(4T_{113}^0 \delta \frac{\partial^2 w}{\partial x_1^2} - T_{113}^0 \delta \frac{\partial^2 w}{\partial x_2^2} - 8T_{113}^3 \delta \frac{\partial \theta_1}{\partial x_1} + 2T_{113}^3 \delta \frac{\partial \theta_2}{\partial x_2} \right) \\
& + \frac{1}{5}\left(-3T_{221}^0 \delta \frac{\partial^2 u}{\partial x_1^2} + 4T_{221}^0 \delta \frac{\partial^2 u}{\partial x_2^2} + 8T_{221}^0 \delta \frac{\partial^2 v}{\partial x_1 \partial x_2} + 3T_{221}^1 \delta \frac{\partial^3 w}{\partial x_1^3} - 12T_{221}^1 \delta \frac{\partial^3 w}{\partial x_1 \partial x_2^2} \right. \\
& \left. - 3T_{221}^2 \delta \frac{\partial^2 \theta_1}{\partial x_1^2} + 4T_{221}^2 \delta \frac{\partial^2 \theta_1}{\partial x_2^2} - T_{221}^4 \delta \theta_1 + 8T_{221}^2 \delta \frac{\partial^2 \theta_2}{\partial x_1 \partial x_2} \right) \\
& - \frac{1}{5}\left(-T_{223}^0 \delta \frac{\partial^2 w}{\partial x_1^2} + 4T_{223}^0 \delta \frac{\partial^2 w}{\partial x_2^2} + 2T_{223}^3 \delta \frac{\partial \theta_1}{\partial x_1} - 8T_{223}^3 \delta \frac{\partial \theta_2}{\partial x_2} \right) \\
& - \frac{1}{5}\left(3T_{331}^0 \delta \frac{\partial^2 u}{\partial x_1^2} + T_{331}^0 \delta \frac{\partial^2 u}{\partial x_2^2} + 2T_{331}^0 \delta \frac{\partial^2 v}{\partial x_1 \partial x_2} - 3T_{331}^1 \delta \frac{\partial^3 w}{\partial x_1^3} - 3T_{331}^1 \delta \frac{\partial^3 w}{\partial x_1 \partial x_2^2} \right. \\
& \left. + 3T_{331}^2 \delta \frac{\partial^2 \theta_1}{\partial x_1^2} + T_{331}^2 \delta \frac{\partial^2 \theta_1}{\partial x_2^2} - 4T_{331}^4 \delta \theta_1 + 2T_{331}^2 \delta \frac{\partial^2 \theta_2}{\partial x_1 \partial x_2} \right) \\
& - \frac{1}{5}\left(2T_{332}^0 \delta \frac{\partial^2 u}{\partial x_1 \partial x_2} + T_{332}^0 \delta \frac{\partial^2 v}{\partial x_1^2} + 3T_{332}^0 \delta \frac{\partial^2 v}{\partial x_2^2} - 3T_{332}^1 \delta \frac{\partial^3 w}{\partial x_2^3} - 3T_{332}^1 \delta \frac{\partial^3 w}{\partial x_1^2 \partial x_2} \right. \\
& \left. + 2T_{332}^2 \delta \frac{\partial^2 \theta_1}{\partial x_1 \partial x_2} + T_{332}^2 \delta \frac{\partial^2 \theta_2}{\partial x_1^2} + 3T_{332}^2 \delta \frac{\partial^2 \theta_2}{\partial x_2^2} - 4T_{332}^4 \delta \theta_2 \right) \\
& + 2\left(-T_{123}^0 \delta \frac{\partial^2 w}{\partial x_1 \partial x_2} + T_{123}^3 \delta \frac{\partial \theta_1}{\partial x_2} + T_{123}^3 \delta \frac{\partial \theta_2}{\partial x_1} \right) \Bigg\} dA,
\end{aligned}
\tag{39}
$$

$$\delta U_4=\int_A \frac{1}{2}\left\{\left(2Y_{11}^0\delta\frac{\partial^2 w}{\partial x_1\partial x_2}-Y_{11}^3\delta\frac{\partial\theta_2}{\partial x_1}\right)-\left(2Y_{22}^0\delta\frac{\partial^2 w}{\partial x_1\partial x_2}-Y_{22}^3\delta\frac{\partial\theta_1}{\partial x_2}\right)-\left(Y_{33}^3\delta\frac{\partial\theta_1}{\partial x_2}-Y_{33}^3\delta\frac{\partial\theta_2}{\partial x_1}\right)\right.$$
$$+\left(-2Y_{12}^0\delta\frac{\partial^2 w}{\partial x_1^2}+2Y_{12}^0\delta\frac{\partial^2 w}{\partial x_2^2}+Y_{12}^3\delta\frac{\partial\theta_1}{\partial x_1}-Y_{12}^3\delta\frac{\partial\theta_2}{\partial x_2}\right)$$
$$+\left(-Y_{13}^0\delta\frac{\partial^2 u}{\partial x_1\partial x_2}+Y_{13}^0\delta\frac{\partial^2 v}{\partial x_1^2}-Y_{13}^2\delta\frac{\partial^2\theta_1}{\partial x_1\partial x_2}+Y_{13}^2\delta\frac{\partial^2\theta_2}{\partial x_1^2}-Y_{13}^4\delta\theta_2\right)$$
$$\left.+\left(-Y_{23}^0\delta\frac{\partial^2 u}{\partial x_2^2}+Y_{23}^0\delta\frac{\partial^2 v}{\partial x_1\partial x_2}-Y_{23}^2\delta\frac{\partial^2\theta_1}{\partial x_2^2}+Y_{23}^4\delta\theta_1+Y_{23}^2\delta\frac{\partial^2\theta_2}{\partial x_1\partial x_2}\right)\right\}dA.$$

(40)

u, v, w, θ_1 ve θ_2'nin varyasyonlarını kullanarak hareket denklemlerini elde edebilmek için, Green teoremi vasıtasıyla yüzey integrali eğri integraline dönüştürülmüştür. Green teoremi Γ sınır çizgisi ile çevrelenmiş bir alan için aşağıdaki gibi ifade edilmektedir:

$$\iint_A\left(\vec{\nabla}.\vec{F}\right)dA=\oint_\Gamma \vec{F}.\vec{n}d\Gamma$$
$$\xrightarrow{\vec{F}=(M,-L)\&\vec{n}=(n_{x1},n_{x2})}\iint_A\left(\frac{\partial M}{\partial x_1}-\frac{\partial L}{\partial x_2}\right)dA=\oint_\Gamma\left(Mn_{x_1}-Ln_{x_2}\right)d\Gamma. \quad (41)$$

n_{x_1} ve n_{x_2} orta düzlemin sınırının birim normalinin yön kosinüsleridir. Rasgele bir yüzey sınırının lokal normal $\vec{n}$ ve teğet $\vec{t}$ vektörleri şu şekilde yazılır:

$$\vec{n}=n_{x_1}\vec{e_1}+n_{x_2}\vec{e_2},$$
$$\vec{t}=-n_{x_1}\vec{e_1}+n_{x_2}\vec{e_2}, \quad (42)$$

Green teoreminin uygulaması ise

$$\int_A M_{11}^0\delta\frac{\partial u}{\partial x_1}dA=\int_A\left\{\frac{\partial}{\partial x_1}\left(M_{11}^0\delta u\right)-\frac{\partial M_{11}^0}{\partial x_1}\delta u\right\}dA=\oint_\Gamma M_{11}^0 n_{x_1}\delta u d\Gamma-\int_A\frac{\partial M_{11}^0}{\partial x_1}\delta u dA, \quad (43)$$

formundadır. Bu yöntem (37)-(40) numaralı denklemlere uygulanarak enerji varyasyonları şu şekilde yazılmıştır:

$$
\begin{aligned}
\delta U_1 = \int_A \Bigg\{ & \left(-\frac{\partial M_{11}^0}{\partial x_1}\delta u - \frac{\partial^2 M_{11}^1}{\partial x_1^2}\delta w - \frac{\partial M_{11}^2}{\partial x_1}\delta\theta_1 \right) + \left(-\frac{\partial M_{22}^0}{\partial x_2}\delta v - \frac{\partial^2 M_{22}^1}{\partial x_2^2}\delta w - \frac{\partial M_{22}^2}{\partial x_2}\delta\theta_2 \right) \\
& + \left(-\frac{\partial M_{12}^0}{\partial x_2}\delta u - \frac{\partial M_{12}^0}{\partial x_1}\delta v - 2\frac{\partial^2 M_{12}^1}{\partial x_1 \partial x_2}\delta w - \frac{\partial M_{12}^2}{\partial x_2}\delta\theta_1 - \frac{\partial M_{12}^2}{\partial x_1}\delta\theta_2 \right) + M_{13}^3 \delta\theta_1 + M_{23}^3 \delta\theta_2 \Bigg\} dA \\
& + \oint_\Gamma \Bigg\{ \left(M_{11}^0 n_{x_1} \delta u - M_{11}^1 n_{x_1} \delta \frac{\partial w}{\partial x_1} + \frac{\partial M_{11}^1}{\partial x_1} n_{x_1} \delta w + M_{11}^2 n_{x_1} \delta\theta_1 \right) \\
& + \left(M_{22}^0 n_{x_2} \delta v - M_{22}^1 n_{x_2} \delta \frac{\partial w}{\partial x_2} + \frac{\partial M_{22}^1}{\partial x_2} n_{x_2} \delta w + M_{22}^2 n_{x_2} \delta\theta_2 \right) \\
& + \Bigg(M_{12}^0 n_{x_2} \delta u + M_{12}^0 n_{x_1} \delta v - M_{12}^1 n_{x_1} \delta \frac{\partial w}{\partial x_2} + \frac{\partial M_{12}^1}{\partial x_1} n_{x_2} \delta w - M_{12}^1 n_{x_2} \delta \frac{\partial w}{\partial x_1} + \frac{\partial M_{12}^1}{\partial x_2} n_{x_1} \delta w \\
& + M_{12}^2 n_{x_2} \delta\theta_1 + M_{12}^2 n_{x_1} \delta\theta_2 \Bigg) \Bigg\} d\Gamma,
\end{aligned}
$$

(44)

$$
\begin{aligned}
\delta U_2 = \int_A \Bigg(& \frac{\partial^2 P_1^0}{\partial x_1^2}\delta u + \frac{\partial^2 P_1^0}{\partial x_1 \partial x_2}\delta v + \frac{\partial^3 P_1^1}{\partial x_1^3}\delta w + \frac{\partial^3 P_1^1}{\partial x_1 \partial x_2^2}\delta w + \frac{\partial^2 P_1^2}{\partial x_1^2}\delta\theta_1 + \frac{\partial^2 P_1^2}{\partial x_1 \partial x_2}\delta\theta_2 \Bigg) \\
& + \left(\frac{\partial^2 P_2^0}{\partial x_1 \partial x_2}\delta u + \frac{\partial^2 P_2^0}{\partial x_2^2}\delta v + \frac{\partial^3 P_2^1}{\partial x_2^3}\delta w + \frac{\partial^3 P_2^1}{\partial x_1^2 \partial x_2}\delta w + \frac{\partial^2 P_2^2}{\partial x_1 \partial x_2}\delta\theta_1 + \frac{\partial^2 P_2^2}{\partial x_2^2}\delta\theta_2 \right) \\
& + \left(-\frac{\partial^2 P_3^0}{\partial x_1^2}\delta w - \frac{\partial^2 P_3^0}{\partial x_2^2}\delta w - \frac{\partial P_3^3}{\partial x_1}\delta\theta_1 - \frac{\partial P_3^3}{\partial x_2}\delta\theta_2 \right) \Bigg\} dA \\
& + \oint_\Gamma \Bigg\{ \Bigg(P_1^0 n_{x_1} \delta \frac{\partial u}{\partial x_1} - \frac{\partial P_1^0}{\partial x_1} n_{x_1} \delta u + \frac{1}{2} P_1^0 n_{x_1} \delta \frac{\partial v}{\partial x_2} - \frac{1}{2}\frac{\partial P_1^0}{\partial x_1} n_{x_2} \delta v + \frac{1}{2} P_1^0 n_{x_2} \delta \frac{\partial v}{\partial x_1} - \frac{1}{2}\frac{\partial P_1^0}{\partial x_2} n_{x_1} \delta v \\
& - P_1^1 n_{x_1} \delta \frac{\partial^2 w}{\partial x_1^2} + \frac{\partial P_1^1}{\partial x_1} n_{x_1} \delta \frac{\partial w}{\partial x_1} - \frac{\partial^2 P_1^1}{\partial x_1^2} n_{x_1} \delta w - P_1^1 n_{x_1} \delta \frac{\partial^2 w}{\partial x_2^2} + \frac{\partial P_1^1}{\partial x_1} n_{x_2} \delta \frac{\partial w}{\partial x_2} - \frac{\partial^2 P_1^1}{\partial x_1 \partial x_2} n_{x_2} \delta w \\
& + P_1^2 n_{x_1} \delta \frac{\partial \theta_1}{\partial x_1} - \frac{\partial P_1^2}{\partial x_1} n_{x_1} \delta\theta_1 + \frac{1}{2} P_1^2 n_{x_1} \delta \frac{\partial \theta_2}{\partial x_2} - \frac{1}{2}\frac{\partial P_1^2}{\partial x_1} n_{x_2} \delta\theta_2 + \frac{1}{2} P_1^2 n_{x_2} \delta \frac{\partial \theta_2}{\partial x_1} - \frac{1}{2}\frac{\partial P_1^2}{\partial x_2} n_{x_1} \delta\theta_2 \Bigg) \\
& + \Bigg(\frac{1}{2} P_2^0 n_{x_1} \delta \frac{\partial u}{\partial x_2} - \frac{1}{2}\frac{\partial P_2^0}{\partial x_1} n_{x_2} \delta u + \frac{1}{2} P_2^0 n_{x_2} \delta \frac{\partial u}{\partial x_1} - \frac{1}{2}\frac{\partial P_2^0}{\partial x_2} n_{x_1} \delta u + P_2^0 n_{x_2} \delta \frac{\partial v}{\partial x_2} - \frac{\partial P_2^0}{\partial x_2} n_{x_2} \delta v \\
& - P_2^1 n_{x_2} \delta \frac{\partial^2 w}{\partial x_2^2} + \frac{\partial P_2^1}{\partial x_2} n_{x_2} \delta \frac{\partial w}{\partial x_2} - \frac{\partial^2 P_2^1}{\partial x_2^2} n_{x_2} \delta w - P_2^1 n_{x_2} \delta \frac{\partial^2 w}{\partial x_1^2} + \frac{\partial P_2^1}{\partial x_2} n_{x_1} \delta \frac{\partial w}{\partial x_1} - \frac{\partial^2 P_2^1}{\partial x_1 \partial x_2} n_{x_1} \delta w \\
& + \frac{1}{2} P_2^2 n_{x_1} \delta \frac{\partial \theta_1}{\partial x_2} - \frac{1}{2}\frac{\partial P_2^2}{\partial x_1} n_{x_2} \delta\theta_1 + \frac{1}{2} P_2^2 n_{x_2} \delta \frac{\partial \theta_1}{\partial x_1} - \frac{1}{2}\frac{\partial P_2^2}{\partial x_2} n_{x_1} \delta\theta_1 + P_2^2 n_{x_2} \delta \frac{\partial \theta_2}{\partial x_2} - \frac{\partial P_2^2}{\partial x_2} n_{x_2} \delta\theta_2 \Bigg) \\
& + \left(-P_3^0 n_{x_1} \delta \frac{\partial w}{\partial x_1} + \frac{\partial P_3^0}{\partial x_1} n_{x_1} \delta w - P_3^0 n_{x_2} \delta \frac{\partial w}{\partial x_2} + \frac{\partial P_3^0}{\partial x_2} n_{x_2} \delta w + P_3^3 n_{x_1} \delta\theta_1 + P_3^3 n_{x_2} \delta\theta_2 \right) \Bigg\} d\Gamma,
\end{aligned}
$$

(45)

$$
\begin{aligned}
\delta U_3 = \int_A \Bigg\{ & \frac{2}{5}\left(\frac{\partial^2 T_{111}^0}{\partial x_1^2}\delta u - \frac{1}{2}\frac{\partial^2 T_{111}^0}{\partial x_2^2}\delta u - \frac{\partial^2 T_{111}^0}{\partial x_1 \partial x_2}\delta v + \frac{\partial^3 T_{111}^1}{\partial x_1^3}\delta w - \frac{3}{2}\frac{\partial^3 T_{111}^1}{\partial x_1 \partial x_2^2}\delta w \right. \\
& \left. + \frac{\partial^2 T_{111}^2}{\partial x_1^2}\delta\theta_1 - \frac{1}{2}\frac{\partial^2 T_{111}^2}{\partial x_2^2}\delta\theta_1 - \frac{1}{2}T_{111}^4\delta\theta_1 - \frac{\partial^2 T_{111}^2}{\partial x_1 \partial x_2}\delta\theta_2 \right) \\
& + \frac{2}{5}\left(-\frac{\partial^2 T_{222}^0}{\partial x_1 \partial x_2}\delta u - \frac{1}{2}\frac{\partial^2 T_{222}^0}{\partial x_1^2}\delta v + \frac{\partial^2 T_{222}^0}{\partial x_2^2}\delta v + \frac{\partial^3 T_{222}^1}{\partial x_2^3}\delta w - \frac{3}{2}\frac{\partial^3 T_{222}^1}{\partial x_1^2 \partial x_2}\delta w \right. \\
& \left. - \frac{\partial^2 T_{222}^2}{\partial x_1 \partial x_2}\delta\theta_1 - \frac{1}{2}\frac{\partial^2 T_{222}^2}{\partial x_1^2}\delta\theta_2 + \frac{\partial^2 T_{222}^2}{\partial x_2^2}\delta\theta_2 - \frac{1}{2}T_{222}^4\delta\theta_2 \right) \\
& + \frac{1}{5}\left(\frac{\partial^2 T_{333}^0}{\partial x_1^2}\delta w + \frac{\partial^2 T_{333}^0}{\partial x_2^2}\delta w + 2\frac{\partial T_{333}^3}{\partial x_1}\delta\theta_1 + 2\frac{\partial T_{333}^3}{\partial x_2}\delta\theta_2 \right) \\
& + \frac{1}{5}\left(8\frac{\partial^2 T_{112}^0}{\partial x_1 \partial x_2}\delta u + 4\frac{\partial^2 T_{112}^0}{\partial x_1^2}\delta v - 3\frac{\partial^2 T_{112}^0}{\partial x_2^2}\delta v - 3\frac{\partial^3 T_{112}^1}{\partial x_2^3}\delta w + 12\frac{\partial^3 T_{112}^1}{\partial x_1^2 \partial x_2}\delta w \right. \\
& \left. + 8\frac{\partial^2 T_{112}^2}{\partial x_1 \partial x_2}\delta\theta_1 + 4\frac{\partial^2 T_{112}^2}{\partial x_1^2}\delta\theta_2 - 3\frac{\partial^2 T_{112}^2}{\partial x_2^2}\delta\theta_2 - T_{112}^4\delta\theta_2 \right) \\
& - \frac{1}{5}\left(4\frac{\partial^2 T_{113}^0}{\partial x_1^2}\delta w - \frac{\partial^2 T_{113}^0}{\partial x_2^2}\delta w + 8\frac{\partial T_{113}^3}{\partial x_1}\delta\theta_1 - 2\frac{\partial T_{113}^3}{\partial x_2}\delta\theta_2 \right) \\
& + \frac{1}{5}\left(-3\frac{\partial^2 T_{221}^0}{\partial x_1^2}\delta u + 4\frac{\partial^2 T_{221}^0}{\partial x_2^2}\delta u + 8\frac{\partial^2 T_{221}^0}{\partial x_1 \partial x_2}\delta v - 3\frac{\partial^3 T_{221}^1}{\partial x_1^3}\delta w + 12\frac{\partial^3 T_{221}^1}{\partial x_1 \partial x_2^2}\delta w \right. \\
& \left. - 3\frac{\partial^2 T_{221}^2}{\partial x_1^2}\delta\theta_1 + 4\frac{\partial^2 T_{221}^2}{\partial x_2^2}\delta\theta_1 - T_{221}^4\delta\theta_1 + 8\frac{\partial^2 T_{221}^2}{\partial x_1 \partial x_2}\delta\theta_2 \right) \\
& - \frac{1}{5}\left(-\frac{\partial^2 T_{223}^0}{\partial x_1^2}\delta w + 4\frac{\partial^2 T_{223}^0}{\partial x_2^2}\delta w - 2\frac{\partial T_{223}^3}{\partial x_1}\delta\theta_1 + 8\frac{\partial T_{223}^3}{\partial x_2}\delta\theta_2 \right) \\
& - \frac{1}{5}\left(3\frac{\partial^2 T_{331}^0}{\partial x_1^2}\delta u + \frac{\partial^2 T_{331}^0}{\partial x_2^2}\delta u + 2\frac{\partial^2 T_{331}^0}{\partial x_1 \partial x_2}\delta v + 3\frac{\partial^3 T_{331}^1}{\partial x_1^3}\delta w + 3\frac{\partial^3 T_{331}^1}{\partial x_1 \partial x_2^2}\delta w \right. \\
& \left. + 3\frac{\partial^2 T_{331}^2}{\partial x_1^2}\delta\theta_1 + \frac{\partial^2 T_{331}^2}{\partial x_2^2}\delta\theta_1 - 4T_{331}^4\delta\theta_1 + 2\frac{\partial^2 T_{331}^2}{\partial x_1 \partial x_2}\delta\theta_2 \right) \\
& - \frac{1}{5}\left(2\frac{\partial^2 T_{332}^0}{\partial x_1 \partial x_2}\delta u + \frac{\partial^2 T_{332}^0}{\partial x_1^2}\delta v + 3\frac{\partial^2 T_{332}^0}{\partial x_2^2}\delta v + 3\frac{\partial^3 T_{332}^1}{\partial x_2^3}\delta w + 3\frac{\partial^3 T_{332}^1}{\partial x_1^2 \partial x_2}\delta w \right. \\
& \left. + 2\frac{\partial^2 T_{332}^2}{\partial x_1 \partial x_2}\delta\theta_1 + \frac{\partial^2 T_{332}^2}{\partial x_1^2}\delta\theta_2 + 3\frac{\partial^2 T_{332}^2}{\partial x_2^2}\delta\theta_2 - 4T_{332}^4\delta\theta_2 \right) \\
& + 2\left(-\frac{\partial^2 T_{123}^0}{\partial x_1 \partial x_2}\delta w - \frac{\partial T_{123}^3}{\partial x_2}\delta\theta_1 - \frac{\partial T_{123}^3}{\partial x_1}\delta\theta_2 \right) \Bigg\} dA
\end{aligned}
$$

$$+\oint_{\Gamma}\Bigg\{\Bigg\{\frac{2}{5}\Bigg(T_{111}^{0}n_{x_1}\delta\frac{\partial u}{\partial x_1}-\frac{\partial T_{111}^{0}}{\partial x_1}n_{x_1}\delta u-\frac{1}{2}T_{111}^{0}n_{x_2}\delta\frac{\partial u}{\partial x_2}+\frac{1}{2}\frac{\partial T_{111}^{0}}{\partial x_2}n_{x_2}\delta u-\frac{1}{2}T_{111}^{0}n_{x_1}\delta\frac{\partial v}{\partial x_2}+\frac{1}{2}\frac{\partial T_{111}^{0}}{\partial x_1}n_{x_2}\delta v$$

$$-\frac{1}{2}T_{111}^{0}n_{x_2}\delta\frac{\partial v}{\partial x_1}+\frac{1}{2}\frac{\partial T_{111}^{0}}{\partial x_2}n_{x_1}\delta v-T_{111}^{1}n_{x_1}\delta\frac{\partial^2 w}{\partial x_1^2}+\frac{\partial T_{111}^{1}}{\partial x_1}n_{x_1}\delta\frac{\partial w}{\partial x_1}-\frac{\partial^2 T_{111}^{1}}{\partial x_1^2}n_{x_1}\delta w$$

$$+\frac{3}{2}T_{111}^{1}n_{x_1}\delta\frac{\partial^2 w}{\partial x_2^2}-\frac{3}{2}\frac{\partial T_{111}^{1}}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_2}+\frac{3}{2}\frac{\partial^2 T_{111}^{1}}{\partial x_1\partial x_2}n_{x_2}\delta w+T_{111}^{2}n_{x_1}\delta\frac{\partial\theta_1}{\partial x_1}-\frac{\partial T_{111}^{2}}{\partial x_1}n_{x_1}\delta\theta_1-\frac{1}{2}T_{111}^{2}n_{x_2}\delta\frac{\partial\theta_1}{\partial x_2}$$

$$+\frac{1}{2}\frac{\partial T_{111}^{2}}{\partial x_2}n_{x_2}\delta\theta_1-\frac{1}{2}T_{111}^{2}n_{x_1}\delta\frac{\partial\theta_2}{\partial x_2}+\frac{1}{2}\frac{\partial T_{111}^{2}}{\partial x_1}n_{x_2}\delta\theta_2-\frac{1}{2}T_{111}^{2}n_{x_2}\delta\frac{\partial\theta_2}{\partial x_1}+\frac{1}{2}\frac{\partial T_{111}^{2}}{\partial x_2}n_{x_1}\delta\theta_2\Bigg)$$

$$+\frac{2}{5}\Bigg(-\frac{1}{2}T_{222}^{0}n_{x_1}\delta\frac{\partial u}{\partial x_2}+\frac{1}{2}\frac{\partial T_{222}^{0}}{\partial x_1}n_{x_2}\delta u-\frac{1}{2}T_{222}^{0}n_{x_2}\delta\frac{\partial u}{\partial x_1}+\frac{1}{2}\frac{\partial T_{222}^{0}}{\partial x_2}n_{x_1}\delta u-\frac{1}{2}T_{222}^{0}n_{x_1}\delta\frac{\partial v}{\partial x_1}+\frac{1}{2}\frac{\partial T_{222}^{0}}{\partial x_1}n_{x_1}\delta v$$

$$+T_{222}^{0}n_{x_2}\delta\frac{\partial v}{\partial x_2}-\frac{\partial T_{222}^{0}}{\partial x_2}n_{x_2}\delta v-T_{222}^{1}n_{x_2}\delta\frac{\partial^2 w}{\partial x_2^2}+\frac{\partial T_{222}^{1}}{\partial x_2}n_{x_2}\delta\frac{\partial w}{\partial x_2}-\frac{\partial^2 T_{222}^{1}}{\partial x_2^2}n_{x_2}\delta w$$

$$+\frac{3}{2}T_{222}^{1}n_{x_2}\delta\frac{\partial^2 w}{\partial x_1^2}-\frac{3}{2}\frac{\partial T_{222}^{1}}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_1}+\frac{3}{2}\frac{\partial T_{222}^{1}}{\partial x_1\partial x_2}n_{x_1}\delta w-\frac{1}{2}T_{222}^{2}n_{x_1}\delta\frac{\partial\theta_1}{\partial x_2}+\frac{1}{2}\frac{\partial T_{222}^{2}}{\partial x_1}n_{x_2}\delta\theta_1$$

$$-\frac{1}{2}T_{222}^{2}n_{x_2}\delta\frac{\partial\theta_1}{\partial x_1}+\frac{1}{2}\frac{\partial T_{222}^{2}}{\partial x_2}n_{x_1}\delta\theta_1-\frac{1}{2}T_{222}^{2}n_{x_1}\delta\frac{\partial\theta_2}{\partial x_1}+\frac{1}{2}\frac{\partial T_{222}^{2}}{\partial x_1}n_{x_1}\delta\theta_2+T_{222}^{2}n_{x_2}\delta\frac{\partial\theta_2}{\partial x_2}-\frac{\partial T_{222}^{2}}{\partial x_2}n_{x_2}\delta\theta_2\Bigg)$$

$$+\frac{1}{5}\Bigg(T_{333}^{0}n_{x_1}\delta\frac{\partial w}{\partial x_1}-\frac{\partial T_{333}^{0}}{\partial x_1}n_{x_1}\delta w+T_{333}^{0}n_{x_2}\delta\frac{\partial w}{\partial x_2}-\frac{\partial T_{333}^{0}}{\partial x_2}n_{x_2}\delta w-2T_{333}^{3}n_{x_1}\delta\theta_1-2T_{333}^{3}n_{x_2}\delta\theta_2\Bigg)$$

$$+\frac{1}{5}\Bigg(4T_{112}^{0}n_{x_1}\delta\frac{\partial u}{\partial x_2}-4\frac{\partial T_{112}^{0}}{\partial x_1}n_{x_2}\delta u+4T_{112}^{0}n_{x_2}\delta\frac{\partial u}{\partial x_1}-4\frac{\partial T_{112}^{0}}{\partial x_2}n_{x_1}\delta u+4T_{112}^{0}n_{x_1}\delta\frac{\partial v}{\partial x_1}-4\frac{\partial T_{112}^{0}}{\partial x_1}n_{x_1}\delta v$$

$$-3T_{112}^{0}n_{x_2}\delta\frac{\partial v}{\partial x_2}+3\frac{\partial T_{112}^{0}}{\partial x_2}n_{x_2}\delta v+3T_{112}^{1}n_{x_2}\delta\frac{\partial^2 w}{\partial x_2^2}-3\frac{\partial T_{112}^{1}}{\partial x_2}n_{x_2}\delta\frac{\partial w}{\partial x_2}+3\frac{\partial^2 T_{112}^{1}}{\partial x_2^2}n_{x_2}\delta w$$

$$-12T_{112}^{1}n_{x_2}\delta\frac{\partial^2 w}{\partial x_1^2}+12\frac{\partial T_{112}^{1}}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_1}-12\frac{\partial^2 T_{112}^{1}}{\partial x_1\partial x_2}n_{x_1}\delta w+4T_{112}^{2}n_{x_1}\delta\frac{\partial\theta_1}{\partial x_2}-4\frac{\partial T_{112}^{2}}{\partial x_1}n_{x_2}\delta\theta_1$$

$$+4T_{112}^{2}n_{x_2}\delta\frac{\partial\theta_1}{\partial x_1}-4\frac{\partial T_{112}^{2}}{\partial x_2}n_{x_1}\delta\theta_1+4T_{112}^{2}n_{x_1}\delta\frac{\partial\theta_2}{\partial x_1}-4\frac{\partial T_{112}^{2}}{\partial x_1}n_{x_1}\delta\theta_2-3T_{112}^{2}n_{x_2}\delta\frac{\partial\theta_2}{\partial x_2}+3\frac{\partial T_{112}^{2}}{\partial x_2}n_{x_2}\delta\theta_2\Bigg)$$

$$-\frac{1}{5}\Bigg(4T_{113}^{0}n_{x_1}\delta\frac{\partial w}{\partial x_1}-4\frac{\partial T_{113}^{0}}{\partial x_1}n_{x_1}\delta w-T_{113}^{0}n_{x_2}\delta\frac{\partial w}{\partial x_2}+\frac{\partial T_{113}^{0}}{\partial x_2}n_{x_2}\delta w-8T_{113}^{3}n_{x_1}\delta\theta_1+2T_{113}^{3}n_{x_2}\delta\theta_2\Bigg)$$

$$+\frac{1}{5}\Bigg(-3T_{221}^{0}n_{x_1}\delta\frac{\partial u}{\partial x_1}+3\frac{\partial T_{221}^{0}}{\partial x_1}n_{x_1}\delta u+4T_{221}^{0}n_{x_2}\delta\frac{\partial u}{\partial x_2}-4\frac{\partial T_{221}^{0}}{\partial x_2}n_{x_2}\delta u+4T_{221}^{0}n_{x_1}\delta\frac{\partial v}{\partial x_2}-4\frac{\partial T_{221}^{0}}{\partial x_1}n_{x_2}\delta v$$

$$+4T_{221}^{0}n_{x_2}\delta\frac{\partial v}{\partial x_1}-4\frac{\partial T_{221}^{0}}{\partial x_2}n_{x_1}\delta v+3T_{221}^{1}n_{x_1}\delta\frac{\partial^2 w}{\partial x_1^2}-3\frac{\partial T_{221}^{1}}{\partial x_1}n_{x_1}\delta\frac{\partial w}{\partial x_1}+3\frac{\partial^2 T_{221}^{1}}{\partial x_1^2}n_{x_1}\delta w$$

$$-12T_{221}^{1}n_{x_1}\delta\frac{\partial^2 w}{\partial x_2^2}+12\frac{\partial T_{221}^{1}}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_2}-12\frac{\partial^2 T_{221}^{1}}{\partial x_1\partial x_2}n_{x_2}\delta w-3T_{221}^{2}n_{x_1}\delta\frac{\partial\theta_1}{\partial x_1}+3\frac{\partial T_{221}^{2}}{\partial x_1}n_{x_1}\delta\theta_1$$

$$
+4T_{221}^{2}n_{x_2}\delta\frac{\partial\theta_1}{\partial x_2}-4\frac{\partial T_{221}^{2}}{\partial x_2}n_{x_2}\delta\theta_1+4T_{221}^{2}n_{x_1}\delta\frac{\partial\theta_2}{\partial x_2}-4\frac{\partial T_{221}^{2}}{\partial x_1}n_{x_2}\delta\theta_2+4T_{221}^{2}n_{x_2}\delta\frac{\partial\theta_2}{\partial x_1}-4\frac{\partial T_{221}^{2}}{\partial x_2}n_{x_1}\delta\theta_2\Bigg)
$$

$$
-\frac{1}{5}\Bigg(-T_{223}^{0}n_{x_1}\delta\frac{\partial w}{\partial x_1}+\frac{\partial T_{223}^{0}}{\partial x_1}n_{x_1}\delta w+4T_{223}^{0}n_{x_2}\delta\frac{\partial w}{\partial x_2}-4\frac{\partial T_{223}^{0}}{\partial x_2}n_{x_2}\delta w+2T_{223}^{3}n_{x_1}\delta\theta_1-8T_{223}^{3}n_{x_2}\delta\theta_2\Bigg)
$$

$$
-\frac{1}{5}\Bigg(3T_{331}^{0}n_{x_1}\delta\frac{\partial u}{\partial x_1}-3\frac{\partial T_{331}^{0}}{\partial x_1}n_{x_1}\delta u+T_{331}^{0}n_{x_2}\delta\frac{\partial u}{\partial x_2}-\frac{\partial T_{331}^{0}}{\partial x_2}n_{x_2}\delta u+T_{331}^{0}n_{x_1}\delta\frac{\partial v}{\partial x_2}-\frac{\partial T_{331}^{0}}{\partial x_1}n_{x_2}\delta v
$$

$$
+T_{331}^{0}n_{x_2}\delta\frac{\partial v}{\partial x_1}-\frac{\partial T_{331}^{0}}{\partial x_2}n_{x_1}\delta v-3T_{331}^{1}n_{x_1}\delta\frac{\partial^2 w}{\partial x_1^2}+3\frac{\partial T_{331}^{1}}{\partial x_1}n_{x_1}\delta\frac{\partial w}{\partial x_1}-3\frac{\partial^2 T_{331}^{1}}{\partial x_1^2}n_{x_1}\delta w
$$

$$
-3T_{331}^{1}n_{x_1}\delta\frac{\partial^2 w}{\partial x_2^2}+3\frac{\partial T_{331}^{1}}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_2}-3\frac{\partial^2 T_{331}^{1}}{\partial x_1\partial x_2}n_{x_2}\delta w+3T_{331}^{2}n_{x_1}\delta\frac{\partial\theta_1}{\partial x_1}-3\frac{\partial T_{331}^{2}}{\partial x_1}n_{x_1}\delta\theta_1
$$

$$
+T_{331}^{2}n_{x_2}\delta\frac{\partial\theta_1}{\partial x_2}-\frac{\partial T_{331}^{2}}{\partial x_2}n_{x_2}\delta\theta_1+T_{331}^{2}n_{x_1}\delta\frac{\partial\theta_2}{\partial x_2}-\frac{\partial T_{331}^{2}}{\partial x_1}n_{x_2}\delta\theta_2+T_{331}^{2}n_{x_2}\delta\frac{\partial\theta_2}{\partial x_1}-\frac{\partial T_{331}^{2}}{\partial x_2}n_{x_1}\delta\theta_2\Bigg)
$$

$$
-\frac{1}{5}\Bigg(T_{332}^{0}n_{x_1}\delta\frac{\partial u}{\partial x_2}-\frac{\partial T_{332}^{0}}{\partial x_1}n_{x_2}\delta u+T_{332}^{0}n_{x_2}\delta\frac{\partial u}{\partial x_1}-\frac{\partial T_{332}^{0}}{\partial x_2}n_{x_1}\delta u+T_{332}^{0}n_{x_1}\delta\frac{\partial v}{\partial x_1}-\frac{\partial T_{332}^{0}}{\partial x_1}n_{x_1}\delta v
$$

$$
+3T_{332}^{0}n_{x_2}\delta\frac{\partial v}{\partial x_2}-3\frac{\partial T_{332}^{0}}{\partial x_2}n_{x_2}\delta v-3T_{332}^{1}n_{x_2}\delta\frac{\partial^2 w}{\partial x_2^2}+3\frac{\partial T_{332}^{1}}{\partial x_2}n_{x_2}\delta\frac{\partial w}{\partial x_2}-3\frac{\partial^2 T_{332}^{1}}{\partial x_2^2}n_{x_2}\delta w
$$

$$
-3T_{332}^{1}n_{x_2}\delta\frac{\partial^2 w}{\partial x_1^2}+3\frac{\partial T_{332}^{1}}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_1}-3\frac{\partial^2 T_{332}^{1}}{\partial x_1\partial x_2}n_{x_1}\delta w+T_{332}^{2}n_{x_1}\delta\frac{\partial\theta_1}{\partial x_2}-\frac{\partial T_{332}^{2}}{\partial x_1}n_{x_2}\delta\theta_1
$$

$$
+T_{332}^{2}n_{x_2}\delta\frac{\partial\theta_1}{\partial x_1}-\frac{\partial T_{332}^{2}}{\partial x_2}n_{x_1}\delta\theta_1+T_{332}^{2}n_{x_1}\delta\frac{\partial\theta_2}{\partial x_1}-\frac{\partial T_{332}^{2}}{\partial x_1}n_{x_1}\delta\theta_2+3T_{332}^{2}n_{x_2}\delta\frac{\partial\theta_2}{\partial x_2}-3\frac{\partial T_{332}^{2}}{\partial x_2}n_{x_2}\delta\theta_2\Bigg)
$$

$$
+2\Bigg(-\frac{1}{2}T_{123}^{0}n_{x_1}\delta\frac{\partial w}{\partial x_2}+\frac{1}{2}\frac{\partial T_{123}^{0}}{\partial x_1}n_{x_2}\delta w-\frac{1}{2}T_{123}^{0}n_{x_2}\delta\frac{\partial w}{\partial x_1}+\frac{1}{2}\frac{\partial T_{123}^{0}}{\partial x_2}n_{x_1}\delta w+T_{123}^{3}n_{x_2}\delta\theta_1+T_{123}^{3}n_{x_1}\delta\theta_2\Bigg)\Bigg\}d\Gamma, \qquad (46)
$$

$$
\begin{aligned}
\delta U_4 = \int_A \frac{1}{2} \Bigg\{ & \left(2\frac{\partial^2 Y_{11}^0}{\partial x_1 \partial x_2}\delta w + \frac{\partial Y_{11}^3}{\partial x_1}\delta\theta_2 \right) - \left(2\frac{\partial^2 Y_{22}^0}{\partial x_1 \partial x_2}\delta w + \frac{\partial Y_{22}^3}{\partial x_2}\delta\theta_1 \right) - \left(-\frac{\partial Y_{33}^3}{\partial x_2}\delta\theta_1 + \frac{\partial Y_{33}^3}{\partial x_1}\delta\theta_2 \right) \\
& + \left(-2\frac{\partial^2 Y_{12}^0}{\partial x_1^2}\delta w + 2\frac{\partial^2 Y_{12}^0}{\partial x_2^2}\delta w - \frac{\partial Y_{12}^3}{\partial x_1}\delta\theta_1 + \frac{\partial Y_{12}^3}{\partial x_2}\delta\theta_2 \right) \\
& + \left(-\frac{\partial^2 Y_{13}^0}{\partial x_1 \partial x_2}\delta u + \frac{\partial^2 Y_{13}^0}{\partial x_1^2}\delta v - \frac{\partial^2 Y_{13}^2}{\partial x_1 \partial x_2}\delta\theta_1 + \frac{\partial^2 Y_{13}^2}{\partial x_1^2}\delta\theta_2 - Y_{13}^4\delta\theta_2 \right) \\
& + \left(-\frac{\partial^2 Y_{23}^0}{\partial x_2^2}\delta u + \frac{\partial^2 Y_{23}^0}{\partial x_1 \partial x_2}\delta v - \frac{\partial^2 Y_{23}^2}{\partial x_2^2}\delta\theta_1 + Y_{23}^4\delta\theta_1 + \frac{\partial^2 Y_{23}^2}{\partial x_1 \partial x_2}\delta\theta_2 \right) \Bigg\} dA \\
+ \oint_\Gamma \frac{1}{2} \Bigg\{ & \left(Y_{11}^0 n_{x_1}\delta\frac{\partial w}{\partial x_2} - \frac{\partial Y_{11}^0}{\partial x_1} n_{x_2}\delta w + Y_{11}^0 n_{x_2}\delta\frac{\partial w}{\partial x_1} - \frac{\partial Y_{11}^0}{\partial x_2} n_{x_1}\delta w - Y_{11}^3 n_{x_1}\delta\theta_2 \right) \\
& - \left(Y_{22}^0 n_{x_1}\delta\frac{\partial w}{\partial x_2} - \frac{\partial Y_{22}^0}{\partial x_1} n_{x_2}\delta w + Y_{22}^0 n_{x_2}\delta\frac{\partial w}{\partial x_1} - \frac{\partial Y_{22}^0}{\partial x_2} n_{x_1}\delta w - Y_{22}^3 n_{x_2}\delta\theta_1 \right) \\
& - \left(Y_{33}^3 n_{x_2}\delta\theta_1 - Y_{33}^3 n_{x_1}\delta\theta_2 \right) \\
& + \left(-2Y_{12}^0 n_{x_1}\delta\frac{\partial w}{\partial x_1} + 2\frac{\partial Y_{12}^0}{\partial x_1} n_{x_1}\delta w + 2Y_{12}^0 n_{x_2}\delta\frac{\partial w}{\partial x_2} - 2\frac{\partial Y_{12}^0}{\partial x_2} n_{x_2}\delta w + Y_{12}^3 n_{x_1}\delta\theta_1 - Y_{12}^3 n_{x_2}\delta\theta_2 \right) \\
& + \Bigg(-Y_{13}^0 n_{x_1}\delta\frac{\partial u}{\partial x_2} + \frac{\partial Y_{13}^0}{\partial x_1} n_{x_2}\delta u + Y_{13}^0 n_{x_1}\delta\frac{\partial v}{\partial x_1} - \frac{\partial Y_{13}^0}{\partial x_1} n_{x_1}\delta v \\
& - \frac{1}{2}Y_{13}^2 n_{x_1}\delta\frac{\partial\theta_1}{\partial x_2} + \frac{1}{2}\frac{\partial Y_{13}^2}{\partial x_1} n_{x_2}\delta\theta_1 - \frac{1}{2}Y_{13}^2 n_{x_2}\delta\frac{\partial\theta_1}{\partial x_1} + \frac{1}{2}\frac{\partial Y_{13}^2}{\partial x_2} n_{x_1}\delta\theta_1 + Y_{13}^2 n_{x_1}\delta\frac{\partial\theta_2}{\partial x_1} - \frac{\partial Y_{13}^2}{\partial x_1} n_{x_1}\delta\theta_2 \Bigg) \\
& + \Bigg(-Y_{23}^0 n_{x_2}\delta\frac{\partial u}{\partial x_2} + \frac{\partial Y_{23}^0}{\partial x_2} n_{x_2}\delta u + Y_{23}^0 n_{x_2}\delta\frac{\partial v}{\partial x_1} - \frac{\partial Y_{23}^0}{\partial x_2} n_{x_1}\delta v - Y_{23}^2 n_{x_2}\delta\frac{\partial\theta_1}{\partial x_2} + \frac{\partial Y_{23}^2}{\partial x_2} n_{x_2}\delta\theta_1 \\
& + \frac{1}{2}Y_{23}^2 n_{x_1}\delta\frac{\partial\theta_2}{\partial x_2} - \frac{1}{2}\frac{\partial Y_{23}^2}{\partial x_1} n_{x_2}\delta\theta_2 + \frac{1}{2}Y_{23}^2 n_{x_2}\delta\frac{\partial\theta_2}{\partial x_1} - \frac{1}{2}\frac{\partial Y_{23}^2}{\partial x_2} n_{x_1}\delta\theta_2 \Bigg) \Bigg\} d\Gamma.
\end{aligned}
\tag{47}
$$

Mikro-plağın kinetik enerjisinin açık formu

$$
K = \frac{1}{2}\int_A \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \left\{ \left(\frac{\partial u}{\partial t} - x_3\frac{\partial^2 w}{\partial x_1 \partial t} + f\frac{\partial\theta_1}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} - x_3\frac{\partial^2 w}{\partial x_2 \partial t} + f\frac{\partial\theta_2}{\partial t} \right)^2 + \left(\frac{\partial w}{\partial t} \right)^2 \right\} dx_3 dA, \tag{48}
$$

olarak yazılır. Bu durumda kinetik enerjinin varyasyonu

$$
\begin{aligned}
\delta K = \int_A \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \Bigg\{ & \left(\frac{\partial u}{\partial t} - x_3 \frac{\partial^2 w}{\partial x_1 \partial t} + f \frac{\partial \theta_1}{\partial t} \right) \left(\delta \frac{\partial u}{\partial t} - x_3 \delta \frac{\partial^2 w}{\partial x_1 \partial t} + f \delta \frac{\partial \theta_1}{\partial t} \right) \\
& + \left(\frac{\partial v}{\partial t} - x_3 \frac{\partial^2 w}{\partial x_2 \partial t} + f \frac{\partial \theta_2}{\partial t} \right) \left(\delta \frac{\partial v}{\partial t} - x_3 \delta \frac{\partial^2 w}{\partial x_2 \partial t} + f \delta \frac{\partial \theta_2}{\partial t} \right) + \left(\frac{\partial w}{\partial t} \right) \left(\delta \frac{\partial w}{\partial t} \right) \Bigg\} dx_3 dA,
\end{aligned} \tag{49}
$$

şeklindedir. Atalet katsayıları

$$
\{I_0, I_1, I_2, I_3, I_4, I_5\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(x_3) \{1, x_3, x_3^2, f, x_3 f, f^2\} dx_3, \tag{50}
$$

şeklinde tanımlanmaktadır. Böylece kinetik enerji varyasyonu

$$
\begin{aligned}
\delta K = \int_A \Bigg\{ & \left(I_0 \frac{\partial u}{\partial t} - I_1 \frac{\partial^2 w}{\partial x_1 \partial t} + I_3 \frac{\partial \theta_1}{\partial t} \right) \left(\delta \frac{\partial u}{\partial t} \right) - \left(I_1 \frac{\partial u}{\partial t} - I_2 \frac{\partial^2 w}{\partial x_1 \partial t} + I_4 \frac{\partial \theta_1}{\partial t} \right) \left(\delta \frac{\partial^2 w}{\partial x_1 \partial t} \right) \\
& + \left(I_3 \frac{\partial u}{\partial t} - I_4 \frac{\partial^2 w}{\partial x_1 \partial t} + I_5 \frac{\partial \theta_1}{\partial t} \right) \left(\delta \frac{\partial \theta_1}{\partial t} \right) \\
& + \left(I_0 \frac{\partial v}{\partial t} - I_1 \frac{\partial^2 w}{\partial x_2 \partial t} + I_3 \frac{\partial \theta_2}{\partial t} \right) \left(\delta \frac{\partial v}{\partial t} \right) - \left(I_1 \frac{\partial v}{\partial t} - I_2 \frac{\partial^2 w}{\partial x_2 \partial t} + I_4 \frac{\partial \theta_2}{\partial t} \right) \left(\delta \frac{\partial^2 w}{\partial x_2 \partial t} \right) \\
& + \left(I_3 \frac{\partial v}{\partial t} - I_4 \frac{\partial^2 w}{\partial x_2 \partial t} + I_5 \frac{\partial \theta_2}{\partial t} \right) \left(\delta \frac{\partial \theta_2}{\partial t} \right) + \left(I_0 \frac{\partial w}{\partial t} \right) \left(\delta \frac{\partial w}{\partial t} \right) \Bigg\} dA,
\end{aligned} \tag{51}
$$

formunda bulunmuştur. Zaman integrali için kısmi integral metotu kullanılarak aşağıdaki ifade elde edilmiştir:

$$
\begin{aligned}
\delta \int_{t_1}^{t_2} K dt = \int_{t_1}^{t_2} \int_A \Bigg\{ & - \left(I_0 \frac{\partial^2 u}{\partial t^2} - I_1 \frac{\partial^3 w}{\partial x_1 \partial t^2} + I_3 \frac{\partial^2 \theta_1}{\partial t^2} \right) \delta u + \left(I_1 \frac{\partial^2 u}{\partial t^2} - I_2 \frac{\partial^3 w}{\partial x_1 \partial t^2} + I_4 \frac{\partial^2 \theta_1}{\partial t^2} \right) \delta \frac{\partial w}{\partial x_1} \\
& - \left(I_3 \frac{\partial u^2}{\partial t^2} - I_4 \frac{\partial^3 w}{\partial x_1 \partial t^2} + I_5 \frac{\partial^2 \theta_1}{\partial t^2} \right) \delta \theta_1 \\
& - \left(I_0 \frac{\partial^2 v}{\partial t^2} - I_1 \frac{\partial^3 w}{\partial x_2 \partial t^2} + I_3 \frac{\partial^2 \theta_2}{\partial t^2} \right) \delta v + \left(I_1 \frac{\partial^2 v}{\partial t^2} - I_2 \frac{\partial^3 w}{\partial x_2 \partial t^2} + I_4 \frac{\partial^2 \theta_2}{\partial t^2} \right) \delta \frac{\partial w}{\partial x_2} \\
& - \left(I_3 \frac{\partial^2 v}{\partial t^2} - I_4 \frac{\partial^3 w}{\partial x_2 \partial t^2} + I_5 \frac{\partial^2 \theta_2}{\partial t^2} \right) \delta \theta_2 - \left(I_0 \frac{\partial^2 w}{\partial t^2} \right) \delta w \Bigg\} dA dt.
\end{aligned} \tag{52}
$$

Uzaysal integral için ise kısmi integral yöntemi ile Green teoremi kullanıldıktan sonra kinetik enerjisinin varyasyonu:

$$\delta\int_{t_1}^{t_2} K dt = \int_{t_1}^{t_2}\int_{A}\Bigg\{-\left(I_0\frac{\partial^2 u}{\partial t^2} - I_1\frac{\partial^3 w}{\partial x_1\partial t^2} + I_3\frac{\partial^2\theta_1}{\partial t^2}\right)\delta u - \left(I_1\frac{\partial^3 u}{\partial x_1\partial t^2} - I_2\frac{\partial^4 w}{\partial x_1^2\partial t^2} + I_4\frac{\partial^3\theta_1}{\partial x_1\partial t^2}\right)\delta w$$
$$-\left(I_3\frac{\partial u^2}{\partial t^2} - I_4\frac{\partial^3 w}{\partial x_1\partial t^2} + I_5\frac{\partial^2\theta_1}{\partial t^2}\right)\delta\theta_1$$
$$-\left(I_0\frac{\partial^2 v}{\partial t^2} - I_1\frac{\partial^3 w}{\partial x_2\partial t^2} + I_3\frac{\partial^2\theta_2}{\partial t^2}\right)\delta v - \left(I_1\frac{\partial^3 v}{\partial x_2\partial t^2} - I_2\frac{\partial^4 w}{\partial x_2^2\partial t^2} + I_4\frac{\partial^3\theta_2}{\partial x_2\partial t^2}\right)\delta w$$
$$-\left(I_3\frac{\partial^2 v}{\partial t^2} - I_4\frac{\partial^3 w}{\partial x_2\partial t^2} + I_5\frac{\partial^2\theta_2}{\partial t^2}\right)\delta\theta_2 - \left(I_0\frac{\partial^2 w}{\partial t^2}\right)\delta w\Bigg\} dAdt$$
$$+\oint_{\Gamma}\left(\left(I_1\frac{\partial^2 u}{\partial t^2} - I_2\frac{\partial^3 w}{\partial x_1\partial t^2} + I_4\frac{\partial^2\theta_1}{\partial t^2}\right)n_{x_1}\delta w + \left(I_1\frac{\partial^2 v}{\partial t^2} - I_2\frac{\partial^3 w}{\partial x_2\partial t^2} + I_4\frac{\partial^2\theta_2}{\partial t^2}\right)n_{x_2}\delta w\right)d\Gamma, \tag{53}$$

şeklinde bulunmuştur. Dış kuvvetlerin yaptığı iş ise aşağıdaki gibi hesaplanmaktadır

$$W = \int_V \left(f_1 u_1 + f_2 u_2 + f_3 u_3 + c_1\omega_1 + c_2\omega_2 + c_3\omega_3\right)dV$$
$$+\int_{\Omega^+}\left(q_1^t u_1 + q_2^t u_2 + q_3^t u_3\right)dA + \int_{\Omega^-}\left(q_1^b u_1 + q_2^b u_2 + q_3^b u_3\right)dA \tag{54}$$
$$+\int_S\left(t_1 u_1 + t_2 u_2 + t_3 u_3\right)dS + \frac{1}{2}\int_A\left(P_{x_1}\left(\frac{\partial w}{\partial x_1}\right)^2 + P_{x_2}\left(\frac{\partial w}{\partial x_2}\right)^2\right)dA.$$

Bu eşitlikte f_1, f_2 ve f_3 birim hacime düşen cisim kuvvetlerini; c_1, c_2 ve c_3 birim hacime düşen cisim momentlerini; q_1, q_2 ve q_3 üst ve alt yüzeylerdeki yayılı kuvvetleri; ve t_1, t_2 ve t_3, x_3-eksenine parallel olan sınır yüzeye birim alan başına uygulanan kuvvetleri temsil etmektedir. t ve b üst indisleri sırasıyla üst ve alt yüzeyleri göstermektedir. P_{x_1} ve P_{x_2} ise, x_1 ve x_2 yönlerindeki düzlem içi basma kuvvetleridir.

Mikro-plağın ebatının oldukça küçük olması nedeniyle cisim kuvvetleri ve momentleri ihmal edilebilecek düzeydedir. Çalışmamızda dış kuvvet olarak yayılı kuvvet $q(x_1, x_2)$ ve P_{x_1} ve P_{x_2} ele alınmıştır. Bu kuvvetlerin yaptığı iş ve varyasyonu

$$W=\int_A qwdA+\frac{1}{2}\int_A\left(P_{x_1}\left(\frac{\partial w}{\partial x_1}\right)^2+P_{x_2}\left(\frac{\partial w}{\partial x_2}\right)^2\right)dA, \tag{55}$$

$$\delta W=\int_A q\delta wdA+\frac{1}{2}\int_A\left(P_{x_1}\delta\left(\frac{\partial w}{\partial x_1}\right)^2+P_{x_2}\delta\left(\frac{\partial w}{\partial x_2}\right)^2\right)dA, \tag{56}$$

olarak yazılır. P_{x1} ve P_{x2} sabit varsayılarak , kısmı integral yöntemi ile δW aşağıdaki forma indirgenmiştir:

$$\begin{aligned}\delta W&=\int_A q\delta wdA+\int_A\left(\frac{\partial w}{\partial x_1}P_{x_1}\delta\frac{\partial w}{\partial x_1}+\frac{\partial w}{\partial x_2}P_{x_2}\delta\frac{\partial w}{\partial x_2}\right)dA\\&=\int_A q\delta wdA+\int_A\left(-\frac{\partial^2 w}{\partial x_1^2}P_{x_1}\delta w-\frac{\partial^2 w}{\partial x_2^2}P_{x_2}\delta w\right)dA\\&+\oint_\Gamma\left(\frac{\partial w}{\partial x_1}P_{x_1}n_{x_1}\delta w+\frac{\partial w}{\partial x_2}P_{x_2}n_{x_2}\delta w\right)d\Gamma.\end{aligned} \tag{57}$$

Gerekli enerji varyasyonu terimleri (27) numaralı denklemde yerine konularak kısmi diferansiyel denklemler ile sınır koşulları şu şekilde türetilmiştir:

$\delta u:$

$$\begin{aligned}&\frac{\partial M_{11}^0}{\partial x_1}+\frac{\partial M_{12}^0}{\partial x_2}-\frac{\partial^2 P_1^0}{\partial x_1^2}-\frac{\partial^2 P_2^0}{\partial x_1\partial x_2}-\frac{2}{5}\frac{\partial^2 T_{111}^0}{\partial x_1^2}+\frac{1}{5}\frac{\partial^2 T_{111}^0}{\partial x_2^2}+\frac{2}{5}\frac{\partial^2 T_{222}^0}{\partial x_1\partial x_2}\\&-\frac{8}{5}\frac{\partial^2 T_{112}^0}{\partial x_1\partial x_2}+\frac{3}{5}\frac{\partial^2 T_{221}^0}{\partial x_1^2}-\frac{4}{5}\frac{\partial^2 T_{221}^0}{\partial x_2^2}+\frac{3}{5}\frac{\partial^2 T_{331}^0}{\partial x_1^2}+\frac{1}{5}\frac{\partial^2 T_{331}^0}{\partial x_2^2}+\frac{2}{5}\frac{\partial^2 T_{332}^0}{\partial x_1\partial x_2}+\frac{1}{2}\frac{\partial^2 Y_{13}^0}{\partial x_1\partial x_2}+\frac{1}{2}\frac{\partial^2 Y_{23}^0}{\partial x_2^2}\\&=I_0\frac{\partial^2 u}{\partial t^2}-I_1\frac{\partial^3 w}{\partial x_1\partial t^2}+I_3\frac{\partial^2\theta_1}{\partial t^2},\end{aligned} \tag{58}$$

$\delta v:$

$$\begin{aligned}&\frac{\partial M_{22}^0}{\partial x_2}+\frac{\partial M_{12}^0}{\partial x_1}-\frac{\partial^2 P_1^0}{\partial x_1\partial x_2}-\frac{\partial^2 P_2^0}{\partial x_2^2}+\frac{2}{5}\frac{\partial^2 T_{111}^0}{\partial x_1\partial x_2}+\frac{1}{5}\frac{\partial^2 T_{222}^0}{\partial x_1^2}-\frac{2}{5}\frac{\partial^2 T_{222}^0}{\partial x_2^2}\\&-\frac{4}{5}\frac{\partial^2 T_{112}^0}{\partial x_1^2}+\frac{3}{5}\frac{\partial^2 T_{112}^0}{\partial x_2^2}-\frac{8}{5}\frac{\partial^2 T_{221}^0}{\partial x_1\partial x_2}+\frac{2}{5}\frac{\partial^2 T_{331}^0}{\partial x_1\partial x_2}+\frac{1}{5}\frac{\partial^2 T_{332}^0}{\partial x_1^2}+\frac{3}{5}\frac{\partial^2 T_{332}^0}{\partial x_2^2}-\frac{1}{2}\frac{\partial^2 Y_{13}^0}{\partial x_1^2}-\frac{1}{2}\frac{\partial^2 Y_{23}^0}{\partial x_1\partial x_2}\\&=I_0\frac{\partial^2 v}{\partial t^2}-I_1\frac{\partial^3 w}{\partial x_2\partial t^2}+I_3\frac{\partial^2\theta_2}{\partial t^2},\end{aligned} \tag{59}$$

$\delta w:$

$$
\begin{aligned}
&\frac{\partial^2 M_{11}^1}{\partial x_1^2}+\frac{\partial^2 M_{22}^1}{\partial x_2^2}+2\frac{\partial^2 M_{12}^1}{\partial x_1 \partial x_2}-\frac{\partial^3 P_1^1}{\partial x_1^3}-\frac{\partial^3 P_1^1}{\partial x_1 \partial x_2^2}-\frac{\partial^3 P_2^1}{\partial x_2^3}-\frac{\partial^3 P_2^1}{\partial x_1^2 \partial x_2}+\frac{\partial^2 P_3^0}{\partial x_1^2}+\frac{\partial^2 P_3^0}{\partial x_2^2} \\
&-\frac{2}{5}\frac{\partial^3 T_{111}^1}{\partial x_1^3}+\frac{3}{5}\frac{\partial^3 T_{111}^1}{\partial x_1 \partial x_2^2}-\frac{2}{5}\frac{\partial^3 T_{222}^1}{\partial x_2^3}+\frac{3}{5}\frac{\partial^3 T_{222}^1}{\partial x_1^2 \partial x_2}-\frac{1}{5}\frac{\partial^2 T_{333}^0}{\partial x_1^2}-\frac{1}{5}\frac{\partial^2 T_{333}^0}{\partial x_2^2}+\frac{3}{5}\frac{\partial^3 T_{112}^1}{\partial x_2^3}-\frac{12}{5}\frac{\partial^3 T_{112}^1}{\partial x_1^2 \partial x_2} \\
&+\frac{4}{5}\frac{\partial^2 T_{113}^0}{\partial x_1^2}-\frac{1}{5}\frac{\partial^2 T_{113}^0}{\partial x_2^2}+\frac{3}{5}\frac{\partial^3 T_{221}^1}{\partial x_1^3}-\frac{12}{5}\frac{\partial^3 T_{221}^1}{\partial x_1 \partial x_2^2}-\frac{1}{5}\frac{\partial^2 T_{223}^0}{\partial x_1^2}+\frac{4}{5}\frac{\partial^2 T_{223}^0}{\partial x_2^2}+\frac{3}{5}\frac{\partial^3 T_{331}^1}{\partial x_1^3}+\frac{3}{5}\frac{\partial^3 T_{331}^1}{\partial x_1 \partial x_2^2} \\
&+\frac{3}{5}\frac{\partial^3 T_{332}^1}{\partial x_2^3}+\frac{3}{5}\frac{\partial^3 T_{332}^1}{\partial x_1^2 \partial x_2}+2\frac{\partial^2 T_{123}^0}{\partial x_1 \partial x_2}-\frac{\partial^2 Y_{11}^0}{\partial x_1 \partial x_2}+\frac{\partial^2 Y_{22}^0}{\partial x_1 \partial x_2}+\frac{\partial^2 Y_{12}^0}{\partial x_1^2}-\frac{\partial^2 Y_{12}^0}{\partial x_2^2} \\
&+q-P_{x_1}\frac{\partial^2 w}{\partial x_1^2}-P_{x_2}\frac{\partial^2 w}{\partial x_2^2}+P_{x_1}^0\frac{\partial^2 w}{\partial x_1^2}+P_{x_2}^0\frac{\partial^2 w}{\partial x_2^2}+2P_{x_1x_2}^0\frac{\partial^2 w}{\partial x_1 \partial x_2} \\
&=I_1\frac{\partial^3 u}{\partial x_1 \partial t^2}-I_2\frac{\partial^4 w}{\partial x_1^2 \partial t^2}+I_4\frac{\partial^3 \theta_1}{\partial x_1 \partial t^2}+I_1\frac{\partial^3 v}{\partial x_2 \partial t^2}-I_2\frac{\partial^4 w}{\partial x_2^2 \partial t^2}+I_4\frac{\partial^3 \theta_2}{\partial x_2 \partial t^2}+I_0\frac{\partial^2 w}{\partial t^2},
\end{aligned}
\tag{60}
$$

$\delta\theta_1:$

$$
\begin{aligned}
&\frac{\partial M_{11}^2}{\partial x_1}+\frac{\partial M_{12}^2}{\partial x_2}-M_{13}^3-\frac{\partial^2 P_1^2}{\partial x_1^2}-\frac{\partial^2 P_2^2}{\partial x_1 \partial x_2}+\frac{\partial P_3^3}{\partial x_1}-\frac{2}{5}\frac{\partial^2 T_{111}^2}{\partial x_1^2}+\frac{1}{5}\frac{\partial^2 T_{111}^2}{\partial x_2^2}+\frac{1}{5}T_{111}^4+\frac{2}{5}\frac{\partial^2 T_{222}^2}{\partial x_1 \partial x_2} \\
&-\frac{2}{5}\frac{\partial T_{333}^3}{\partial x_1}-\frac{8}{5}\frac{\partial^2 T_{112}^2}{\partial x_1 \partial x_2}+\frac{8}{5}\frac{\partial T_{113}^3}{\partial x_1}+\frac{3}{5}\frac{\partial^2 T_{221}^2}{\partial x_1^2}-\frac{4}{5}\frac{\partial^2 T_{221}^2}{\partial x_2^2}+\frac{1}{5}T_{221}^4-\frac{2}{5}\frac{\partial T_{223}^3}{\partial x_1} \\
&+\frac{3}{5}\frac{\partial^2 T_{331}^2}{\partial x_1^2}+\frac{1}{5}\frac{\partial^2 T_{331}^2}{\partial x_2^2}-\frac{4}{5}T_{331}^4+\frac{2}{5}\frac{\partial^2 T_{332}^2}{\partial x_1 \partial x_2}+2\frac{\partial T_{123}^3}{\partial x_2}+\frac{1}{2}\frac{\partial Y_{22}^3}{\partial x_2}-\frac{1}{2}\frac{\partial Y_{33}^3}{\partial x_2} \\
&+\frac{1}{2}\frac{\partial Y_{12}^3}{\partial x_1}+\frac{1}{2}\frac{\partial^2 Y_{13}^2}{\partial x_1 \partial x_2}+\frac{1}{2}\frac{\partial^2 Y_{23}^2}{\partial x_2^2}-\frac{1}{2}Y_{23}^4=I_3\frac{\partial^2 u}{\partial t^2}-I_4\frac{\partial^3 w}{\partial x_1 \partial t^2}+I_5\frac{\partial^2 \theta_1}{\partial t^2},
\end{aligned}
\tag{61}
$$

$\delta\theta_2:$

$$
\begin{aligned}
&\frac{\partial M_{22}^2}{\partial x_2}+\frac{\partial M_{12}^2}{\partial x_1}-M_{23}^3-\frac{\partial^2 P_1^2}{\partial x_1 \partial x_2}-\frac{\partial^2 P_2^2}{\partial x_2^2}+\frac{\partial P_3^3}{\partial x_2}+\frac{2}{5}\frac{\partial^2 T_{111}^2}{\partial x_1 \partial x_2}+\frac{1}{5}\frac{\partial^2 T_{222}^2}{\partial x_1^2}-\frac{2}{5}\frac{\partial^2 T_{222}^2}{\partial x_2^2}+\frac{1}{5}T_{222}^4 \\
&-\frac{2}{5}\frac{\partial T_{333}^3}{\partial x_2}-\frac{4}{5}\frac{\partial^2 T_{112}^2}{\partial x_1^2}+\frac{3}{5}\frac{\partial^2 T_{112}^2}{\partial x_2^2}+\frac{1}{5}T_{112}^4-\frac{2}{5}\frac{\partial T_{113}^3}{\partial x_2}-\frac{8}{5}\frac{\partial^2 T_{221}^2}{\partial x_1 \partial x_2}+\frac{8}{5}\frac{\partial T_{223}^3}{\partial x_2} \\
&+\frac{2}{5}\frac{\partial^2 T_{331}^2}{\partial x_1 \partial x_2}+\frac{1}{5}\frac{\partial^2 T_{332}^2}{\partial x_1^2}+\frac{3}{5}\frac{\partial^2 T_{332}^2}{\partial x_2^2}-\frac{4}{5}T_{332}^4+2\frac{\partial T_{123}^3}{\partial x_1}-\frac{1}{2}\frac{\partial Y_{11}^3}{\partial x_1}+\frac{1}{2}\frac{\partial Y_{33}^3}{\partial x_1} \\
&-\frac{1}{2}\frac{\partial Y_{12}^3}{\partial x_2}-\frac{1}{2}\frac{\partial^2 Y_{13}^2}{\partial x_1^2}+\frac{1}{2}Y_{13}^4-\frac{1}{2}\frac{\partial^2 Y_{23}^2}{\partial x_1 \partial x_2}=I_3\frac{\partial^2 v}{\partial t^2}-I_4\frac{\partial^3 w}{\partial x_2 \partial t^2}+I_5\frac{\partial^2 \theta_2}{\partial t^2},
\end{aligned}
\tag{62}
$$

$\delta u = 0$ veya

$$\left(M_{11}^0 - \frac{\partial P_1^0}{\partial x_1} - \frac{1}{2}\frac{\partial P_2^0}{\partial x_2} - \frac{2}{5}\frac{\partial T_{111}^0}{\partial x_1} + \frac{1}{5}\frac{\partial T_{222}^0}{\partial x_2} - \frac{4}{5}\frac{\partial T_{112}^0}{\partial x_2} + \frac{3}{5}\frac{\partial T_{221}^0}{\partial x_1} + \frac{3}{5}\frac{\partial T_{331}^0}{\partial x_1} + \frac{1}{5}\frac{\partial T_{332}^0}{\partial x_2} \right) n_{x_1}$$
$$+ \left(M_{12}^0 - \frac{1}{2}\frac{\partial P_2^0}{\partial x_1} + \frac{1}{5}\frac{\partial T_{111}^0}{\partial x_2} + \frac{1}{5}\frac{\partial T_{222}^0}{\partial x_1} - \frac{4}{5}\frac{\partial T_{112}^0}{\partial x_1} - \frac{4}{5}\frac{\partial T_{221}^0}{\partial x_2} + \frac{1}{5}\frac{\partial T_{331}^0}{\partial x_2} + \frac{1}{5}\frac{\partial T_{332}^0}{\partial x_1} + \frac{1}{2}\frac{\partial Y_{13}^0}{\partial x_1} + \frac{1}{2}\frac{\partial Y_{23}^0}{\partial x_2} \right) n_{x_2} = 0,$$

(63)

$\delta \dfrac{\partial u}{\partial x_1} = 0$ veya

$$\left(P_1^0 + \frac{2}{5}T_{111}^0 - \frac{3}{5}T_{221}^0 - \frac{3}{5}T_{331}^0 \right) n_{x_1} + \left(\frac{1}{2}P_2^0 - \frac{1}{5}T_{222}^0 + \frac{4}{5}T_{112}^0 - \frac{1}{5}T_{332}^0 \right) n_{x_2} = 0,$$

(64)

$\delta \dfrac{\partial u}{\partial x_2} = 0$ veya

$$\left(\frac{1}{2}P_2^0 - \frac{1}{5}T_{222}^0 + \frac{4}{5}T_{112}^0 - \frac{1}{5}T_{332}^0 - \frac{1}{2}Y_{13}^0 \right) n_{x_1} + \left(-\frac{1}{5}T_{111}^0 + \frac{4}{5}T_{221}^0 - \frac{1}{5}T_{331}^0 - \frac{1}{2}Y_{23}^0 \right) n_{x_2} = 0,$$

(65)

$\delta v = 0$ veya

$$\left(M_{12}^0 - \frac{1}{2}\frac{\partial P_1^0}{\partial x_2} + \frac{1}{5}\frac{\partial T_{111}^0}{\partial x_2} + \frac{1}{5}\frac{\partial T_{222}^0}{\partial x_1} - \frac{4}{5}\frac{\partial T_{112}^0}{\partial x_1} - \frac{4}{5}\frac{\partial T_{221}^0}{\partial x_2} + \frac{1}{5}\frac{\partial T_{331}^0}{\partial x_2} + \frac{1}{5}\frac{\partial T_{332}^0}{\partial x_1} - \frac{1}{2}\frac{\partial Y_{13}^0}{\partial x_1} - \frac{1}{2}\frac{\partial Y_{23}^0}{\partial x_2} \right) n_{x_1}$$
$$+ \left(M_{22}^0 - \frac{1}{2}\frac{\partial P_1^0}{\partial x_1} - \frac{\partial P_2^0}{\partial x_2} + \frac{1}{5}\frac{\partial T_{111}^0}{\partial x_1} - \frac{2}{5}\frac{\partial T_{222}^0}{\partial x_2} + \frac{3}{5}\frac{\partial T_{112}^0}{\partial x_2} - \frac{4}{5}\frac{\partial T_{221}^0}{\partial x_1} + \frac{1}{5}\frac{\partial T_{331}^0}{\partial x_1} + \frac{3}{5}\frac{\partial T_{332}^0}{\partial x_2} \right) n_{x_2} = 0,$$

(66)

$\delta \dfrac{\partial v}{\partial x_1} = 0$ veya

$$\left(-\frac{1}{5}T_{222}^0 + \frac{4}{5}T_{112}^0 - \frac{1}{5}T_{332}^0 + \frac{1}{2}Y_{13}^0 \right) n_{x_1} + \left(\frac{1}{2}P_1^0 - \frac{1}{5}T_{111}^0 + \frac{4}{5}T_{221}^0 - \frac{1}{5}T_{331}^0 + \frac{1}{2}Y_{23}^0 \right) n_{x_2} = 0,$$

(67)

$\delta \dfrac{\partial v}{\partial x_2} = 0$ veya

$$\left(\frac{1}{2}P_1^0 - \frac{1}{5}T_{111}^0 + \frac{4}{5}T_{221}^0 - \frac{1}{5}T_{331}^0 \right) n_{x_1} + \left(P_2^0 + \frac{2}{5}T_{222}^0 - \frac{3}{5}T_{112}^0 - \frac{3}{5}T_{332}^0 \right) n_{x_2} = 0,$$

(68)

$\delta w = 0$ veya

$$\left(\frac{\partial M_{11}^1}{\partial x_1}+\frac{\partial M_{12}^1}{\partial x_2}-\frac{\partial^2 P_1^1}{\partial x_1^2}-\frac{\partial^2 P_2^1}{\partial x_1 \partial x_2}+\frac{\partial P_3^0}{\partial x_1}-\frac{2}{5}\frac{\partial^2 T_{111}^1}{\partial x_1^2}+\frac{3}{5}\frac{\partial T_{222}^1}{\partial x_1 \partial x_2}-\frac{1}{5}\frac{\partial T_{333}^0}{\partial x_1}-\frac{12}{5}\frac{\partial^2 T_{112}^1}{\partial x_1 \partial x_2}+\frac{4}{5}\frac{\partial T_{113}^0}{\partial x_1}\right.$$
$$\left.+\frac{3}{5}\frac{\partial^2 T_{221}^1}{\partial x_1^2}-\frac{1}{5}\frac{\partial T_{223}^0}{\partial x_1}+\frac{3}{5}\frac{\partial^2 T_{331}^1}{\partial x_1^2}+\frac{3}{5}\frac{\partial^2 T_{332}^1}{\partial x_1 \partial x_2}+\frac{\partial T_{123}^0}{\partial x_2}-\frac{1}{2}\frac{\partial Y_{11}^0}{\partial x_2}+\frac{1}{2}\frac{\partial Y_{22}^0}{\partial x_2}+\frac{\partial Y_{12}^0}{\partial x_1}-\frac{\partial w}{\partial x_1}P_{x_1}\right)n_{x_1}$$
$$+\left(\frac{\partial M_{22}^1}{\partial x_2}+\frac{\partial M_{12}^1}{\partial x_1}-\frac{\partial^2 P_1^1}{\partial x_1 \partial x_2}-\frac{\partial^2 P_2^1}{\partial x_2^2}+\frac{\partial P_3^0}{\partial x_2}+\frac{3}{5}\frac{\partial^2 T_{111}^1}{\partial x_1 \partial x_2}-\frac{2}{5}\frac{\partial^2 T_{222}^1}{\partial x_2^2}-\frac{1}{5}\frac{\partial T_{333}^0}{\partial x_2}+\frac{3}{5}\frac{\partial^2 T_{112}^1}{\partial x_2^2}-\frac{1}{5}\frac{\partial T_{113}^0}{\partial x_2}\right.$$
$$\left.-\frac{12}{5}\frac{\partial^2 T_{221}^1}{\partial x_1 \partial x_2}+\frac{4}{5}\frac{\partial T_{223}^0}{\partial x_2}+\frac{3}{5}\frac{\partial^2 T_{331}^1}{\partial x_1 \partial x_2}+\frac{3}{5}\frac{\partial^2 T_{332}^1}{\partial x_2^2}+\frac{\partial T_{123}^0}{\partial x_1}-\frac{1}{2}\frac{\partial Y_{11}^0}{\partial x_1}+\frac{1}{2}\frac{\partial Y_{22}^0}{\partial x_1}-\frac{\partial Y_{12}^0}{\partial x_2}-\frac{\partial w}{\partial x_2}P_{x_2}\right)n_{x_2}$$
$$=\left(I_1\frac{\partial^2 u}{\partial t^2}-I_2\frac{\partial^3 w}{\partial x_1 \partial t^2}+I_4\frac{\partial^2 \theta_1}{\partial t^2}\right)n_{x_1}+\left(I_1\frac{\partial^2 v}{\partial t^2}-I_2\frac{\partial^3 w}{\partial x_2 \partial t^2}+I_4\frac{\partial^2 \theta_2}{\partial t^2}\right)n_{x_2}, \tag{69}$$

$\delta \dfrac{\partial w}{\partial x_1} = 0$ veya

$$\left(-M_{11}^1+\frac{\partial P_1^1}{\partial x_1}+\frac{\partial P_2^1}{\partial x_2}-P_3^0+\frac{2}{5}\frac{\partial T_{111}^1}{\partial x_1}-\frac{3}{5}\frac{\partial T_{222}^1}{\partial x_2}+\frac{1}{5}T_{333}^0+\frac{12}{5}\frac{\partial T_{112}^1}{\partial x_2}-\frac{4}{5}T_{113}^0\right.$$
$$\left.-\frac{3}{5}\frac{\partial T_{221}^1}{\partial x_1}+\frac{1}{5}T_{223}^0-\frac{3}{5}\frac{\partial T_{331}^1}{\partial x_1}-\frac{3}{5}\frac{\partial T_{332}^1}{\partial x_2}-Y_{12}^0\right)n_{x_1}+\left(-M_{12}^1-T_{123}^0+\frac{1}{2}Y_{11}^0-\frac{1}{2}Y_{22}^0\right)n_{x_2}=0, \tag{70}$$

$\delta \dfrac{\partial w}{\partial x_2} = 0$ veya

$$\left(-M_{12}^1-T_{123}^0+\frac{1}{2}Y_{11}^0-\frac{1}{2}Y_{22}^0\right)n_{x_1}+\left(-M_{22}^1+\frac{\partial P_1^1}{\partial x_1}+\frac{\partial P_2^1}{\partial x_2}-P_3^0-\frac{3}{5}\frac{\partial T_{111}^1}{\partial x_1}+\frac{2}{5}\frac{\partial T_{222}^1}{\partial x_2}+\frac{1}{5}T_{333}^0\right.$$
$$\left.-\frac{3}{5}\frac{\partial T_{112}^1}{\partial x_2}+\frac{1}{5}T_{113}^0+\frac{12}{5}\frac{\partial T_{221}^1}{\partial x_1}-\frac{4}{5}T_{223}^0-\frac{3}{5}\frac{\partial T_{331}^1}{\partial x_1}-\frac{3}{5}\frac{\partial T_{332}^1}{\partial x_2}+Y_{12}^0\right)n_{x_2}=0, \tag{71}$$

$\delta \dfrac{\partial^2 w}{\partial x_1^2} = 0$ veya

$$\left(-P_1^1-\frac{2}{5}T_{111}^1+\frac{3}{5}T_{221}^1+\frac{3}{5}T_{331}^1\right)n_{x_1}+\left(-P_2^1+\frac{3}{5}T_{222}^1-\frac{12}{5}T_{112}^1+\frac{3}{5}T_{332}^1\right)n_{x_2}=0, \tag{72}$$

$$\delta \frac{\partial^2 w}{\partial x_2^2} = 0 \quad \text{veya}$$

$$\left(-P_1^1 + \frac{3}{5}T_{111}^1 - \frac{12}{5}T_{221}^1 + \frac{3}{5}T_{331}^1\right)n_{x_1} + \left(-P_2^1 - \frac{2}{5}T_{222}^1 + \frac{3}{5}T_{112}^1 + \frac{3}{5}T_{332}^1\right)n_{x_2} = 0, \tag{73}$$

$$\delta\theta_1 = 0 \quad \text{veya}$$

$$\left(M_{11}^2 - \frac{\partial P_1^2}{\partial x_1} - \frac{1}{2}\frac{\partial P_2^2}{\partial x_2} + P_3^3 - \frac{2}{5}\frac{\partial T_{111}^2}{\partial x_1} + \frac{1}{5}\frac{\partial T_{222}^2}{\partial x_2} - \frac{2}{5}T_{333}^3 - \frac{4}{5}\frac{\partial T_{112}^2}{\partial x_2} + \frac{8}{5}T_{113}^3 + \frac{3}{5}\frac{\partial T_{221}^2}{\partial x_1} - \frac{2}{5}T_{223}^3\right.$$
$$\left. + \frac{3}{5}\frac{\partial T_{331}^2}{\partial x_1} + \frac{1}{5}\frac{\partial T_{332}^2}{\partial x_2} + \frac{1}{2}Y_{12}^3 + \frac{1}{4}\frac{\partial Y_{13}^2}{\partial x_2}\right)n_{x_1} + \left(M_{12}^2 - \frac{1}{2}\frac{\partial P_2^2}{\partial x_1} + \frac{1}{5}\frac{\partial T_{111}^2}{\partial x_2} + \frac{1}{5}\frac{\partial T_{222}^2}{\partial x_1} - \frac{4}{5}\frac{\partial T_{112}^2}{\partial x_1} - \frac{4}{5}\frac{\partial T_{221}^2}{\partial x_2}\right.$$
$$\left. + \frac{1}{5}\frac{\partial T_{331}^2}{\partial x_2} + \frac{1}{5}\frac{\partial T_{332}^2}{\partial x_1} + 2T_{123}^3 + \frac{1}{2}Y_{22}^3 - \frac{1}{2}Y_{33}^3 + \frac{1}{4}\frac{\partial Y_{13}^2}{\partial x_1} + \frac{1}{2}\frac{\partial Y_{23}^2}{\partial x_2}\right)n_{x_2} = 0, \tag{74}$$

$$\delta \frac{\partial \theta_1}{\partial x_1} = 0 \quad \text{veya}$$

$$\left(P_1^2 + \frac{2}{5}T_{111}^2 - \frac{3}{5}T_{221}^2 - \frac{3}{5}T_{331}^2\right)n_{x_1} + \left(\frac{1}{2}P_2^2 - \frac{1}{5}T_{222}^2 + \frac{4}{5}T_{112}^2 - \frac{1}{5}T_{332}^2 - \frac{1}{4}Y_{13}^2\right)n_{x_2} = 0, \tag{75}$$

$$\delta \frac{\partial \theta_1}{\partial x_2} = 0 \quad \text{veya}$$

$$\left(\frac{1}{2}P_2^2 - \frac{1}{5}T_{222}^2 + \frac{4}{5}T_{112}^2 - \frac{1}{5}T_{332}^2 - \frac{1}{4}Y_{13}^2\right)n_{x_1} + \left(-\frac{1}{5}T_{111}^2 + \frac{4}{5}T_{221}^2 - \frac{1}{5}T_{331}^2 - \frac{1}{2}Y_{23}^2\right)n_{x_2} = 0, \tag{76}$$

$$\delta\theta_2 = 0 \quad \text{veya}$$

$$\left(M_{12}^2 - \frac{1}{2}\frac{\partial P_1^2}{\partial x_2} + \frac{1}{5}\frac{\partial T_{111}^2}{\partial x_2} + \frac{1}{5}\frac{\partial T_{222}^2}{\partial x_1} - \frac{4}{5}\frac{\partial T_{112}^2}{\partial x_1} - \frac{4}{5}\frac{\partial T_{221}^2}{\partial x_2} + \frac{1}{5}\frac{\partial T_{331}^2}{\partial x_2} + \frac{1}{5}\frac{\partial T_{332}^2}{\partial x_1} + 2T_{123}^3\right)$$
$$\left. - \frac{1}{2}Y_{11}^3 + \frac{1}{2}Y_{33}^3 - \frac{1}{2}\frac{\partial Y_{13}^2}{\partial x_1} - \frac{1}{4}\frac{\partial Y_{23}^2}{\partial x_2}\right)n_{x_1} + \left(M_{22}^2 - \frac{1}{2}\frac{\partial P_1^2}{\partial x_1} - \frac{\partial P_2^2}{\partial x_2} + P_3^3 + \frac{1}{5}\frac{\partial T_{111}^2}{\partial x_1} - \frac{2}{5}\frac{\partial T_{222}^2}{\partial x_2} - \frac{2}{5}T_{333}^3\right.$$
$$\left. + \frac{3}{5}\frac{\partial T_{112}^2}{\partial x_2} - \frac{2}{5}T_{113}^3 - \frac{4}{5}\frac{\partial T_{221}^2}{\partial x_1} + \frac{8}{5}T_{223}^3 + \frac{1}{5}\frac{\partial T_{331}^2}{\partial x_1} + \frac{3}{5}\frac{\partial T_{332}^2}{\partial x_2} - \frac{1}{2}Y_{12}^3 - \frac{1}{4}\frac{\partial Y_{23}^2}{\partial x_1}\right)n_{x_2} = 0, \tag{77}$$

$$\delta \frac{\partial \theta_2}{\partial x_1} = 0 \quad \text{veya}$$

$$\left(-\frac{1}{5} T_{222}^2 + \frac{4}{5} T_{112}^2 - \frac{1}{5} T_{332}^2 + \frac{1}{2} Y_{13}^2 \right) n_{x_1} + \left(\frac{1}{2} P_1^2 - \frac{1}{5} T_{111}^2 + \frac{4}{5} T_{221}^2 - \frac{1}{5} T_{331}^2 + \frac{1}{4} Y_{23}^2 \right) n_{x_2} = 0, \tag{78}$$

$$\delta \frac{\partial \theta_2}{\partial x_2} = 0 \quad \text{veya}$$

$$\left(\frac{1}{2} P_1^2 - \frac{1}{5} T_{111}^2 + \frac{4}{5} T_{221}^2 - \frac{1}{5} T_{331}^2 + \frac{1}{4} Y_{23}^2 \right) n_{x_1} + \left(P_2^2 + \frac{2}{5} T_{222}^2 - \frac{3}{5} T_{112}^2 - \frac{3}{5} T_{332}^2 \right) n_{x_2} = 0, \tag{79}$$

(60) numaralı eşitlikteki $P_{x_1}^0$, $P_{x_2}^0$ ve $P_{x_1 x_2}^0$ termal değişimden dolayı oluşan düzlem içi kuvvetlerdir ve statik termal eğilme analizi kullanılarak bulunur.

3.2.1. Denklemlerin ϕ_1 ve ϕ_2 Cinsinden İfadeleri

Bir önceki bölümde kısmi diferansiyel denklemler ve sınır koşulları orta düzlemin kayma gerinimleri θ_1 ve θ_2 cinsinden türetilmiştir. Yerdeğiştirme alanında (denklem (1) - (3)) θ_1 ve θ_2 yerine orta düzlem normalinin dönmeleri ϕ_1 ve ϕ_2 kullanılarak denklemler ϕ_1 ve ϕ_2 cinsinden ifade edilebilir. (4) ve (5) numaralı eşitlikler aracılığıyla, ϕ_1 ve ϕ_2 cinsinden kısmi diferansiyel denklemler şu şekilde bulunmuştur:

$\delta u :$

$$\begin{aligned}
&\frac{\partial M_{11}^0}{\partial x_1}+\frac{\partial M_{12}^0}{\partial x_2}-\frac{\partial^2 P_1^0}{\partial x_1^2}-\frac{\partial^2 P_2^0}{\partial x_1 \partial x_2}-\frac{2}{5}\frac{\partial^2 T_{111}^0}{\partial x_1^2}+\frac{1}{5}\frac{\partial^2 T_{111}^0}{\partial x_2^2}+\frac{2}{5}\frac{\partial^2 T_{222}^0}{\partial x_1 \partial x_2} \\
&-\frac{8}{5}\frac{\partial^2 T_{112}^0}{\partial x_1 \partial x_2}+\frac{3}{5}\frac{\partial^2 T_{221}^0}{\partial x_1^2}-\frac{4}{5}\frac{\partial^2 T_{221}^0}{\partial x_2^2}+\frac{3}{5}\frac{\partial^2 T_{331}^0}{\partial x_1^2}+\frac{1}{5}\frac{\partial^2 T_{331}^0}{\partial x_2^2}+\frac{2}{5}\frac{\partial^2 T_{332}^0}{\partial x_1 \partial x_2}+\frac{1}{2}\frac{\partial^2 Y_{13}^0}{\partial x_1 \partial x_2}+\frac{1}{2}\frac{\partial^2 Y_{23}^0}{\partial x_2^2} \\
&=I_0\frac{\partial^2 u}{\partial t^2}+\left(I_3-I_1\right)\frac{\partial^3 w}{\partial x_1 \partial t^2}+I_3\frac{\partial^2 \phi_1}{\partial t^2},
\end{aligned} \tag{80}$$

$\delta v :$

$$\begin{aligned}
&\frac{\partial M_{22}^0}{\partial x_2}+\frac{\partial M_{12}^0}{\partial x_1}-\frac{\partial^2 P_1^0}{\partial x_1 \partial x_2}-\frac{\partial^2 P_2^0}{\partial x_2^2}+\frac{2}{5}\frac{\partial^2 T_{111}^0}{\partial x_1 \partial x_2}+\frac{1}{5}\frac{\partial^2 T_{222}^0}{\partial x_1^2}-\frac{2}{5}\frac{\partial^2 T_{222}^0}{\partial x_2^2} \\
&-\frac{4}{5}\frac{\partial^2 T_{112}^0}{\partial x_1^2}+\frac{3}{5}\frac{\partial^2 T_{112}^0}{\partial x_2^2}-\frac{8}{5}\frac{\partial^2 T_{221}^0}{\partial x_1 \partial x_2}+\frac{2}{5}\frac{\partial^2 T_{331}^0}{\partial x_1 \partial x_2}+\frac{1}{5}\frac{\partial^2 T_{332}^0}{\partial x_1^2}+\frac{3}{5}\frac{\partial^2 T_{332}^0}{\partial x_2^2}-\frac{1}{2}\frac{\partial^2 Y_{13}^0}{\partial x_1^2}-\frac{1}{2}\frac{\partial^2 Y_{23}^0}{\partial x_1 \partial x_2} \\
&=I_0\frac{\partial^2 v}{\partial t^2}+\left(I_3-I_1\right)\frac{\partial^3 w}{\partial x_2 \partial t^2}+I_3\frac{\partial^2 \phi_2}{\partial t^2},
\end{aligned} \tag{81}$$

$\delta w:$

$$
\begin{aligned}
&\frac{\partial^2 M_{11}^1}{\partial x_1^2}-\frac{\partial^2 M_{11}^2}{\partial x_1^2}+\frac{\partial^2 M_{22}^1}{\partial x_2^2}-\frac{\partial^2 M_{22}^2}{\partial x_2^2}+2\frac{\partial^2 M_{12}^1}{\partial x_1\partial x_2}-2\frac{\partial^2 M_{12}^2}{\partial x_1\partial x_2}+\frac{\partial M_{13}^3}{\partial x_1}+\frac{\partial M_{23}^3}{\partial x_2}\\
&-\frac{\partial^3 P_1^1}{\partial x_1^3}+\frac{\partial^3 P_1^2}{\partial x_1^3}-\frac{\partial^3 P_1^1}{\partial x_1\partial x_2^2}+\frac{\partial^3 P_1^2}{\partial x_1\partial x_2^2}-\frac{\partial^3 P_2^1}{\partial x_2^3}+\frac{\partial^3 P_2^2}{\partial x_2^3}-\frac{\partial^3 P_2^1}{\partial x_1^2\partial x_2}+\frac{\partial^3 P_2^2}{\partial x_1^2\partial x_2}+\frac{\partial^2 P_3^0}{\partial x_1^2}-\frac{\partial^2 P_3^3}{\partial x_1^2}+\frac{\partial^2 P_3^0}{\partial x_2^2}-\frac{\partial^2 P_3^3}{\partial x_2^2}\\
&-\frac{2}{5}\frac{\partial^3 T_{111}^1}{\partial x_1^3}+\frac{2}{5}\frac{\partial^3 T_{111}^2}{\partial x_1^3}+\frac{3}{5}\frac{\partial^3 T_{111}^1}{\partial x_1\partial x_2^2}-\frac{2}{5}\frac{\partial^3 T_{111}^2}{\partial x_1\partial x_2^2}-\frac{1}{5}\frac{\partial^3 T_{111}^2}{\partial x_1\partial x_2^2}-\frac{1}{5}\frac{\partial T_{111}^4}{\partial x_1}\\
&-\frac{2}{5}\frac{\partial^3 T_{222}^1}{\partial x_2^3}+\frac{2}{5}\frac{\partial^3 T_{222}^2}{\partial x_2^3}+\frac{3}{5}\frac{\partial^3 T_{222}^1}{\partial x_1^2\partial x_2}-\frac{1}{5}\frac{\partial^3 T_{222}^2}{\partial x_1^2\partial x_2}-\frac{2}{5}\frac{\partial^3 T_{222}^2}{\partial x_1^2\partial x_2}-\frac{1}{5}\frac{\partial T_{222}^4}{\partial x_2}\\
&-\frac{1}{5}\frac{\partial^2 T_{333}^0}{\partial x_1^2}+\frac{2}{5}\frac{\partial^2 T_{333}^3}{\partial x_1^2}-\frac{1}{5}\frac{\partial^2 T_{333}^0}{\partial x_2^2}+\frac{2}{5}\frac{\partial^2 T_{333}^3}{\partial x_2^2}+\frac{3}{5}\frac{\partial^3 T_{112}^1}{\partial x_2^3}-\frac{3}{5}\frac{\partial^3 T_{112}^2}{\partial x_2^3}-\frac{12}{5}\frac{\partial^3 T_{112}^1}{\partial x_1^2\partial x_2}+\frac{4}{5}\frac{\partial^3 T_{112}^2}{\partial x_1^2\partial x_2}+\frac{8}{5}\frac{\partial^3 T_{112}^2}{\partial x_1^2\partial x_2}-\frac{1}{5}\frac{\partial T_{112}^4}{\partial x_2}\\
&+\frac{4}{5}\frac{\partial^2 T_{113}^0}{\partial x_1^2}-\frac{8}{5}\frac{\partial^2 T_{113}^3}{\partial x_1^2}-\frac{1}{5}\frac{\partial^2 T_{113}^0}{\partial x_2^2}+\frac{2}{5}\frac{\partial^2 T_{113}^3}{\partial x_2^2}+\frac{3}{5}\frac{\partial^3 T_{221}^1}{\partial x_1^3}-\frac{3}{5}\frac{\partial^3 T_{221}^2}{\partial x_1^3}-\frac{12}{5}\frac{\partial^3 T_{221}^1}{\partial x_1\partial x_2^2}+\frac{8}{5}\frac{\partial^3 T_{221}^2}{\partial x_1\partial x_2^2}+\frac{4}{5}\frac{\partial^3 T_{221}^2}{\partial x_1\partial x_2^2}-\frac{1}{5}\frac{\partial T_{221}^4}{\partial x_1}\\
&-\frac{1}{5}\frac{\partial^2 T_{223}^0}{\partial x_1^2}+\frac{2}{5}\frac{\partial^2 T_{223}^3}{\partial x_1^2}+\frac{4}{5}\frac{\partial^2 T_{223}^0}{\partial x_2^2}-\frac{8}{5}\frac{\partial^2 T_{223}^3}{\partial x_2^2}+\frac{3}{5}\frac{\partial^3 T_{331}^1}{\partial x_1^3}-\frac{3}{5}\frac{\partial^3 T_{331}^2}{\partial x_1^3}+\frac{3}{5}\frac{\partial^3 T_{331}^1}{\partial x_1\partial x_2^2}-\frac{2}{5}\frac{\partial^3 T_{331}^2}{\partial x_1\partial x_2^2}-\frac{1}{5}\frac{\partial^3 T_{331}^2}{\partial x_1\partial x_2^2}+\frac{4}{5}\frac{\partial T_{331}^4}{\partial x_1}\\
&+\frac{3}{5}\frac{\partial^3 T_{332}^1}{\partial x_2^3}-\frac{3}{5}\frac{\partial^3 T_{332}^2}{\partial x_2^3}+\frac{3}{5}\frac{\partial^3 T_{332}^1}{\partial x_1^2\partial x_2}-\frac{1}{5}\frac{\partial^3 T_{332}^2}{\partial x_1^2\partial x_2}-\frac{2}{5}\frac{\partial^3 T_{332}^2}{\partial x_1^2\partial x_2}+\frac{4}{5}\frac{\partial T_{332}^4}{\partial x_2}+2\frac{\partial^2 T_{123}^0}{\partial x_1\partial x_2}-4\frac{\partial^2 T_{123}^3}{\partial x_1\partial x_2}\\
&-\frac{\partial^2 Y_{11}^0}{\partial x_1\partial x_2}+\frac{1}{2}\frac{\partial^2 Y_{11}^3}{\partial x_1\partial x_2}+\frac{\partial^2 Y_{22}^0}{\partial x_1\partial x_2}-\frac{1}{2}\frac{\partial^2 Y_{22}^3}{\partial x_1\partial x_2}+\frac{\partial^2 Y_{12}^0}{\partial x_1^2}-\frac{1}{2}\frac{\partial^2 Y_{12}^3}{\partial x_1^2}-\frac{\partial^2 Y_{12}^0}{\partial x_2^2}+\frac{1}{2}\frac{\partial^2 Y_{12}^3}{\partial x_2^2}-\frac{1}{2}\frac{\partial Y_{13}^4}{\partial x_2}+\frac{1}{2}\frac{\partial Y_{23}^4}{\partial x_1}\\
&+q-P_{x_1}\frac{\partial^2 w}{\partial x_1^2}-P_{x_2}\frac{\partial^2 w}{\partial x_2^2}+P_{x_1}^0\frac{\partial^2 w}{\partial x_1^2}+P_{x_2}^0\frac{\partial^2 w}{\partial x_2^2}+2P_{x_1x_2}^0\frac{\partial^2 w}{\partial x_1\partial x_2}\\
&=\left(I_1-I_3\right)\frac{\partial^3 u}{\partial x_1\partial t^2}+\left(I_1-I_3\right)\frac{\partial^3 v}{\partial x_2\partial t^2}+\left(2I_4-I_2-I_5\right)\frac{\partial^4 w}{\partial x_1^2\partial t^2}+\left(2I_4-I_2-I_5\right)\frac{\partial^4 w}{\partial x_2^2\partial t^2}+I_0\frac{\partial^2 w}{\partial t^2}\\
&+\left(I_4-I_5\right)\frac{\partial^3\phi_1}{\partial x_1\partial t^2}+\left(I_4-I_5\right)\frac{\partial^3\phi_2}{\partial x_2\partial t^2},
\end{aligned}
\tag{82}
$$

$\delta\phi_1:$

$$
\begin{aligned}
&\frac{\partial M_{11}^2}{\partial x_1}+\frac{\partial M_{12}^2}{\partial x_2}-M_{13}^3-\frac{\partial^2 P_1^2}{\partial x_1^2}-\frac{\partial^2 P_2^2}{\partial x_1\partial x_2}+\frac{\partial P_3^3}{\partial x_1}-\frac{2}{5}\frac{\partial^2 T_{111}^2}{\partial x_1^2}+\frac{1}{5}\frac{\partial^2 T_{111}^2}{\partial x_2^2}+\frac{1}{5}T_{111}^4+\frac{2}{5}\frac{\partial^2 T_{222}^2}{\partial x_1\partial x_2}\\
&-\frac{2}{5}\frac{\partial T_{333}^3}{\partial x_1}-\frac{8}{5}\frac{\partial^2 T_{112}^2}{\partial x_1\partial x_2}+\frac{8}{5}\frac{\partial T_{113}^3}{\partial x_1}+\frac{3}{5}\frac{\partial^2 T_{221}^2}{\partial x_1^2}-\frac{4}{5}\frac{\partial^2 T_{221}^2}{\partial x_2^2}+\frac{1}{5}T_{221}^4-\frac{2}{5}\frac{\partial T_{223}^3}{\partial x_1}\\
&+\frac{3}{5}\frac{\partial^2 T_{331}^2}{\partial x_1^2}+\frac{1}{5}\frac{\partial^2 T_{331}^2}{\partial x_2^2}-\frac{4}{5}T_{331}^4+\frac{2}{5}\frac{\partial^2 T_{332}^2}{\partial x_1\partial x_2}+2\frac{\partial T_{123}^3}{\partial x_2}\\
&+\frac{1}{2}\frac{\partial Y_{22}^3}{\partial x_2}-\frac{1}{2}\frac{\partial Y_{33}^3}{\partial x_2}+\frac{1}{2}\frac{\partial Y_{12}^3}{\partial x_1}+\frac{1}{2}\frac{\partial^2 Y_{13}^2}{\partial x_1\partial x_2}+\frac{1}{2}\frac{\partial^2 Y_{23}^2}{\partial x_2^2}-\frac{1}{2}Y_{23}^4\\
&=I_3\frac{\partial^2 u}{\partial t^2}+\left(I_5-I_4\right)\frac{\partial^3 w}{\partial x_1\partial t^2}+I_5\frac{\partial^2\phi_1}{\partial t^2},
\end{aligned}
\tag{83}
$$

$\delta\phi_2$:

$$\frac{\partial M_{22}^2}{\partial x_2}+\frac{\partial M_{12}^2}{\partial x_1}-M_{23}^3-\frac{\partial^2 P_1^2}{\partial x_1 \partial x_2}-\frac{\partial^2 P_2^2}{\partial x_2^2}+\frac{\partial P_3^3}{\partial x_2}$$
$$+\frac{2}{5}\frac{\partial^2 T_{111}^2}{\partial x_1 \partial x_2}+\frac{1}{5}\frac{\partial^2 T_{222}^2}{\partial x_1^2}-\frac{2}{5}\frac{\partial^2 T_{222}^2}{\partial x_2^2}+\frac{1}{5}T_{222}^4-\frac{2}{5}\frac{\partial T_{333}^3}{\partial x_2}-\frac{4}{5}\frac{\partial^2 T_{112}^2}{\partial x_1^2}+\frac{3}{5}\frac{\partial^2 T_{112}^2}{\partial x_2^2}+\frac{1}{5}T_{112}^4-\frac{2}{5}\frac{\partial T_{113}^3}{\partial x_2} \quad (84)$$
$$-\frac{8}{5}\frac{\partial^2 T_{221}^2}{\partial x_1 \partial x_2}+\frac{8}{5}\frac{\partial T_{223}^3}{\partial x_2}+\frac{2}{5}\frac{\partial^2 T_{331}^2}{\partial x_1 \partial x_2}+\frac{1}{5}\frac{\partial^2 T_{332}^2}{\partial x_1^2}+\frac{3}{5}\frac{\partial^2 T_{332}^2}{\partial x_2^2}-\frac{4}{5}T_{332}^4+2\frac{\partial T_{123}^3}{\partial x_1}$$
$$-\frac{1}{2}\frac{\partial Y_{11}^3}{\partial x_1}+\frac{1}{2}\frac{\partial Y_{33}^3}{\partial x_1}-\frac{1}{2}\frac{\partial Y_{12}^3}{\partial x_2}-\frac{1}{2}\frac{\partial^2 Y_{13}^2}{\partial x_1^2}+\frac{1}{2}Y_{13}^4-\frac{1}{2}\frac{\partial^2 Y_{23}^2}{\partial x_1 \partial x_2}=I_3\frac{\partial^2 v}{\partial t^2}+\left(I_5-I_4\right)\frac{\partial^3 w}{\partial x_2 \partial t^2}+I_5\frac{\partial^2 \phi_2}{\partial t^2}.$$

Sınır koşullarını da bu şekilde ifade edebilmek için öncelikle aşağıda verilen eşitlikler kullanılmıştır:

$$\frac{1}{2}P_2^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_1\partial x_2}-\frac{1}{2}\frac{\partial P_2^2}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_1}+\frac{1}{2}P_2^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1^2}-\frac{1}{2}\frac{\partial P_2^2}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_1}=P_2^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1^2}-\frac{\partial P_2^2}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_1}$$

$$-\frac{1}{5}T_{222}^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_1\partial x_2}+\frac{1}{5}\frac{\partial T_{222}^2}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_1}-\frac{1}{5}T_{222}^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1^2}+\frac{1}{5}\frac{\partial T_{222}^2}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_1}$$
$$=-\frac{2}{5}T_{222}^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1^2}+\frac{2}{5}\frac{\partial T_{222}^2}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_1}$$

$$-\frac{1}{5}T_{222}^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_1\partial x_2}+\frac{1}{5}\frac{\partial T_{222}^2}{\partial x_1}n_{x_1}\delta\frac{\partial w}{\partial x_2}=-\frac{1}{5}T_{222}^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1^2}+\frac{1}{5}\frac{\partial T_{222}^2}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_1}$$

$$+\frac{4}{5}T_{112}^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_1\partial x_2}-\frac{4}{5}\frac{\partial T_{112}^2}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_1}+\frac{4}{5}T_{112}^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1^2}-\frac{4}{5}\frac{\partial T_{112}^2}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_1}$$
$$=+\frac{8}{5}T_{112}^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1^2}-\frac{8}{5}\frac{\partial T_{112}^2}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_1}$$

$$\frac{4}{5}T_{112}^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_1\partial x_2}-\frac{4}{5}\frac{\partial T_{112}^2}{\partial x_1}n_{x_1}\delta\frac{\partial w}{\partial x_2}=\frac{4}{5}T_{112}^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1^2}-\frac{4}{5}\frac{\partial T_{112}^2}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_1}$$

$$-\frac{1}{5}T_{331}^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1\partial x_2}+\frac{1}{5}\frac{\partial T_{331}^2}{\partial x_2}n_{x_2}\delta\frac{\partial w}{\partial x_1}=-\frac{1}{5}T_{331}^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_2^2}+\frac{1}{5}\frac{\partial T_{331}^2}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_2}$$

$$-\frac{1}{5}T_{331}^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_2^2}+\frac{1}{5}\frac{\partial T_{331}^2}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_2}-\frac{1}{5}T_{331}^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1\partial x_2}+\frac{1}{5}\frac{\partial T_{331}^2}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_2}=-\frac{2}{5}T_{331}^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_2^2}+\frac{2}{5}\frac{\partial T_{331}^2}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_2}$$

$$-\frac{1}{5}T_{332}^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_1\partial x_2}+\frac{1}{5}\frac{\partial T_{332}^2}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_1}-\frac{1}{5}T_{332}^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1^2}+\frac{1}{5}\frac{\partial T_{332}^2}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_1}=-\frac{2}{5}T_{332}^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1^2}+\frac{2}{5}\frac{\partial T_{332}^2}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_1}$$

$$-\frac{1}{5}T_{332}^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_1\partial x_2}+\frac{1}{5}\frac{\partial T_{332}^2}{\partial x_1}n_{x_1}\delta\frac{\partial w}{\partial x_2}=-\frac{1}{5}T_{332}^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1^2}+\frac{1}{5}\frac{\partial T_{332}^2}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_1}$$

$$-\frac{1}{4}Y_{13}^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_1\partial x_2}+\frac{1}{4}\frac{\partial Y_{13}^2}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_1}-\frac{1}{4}Y_{13}^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1^2}+\frac{1}{4}\frac{\partial Y_{13}^2}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_1}=-\frac{1}{2}Y_{13}^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1^2}+\frac{1}{2}\frac{\partial Y_{13}^2}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_1}$$

$$\frac{1}{2}Y_{13}^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_1\partial x_2}-\frac{1}{2}\frac{\partial Y_{13}^2}{\partial x_1}n_{x_1}\delta\frac{\partial w}{\partial x_2}=\frac{1}{2}Y_{13}^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1^2}-\frac{1}{2}\frac{\partial Y_{13}^2}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_1}$$

$$\frac{1}{2}P_1^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_2^2}-\frac{1}{2}\frac{\partial P_1^2}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_2}+\frac{1}{2}P_1^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1\partial x_2}-\frac{1}{2}\frac{\partial P_1^2}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_2}=P_1^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_2^2}-\frac{\partial P_1^2}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_2}$$

$$-\frac{1}{5}T_{111}^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1\partial x_2}+\frac{1}{5}\frac{\partial T_{111}^2}{\partial x_2}n_{x_2}\delta\frac{\partial w}{\partial x_1}=-\frac{1}{5}T_{111}^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_2^2}+\frac{1}{5}\frac{\partial T_{111}^2}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_2}$$

$$-\frac{1}{5}T_{111}^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_2^2}+\frac{1}{5}\frac{\partial T_{111}^2}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_2}-\frac{1}{5}T_{111}^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1\partial x_2}+\frac{1}{5}\frac{\partial T_{111}^2}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_2}$$
$$=-\frac{2}{5}T_{111}^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_2^2}+\frac{2}{5}\frac{\partial T_{111}^2}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_2}$$

$$\frac{4}{5}T_{221}^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1\partial x_2}-\frac{4}{5}\frac{\partial T_{221}^2}{\partial x_2}n_{x_2}\delta\frac{\partial w}{\partial x_1}=\frac{4}{5}T_{221}^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_2^2}-\frac{4}{5}\frac{\partial T_{221}^2}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_2}$$

$$\frac{4}{5}T_{221}^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_2^2}-\frac{4}{5}\frac{\partial T_{221}^2}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_2}+\frac{4}{5}T_{221}^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1\partial x_2}-\frac{4}{5}\frac{\partial T_{221}^2}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_2}$$
$$=\frac{8}{5}T_{221}^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_2^2}-\frac{8}{5}\frac{\partial T_{221}^2}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_2}$$

$$-\frac{1}{2}Y_{23}^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1\partial x_2}+\frac{1}{2}\frac{\partial Y_{23}^2}{\partial x_2}n_{x_2}\delta\frac{\partial w}{\partial x_1}=-\frac{1}{2}Y_{23}^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_2^2}+\frac{1}{2}\frac{\partial Y_{23}^2}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_2}$$

$$\frac{1}{4}Y_{23}^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_2^2}-\frac{1}{4}\frac{\partial Y_{23}^2}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_2}+\frac{1}{4}Y_{23}^2 n_{x_2}\delta\frac{\partial^2 w}{\partial x_1\partial x_2}-\frac{1}{4}\frac{\partial Y_{23}^2}{\partial x_2}n_{x_1}\delta\frac{\partial w}{\partial x_2}$$
$$=\frac{1}{2}Y_{23}^2 n_{x_1}\delta\frac{\partial^2 w}{\partial x_2^2}-\frac{1}{2}\frac{\partial Y_{23}^2}{\partial x_1}n_{x_2}\delta\frac{\partial w}{\partial x_2}$$

$$-\frac{1}{2}\left(Y_{33}^3 n_{x_1}\delta\frac{\partial w}{\partial x_2}-\frac{\partial Y_{33}^3}{\partial x_1}n_{x_2}\delta w-Y_{33}^3 n_{x_2}\delta\frac{\partial w}{\partial x_1}+\frac{\partial Y_{33}^3}{\partial x_2}n_{x_1}\delta w\right)=-\frac{1}{4}\left(Y_{33}^3 n_{x_1}\delta\frac{\partial w}{\partial x_2}-\delta\frac{\partial Y_{33}^3}{\partial x_1}n_{x_2}\delta w\right.$$
$$\left.+Y_{33}^3 n_{x_2}\delta\frac{\partial w}{\partial x_1}-\frac{\partial Y_{33}^3}{\partial x_2}n_{x_1}\delta w-Y_{33}^3 n_{x_2}\delta\frac{\partial w}{\partial x_1}+\frac{\partial Y_{33}^3}{\partial x_2}n_{x_1}\delta w-Y_{33}^3 n_{x_1}\delta\frac{\partial w}{\partial x_2}+\frac{\partial Y_{33}^3}{\partial x_1}n_{x_2}\delta w\right)=0$$

$$-\frac{1}{2}\frac{\partial^2 Y_{23}^2}{\partial x_2^2}n_{x_1}\delta w+\frac{1}{2}\frac{\partial^2 Y_{23}^2}{\partial x_1\partial x_2}n_{x_2}\delta w=0$$

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θ_1 ve θ_2 yerine ϕ_1 ve ϕ_2 kullanılarak sınır koşulları şu şekilde yeniden düzenlenmiştir:

$\delta u = 0$ veya

$$\left(M_{11}^{0} - \frac{\partial P_1^0}{\partial x_1} - \frac{1}{2}\frac{\partial P_2^0}{\partial x_2} - \frac{2}{5}\frac{\partial T_{111}^0}{\partial x_1} + \frac{1}{5}\frac{\partial T_{222}^0}{\partial x_2} - \frac{4}{5}\frac{\partial T_{112}^0}{\partial x_2} + \frac{3}{5}\frac{\partial T_{221}^0}{\partial x_1} + \frac{3}{5}\frac{\partial T_{331}^0}{\partial x_1} + \frac{1}{5}\frac{\partial T_{332}^0}{\partial x_2} \right) n_{x_1}$$
$$+ \left(M_{12}^{0} - \frac{1}{2}\frac{\partial P_2^0}{\partial x_1} + \frac{1}{5}\frac{\partial T_{111}^0}{\partial x_2} + \frac{1}{5}\frac{\partial T_{222}^0}{\partial x_1} - \frac{4}{5}\frac{\partial T_{112}^0}{\partial x_1} - \frac{4}{5}\frac{\partial T_{221}^0}{\partial x_2} + \frac{1}{5}\frac{\partial T_{331}^0}{\partial x_2} + \frac{1}{5}\frac{\partial T_{332}^0}{\partial x_1} + \frac{1}{2}\frac{\partial Y_{13}^0}{\partial x_1} + \frac{1}{2}\frac{\partial Y_{23}^0}{\partial x_2} \right) n_{x_2} = 0, \tag{86}$$

$\delta \dfrac{\partial u}{\partial x_1} = 0$ veya

$$\left(P_1^0 + \frac{2}{5}T_{111}^0 - \frac{3}{5}T_{221}^0 - \frac{3}{5}T_{331}^0 \right) n_{x_1} + \left(\frac{1}{2}P_2^0 - \frac{1}{5}T_{222}^0 + \frac{4}{5}T_{112}^0 - \frac{1}{5}T_{332}^0 \right) n_{x_2} = 0, \tag{87}$$

$\delta \dfrac{\partial u}{\partial x_2} = 0$ veya

$$\left(\frac{1}{2}P_2^0 - \frac{1}{5}T_{222}^0 + \frac{4}{5}T_{112}^0 - \frac{1}{5}T_{332}^0 - \frac{1}{2}Y_{13}^0 \right) n_{x_1} + \left(-\frac{1}{5}T_{111}^0 + \frac{4}{5}T_{221}^0 - \frac{1}{5}T_{331}^0 - \frac{1}{2}Y_{23}^0 \right) n_{x_2} = 0, \tag{88}$$

$\delta v = 0$ veya

$$\left(M_{12}^{0} - \frac{1}{2}\frac{\partial P_1^0}{\partial x_2} + \frac{1}{5}\frac{\partial T_{111}^0}{\partial x_2} + \frac{1}{5}\frac{\partial T_{222}^0}{\partial x_1} - \frac{4}{5}\frac{\partial T_{112}^0}{\partial x_1} - \frac{4}{5}\frac{\partial T_{221}^0}{\partial x_2} + \frac{1}{5}\frac{\partial T_{331}^0}{\partial x_2} + \frac{1}{5}\frac{\partial T_{332}^0}{\partial x_1} - \frac{1}{2}\frac{\partial Y_{13}^0}{\partial x_1} - \frac{1}{2}\frac{\partial Y_{23}^0}{\partial x_2} \right) n_{x_1}$$
$$+ \left(M_{22}^{0} - \frac{1}{2}\frac{\partial P_1^0}{\partial x_1} - \frac{\partial P_2^0}{\partial x_2} + \frac{1}{5}\frac{\partial T_{111}^0}{\partial x_1} - \frac{2}{5}\frac{\partial T_{222}^0}{\partial x_2} + \frac{3}{5}\frac{\partial T_{112}^0}{\partial x_2} - \frac{4}{5}\frac{\partial T_{221}^0}{\partial x_1} + \frac{1}{5}\frac{\partial T_{331}^0}{\partial x_1} + \frac{3}{5}\frac{\partial T_{332}^0}{\partial x_2} \right) n_{x_2} = 0, \tag{89}$$

$\delta \dfrac{\partial v}{\partial x_1} = 0$ veya

$$\left(-\frac{1}{5}T_{222}^0 + \frac{4}{5}T_{112}^0 - \frac{1}{5}T_{332}^0 + \frac{1}{2}Y_{13}^0 \right) n_{x_1} + \left(\frac{1}{2}P_1^0 - \frac{1}{5}T_{111}^0 + \frac{4}{5}T_{221}^0 - \frac{1}{5}T_{331}^0 + \frac{1}{2}Y_{23}^0 \right) n_{x_2} = 0, \tag{90}$$

$\delta \dfrac{\partial v}{\partial x_2} = 0$ veya

$$\left(\frac{1}{2}P_1^0 - \frac{1}{5}T_{111}^0 + \frac{4}{5}T_{221}^0 - \frac{1}{5}T_{331}^0 \right) n_{x_1} + \left(P_2^0 + \frac{2}{5}T_{222}^0 - \frac{3}{5}T_{112}^0 - \frac{3}{5}T_{332}^0 \right) n_{x_2} = 0, \tag{91}$$

$\delta w = 0$ veya

$$
\begin{aligned}
&\left(\frac{\partial M_{11}^{1}}{\partial x_1}-\frac{\partial M_{11}^{2}}{\partial x_1}+\frac{\partial M_{12}^{1}}{\partial x_2}-\frac{\partial M_{12}^{2}}{\partial x_2}+M_{13}^{3}-\frac{\partial^2 P_1^1}{\partial x_1^2}+\frac{\partial^2 P_1^2}{\partial x_1^2}-\frac{\partial^2 P_2^1}{\partial x_1 \partial x_2}+\frac{\partial^2 P_2^2}{\partial x_1 \partial x_2}+\frac{\partial P_3^0}{\partial x_1}-\frac{\partial P_3^3}{\partial x_1}\right.\\
&-\frac{2}{5}\frac{\partial^2 T_{111}^{1}}{\partial x_1^2}+\frac{2}{5}\frac{\partial^2 T_{111}^{2}}{\partial x_1^2}-\frac{1}{5}\frac{\partial^2 T_{111}^{2}}{\partial x_2^2}-\frac{1}{5}T_{111}^{4}+\frac{3}{5}\frac{\partial T_{222}^{1}}{\partial x_1 \partial x_2}-\frac{2}{5}\frac{\partial^2 T_{222}^{2}}{\partial x_1 \partial x_2}-\frac{1}{5}\frac{\partial T_{333}^{0}}{\partial x_1}+\frac{2}{5}\frac{\partial T_{333}^{3}}{\partial x_1}\\
&-\frac{12}{5}\frac{\partial^2 T_{112}^{1}}{\partial x_1 \partial x_2}+\frac{8}{5}\frac{\partial^2 T_{112}^{2}}{\partial x_1 \partial x_2}+\frac{4}{5}\frac{\partial T_{113}^{0}}{\partial x_1}-\frac{8}{5}\frac{\partial T_{113}^{3}}{\partial x_1}+\frac{3}{5}\frac{\partial^2 T_{221}^{1}}{\partial x_1^2}-\frac{3}{5}\frac{\partial^2 T_{221}^{2}}{\partial x_1^2}+\frac{4}{5}\frac{\partial^2 T_{221}^{2}}{\partial x_2^2}-\frac{1}{5}T_{221}^{4}\\
&-\frac{1}{5}\frac{\partial T_{223}^{0}}{\partial x_1}+\frac{2}{5}\frac{\partial T_{223}^{3}}{\partial x_1}+\frac{3}{5}\frac{\partial^2 T_{331}^{1}}{\partial x_1^2}-\frac{3}{5}\frac{\partial^2 T_{331}^{2}}{\partial x_1^2}-\frac{1}{5}\frac{\partial^2 T_{331}^{2}}{\partial x_2^2}+\frac{4}{5}T_{331}^{4}+\frac{3}{5}\frac{\partial^2 T_{332}^{1}}{\partial x_1 \partial x_2}-\frac{2}{5}\frac{\partial^2 T_{332}^{2}}{\partial x_1 \partial x_2}+\frac{\partial T_{123}^{0}}{\partial x_2}-2\frac{\partial T_{123}^{3}}{\partial x_2}\\
&\left.-\frac{1}{2}\frac{\partial Y_{11}^{0}}{\partial x_2}+\frac{1}{2}\frac{\partial Y_{22}^{0}}{\partial x_2}-\frac{1}{2}\frac{\partial Y_{22}^{3}}{\partial x_2}+\frac{\partial Y_{12}^{0}}{\partial x_1}-\frac{1}{2}\frac{\partial Y_{12}^{3}}{\partial x_1}+\frac{1}{2}Y_{23}^{4}-\frac{\partial w}{\partial x_1}P_{x_1}\right)n_{x_1}\\
&+\left(\frac{\partial M_{22}^{1}}{\partial x_2}-\frac{\partial M_{22}^{2}}{\partial x_2}+\frac{\partial M_{12}^{1}}{\partial x_1}-\frac{\partial M_{12}^{2}}{\partial x_1}+M_{23}^{3}-\frac{\partial^2 P_1^1}{\partial x_1 \partial x_2}+\frac{\partial^2 P_1^2}{\partial x_1 \partial x_2}-\frac{\partial^2 P_2^1}{\partial x_2^2}+\frac{\partial^2 P_2^2}{\partial x_2^2}+\frac{\partial P_3^0}{\partial x_2}-\frac{\partial P_3^3}{\partial x_2}\right.\\
&+\frac{3}{5}\frac{\partial^2 T_{111}^{1}}{\partial x_1 \partial x_2}-\frac{2}{5}\frac{\partial^2 T_{111}^{2}}{\partial x_1 \partial x_2}-\frac{2}{5}\frac{\partial^2 T_{222}^{1}}{\partial x_2^2}-\frac{1}{5}\frac{\partial^2 T_{222}^{2}}{\partial x_1^2}-\frac{1}{5}T_{222}^{4}+\frac{2}{5}\frac{\partial^2 T_{222}^{2}}{\partial x_2^2}-\frac{1}{5}\frac{\partial T_{333}^{0}}{\partial x_2}+\frac{2}{5}\frac{\partial T_{333}^{3}}{\partial x_2}\\
&+\frac{3}{5}\frac{\partial^2 T_{112}^{1}}{\partial x_2^2}+\frac{4}{5}\frac{\partial^2 T_{112}^{2}}{\partial x_1^2}-\frac{3}{5}\frac{\partial^2 T_{112}^{2}}{\partial x_2^2}-\frac{1}{5}T_{112}^{4}-\frac{1}{5}\frac{\partial T_{113}^{0}}{\partial x_2}+\frac{2}{5}\frac{\partial T_{113}^{3}}{\partial x_2}-\frac{12}{5}\frac{\partial^2 T_{221}^{1}}{\partial x_1 \partial x_2}+\frac{8}{5}\frac{\partial^2 T_{221}^{2}}{\partial x_1 \partial x_2}\\
&+\frac{4}{5}\frac{\partial T_{223}^{0}}{\partial x_2}-\frac{8}{5}\frac{\partial T_{223}^{3}}{\partial x_2}+\frac{3}{5}\frac{\partial^2 T_{331}^{1}}{\partial x_1 \partial x_2}-\frac{2}{5}\frac{\partial^2 T_{331}^{2}}{\partial x_1 \partial x_2}+\frac{3}{5}\frac{\partial^2 T_{332}^{1}}{\partial x_2^2}-\frac{1}{5}\frac{\partial^2 T_{332}^{2}}{\partial x_1^2}-\frac{3}{5}\frac{\partial^2 T_{332}^{2}}{\partial x_2^2}+\frac{4}{5}T_{332}^{4}+\frac{\partial T_{123}^{0}}{\partial x_1}-2\frac{\partial T_{123}^{3}}{\partial x_1}\\
&\left.-\frac{1}{2}\frac{\partial Y_{11}^{0}}{\partial x_1}+\frac{1}{2}\frac{\partial Y_{11}^{3}}{\partial x_1}+\frac{1}{2}\frac{\partial Y_{22}^{0}}{\partial x_1}-\frac{\partial Y_{12}^{0}}{\partial x_2}+\frac{1}{2}\frac{\partial Y_{12}^{3}}{\partial x_2}-\frac{1}{2}Y_{13}^{4}-\frac{\partial w}{\partial x_2}P_{x_2}\right)n_{x_2}\\
&=\left(\left(I_1-I_3\right)\frac{\partial^2 u}{\partial t^2}+\left(2I_4-I_2-I_5\right)\frac{\partial^3 w}{\partial x_1 \partial t^2}+\left(I_4-I_5\right)\frac{\partial^2 \phi_1}{\partial t^2}\right)n_{x_1}\\
&+\left(\left(I_1-I_3\right)\frac{\partial^2 v}{\partial t^2}+\left(2I_4-I_2-I_5\right)\frac{\partial^3 w}{\partial x_2 \partial t^2}+\left(I_4-I_5\right)\frac{\partial^2 \phi_2}{\partial t^2}\right)n_{x_2},
\end{aligned}
\tag{92}
$$

$$\delta\frac{\partial w}{\partial x_1}=0\quad \text{veya}$$

$$\left(-M_{11}^1+M_{11}^2+\frac{\partial P_1^1}{\partial x_1}-\frac{\partial P_1^2}{\partial x_1}+\frac{\partial P_2^1}{\partial x_2}-\frac{\partial P_2^2}{\partial x_2}-P_3^0+P_3^3+\frac{2}{5}\frac{\partial T_{111}^1}{\partial x_1}-\frac{2}{5}\frac{\partial T_{111}^2}{\partial x_1}-\frac{3}{5}\frac{\partial T_{222}^1}{\partial x_2}+\frac{3}{5}\frac{\partial T_{222}^2}{\partial x_2}\right.$$
$$+\frac{1}{5}T_{333}^0-\frac{2}{5}T_{333}^3+\frac{12}{5}\frac{\partial T_{112}^1}{\partial x_2}-\frac{12}{5}\frac{\partial T_{112}^2}{\partial x_2}-\frac{4}{5}T_{113}^0+\frac{8}{5}T_{113}^3-\frac{3}{5}\frac{\partial T_{221}^1}{\partial x_1}+\frac{3}{5}\frac{\partial T_{221}^2}{\partial x_1}+\frac{1}{5}T_{223}^0-\frac{2}{5}T_{223}^3 \qquad (93)$$
$$\left.-\frac{3}{5}\frac{\partial T_{331}^1}{\partial x_1}+\frac{3}{5}\frac{\partial T_{331}^2}{\partial x_1}-\frac{3}{5}\frac{\partial T_{332}^1}{\partial x_2}+\frac{3}{5}\frac{\partial T_{332}^2}{\partial x_2}-Y_{12}^0+\frac{1}{2}Y_{12}^3\right)n_{x_1}$$
$$+\left(-M_{12}^1+M_{12}^2-T_{123}^0+2T_{123}^3+\frac{1}{2}Y_{11}^0-\frac{1}{2}Y_{22}^0+\frac{1}{2}Y_{22}^3\right)n_{x_2}=0,$$

$$\delta\frac{\partial w}{\partial x_2}=0\quad \text{veya}$$

$$\left(-M_{12}^1+M_{12}^2-T_{123}^0+2T_{123}^3+\frac{1}{2}Y_{11}^0-\frac{1}{2}Y_{11}^3-\frac{1}{2}Y_{22}^0\right)n_{x_1}$$
$$+\left(-M_{22}^1+M_{22}^2+\frac{\partial P_1^1}{\partial x_1}-\frac{\partial P_1^2}{\partial x_1}+\frac{\partial P_2^1}{\partial x_2}-\frac{\partial P_2^2}{\partial x_2}-P_3^0+P_3^3-\frac{3}{5}\frac{\partial T_{111}^1}{\partial x_1}+\frac{3}{5}\frac{\partial T_{111}^2}{\partial x_1}\right.$$
$$+\frac{2}{5}\frac{\partial T_{222}^1}{\partial x_2}-\frac{2}{5}\frac{\partial T_{222}^2}{\partial x_2}+\frac{1}{5}T_{333}^0-\frac{2}{5}T_{333}^3-\frac{3}{5}\frac{\partial T_{112}^1}{\partial x_2}+\frac{3}{5}\frac{\partial T_{112}^2}{\partial x_2}+\frac{1}{5}T_{113}^0-\frac{2}{5}T_{113}^3+\frac{12}{5}\frac{\partial T_{221}^1}{\partial x_1}-\frac{12}{5}\frac{\partial T_{221}^2}{\partial x_1}$$
$$\left.-\frac{4}{5}T_{223}^0+\frac{8}{5}T_{223}^3-\frac{3}{5}\frac{\partial T_{331}^1}{\partial x_1}+\frac{3}{5}\frac{\partial T_{331}^2}{\partial x_1}-\frac{3}{5}\frac{\partial T_{332}^1}{\partial x_2}+\frac{3}{5}\frac{\partial T_{332}^2}{\partial x_2}+Y_{12}^0-\frac{1}{2}Y_{12}^3\right)n_{x_2}=0,$$

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$$\delta\frac{\partial^2 w}{\partial x_1^2}=0\quad \text{veya}$$

$$\left(-P_1^1+P_1^2-\frac{2}{5}T_{111}^1+\frac{2}{5}T_{111}^2+\frac{3}{5}T_{221}^1-\frac{3}{5}T_{221}^2+\frac{3}{5}T_{331}^1-\frac{3}{5}T_{331}^2\right)n_{x_1} \qquad (95)$$
$$+\left(-P_2^1+P_2^2+\frac{3}{5}T_{222}^1-\frac{3}{5}T_{222}^2-\frac{12}{5}T_{112}^1+\frac{12}{5}T_{112}^2+\frac{3}{5}T_{332}^1-\frac{3}{5}T_{332}^2\right)n_{x_2}=0,$$

$$\delta\frac{\partial^2 w}{\partial x_2^2}=0\quad \text{veya}$$

$$\left(-P_1^1+P_1^2+\frac{3}{5}T_{111}^1-\frac{3}{5}T_{111}^2-\frac{12}{5}T_{221}^1+\frac{12}{5}T_{221}^2+\frac{3}{5}T_{331}^1-\frac{3}{5}T_{331}^2\right)n_{x_1} \qquad (96)$$
$$+\left(-P_2^1+P_2^2-\frac{2}{5}T_{222}^1+\frac{2}{5}T_{222}^2+\frac{3}{5}T_{112}^1-\frac{3}{5}T_{112}^2+\frac{3}{5}T_{332}^1-\frac{3}{5}T_{332}^2\right)n_{x_2}=0,$$

$\delta\phi_1 = 0$ veya

$$\left(M_{11}^2 - \frac{\partial P_1^2}{\partial x_1} - \frac{1}{2}\frac{\partial P_2^2}{\partial x_2} + P_3^3 - \frac{2}{5}\frac{\partial T_{111}^2}{\partial x_1} + \frac{1}{5}\frac{\partial T_{222}^2}{\partial x_2} - \frac{1}{5}2T_{333}^3 - \frac{4}{5}\frac{\partial T_{112}^2}{\partial x_2} + \frac{8}{5}T_{113}^3 + \frac{3}{5}\frac{\partial T_{221}^2}{\partial x_1} - \frac{2}{5}T_{223}^3\right.$$
$$\left. + \frac{3}{5}\frac{\partial T_{331}^2}{\partial x_1} + \frac{1}{5}\frac{\partial T_{332}^2}{\partial x_2} + \frac{1}{2}Y_{12}^3 + \frac{1}{4}\frac{\partial Y_{13}^2}{\partial x_2}\right)n_{x_1} + \left(M_{12}^2 - \frac{1}{2}\frac{\partial P_2^2}{\partial x_1} + \frac{1}{5}\frac{\partial T_{111}^2}{\partial x_2} + \frac{1}{5}\frac{\partial T_{222}^2}{\partial x_1}\right. \tag{97}$$
$$\left. - \frac{4}{5}\frac{\partial T_{112}^2}{\partial x_1} - \frac{4}{5}\frac{\partial T_{221}^2}{\partial x_2} + \frac{1}{5}\frac{\partial T_{331}^2}{\partial x_2} + \frac{1}{5}\frac{\partial T_{332}^2}{\partial x_1} + 2T_{123}^3 + \frac{1}{2}Y_{22}^3 - \frac{1}{2}Y_{33}^3 + \frac{1}{4}\frac{\partial Y_{13}^2}{\partial x_1} + \frac{1}{2}\frac{\partial Y_{23}^2}{\partial x_2}\right)n_{x_2} = 0,$$

$\delta\dfrac{\partial\phi_1}{\partial x_1} = 0$ veya

$$\left(P_1^2 + \frac{2}{5}T_{111}^2 - \frac{3}{5}T_{221}^2 - \frac{3}{5}T_{331}^2\right)n_{x_1} + \left(\frac{1}{2}P_2^2 - \frac{1}{5}T_{222}^2 + \frac{4}{5}T_{112}^2 - \frac{1}{5}T_{332}^2 - \frac{1}{4}Y_{13}^2\right)n_{x_2} = 0, \tag{98}$$

$\delta\dfrac{\partial\phi_1}{\partial x_2} = 0$ veya

$$\left(\frac{1}{2}P_2^2 - \frac{1}{5}T_{222}^2 + \frac{4}{5}T_{112}^2 - \frac{1}{5}T_{332}^2 - \frac{1}{4}Y_{13}^2\right)n_{x_1} + \left(-\frac{1}{5}T_{111}^2 + \frac{4}{5}T_{221}^2 - \frac{1}{5}T_{331}^2 - \frac{1}{2}Y_{23}^2\right)n_{x_2} = 0, \tag{99}$$

$\delta\phi_2 = 0$ veya

$$\left(M_{12}^2 - \frac{1}{2}\frac{\partial P_1^2}{\partial x_2} + \frac{1}{5}\frac{\partial T_{111}^2}{\partial x_2} + \frac{1}{5}\frac{\partial T_{222}^2}{\partial x_1} - \frac{4}{5}\frac{\partial T_{112}^2}{\partial x_1} - \frac{4}{5}\frac{\partial T_{221}^2}{\partial x_2} + \frac{1}{5}\frac{\partial T_{331}^2}{\partial x_2} + \frac{1}{5}\frac{\partial T_{332}^2}{\partial x_1} + 2T_{123}^3\right.$$
$$\left. - \frac{1}{2}Y_{11}^3 + \frac{1}{2}Y_{33}^3 - \frac{1}{2}\frac{\partial Y_{13}^2}{\partial x_1} - \frac{1}{4}\frac{\partial Y_{23}^2}{\partial x_2}\right)n_{x_1} + \left(M_{22}^2 - \frac{1}{2}\frac{\partial P_1^2}{\partial x_1} - \frac{\partial P_2^2}{\partial x_2} + P_3^3 + \frac{1}{5}\frac{\partial T_{111}^2}{\partial x_1} - \frac{2}{5}\frac{\partial T_{222}^2}{\partial x_2}\right. \tag{100}$$
$$\left. - \frac{2}{5}T_{333}^3 + \frac{3}{5}\frac{\partial T_{112}^2}{\partial x_2} - \frac{2}{5}T_{113}^3 - \frac{4}{5}\frac{\partial T_{221}^2}{\partial x_1} + \frac{8}{5}T_{223}^3 + \frac{1}{5}\frac{\partial T_{331}^2}{\partial x_1} + \frac{3}{5}\frac{\partial T_{332}^2}{\partial x_2} - \frac{1}{2}Y_{12}^3 - \frac{1}{4}\frac{\partial Y_{23}^2}{\partial x_1}\right)n_{x_2} = 0,$$

$\delta\dfrac{\partial\phi_2}{\partial x_1} = 0$ veya

$$\left(-\frac{1}{5}T_{222}^2 + \frac{4}{5}T_{112}^2 - \frac{1}{5}T_{332}^2 + \frac{1}{2}Y_{13}^2\right)n_{x_1} + \left(\frac{1}{2}P_1^2 - \frac{1}{5}T_{111}^2 + \frac{4}{5}T_{221}^2 - \frac{1}{5}T_{331}^2 + \frac{1}{4}Y_{23}^2\right)n_{x_2} = 0, \tag{101}$$

$\delta\dfrac{\partial\phi_2}{\partial x_2} = 0$ veya

$$\left(\frac{1}{2}P_1^2 - \frac{1}{5}T_{111}^2 + \frac{4}{5}T_{221}^2 - \frac{1}{5}T_{331}^2 + \frac{1}{4}Y_{23}^2\right)n_{x_1} + \left(P_2^2 + \frac{2}{5}T_{222}^2 - \frac{3}{5}T_{112}^2 - \frac{3}{5}T_{332}^2\right)n_{x_2} = 0. \tag{102}$$

3.2.2. Kuvvet ve Moment Katsayılarının Atalet ve Mukavemet Katsayıları Cinsinden İfadeleri ve Denklemlerin Son Formu

Bir önceki bölümde verilen denklemler kuvvet ve moment katsayılarını ($M_{11}^{i}, M_{22}^{i}, M_{12}^{i}, M_{13}^{i}, P_{1}^{i}, P_{2}^{i}, P_{3}^{i}, T_{111}^{i}, T_{222}^{i}, T_{333}^{i}, T_{112}^{i}, T_{221}^{i}, T_{113}^{i}, T_{223}^{i}, T_{331}^{i}, T_{332}^{i}, T_{123}^{i}, Y_{11}^{i}, Y_{22}^{i}, Y_{33}^{i}$, $Y_{12}^{i}, Y_{13}^{i}, Y_{23}^{i}$) içermektedir. Bu katsayıların hesabında aşağıda verilen tanımlar kullanılmıştır:

$$\left\{A_{11}, B_{11}, D_{11}, F_{11}, F_{22}, F_{33}\right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E\left(x_3, T\right)}{1-\upsilon\left(x_3, T\right)^2}\left\{1, x_3, x_3^2, f, x_3 f, f^2\right\} \mathrm{d}x_3$$

$$\left\{A_{L11}, B_{L11}, D_{L11}, F_{L11}, F_{L22}, F_{L33}\right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E\left(x_3, T\right)\upsilon\left(x_3, T\right)}{1-\upsilon\left(x_3, T\right)^2}\left\{1, x_3, x_3^2, f, x_3 f, f^2\right\} \mathrm{d}x_3$$

$$\left\{A_{T11}, B_{T11}, F_{T11}\right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E\left(x_3, T\right)}{\left(1-\upsilon\left(x_3, T\right)\right)}\alpha\left(x_3, T\right)\left\{1, x_3, f\right\} \mathrm{d}x_3$$

$$\left\{A_{55}, B_{55}, D_{55}, F_{44}, F_{46}, F_{47}, F_{48}, F_{55}, F_{57}, F_{66}, F_{67}, F_{68}\right\}$$
$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E\left(x_3\right)}{2\left(1+\upsilon\left(x_3\right)\right)}\left\{1, x_3, x_3^2, f^2, ff'', f, x_3 f, f'^2, f', f''^2, f'', x_3 f''\right\} \mathrm{d}x_3,$$

$$\left\{A_{550}, B_{550}, D_{550}, F_{440}, F_{460}, F_{470}, F_{480}, F_{550}, F_{570}, F_{660}, F_{670}, F_{680}\right\}$$
$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E\left(x_3\right).l_0^2}{2\left(1+\upsilon\left(x_3\right)\right)}\left\{1, x_3, x_3^2, f^2, ff'', f, x_3 f, f'^2, f', f''^2, f'', x_3 f''\right\} \mathrm{d}x_3,$$

$$\left\{A_{551}, B_{551}, D_{551}, F_{441}, F_{461}, F_{471}, F_{481}, F_{551}, F_{571}, F_{661}, F_{671}, F_{681}\right\}$$
$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E\left(x_3\right).l_1^2}{2\left(1+\upsilon\left(x_3\right)\right)}\left\{1, x_3, x_3^2, f^2, ff'', f, x_3 f, f'^2, f', f''^2, f'', x_3 f''\right\} \mathrm{d}x_3,$$

$$\left\{A_{552}, B_{552}, D_{552}, F_{442}, F_{462}, F_{472}, F_{482}, F_{552}, F_{572}, F_{662}, F_{672}, F_{682}\right\}$$
$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E\left(x_3\right).l_2^2}{2\left(1+\upsilon\left(x_3\right)\right)}\left\{1, x_3, x_3^2, f^2, ff'', f, x_3 f, f'^2, f', f''^2, f'', x_3 f''\right\} \mathrm{d}x_3.$$

$$\left\{I_0, I_1, I_2, I_3, I_4, I_5\right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho\left(x_3\right)\left\{1, x_3, x_3^2, f, x_3 f, f^2\right\} dx_3 \qquad (103)$$

Kuvvet ve moment katsayıları bu şekilde ifade edilerek, kısmi diferansiyel denklemler ve sınır koşulları aşağıda sunulan formlarıyla yeniden düzenlenmiştir:

$\delta u:$

$$
\begin{aligned}
&\left(-2A_{550}-\frac{4}{5}A_{551}\right)\frac{\partial^4 u}{\partial x_1^4}+\left(-\frac{8}{15}A_{551}-\frac{1}{4}A_{552}\right)\frac{\partial^4 u}{\partial x_2^4}+\left(-2A_{550}-\frac{4}{3}A_{551}-\frac{1}{4}A_{552}\right)\frac{\partial^4 u}{\partial x_1^2\partial x_2^2}\\
&+A_{11}\frac{\partial^2 u}{\partial x_1^2}+A_{55}\frac{\partial^2 u}{\partial x_2^2}\\
&+\left(-2A_{550}-\frac{4}{15}A_{551}+\frac{1}{4}A_{552}\right)\frac{\partial^4 v}{\partial x_1^3\partial x_2}+\left(-2A_{550}-\frac{4}{15}A_{551}+\frac{1}{4}A_{552}\right)\frac{\partial^4 v}{\partial x_1\partial x_2^3}\\
&+\left(A_{55}+A_{L11}\right)\frac{\partial^2 v}{\partial x_1\partial x_2}\\
&+\left(-2F_{470}+2B_{550}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\right)\frac{\partial^5 w}{\partial x_1^5}+\left(4B_{550}-4F_{470}+\frac{8}{5}B_{551}-\frac{8}{5}F_{471}\right)\frac{\partial^5 w}{\partial x_1^3\partial x_2^2}\\
&+\left(-2F_{470}+2B_{550}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\right)\frac{\partial^5 w}{\partial x_1\partial x_2^4}\\
&+\left(F_{11}-B_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^3 w}{\partial x_1^3}+\left(F_{11}-B_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^3 w}{\partial x_1\partial x_2^2}\\
&+\left(-2F_{470}-\frac{4}{5}F_{471}\right)\frac{\partial^4 \phi_1}{\partial x_1^4}+\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^4 \phi_1}{\partial x_2^4}+\left(-2F_{470}-\frac{4}{3}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^4 \phi_1}{\partial x_1^2\partial x_2^2}\\
&+\left(F_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^2 \phi_1}{\partial x_1^2}+\left(F_{47}+\frac{2}{15}F_{671}+\frac{1}{4}F_{672}\right)\frac{\partial^2 \phi_1}{\partial x_2^2}\\
&+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^4 \phi_2}{\partial x_1^3\partial x_2}+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^4 \phi_2}{\partial x_1\partial x_2^3}\\
&+\left(F_{L11}+F_{47}+\frac{4}{15}F_{671}-\frac{1}{4}F_{672}\right)\frac{\partial^2 \phi_2}{\partial x_1\partial x_2}-A_{T11}\frac{\partial \Delta T}{\partial x_1}\\
&=I_0\frac{\partial^2 u}{\partial t^2}+\left(I_3-I_1\right)\frac{\partial^3 w}{\partial x_1\partial t^2}+I_3\frac{\partial^2 \phi_1}{\partial t^2},
\end{aligned}
\tag{104}
$$

$\delta v:$

$$
\begin{aligned}
&\left(-2A_{550}-\frac{4}{15}A_{551}+\frac{1}{4}A_{552}\right)\frac{\partial^4 u}{\partial x_1^3\partial x_2}+\left(-2A_{550}-\frac{4}{15}A_{551}+\frac{1}{4}A_{552}\right)\frac{\partial^4 u}{\partial x_1\partial x_2^3}\\
&+\left(A_{55}+A_{L11}\right)\frac{\partial^2 u}{\partial x_1\partial x_2}\\
&+\left(-\frac{8}{15}A_{551}-\frac{1}{4}A_{552}\right)\frac{\partial^4 v}{\partial x_1^4}+\left(-2A_{550}-\frac{4}{5}A_{551}\right)\frac{\partial^4 v}{\partial x_2^4}+\left(-2A_{550}-\frac{4}{3}A_{551}-\frac{1}{4}A_{552}\right)\frac{\partial^4 v}{\partial x_1^2\partial x_2^2}\\
&+A_{55}\frac{\partial^2 v}{\partial x_1^2}+A_{11}\frac{\partial^2 v}{\partial x_2^2}\\
&+\left(-2F_{470}+2B_{550}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\right)\frac{\partial^5 w}{\partial x_2^5}+\left(2B_{550}+\frac{4}{5}B_{551}-2F_{470}-\frac{4}{5}F_{471}\right)\frac{\partial^5 w}{\partial x_1^4\partial x_2}\\
&+\left(4B_{550}+\frac{8}{5}B_{551}-4F_{470}-\frac{8}{5}F_{471}\right)\frac{\partial^5 w}{\partial x_1^2\partial x_2^3}\\
&+\left(F_{11}-B_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^3 w}{\partial x_2^3}+\left(F_{11}-B_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^3 w}{\partial x_1^2\partial x_2}\\
&+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^4 \phi_1}{\partial x_1^3\partial x_2}+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^4 \phi_1}{\partial x_1\partial x_2^3}\\
&+\left(F_{L11}+F_{47}+\frac{4}{15}F_{671}-\frac{1}{4}F_{672}\right)\frac{\partial^2 \phi_1}{\partial x_1\partial x_2}\\
&+\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^4 \phi_2}{\partial x_1^4}+\left(-2F_{470}-\frac{4}{5}F_{471}\right)\frac{\partial^4 \phi_2}{\partial x_2^4}+\left(-2F_{470}-\frac{4}{3}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^4 \phi_2}{\partial x_1^2\partial x_2^2}\\
&+\left(F_{47}+\frac{2}{15}F_{671}+\frac{1}{4}F_{672}\right)\frac{\partial^2 \phi_2}{\partial x_1^2}+\left(F_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^2 \phi_2}{\partial x_2^2}\\
&-A_{T11}\frac{\partial \Delta T}{\partial x_2}=I_0\frac{\partial^2 v}{\partial t^2}+\left(I_3-I_1\right)\frac{\partial^3 w}{\partial x_2\partial t^2}+I_3\frac{\partial^2 \phi_2}{\partial t^2},
\end{aligned}
\tag{105}
$$

$\delta w:$

$$\left(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{5}F_{471}\right)\frac{\partial^5 u}{\partial x_1^5}+\left(-4B_{550}+4F_{470}-\frac{8}{5}B_{551}+\frac{8}{5}F_{471}\right)\frac{\partial^5 u}{\partial x_1^3\partial x_2^2}$$

$$+\left(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{5}F_{471}\right)\frac{\partial^5 u}{\partial x_1\partial x_2^4}+\left(B_{11}-F_{11}-\frac{2}{5}F_{671}\right)\frac{\partial^3 u}{\partial x_1^3}+\left(B_{11}-F_{11}-\frac{2}{5}F_{671}\right)\frac{\partial^3 u}{\partial x_1\partial x_2^2}$$

$$+\left(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{5}F_{471}\right)\frac{\partial^5 v}{\partial x_2^5}+\left(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{5}F_{471}\right)\frac{\partial^5 v}{\partial x_1^4\partial x_2}$$

$$+\left(-4B_{550}+4F_{470}-\frac{8}{5}B_{551}+\frac{8}{5}F_{471}\right)\frac{\partial^5 v}{\partial x_1^2\partial x_2^3}+\left(B_{11}-F_{11}-\frac{2}{5}F_{671}\right)\frac{\partial^3 v}{\partial x_2^3}+\left(B_{11}-F_{11}-\frac{2}{5}F_{671}\right)\frac{\partial^3 v}{\partial x_1^2\partial x_2}$$

$$+\left(2D_{550}+2F_{440}-4F_{480}+\frac{4}{5}D_{551}+\frac{4}{5}F_{441}-\frac{8}{5}F_{481}\right)\frac{\partial^6 w}{\partial x_1^6}$$

$$+\left(2D_{550}+2F_{440}-4F_{480}+\frac{4}{5}D_{551}+\frac{4}{5}F_{441}-\frac{8}{5}F_{481}\right)\frac{\partial^6 w}{\partial x_2^6}$$

$$+\left(6D_{550}+6F_{440}-12F_{480}+\frac{12}{5}D_{551}+\frac{12}{5}F_{441}-\frac{24}{5}F_{481}\right)\frac{\partial^6 w}{\partial x_1^4\partial x_2^2}$$

$$+\left(6D_{550}+6F_{440}-12F_{480}+\frac{12}{5}D_{551}+\frac{12}{5}F_{441}-\frac{24}{5}F_{481}\right)\frac{\partial^6 w}{\partial x_1^2\partial x_2^4}+\Big(-D_{11}+2F_{22}-F_{33}$$

$$-2A_{550}-2F_{550}+4F_{570}-\frac{8}{15}A_{551}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{32}{15}F_{571}+\frac{4}{5}F_{681}-A_{552}-\frac{1}{4}F_{552}+F_{572}\Big)\frac{\partial^4 w}{\partial x_1^4}$$

$$+\Big(-D_{11}+2F_{22}-F_{33}-2A_{550}-2F_{550}+4F_{570}-\frac{8}{15}A_{551}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{32}{15}F_{571}+\frac{4}{5}F_{681}$$

$$-A_{552}-\frac{1}{4}F_{552}+F_{572}\Big)\frac{\partial^4 w}{\partial x_2^4}+\Big(-2D_{11}+4F_{22}-2F_{33}-4A_{550}-4F_{550}+8F_{570}$$

$$-\frac{16}{15}A_{551}-\frac{8}{5}F_{461}-\frac{64}{15}F_{551}+\frac{64}{15}F_{571}+\frac{8}{5}F_{681}-2A_{552}-\frac{1}{2}F_{552}+2F_{572}\Big)\frac{\partial^4 w}{\partial x_1^2\partial x_2^2}$$

$$+\left(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\right)\frac{\partial^2 w}{\partial x_1^2}+\left(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\right)\frac{\partial^2 w}{\partial x_2^2}$$

$$+\left(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\right)\frac{\partial^5\phi_1}{\partial x_1^5}+\left(4F_{440}-4F_{480}+\frac{8}{5}F_{441}-\frac{8}{5}F_{481}\right)\frac{\partial^5\phi_1}{\partial x_1^3\partial x_2^2}$$

$$+\left(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\right)\frac{\partial^5\phi_1}{\partial x_1\partial x_2^4}$$

$$+\left(F_{22}-F_{33}-2F_{550}+2F_{570}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{16}{15}F_{571}+\frac{2}{5}F_{681}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\right)\frac{\partial^3\phi_1}{\partial x_1^3}$$

$$+\left(F_{22}-F_{33}-2F_{550}+2F_{570}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{16}{15}F_{571}+\frac{2}{5}F_{681}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\right)\frac{\partial^3\phi_1}{\partial x_1\partial x_2^2}$$

$$+\left(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\right)\frac{\partial\phi_1}{\partial x_1}+\left(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\right)\frac{\partial^5\phi_2}{\partial x_2^5}$$

$$+\left(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\right)\frac{\partial^5\phi_2}{\partial x_1^4\partial x_2}+\left(4F_{440}-4F_{480}+\frac{8}{5}F_{441}-\frac{8}{5}F_{481}\right)\frac{\partial^5\phi_2}{\partial x_1^2\partial x_2^3}$$
$$+\left(F_{22}-F_{33}-2F_{550}+2F_{570}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{16}{15}F_{571}+\frac{2}{5}F_{681}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\right)\frac{\partial^3\phi_2}{\partial x_2^3}$$
$$+\left(F_{22}-F_{33}-2F_{550}+2F_{570}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{16}{15}F_{571}+\frac{2}{5}F_{681}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\right)\frac{\partial^3\phi_2}{\partial x_1^2\partial x_2}$$
$$+\left(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\right)\frac{\partial\phi_2}{\partial x_2} \tag{106}$$
$$+\left(-B_{T11}+F_{T11}\right)\frac{\partial^2\Delta T}{\partial x_1^2}+\left(-B_{T11}+F_{T11}\right)\frac{\partial^2\Delta T}{\partial x_2^2}+P_{x_1}^0\frac{\partial^2 w}{\partial x_1^2}+P_{x_2}^0\frac{\partial^2 w}{\partial x_2^2}+2P_{x_1x_2}^0\frac{\partial^2 w}{\partial x_1\partial x_2}$$
$$+q-P_{x_1}\frac{\partial^2 w}{\partial x_1^2}-P_{x_2}\frac{\partial^2 w}{\partial x_2^2}=\left(I_1-I_3\right)\frac{\partial^3 u}{\partial x_1\partial t^2}+\left(I_1-I_3\right)\frac{\partial^3 v}{\partial x_2\partial t^2}+\left(2I_4-I_2-I_5\right)\frac{\partial^4 w}{\partial x_1^2\partial t^2}$$
$$+\left(2I_4-I_2-I_5\right)\frac{\partial^4 w}{\partial x_2^2\partial t^2}+I_0\frac{\partial^2 w}{\partial t^2}+\left(I_4-I_5\right)\frac{\partial^3\phi_1}{\partial x_1\partial t^2}+\left(I_4-I_5\right)\frac{\partial^3\phi_2}{\partial x_2\partial t^2},$$

$\delta\phi_1$:

$$
\begin{aligned}
&\left(-2F_{470}-\frac{4}{5}F_{471}\right)\frac{\partial^4 u}{\partial x_1^4}+\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^4 u}{\partial x_2^4}+\left(-2F_{470}-\frac{4}{3}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^4 u}{\partial x_1^2\partial x_2^2}\\
&+\left(F_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^2 u}{\partial x_1^2}+\left(F_{47}+\frac{2}{15}F_{671}+\frac{1}{4}F_{672}\right)\frac{\partial^2 u}{\partial x_2^2}\\
&+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^4 v}{\partial x_1^3\partial x_2}+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^4 v}{\partial x_1\partial x_2^3}\\
&+\left(F_{L11}+F_{47}+\frac{4}{15}F_{671}-\frac{1}{4}F_{672}\right)\frac{\partial^2 v}{\partial x_1\partial x_2}\\
&+\left(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_1^5}+\left(-4F_{440}+4F_{480}-\frac{8}{5}F_{441}+\frac{8}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_1^3\partial x_2^2}\\
&+\left(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_1\partial x_2^4}\\
&+\left(-F_{22}+F_{33}+2F_{550}-2F_{570}+\frac{4}{5}F_{461}+\frac{32}{15}F_{551}-\frac{16}{15}F_{571}-\frac{2}{5}F_{681}+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\frac{\partial^3 w}{\partial x_1^3}\\
&+\left(-F_{22}+F_{33}+2F_{550}-2F_{570}+\frac{4}{5}F_{461}+\frac{32}{15}F_{551}-\frac{16}{15}F_{571}-\frac{2}{5}F_{681}+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\frac{\partial^3 w}{\partial x_1\partial x_2^2}\\
&+\left(-k_sF_{55}-\frac{8}{15}F_{661}-\frac{1}{4}F_{662}\right)\frac{\partial w}{\partial x_1}\\
&+\left(-2F_{440}-\frac{4}{5}F_{441}\right)\frac{\partial^4 \phi_1}{\partial x_1^4}+\left(-\frac{8}{15}F_{441}-\frac{1}{4}F_{442}\right)\frac{\partial^4 \phi_1}{\partial x_2^4}+\left(-2F_{440}-\frac{4}{3}F_{441}-\frac{1}{4}F_{442}\right)\frac{\partial^4 \phi_1}{\partial x_1^2\partial x_2^2}\\
&+\left(F_{33}+2F_{550}+\frac{4}{5}F_{461}+\frac{32}{15}F_{551}+\frac{1}{4}F_{552}\right)\frac{\partial^2 \phi_1}{\partial x_1^2}\\
&+\left(F_{44}+\frac{4}{15}F_{461}+\frac{4}{3}F_{551}+\frac{1}{2}F_{462}+F_{552}\right)\frac{\partial^2 \phi_1}{\partial x_2^2}\\
&+\left(-k_sF_{55}-\frac{8}{15}F_{661}-\frac{1}{4}F_{662}\right)\phi_1\\
&+\left(-2F_{440}-\frac{4}{15}F_{441}+\frac{1}{4}F_{442}\right)\frac{\partial^4 \phi_2}{\partial x_1^3\partial x_2}+\left(-2F_{440}-\frac{4}{15}F_{441}+\frac{1}{4}F_{442}\right)\frac{\partial^4 \phi_2}{\partial x_1\partial x_2^3}\\
&+\left(F_{L33}+F_{44}+2F_{550}+\frac{8}{15}F_{461}+\frac{4}{5}F_{551}-\frac{1}{2}F_{462}-\frac{3}{4}F_{552}\right)\frac{\partial^2 \phi_2}{\partial x_1\partial x_2}\\
&-F_{T11}\frac{\partial \Delta T}{\partial x_1}=I_3\frac{\partial^2 u}{\partial t^2}+\left(I_5-I_4\right)\frac{\partial^3 w}{\partial x_1\partial t^2}+I_5\frac{\partial^2 \phi_1}{\partial t^2},
\end{aligned}
\tag{107}
$$

$\delta\phi_2$:

$$\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^4 u}{\partial x_1^3\partial x_2}+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^4 u}{\partial x_1\partial x_2^3}$$
$$+\left(F_{L11}+F_{47}+\frac{4}{15}F_{671}-\frac{1}{4}F_{672}\right)\frac{\partial^2 u}{\partial x_1\partial x_2}$$
$$+\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^4 v}{\partial x_1^4}+\left(-2F_{470}-\frac{4}{5}F_{471}\right)\frac{\partial^4 v}{\partial x_2^4}+\left(-2F_{470}-\frac{4}{3}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^4 v}{\partial x_1^2\partial x_2^2}$$
$$+\left(F_{47}+\frac{2}{15}F_{671}+\frac{1}{4}F_{672}\right)\frac{\partial^2 v}{\partial x_1^2}+\left(F_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^2 v}{\partial x_2^2}$$
$$+\left(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_2^5}+\left(-4F_{440}+4F_{480}-\frac{8}{5}F_{441}+\frac{8}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_1^2\partial x_2^3}$$
$$+\left(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_1^4\partial x_2}$$
$$+\left(-F_{22}+F_{33}+2F_{550}-2F_{570}+\frac{4}{5}F_{461}+\frac{32}{15}F_{551}-\frac{16}{15}F_{571}-\frac{2}{5}F_{681}+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\frac{\partial^3 w}{\partial x_2^3}$$
$$+\left(-F_{22}+F_{33}+2F_{550}-2F_{570}+\frac{4}{5}F_{461}+\frac{32}{15}F_{551}-\frac{16}{15}F_{571}-\frac{2}{5}F_{681}+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\frac{\partial^3 w}{\partial x_1^2\partial x_2}$$
$$+\left(-k_sF_{55}-\frac{8}{15}F_{661}-\frac{1}{4}F_{662}\right)\frac{\partial w}{\partial x_2}$$
$$+\left(-2F_{440}-\frac{4}{15}F_{441}+\frac{1}{4}F_{442}\right)\frac{\partial^4 \phi_1}{\partial x_1^3\partial x_2}+\left(-2F_{440}-\frac{4}{15}F_{441}+\frac{1}{4}F_{442}\right)\frac{\partial^4 \phi_1}{\partial x_1\partial x_2^3}$$
$$+\left(F_{L33}+F_{44}+2F_{550}+\frac{8}{15}F_{461}+\frac{4}{5}F_{551}-\frac{1}{2}F_{462}-\frac{3}{4}F_{552}\right)\frac{\partial^2 \phi_1}{\partial x_1\partial x_2}$$
$$+\left(-\frac{8}{15}F_{441}-\frac{1}{4}F_{442}\right)\frac{\partial^4 \phi_2}{\partial x_1^4}+\left(-2F_{440}-\frac{4}{5}F_{441}\right)\frac{\partial^4 \phi_2}{\partial x_2^4}+\left(-2F_{440}-\frac{4}{3}F_{441}-\frac{1}{4}F_{442}\right)\frac{\partial^4 \phi_2}{\partial x_1^2\partial x_2^2}$$
$$+\left(F_{44}+\frac{4}{15}F_{461}+\frac{4}{3}F_{551}+\frac{1}{2}F_{462}+F_{552}\right)\frac{\partial^2 \phi_2}{\partial x_1^2}$$
$$+\left(F_{33}+2F_{550}+\frac{4}{5}F_{461}+\frac{32}{15}F_{551}+\frac{1}{4}F_{552}\right)\frac{\partial^2 \phi_2}{\partial x_2^2}$$
$$+\left(-k_sF_{55}-\frac{8}{15}F_{661}-\frac{1}{4}F_{662}\right)\phi_2-F_{T11}\frac{\partial \Delta T}{\partial x_2} \qquad (108)$$
$$=I_3\frac{\partial^2 v}{\partial t^2}+\left(I_5-I_4\right)\frac{\partial^3 w}{\partial x_2\partial t^2}+I_5\frac{\partial^2 \phi_2}{\partial t^2},$$

$\delta u = 0$ veya

$$\Bigg(\bigg(\left(-2A_{550}-\frac{4}{5}A_{551}\right)\frac{\partial^3 u}{\partial x_1^3}+\left(-A_{550}-\frac{2}{3}A_{551}-\frac{1}{8}A_{552}\right)\frac{\partial^3 u}{\partial x_1\partial x_2^2}+A_{11}\frac{\partial u}{\partial x_1}$$

$$+\left(-A_{550}+\frac{2}{15}A_{551}\right)\frac{\partial^3 v}{\partial x_2^3}+\left(-2A_{550}+\frac{4}{15}A_{551}+\frac{1}{8}A_{552}\right)\frac{\partial^3 v}{\partial x_1^2\partial x_2}+A_{L11}\frac{\partial v}{\partial x_2}$$

$$+\left(-2F_{470}+2B_{550}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\right)\frac{\partial^4 w}{\partial x_1^4}+\left(-F_{470}+B_{550}-\frac{2}{5}B_{551}+\frac{2}{5}F_{471}\right)\frac{\partial^4 w}{\partial x_2^4}$$

$$+\left(3B_{550}-3F_{470}+\frac{2}{5}B_{551}-\frac{2}{5}F_{471}\right)\frac{\partial^4 w}{\partial x_1^2\partial x_2^2}$$

$$+\left(F_{11}-B_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^2 w}{\partial x_1^2}+\left(F_{L11}-B_{L11}+\frac{2}{15}F_{671}-\frac{1}{8}F_{672}\right)\frac{\partial^2 w}{\partial x_2^2}$$

$$+\left(-2F_{470}-\frac{4}{5}F_{471}\right)\frac{\partial^3 \phi_1}{\partial x_1^3}+\left(-F_{470}-\frac{2}{3}F_{471}-\frac{1}{8}F_{472}\right)\frac{\partial^3 \phi_1}{\partial x_1\partial x_2^2}+\left(F_{11}+\frac{2}{5}F_{671}\right)\frac{\partial \phi_1}{\partial x_1}$$

$$+\left(-F_{470}+\frac{2}{5}F_{471}\right)\frac{\partial^3 \phi_2}{\partial x_2^3}+\left(-2F_{470}+\frac{4}{15}F_{471}+\frac{1}{8}F_{472}\right)\frac{\partial^3 \phi_2}{\partial x_1^2\partial x_2}$$

$$+\left(F_{L11}+\frac{2}{15}F_{671}-\frac{1}{8}F_{672}\right)\frac{\partial \phi_2}{\partial x_2}-A_{T11}\Delta T\bigg)n_{x_1}$$

$$+\bigg(\left(-\frac{8}{15}A_{551}-\frac{1}{4}A_{552}\right)\frac{\partial^3 u}{\partial x_2^3}+\left(-A_{550}-\frac{2}{3}A_{551}-\frac{1}{8}A_{552}\right)\frac{\partial^3 u}{\partial x_1^2\partial x_2}+A_{55}\frac{\partial u}{\partial x_2}$$

$$+\left(-\frac{8}{15}A_{551}+\frac{1}{8}A_{552}\right)\frac{\partial^3 v}{\partial x_1^3}+\left(-A_{550}-\frac{2}{3}A_{551}+\frac{1}{4}A_{552}\right)\frac{\partial^3 v}{\partial x_1\partial x_2^2}+A_{55}\frac{\partial v}{\partial x_1}$$

$$+\left(B_{550}-F_{470}+\frac{6}{5}B_{551}-\frac{6}{5}F_{471}\right)\frac{\partial^4 w}{\partial x_1^3\partial x_2}+\left(-F_{470}+B_{550}+\frac{6}{5}B_{551}-\frac{6}{5}F_{471}\right)\frac{\partial^4 w}{\partial x_1\partial x_2^3}$$

$$+\left(2F_{47}-2B_{55}+\frac{4}{15}F_{671}+\frac{1}{8}F_{672}\right)\frac{\partial^2 w}{\partial x_1\partial x_2}+\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^3 \phi_1}{\partial x_2^3}$$

$$+\left(-F_{470}-\frac{2}{3}F_{471}-\frac{1}{8}F_{472}\right)\frac{\partial^3 \phi_1}{\partial x_1^2\partial x_2}+\left(F_{47}+\frac{2}{15}F_{671}+\frac{1}{4}F_{672}\right)\frac{\partial \phi_1}{\partial x_2}+\left(-\frac{8}{15}F_{471}+\frac{1}{8}F_{472}\right)\frac{\partial^3 \phi_2}{\partial x_1^3}$$

$$+\left(-F_{470}-\frac{2}{3}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^3 \phi_2}{\partial x_1\partial x_2^2}+\left(F_{47}+\frac{2}{15}F_{671}-\frac{1}{8}F_{672}\right)\frac{\partial \phi_2}{\partial x_1}\bigg)n_{x_2}=0, \qquad (109)$$

$$\delta\frac{\partial u}{\partial x_1}=0 \quad \text{veya}$$

$$\begin{aligned}
&\Bigg(\Bigg(\left(2A_{550}+\frac{4}{5}A_{551}\right)\frac{\partial^2 u}{\partial x_1^2}-\frac{2}{5}A_{551}\frac{\partial^2 u}{\partial x_2^2}+\left(2A_{550}-\frac{4}{5}A_{551}\right)\frac{\partial^2 v}{\partial x_1\partial x_2}\\
&+\left(2F_{470}-2B_{550}-\frac{4}{5}B_{551}+\frac{4}{5}F_{471}\right)\frac{\partial^3 w}{\partial x_1^3}+\left(-2B_{550}+2F_{470}+\frac{6}{5}B_{551}-\frac{6}{5}F_{471}\right)\frac{\partial^3 w}{\partial x_1\partial x_2^2}\\
&-\frac{2}{5}F_{671}\frac{\partial w}{\partial x_1}+\left(2F_{470}+\frac{4}{5}F_{471}\right)\frac{\partial^2 \phi_1}{\partial x_1^2}-\frac{2}{5}F_{471}\frac{\partial^2 \phi_1}{\partial x_2^2}-\frac{2}{5}F_{671}\phi_1+\left(2F_{470}-\frac{4}{5}F_{471}\right)\frac{\partial^2 \phi_2}{\partial x_1\partial x_2}\Bigg)n_{x_1}\\
&+\Bigg(\left(A_{550}+\frac{16}{15}A_{551}+\frac{1}{8}A_{552}\right)\frac{\partial^2 u}{\partial x_1\partial x_2}+\left(\frac{8}{15}A_{551}-\frac{1}{8}A_{552}\right)\frac{\partial^2 v}{\partial x_1^2}+\left(A_{550}-\frac{2}{5}A_{551}\right)\frac{\partial^2 v}{\partial x_2^2}\\
&+\left(F_{470}-B_{550}+\frac{2}{5}B_{551}-\frac{2}{5}F_{471}\right)\frac{\partial^3 w}{\partial x_2^3}+\left(-B_{550}+F_{470}-\frac{8}{5}B_{551}+\frac{8}{5}F_{471}\right)\frac{\partial^3 w}{\partial x_1^2\partial x_2}\\
&+\left(-\frac{2}{15}F_{671}+\frac{1}{8}F_{672}\right)\frac{\partial w}{\partial x_2}\\
&+\left(F_{470}+\frac{16}{15}F_{471}+\frac{1}{8}F_{472}\right)\frac{\partial^2 \phi_1}{\partial x_1\partial x_2}+\left(\frac{8}{15}F_{471}-\frac{1}{8}F_{472}\right)\frac{\partial^2 \phi_2}{\partial x_1^2}+\left(F_{470}-\frac{2}{5}F_{471}\right)\frac{\partial^2 \phi_2}{\partial x_2^2}\\
&+\left(-\frac{2}{15}F_{671}+\frac{1}{8}F_{672}\right)\phi_2\Bigg)n_{x_2}=0,
\end{aligned} \tag{110}$$

$$\delta\frac{\partial u}{\partial x_2}=0 \quad \text{veya}$$

$$\begin{aligned}
&\Bigg(\Bigg(\left(A_{550}+\frac{16}{15}A_{551}+\frac{1}{8}A_{552}\right)\frac{\partial^2 u}{\partial x_1\partial x_2}+\left(\frac{8}{15}A_{551}-\frac{1}{8}A_{552}\right)\frac{\partial^2 v}{\partial x_1^2}+\left(A_{550}-\frac{2}{5}A_{551}\right)\frac{\partial^2 v}{\partial x_2^2}\\
&+\left(F_{470}-B_{550}+\frac{2}{5}B_{551}-\frac{2}{5}F_{471}\right)\frac{\partial^3 w}{\partial x_2^3}+\left(-B_{550}+F_{470}-\frac{8}{5}B_{551}+\frac{8}{5}F_{471}\right)\frac{\partial^3 w}{\partial x_1^2\partial x_2}\\
&+\left(-\frac{2}{15}F_{671}+\frac{1}{8}F_{672}\right)\frac{\partial w}{\partial x_2}+\left(F_{470}+\frac{16}{15}F_{471}+\frac{1}{8}F_{472}\right)\frac{\partial^2 \phi_1}{\partial x_1\partial x_2}\\
&+\left(\frac{8}{15}F_{471}-\frac{1}{8}F_{472}\right)\frac{\partial^2 \phi_2}{\partial x_1^2}+\left(F_{470}-\frac{2}{5}F_{471}\right)\frac{\partial^2 \phi_2}{\partial x_2^2}+\left(-\frac{2}{15}F_{671}+\frac{1}{8}F_{672}\right)\phi_2\Bigg)n_{x_1}\\
&+\Bigg(-\frac{2}{5}A_{551}\frac{\partial^2 u}{\partial x_1^2}+\left(\frac{8}{15}A_{551}+\frac{1}{4}A_{552}\right)\frac{\partial^2 u}{\partial x_2^2}+\left(\frac{16}{15}A_{551}-\frac{1}{4}A_{552}\right)\frac{\partial^2 v}{\partial x_1\partial x_2}\\
&+\left(\frac{2}{5}B_{551}-\frac{2}{5}F_{471}\right)\frac{\partial^3 w}{\partial x_1^3}+\left(-\frac{8}{5}B_{551}+\frac{8}{5}F_{471}\right)\frac{\partial^3 w}{\partial x_1\partial x_2^2}+\left(-\frac{2}{15}F_{671}-\frac{1}{4}F_{672}\right)\frac{\partial w}{\partial x_1}-\frac{2}{5}F_{471}\frac{\partial^2 \phi_1}{\partial x_1^2}\\
&+\left(\frac{8}{15}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^2 \phi_1}{\partial x_2^2}+\left(-\frac{2}{15}F_{671}-\frac{1}{4}F_{672}\right)\phi_1+\left(\frac{16}{15}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^2 \phi_2}{\partial x_1\partial x_2}\Bigg)n_{x_2}=0,
\end{aligned} \tag{111}$$

$\delta v = 0$ veya

$$
\begin{aligned}
&\Bigg(\left(\left(-\frac{8}{15}A_{551}+\frac{1}{8}A_{552}\right)\frac{\partial^3 u}{\partial x_2^3}+\left(-A_{550}-\frac{2}{3}A_{551}+\frac{1}{4}A_{552}\right)\frac{\partial^3 u}{\partial x_1^2\partial x_2}+A_{55}\frac{\partial u}{\partial x_2}\right.\\
&+\left(-A_{550}-\frac{2}{3}A_{551}-\frac{1}{8}A_{552}\right)\frac{\partial^3 v}{\partial x_1\partial x_2^2}+\left(-\frac{8}{15}A_{551}-\frac{1}{4}A_{552}\right)\frac{\partial^3 v}{\partial x_1^3}+A_{55}\frac{\partial v}{\partial x_1}\\
&+\left(B_{550}-F_{470}+\frac{6}{5}B_{551}-\frac{6}{5}F_{471}\right)\frac{\partial^4 w}{\partial x_1^3\partial x_2}+\left(B_{550}+\frac{6}{5}B_{551}-F_{470}-\frac{6}{5}F_{471}\right)\frac{\partial^4 w}{\partial x_1\partial x_2^3}\\
&+\left(2F_{47}-2B_{55}+\frac{4}{15}F_{671}+\frac{1}{8}F_{672}\right)\frac{\partial^2 w}{\partial x_1\partial x_2}\\
&+\left(-\frac{8}{15}F_{471}+\frac{1}{8}F_{472}\right)\frac{\partial^3 \phi_1}{\partial x_2^3}+\left(-F_{470}-\frac{2}{3}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^3 \phi_1}{\partial x_1^2\partial x_2}\\
&+\left(F_{47}+\frac{2}{15}F_{671}-\frac{1}{8}F_{672}\right)\frac{\partial \phi_1}{\partial x_2}+\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^3 \phi_2}{\partial x_1^3}\\
&\left.+\left(-F_{470}-\frac{2}{3}F_{471}-\frac{1}{8}F_{472}\right)\frac{\partial^3 \phi_2}{\partial x_1\partial x_2^2}+\left(F_{47}+\frac{2}{15}F_{671}+\frac{1}{4}F_{672}\right)\frac{\partial \phi_2}{\partial x_1}\right)n_{x_1}\\
&+\left(\left(-A_{550}+\frac{2}{5}A_{551}\right)\frac{\partial^3 u}{\partial x_1^3}+\left(-2A_{550}+\frac{4}{15}A_{551}+\frac{1}{8}A_{552}\right)\frac{\partial^3 u}{\partial x_1\partial x_2^2}+A_{L11}\frac{\partial u}{\partial x_1}\right.\\
&+\left(-2A_{550}-\frac{4}{5}A_{551}\right)\frac{\partial^3 v}{\partial x_2^3}+\left(-A_{550}-\frac{2}{3}A_{551}-\frac{1}{8}A_{552}\right)\frac{\partial^3 v}{\partial x_1^2\partial x_2}+A_{11}\frac{\partial v}{\partial x_2}\\
&+\left(B_{550}-F_{470}-\frac{2}{5}B_{551}+\frac{2}{5}F_{471}\right)\frac{\partial^4 w}{\partial x_1^4}+\left(-2F_{470}+2B_{550}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\right)\frac{\partial^4 w}{\partial x_2^4}\\
&+\left(3B_{550}+\frac{2}{5}B_{551}-3F_{470}-\frac{2}{5}F_{471}\right)\frac{\partial^4 w}{\partial x_1^2\partial x_2^2}\\
&+\left(F_{L11}-B_{L11}+\frac{2}{15}F_{671}-\frac{1}{8}F_{672}\right)\frac{\partial^2 w}{\partial x_1^2}+\left(F_{11}-B_{11}+\frac{2}{5}F_{671}\right)\frac{\partial^2 w}{\partial x_2^2}\\
&+\left(-F_{470}+\frac{2}{5}F_{471}\right)\frac{\partial^3 \phi_1}{\partial x_1^3}+\left(-2F_{470}+\frac{4}{15}F_{471}+\frac{1}{8}F_{472}\right)\frac{\partial^3 \phi_1}{\partial x_1\partial x_2^2}\\
&+\left(F_{L11}+\frac{2}{15}F_{671}-\frac{1}{8}F_{672}\right)\frac{\partial \phi_1}{\partial x_1}+\left(-2F_{470}-\frac{4}{5}F_{471}\right)\frac{\partial^3 \phi_2}{\partial x_2^3}\\
&\left.+\left(-F_{470}-\frac{2}{3}F_{471}-\frac{1}{8}F_{472}\right)\frac{\partial^3 \phi_2}{\partial x_1^2\partial x_2}+\left(F_{11}+\frac{2}{5}F_{671}\right)\frac{\partial \phi_2}{\partial x_2}-A_{T11}\Delta T\right)n_{x_2}=0,
\end{aligned}
\tag{112}
$$

$$\delta \frac{\partial v}{\partial x_1} = 0 \quad \text{veya}$$

$$\Bigg(\Bigg(\left(\frac{16}{15}A_{551} - \frac{1}{4}A_{552}\right)\frac{\partial^2 u}{\partial x_1 \partial x_2} + \left(\frac{8}{15}A_{551} + \frac{1}{4}A_{552}\right)\frac{\partial^2 v}{\partial x_1^2} - \frac{2}{5}A_{551}\frac{\partial^2 v}{\partial x_2^2}$$

$$+\left(\frac{2}{5}B_{551} - \frac{2}{5}F_{471}\right)\frac{\partial^3 w}{\partial x_2^3} + \left(-\frac{8}{5}B_{551} + \frac{8}{5}F_{471}\right)\frac{\partial^3 w}{\partial x_1^2 \partial x_2} + \left(-\frac{2}{15}F_{671} - \frac{1}{4}F_{672}\right)\frac{\partial w}{\partial x_2}$$

$$+\left(\frac{16}{15}F_{471} - \frac{1}{4}F_{472}\right)\frac{\partial^2 \phi_1}{\partial x_1 \partial x_2} + \left(\frac{8}{15}F_{471} + \frac{1}{4}F_{472}\right)\frac{\partial^2 \phi_2}{\partial x_1^2} - \frac{2}{5}F_{471}\frac{\partial^2 \phi_2}{\partial x_2^2} + \left(-\frac{2}{15}F_{671} - \frac{1}{4}F_{672}\right)\phi_2\Bigg)n_{x_1}$$

$$+\Bigg(\left(A_{550} - \frac{2}{5}A_{551}\right)\frac{\partial^2 u}{\partial x_1^2} + \left(\frac{8}{15}A_{551} - \frac{1}{8}A_{552}\right)\frac{\partial^2 u}{\partial x_2^2} + \left(A_{550} + \frac{16}{15}A_{551} + \frac{1}{8}A_{552}\right)\frac{\partial^2 v}{\partial x_1 \partial x_2}$$

$$+\left(-B_{550} + F_{470} + \frac{2}{5}B_{551} - \frac{2}{5}F_{471}\right)\frac{\partial^3 w}{\partial x_1^3} + \left(-B_{550} + F_{470} - \frac{8}{5}B_{551} + \frac{8}{5}F_{471}\right)\frac{\partial^3 w}{\partial x_1 \partial x_2^2}$$

$$+\left(-\frac{2}{15}F_{671} + \frac{1}{8}F_{672}\right)\frac{\partial w}{\partial x_1} + \left(F_{470} - \frac{2}{5}F_{471}\right)\frac{\partial^2 \phi_1}{\partial x_1^2} + \left(\frac{8}{15}F_{471} - \frac{1}{8}F_{472}\right)\frac{\partial^2 \phi_1}{\partial x_2^2} + \left(-\frac{2}{15}F_{671} + \frac{1}{8}F_{672}\right)\phi_1$$

$$+\left(F_{470} + \frac{16}{15}F_{471} + \frac{1}{8}F_{472}\right)\frac{\partial^2 \phi_2}{\partial x_1 \partial x_2}\Bigg)n_{x_2} = 0,$$

(113)

$$\delta \frac{\partial v}{\partial x_2} = 0 \quad \text{veya}$$

$$\Bigg(\Bigg(\left(A_{550} - \frac{2}{5}A_{551}\right)\frac{\partial^2 u}{\partial x_1^2} + \left(\frac{8}{15}A_{551} - \frac{1}{8}A_{552}\right)\frac{\partial^2 u}{\partial x_2^2} + \left(A_{550} + \frac{16}{15}A_{551} + \frac{1}{8}A_{552}\right)\frac{\partial^2 v}{\partial x_1 \partial x_2}$$

$$+\left(-B_{550} + F_{470} + \frac{2}{5}B_{551} - \frac{2}{5}F_{471}\right)\frac{\partial^3 w}{\partial x_1^3} + \left(-B_{550} + F_{470} - \frac{8}{5}B_{551} + \frac{8}{5}F_{471}\right)\frac{\partial^3 w}{\partial x_1 \partial x_2^2}$$

$$+\left(-\frac{2}{15}F_{671} + \frac{1}{8}F_{672}\right)\frac{\partial w}{\partial x_1} + \left(F_{470} - \frac{2}{5}F_{471}\right)\frac{\partial^2 \phi_1}{\partial x_1^2} + \left(\frac{8}{15}F_{471} - \frac{1}{8}F_{472}\right)\frac{\partial^2 \phi_1}{\partial x_2^2}$$

$$+\left(-\frac{2}{15}F_{671} + \frac{1}{8}F_{672}\right)\phi_1 + \left(F_{470} + \frac{16}{15}F_{471} + \frac{1}{8}F_{472}\right)\frac{\partial^2 \phi_2}{\partial x_1 \partial x_2}\Bigg)n_{x_1}$$

$$+\Bigg(\left(2A_{550} - \frac{4}{5}A_{551}\right)\frac{\partial^2 u}{\partial x_1 \partial x_2} - \frac{2}{5}A_{551}\frac{\partial^2 v}{\partial x_1^2} + \left(2A_{550} + \frac{4}{5}A_{551}\right)\frac{\partial^2 v}{\partial x_2^2}$$

$$+\left(-2B_{550} + 2F_{470} + \frac{6}{5}B_{551} - \frac{6}{5}F_{471}\right)\frac{\partial^3 w}{\partial x_1^2 \partial x_2} + \left(2F_{470} - 2B_{550} - \frac{4}{5}B_{551} + \frac{4}{5}F_{471}\right)\frac{\partial^3 w}{\partial x_2^3} - \frac{2}{5}F_{671}\frac{\partial w}{\partial x_2}$$

$$+\left(2F_{470} - \frac{4}{5}F_{471}\right)\frac{\partial^2 \phi_1}{\partial x_1 \partial x_2} - \frac{2}{5}F_{471}\frac{\partial^2 \phi_2}{\partial x_1^2} + \left(2F_{470} + \frac{4}{5}F_{471}\right)\frac{\partial^2 \phi_2}{\partial x_2^2} - \frac{2}{5}F_{671}\phi_2\Bigg)n_{x_2} = 0,$$

(114)

$\delta w = 0$ veya

$$\Bigg(\bigg(\left(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{5}F_{471}\right)\frac{\partial^4 u}{\partial x_1^4}+\frac{8}{15}F_{471}\frac{\partial^4 u}{\partial x_2^4}$$

$$+\left(-2B_{550}+2F_{470}-\frac{14}{5}B_{551}+\frac{4}{3}F_{471}\right)\frac{\partial^4 u}{\partial x_1^2\partial x_2^2}$$

$$+\left(B_{11}-F_{11}-\frac{2}{5}F_{671}\right)\frac{\partial^2 u}{\partial x_1^2}+\left(B_{55}-F_{47}-\frac{2}{15}F_{671}-\frac{1}{4}F_{672}\right)\frac{\partial^2 u}{\partial x_2^2}$$

$$+\left(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{15}F_{471}\right)\frac{\partial^4 v}{\partial x_1^3\partial x_2}+\left(-2B_{550}+2F_{470}+\frac{6}{5}B_{551}+\frac{4}{15}F_{471}\right)\frac{\partial^4 v}{\partial x_1\partial x_2^3}$$

$$+\left(B_{55}+B_{L11}-F_{47}-F_{L11}-\frac{4}{15}F_{671}+\frac{1}{4}F_{672}\right)\frac{\partial^2 v}{\partial x_1\partial x_2}$$

$$+\left(2D_{550}+2F_{440}-4F_{480}+\frac{4}{5}D_{551}+\frac{4}{5}F_{441}-\frac{8}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_1^5}$$

$$+\left(4D_{550}+4F_{440}-8F_{480}+\frac{18}{5}D_{551}+\frac{8}{5}F_{441}-\frac{26}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_1^3\partial x_2^2}$$

$$+\left(2D_{550}+2F_{440}-4F_{480}-\frac{6}{5}D_{551}+\frac{4}{5}F_{441}+\frac{2}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_1\partial x_2^4}$$

$$+\bigg(-D_{11}+2F_{22}-F_{33}-2A_{550}-2F_{550}+4F_{570}-\frac{8}{15}A_{551}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{32}{15}F_{571}+\frac{4}{5}F_{681}-A_{552}$$

$$-\frac{1}{4}F_{552}+F_{572}\bigg)\frac{\partial^3 w}{\partial x_1^3}+\bigg(-D_{11}+2F_{22}-F_{33}-2A_{550}-2F_{550}+4F_{570}-\frac{8}{15}A_{551}$$

$$-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{32}{15}F_{571}+\frac{4}{5}F_{681}-A_{552}-\frac{1}{4}F_{552}+F_{572}\bigg)\frac{\partial^3 w}{\partial x_1\partial x_2^2}+\left(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\right)\frac{\partial w}{\partial x_1}$$

$$+\left(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\right)\frac{\partial^4\phi_1}{\partial x_1^4}+\frac{8}{15}F_{441}\frac{\partial^4\phi_1}{\partial x_2^4}+\left(2F_{440}-2F_{480}+\frac{4}{3}F_{441}-\frac{14}{5}F_{481}\right)\frac{\partial^4\phi_1}{\partial x_1^2\partial x_2^2}$$

$$+\left(F_{22}-F_{33}-2F_{550}+2F_{570}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{16}{15}F_{571}+\frac{2}{5}F_{681}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\right)\frac{\partial^2\phi_1}{\partial x_1^2}$$

$$+\left(-F_{44}+F_{48}-\frac{4}{15}F_{461}-\frac{4}{3}F_{551}+\frac{2}{3}F_{571}-\frac{1}{4}F_{462}-\frac{1}{2}F_{552}+\frac{1}{2}F_{572}\right)\frac{\partial^2\phi_1}{\partial x_2^2}$$

$$+\left(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\right)\phi_1+\left(2F_{440}-2F_{480}+\frac{4}{15}F_{441}-\frac{4}{5}F_{481}\right)\frac{\partial^4\phi_2}{\partial x_1^3\partial x_2}$$

$$+\left(2F_{440}-2F_{480}+\frac{4}{15}F_{441}+\frac{6}{5}F_{481}\right)\frac{\partial^4\phi_2}{\partial x_1\partial x_2^3}+\bigg(F_{48}+F_{L22}-F_{44}-F_{L33}-2F_{550}+2F_{570}-\frac{8}{15}F_{461}$$

$$-\frac{4}{5}F_{551}+\frac{2}{5}F_{571}+\frac{2}{5}F_{681}+\frac{1}{4}F_{462}+\frac{1}{4}F_{552}\bigg)\frac{\partial^2\phi_2}{\partial x_1\partial x_2}+\left(-B_{T11}+F_{T11}\right)\frac{\partial\Delta T}{\partial x_1}-\frac{\partial w}{\partial x_1}P_{x_1}\Bigg)n_{x_1}$$

$$
+\left(\left(-2B_{550}+2F_{470}+\frac{6}{5}B_{551}+\frac{4}{15}F_{471}\right)\frac{\partial^4 u}{\partial x_1^3\partial x_2}\right.
$$

$$
+\left(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{15}F_{471}\right)\frac{\partial^4 u}{\partial x_1\partial x_2^3}+\left(B_{55}+B_{L11}-F_{47}-F_{L11}-\frac{4}{15}F_{671}+\frac{1}{4}F_{672}\right)\frac{\partial^2 u}{\partial x_1\partial x_2}
$$

$$
+\frac{8}{15}F_{471}\frac{\partial^4 v}{\partial x_1^4}+\left(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{5}F_{471}\right)\frac{\partial^4 v}{\partial x_2^4}
$$

$$
+\left(-2B_{550}+2F_{470}-\frac{14}{5}B_{551}+\frac{4}{3}F_{471}\right)\frac{\partial^4 v}{\partial x_1^2\partial x_2^2}+\left(B_{11}-F_{11}-\frac{2}{5}F_{671}\right)\frac{\partial^2 v}{\partial x_2^2}
$$

$$
+\left(B_{55}-F_{47}-\frac{2}{15}F_{671}-\frac{1}{4}F_{672}\right)\frac{\partial^2 v}{\partial x_1^2}+\left(2D_{550}+2F_{440}-4F_{480}+\frac{4}{5}D_{551}+\frac{4}{5}F_{441}-\frac{8}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_2^5}
$$

$$
+\left(2D_{550}+2F_{440}-4F_{480}-\frac{6}{5}D_{551}+\frac{4}{5}F_{441}+\frac{2}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_1^4\partial x_2}
$$

$$
+\left(4D_{550}+4F_{440}-8F_{480}+\frac{18}{5}D_{551}+\frac{8}{5}F_{441}-\frac{26}{5}F_{481}\right)\frac{\partial^5 w}{\partial x_1^2\partial x_2^3}
$$

$$
+\left(-D_{11}+2F_{22}-F_{33}-2A_{550}-2F_{550}+4F_{570}-\frac{8}{15}A_{551}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{32}{15}F_{571}+\frac{4}{5}F_{681}-A_{552}\right.
$$

$$
\left.-\frac{1}{4}F_{552}+F_{572}\right)\frac{\partial^3 w}{\partial x_2^3}+\left(-D_{11}+2F_{22}-F_{33}-2A_{550}-2F_{550}+4F_{570}-\frac{8}{15}A_{551}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}\right.
$$

$$
\left.+\frac{32}{15}F_{571}+\frac{4}{5}F_{681}-A_{552}-\frac{1}{4}F_{552}+F_{572}\right)\frac{\partial^3 w}{\partial x_1^2\partial x_2}+\left(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\right)\frac{\partial w}{\partial x_2}
$$

$$
+\left(2F_{440}-2F_{480}+\frac{4}{15}F_{441}+\frac{6}{5}F_{481}\right)\frac{\partial^4\phi_1}{\partial x_1^3\partial x_2}+\left(2F_{440}-2F_{480}+\frac{4}{15}F_{441}-\frac{4}{5}F_{481}\right)\frac{\partial^4\phi_1}{\partial x_1\partial x_2^3}
$$

$$
+\left(F_{48}+F_{L22}-F_{44}-F_{L33}-2F_{550}+2F_{570}-\frac{8}{15}F_{461}-\frac{4}{5}F_{551}+\frac{2}{5}F_{571}+\frac{2}{5}F_{681}+\frac{1}{4}F_{462}+\frac{1}{4}F_{552}\right)\frac{\partial^2\phi_1}{\partial x_1\partial x_2}
$$

$$
+\frac{8}{15}F_{441}\frac{\partial^4\phi_2}{\partial x_1^4}+\left(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\right)\frac{\partial^4\phi_2}{\partial x_2^4}+\left(2F_{440}-2F_{480}+\frac{4}{3}F_{441}-\frac{14}{5}F_{481}\right)\frac{\partial^4\phi_2}{\partial x_1^2\partial x_2^2}
$$

$$
+\left(-F_{44}+F_{48}-\frac{4}{15}F_{461}-\frac{4}{3}F_{551}+\frac{2}{3}F_{571}-\frac{1}{4}F_{462}-\frac{1}{2}F_{552}+\frac{1}{2}F_{572}\right)\frac{\partial^2\phi_2}{\partial x_1^2}
$$

$$
+\left(F_{22}-F_{33}-2F_{550}+2F_{570}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{16}{15}F_{571}+\frac{2}{5}F_{681}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\right)\frac{\partial^2\phi_2}{\partial x_2^2}
$$

$$
\left.+\left(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\right)\phi_2+\left(-B_{T11}+F_{T11}\right)\frac{\partial\Delta T}{\partial x_2}-\frac{\partial w}{\partial x_2}P_{x_2}\right)n_{x_2}
$$

$$
=\left(\left(I_1-I_3\right)\frac{\partial^2 u}{\partial t^2}+\left(2I_4-I_2-I_5\right)\frac{\partial^3 w}{\partial x_1\partial t^2}+\left(I_4-I_5\right)\frac{\partial^2\phi_1}{\partial t^2}\right)n_{x_1}
$$

$$
+\left(\left(I_1-I_3\right)\frac{\partial^2 v}{\partial t^2}+\left(2I_4-I_2-I_5\right)\frac{\partial^3 w}{\partial x_2\partial t^2}+\left(I_4-I_5\right)\frac{\partial^2\phi_2}{\partial t^2}\right)n_{x_2},
$$

(115)

$$\delta \frac{\partial w}{\partial x_1} = 0 \quad \text{veya}$$

$$\Bigg(\bigg(\left(2B_{550} - 2F_{470} + \frac{4}{5}B_{551} - \frac{4}{5}F_{471}\right)\frac{\partial^3 u}{\partial x_1^3} + \left(2B_{550} - 2F_{470} + \frac{14}{5}B_{551} - \frac{14}{5}F_{471}\right)\frac{\partial^3 u}{\partial x_1 \partial x_2^2} + \left(-B_{11} + F_{11}\right)\frac{\partial u}{\partial x_1}$$

$$+\left(2B_{550} - 2F_{470} - \frac{6}{5}B_{551} + \frac{6}{5}F_{471}\right)\frac{\partial^3 v}{\partial x_2^3} + \left(2B_{550} - 2F_{470} + \frac{4}{5}B_{551} - \frac{4}{5}F_{471}\right)\frac{\partial^3 v}{\partial x_1^2 \partial x_2} + \left(-B_{L11} + F_{L11}\right)\frac{\partial v}{\partial x_2}$$

$$+\left(-2D_{550} - 2F_{440} + 4F_{480} - \frac{4}{5}D_{551} - \frac{4}{5}F_{441} + \frac{8}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1^4}$$

$$+\left(-2D_{550} + 4F_{480} - 2F_{440} + \frac{6}{5}D_{551} + \frac{6}{5}F_{441} - \frac{12}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_2^4}$$

$$+\left(-4D_{550} + 8F_{480} - 4F_{440} - \frac{18}{5}D_{551} - \frac{18}{5}F_{441} + \frac{36}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2}$$

$$+\bigg(D_{11} + F_{33} - 2F_{22} + 2A_{550} + 2F_{550} - 4F_{570} + \frac{8}{15}A_{551} + \frac{2}{5}F_{461} + \frac{32}{15}F_{551} - \frac{32}{15}F_{571} - \frac{2}{5}F_{681}$$

$$+A_{552} + \frac{1}{4}F_{552} - F_{572}\bigg)\frac{\partial^2 w}{\partial x_1^2} + \big(D_{L11} - 2F_{L22} + F_{L33} + 2A_{550} + 2F_{550} - 4F_{570}$$

$$-\frac{2}{15}A_{551} + \frac{2}{5}F_{461} - \frac{8}{15}F_{551} + \frac{8}{15}F_{571} - \frac{2}{5}F_{681} - A_{552} - \frac{1}{4}F_{552} + F_{572}\big)\frac{\partial^2 w}{\partial x_2^2}$$

$$+\left(-2F_{440} + 2F_{480} - \frac{4}{5}F_{441} + \frac{4}{5}F_{481}\right)\frac{\partial^3 \phi_1}{\partial x_1^3} + \left(-2F_{440} + 2F_{480} - \frac{14}{5}F_{441} + \frac{14}{5}F_{481}\right)\frac{\partial^3 \phi_1}{\partial x_1 \partial x_2^2}$$

$$+\left(-F_{22} + F_{33} + 2F_{550} - 2F_{570} + \frac{2}{5}F_{461} + \frac{32}{15}F_{551} - \frac{16}{15}F_{571} - \frac{2}{5}F_{681} + \frac{1}{4}F_{552} - \frac{1}{2}F_{572}\right)\frac{\partial \phi_1}{\partial x_1}$$

$$+\left(-2F_{440} + 2F_{480} + \frac{6}{5}F_{441} - \frac{6}{5}F_{481}\right)\frac{\partial^3 \phi_2}{\partial x_2^3} + \left(-2F_{440} + 2F_{480} - \frac{4}{5}F_{441} + \frac{4}{5}F_{481}\right)\frac{\partial^3 \phi_2}{\partial x_1^2 \partial x_2}$$

$$+\left(-F_{L22} + F_{L33} + 2F_{550} - 2F_{570} + \frac{2}{5}F_{461} - \frac{8}{15}F_{551} + \frac{4}{15}F_{571} - \frac{2}{5}F_{681} - \frac{1}{4}F_{552} + \frac{1}{2}F_{572}\right)\frac{\partial \phi_2}{\partial x_2}$$

$$+\left(B_{T11} - F_{T11}\right)\Delta T\bigg)n_{x_1}$$

$$+\bigg(\left(-B_{55} + F_{47}\right)\frac{\partial u}{\partial x_2} + \left(-B_{55} + F_{47}\right)\frac{\partial v}{\partial x_1}$$

$$+\left(2D_{55} + 2F_{44} - 4F_{48} + \frac{2}{3}A_{551} + \frac{8}{3}F_{551} - \frac{8}{3}F_{571} + 2A_{552} + \frac{1}{2}F_{552} - 2F_{572}\right)\frac{\partial^2 w}{\partial x_1 \partial x_2}$$

$$+\left(F_{44} - F_{48} + \frac{4}{3}F_{551} - \frac{2}{3}F_{571} + \frac{1}{4}F_{552} - \frac{1}{2}F_{572}\right)\frac{\partial \phi_1}{\partial x_2}$$

$$+\left(F_{44} - F_{48} + \frac{4}{3}F_{551} - \frac{2}{3}F_{571} + \frac{1}{4}F_{552} - \frac{1}{2}F_{572}\right)\frac{\partial \phi_2}{\partial x_1}\bigg)n_{x_2} = 0, \tag{116}$$

$$\delta \frac{\partial w}{\partial x_2} = 0 \quad \text{veya}$$

$$\Bigg(\left(-B_{55} + F_{47}\right) \frac{\partial u}{\partial x_2} + \left(-B_{55} + F_{47}\right) \frac{\partial v}{\partial x_1} + \Bigg(2D_{55} + 2F_{44} - 4F_{48} + \frac{2}{3} A_{551} + \frac{8}{3} F_{551}$$

$$- \frac{8}{3} F_{571} + 2A_{552} + \frac{1}{2} F_{552} - 2F_{572} \Bigg) \frac{\partial^2 w}{\partial x_1 \partial x_2} + \left(F_{44} - F_{48} + \frac{4}{3} F_{551} - \frac{2}{3} F_{571} + \frac{1}{4} F_{552} - \frac{1}{2} F_{572} \right) \frac{\partial \phi_1}{\partial x_2}$$

$$+ \left(F_{44} - F_{48} + \frac{4}{3} F_{551} - \frac{2}{3} F_{571} + \frac{1}{4} F_{552} - \frac{1}{2} F_{572} \right) \frac{\partial \phi_2}{\partial x_1} \Bigg) n_{x_1}$$

$$+ \Bigg(\left(2B_{550} - 2F_{470} - \frac{6}{5} B_{551} + \frac{6}{5} F_{471} \right) \frac{\partial^3 u}{\partial x_1^3} + \left(2B_{550} - 2F_{470} + \frac{4}{5} B_{551} - \frac{4}{5} F_{471} \right) \frac{\partial^3 u}{\partial x_1 \partial x_2^2}$$

$$+ \left(-B_{L11} + F_{L11}\right) \frac{\partial u}{\partial x_1} + \left(2B_{550} - 2F_{470} + \frac{4}{5} B_{551} - \frac{4}{5} F_{471} \right) \frac{\partial^3 v}{\partial x_2^3}$$

$$+ \left(2B_{550} - 2F_{470} + \frac{14}{5} B_{551} - \frac{14}{5} F_{471} \right) \frac{\partial^3 v}{\partial x_1^2 \partial x_2} + \left(-B_{11} + F_{11}\right) \frac{\partial v}{\partial x_2}$$

$$+ \left(-2D_{550} + 4F_{480} - 2F_{440} + \frac{6}{5} D_{551} + \frac{6}{5} F_{441} - \frac{12}{5} F_{481} \right) \frac{\partial^4 w}{\partial x_1^4}$$

$$+ \left(-2D_{550} - 2F_{440} + 4F_{480} - \frac{4}{5} D_{551} - \frac{4}{5} F_{441} + \frac{8}{5} F_{481} \right) \frac{\partial^4 w}{\partial x_2^4}$$

$$+ \left(-4D_{550} + 8F_{480} - 4F_{440} - \frac{18}{5} D_{551} - \frac{18}{5} F_{441} + \frac{36}{5} F_{481} \right) \frac{\partial^4 w}{\partial x_1^2 \partial x_2^2}$$

$$+ \Bigg(D_{L11} - 2F_{L22} + F_{L33} + 2A_{550} + 2F_{550} - 4F_{570} - \frac{2}{15} A_{551} + \frac{2}{5} F_{461} - \frac{8}{15} F_{551} + \frac{8}{15} F_{571} - \frac{2}{5} F_{681}$$

$$- A_{552} - \frac{1}{4} F_{552} + F_{572} \Bigg) \frac{\partial^2 w}{\partial x_1^2} + \Big(D_{11} + F_{33} - 2F_{22} + 2A_{550} + 2F_{550} - 4F_{570}$$

$$+ \frac{8}{15} A_{551} + \frac{2}{5} F_{461} + \frac{32}{15} F_{551} - \frac{32}{15} F_{571} - \frac{2}{5} F_{681} + A_{552} + \frac{1}{4} F_{552} - F_{572} \Big) \frac{\partial^2 w}{\partial x_2^2}$$

$$+ \left(-2F_{440} + 2F_{480} + \frac{6}{5} F_{441} - \frac{6}{5} F_{481} \right) \frac{\partial^3 \phi_1}{\partial x_1^3} + \left(-2F_{440} + 2F_{480} - \frac{4}{5} F_{441} + \frac{4}{5} F_{481} \right) \frac{\partial^3 \phi_1}{\partial x_1 \partial x_2^2}$$

$$+ \left(-F_{L22} + F_{L33} + 2F_{550} - 2F_{570} + \frac{2}{5} F_{461} - \frac{8}{15} F_{551} + \frac{4}{15} F_{571} - \frac{2}{5} F_{681} - \frac{1}{4} F_{552} + \frac{1}{2} F_{572} \right) \frac{\partial \phi_1}{\partial x_1}$$

$$+ \left(-2F_{440} + 2F_{480} - \frac{4}{5} F_{441} + \frac{4}{5} F_{481} \right) \frac{\partial^3 \phi_2}{\partial x_2^3} + \left(-2F_{440} + 2F_{480} - \frac{14}{5} F_{441} + \frac{14}{5} F_{481} \right) \frac{\partial^3 \phi_2}{\partial x_1^2 \partial x_2}$$

$$+ \left(-F_{22} + F_{33} + 2F_{550} - 2F_{570} + \frac{2}{5} F_{461} + \frac{32}{15} F_{551} - \frac{16}{15} F_{571} - \frac{2}{5} F_{681} + \frac{1}{4} F_{552} - \frac{1}{2} F_{572} \right) \frac{\partial \phi_2}{\partial x_2}$$

$$+ \left(B_{T11} - F_{T11}\right) \Delta T \Bigg) n_{x_2} = 0,$$

(117)

$$\delta \frac{\partial^2 w}{\partial x_1^2} = 0 \quad \text{veya}$$

$$\Bigg(\Bigg(\left(-2B_{550} + 2F_{470} - \frac{4}{5}B_{551} + \frac{4}{5}F_{471}\right)\frac{\partial^2 u}{\partial x_1^2} + \left(+\frac{2}{5}B_{551} - \frac{2}{5}F_{471}\right)\frac{\partial^2 u}{\partial x_2^2}$$

$$+\left(-2B_{550} + 2F_{470} + \frac{4}{5}B_{551} - \frac{4}{5}F_{471}\right)\frac{\partial^2 v}{\partial x_1 \partial x_2} + \left(2D_{550} + 2F_{440} - 4F_{480} + \frac{4}{5}D_{551} + \frac{4}{5}F_{441} - \frac{8}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_1^3}$$

$$+\left(2D_{550} + 2F_{440} - 4F_{480} - \frac{6}{5}D_{551} - \frac{6}{5}F_{441} + \frac{12}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_1 \partial x_2^2}$$

$$+\left(-\frac{2}{5}F_{461} + \frac{2}{5}F_{681}\right)\frac{\partial w}{\partial x_1} + \left(2F_{440} - 2F_{480} + \frac{4}{5}F_{441} - \frac{4}{5}F_{481}\right)\frac{\partial^2 \phi_1}{\partial x_1^2} + \left(-\frac{2}{5}F_{441} + \frac{2}{5}F_{481}\right)\frac{\partial^2 \phi_1}{\partial x_2^2}$$

$$+\left(-\frac{2}{5}F_{461} + \frac{2}{5}F_{681}\right)\phi_1 + \left(2F_{440} - 2F_{480} - \frac{4}{5}F_{441} + \frac{4}{5}F_{481}\right)\frac{\partial^2 \phi_2}{\partial x_1 \partial x_2}\Bigg) n_{x_1}$$

$$+\Bigg(\left(-2B_{550} + 2F_{470} - \frac{16}{5}B_{551} + \frac{16}{5}F_{471}\right)\frac{\partial^2 u}{\partial x_1 \partial x_2}$$

$$+\left(-\frac{8}{5}B_{551} + \frac{8}{5}F_{471}\right)\frac{\partial^2 v}{\partial x_1^2} + \left(-2B_{550} + 2F_{470} + \frac{6}{5}B_{551} - \frac{6}{5}F_{471}\right)\frac{\partial^2 v}{\partial x_2^2}$$

$$+\left(2D_{550} + 2F_{440} - 4F_{480} - \frac{6}{5}D_{551} - \frac{6}{5}F_{441} + \frac{12}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_2^3}$$

$$+\left(2D_{550} + 2F_{440} - 4F_{480} + \frac{24}{5}D_{551} + \frac{24}{5}F_{441} - \frac{48}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_1^2 \partial x_2}$$

$$+\left(-\frac{2}{5}F_{461} + \frac{2}{5}F_{681}\right)\frac{\partial w}{\partial x_2} + \left(2F_{440} - 2F_{480} + \frac{16}{5}F_{441} - \frac{16}{5}F_{481}\right)\frac{\partial^2 \phi_1}{\partial x_1 \partial x_2}$$

$$+\left(\frac{8}{5}F_{441} - \frac{8}{5}F_{481}\right)\frac{\partial^2 \phi_2}{\partial x_1^2} + \left(2F_{440} - 2F_{480} - \frac{6}{5}F_{441} + \frac{6}{5}F_{481}\right)\frac{\partial^2 \phi_2}{\partial x_2^2} + \left(-\frac{2}{5}F_{461} + \frac{2}{5}F_{681}\right)\phi_2\Bigg) n_{x_2} = 0,$$

(118)

$$\delta \frac{\partial^2 w}{\partial x_2^2} = 0 \quad \text{veya}$$

$$\left(\left(-2B_{550} + 2F_{470} + \frac{6}{5}B_{551} - \frac{6}{5}F_{471} \right) \frac{\partial^2 u}{\partial x_1^2} + \left(-\frac{8}{5}B_{551} + \frac{8}{5}F_{471} \right) \frac{\partial^2 u}{\partial x_2^2} \right.$$

$$+ \left(-2B_{550} + 2F_{470} - \frac{16}{5}B_{551} + \frac{16}{5}F_{471} \right) \frac{\partial^2 v}{\partial x_1 \partial x_2}$$

$$+ \left(2D_{550} + 2F_{440} - 4F_{480} - \frac{6}{5}D_{551} - \frac{6}{5}F_{441} + \frac{12}{5}F_{481} \right) \frac{\partial^3 w}{\partial x_1^3}$$

$$+ \left(2D_{550} + 2F_{440} - 4F_{480} + \frac{24}{5}D_{551} + \frac{24}{5}F_{441} - \frac{48}{5}F_{481} \right) \frac{\partial^3 w}{\partial x_1 \partial x_2^2} + \left(-\frac{2}{5}F_{461} + \frac{2}{5}F_{681} \right) \frac{\partial w}{\partial x_1}$$

$$+ \left(2F_{440} - 2F_{480} - \frac{6}{5}F_{441} + \frac{6}{5}F_{481} \right) \frac{\partial^2 \phi_1}{\partial x_1^2} + \left(\frac{8}{5}F_{441} - \frac{8}{5}F_{481} \right) \frac{\partial^2 \phi_1}{\partial x_2^2} + \left(-\frac{2}{5}F_{461} + \frac{2}{5}F_{681} \right) \phi_1$$

$$\left. + \left(2F_{440} - 2F_{480} + \frac{16}{5}F_{441} - \frac{16}{5}F_{481} \right) \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} \right) n_{x_1}$$

$$+ \left(\left(-2B_{550} + 2F_{470} + \frac{4}{5}B_{551} - \frac{4}{5}F_{471} \right) \frac{\partial^2 u}{\partial x_1 \partial x_2} + \left(\frac{2}{5}B_{551} - \frac{2}{5}F_{471} \right) \frac{\partial^2 v}{\partial x_1^2} \right.$$

$$+ \left(-2B_{550} + 2F_{470} - \frac{4}{5}B_{551} + \frac{4}{5}F_{471} \right) \frac{\partial^2 v}{\partial x_2^2} + \left(2D_{550} + 2F_{440} - 4F_{480} + \frac{4}{5}D_{551} + \frac{4}{5}F_{441} - \frac{8}{5}F_{481} \right) \frac{\partial^3 w}{\partial x_2^3}$$

$$+ \left(2D_{550} + 2F_{440} - 4F_{480} - \frac{6}{5}D_{551} - \frac{6}{5}F_{441} + \frac{12}{5}F_{481} \right) \frac{\partial^3 w}{\partial x_1^2 \partial x_2}$$

$$+ \left(-\frac{2}{5}F_{461} + \frac{2}{5}F_{681} \right) \frac{\partial w}{\partial x_2} + \left(2F_{440} - 2F_{480} - \frac{4}{5}F_{441} + \frac{4}{5}F_{481} \right) \frac{\partial^2 \phi_1}{\partial x_1 \partial x_2}$$

$$\left. + \left(-\frac{2}{5}F_{441} + \frac{2}{5}F_{481} \right) \frac{\partial^2 \phi_2}{\partial x_1^2} + \left(2F_{440} - 2F_{480} + \frac{4}{5}F_{441} - \frac{4}{5}F_{481} \right) \frac{\partial^2 \phi_2}{\partial x_2^2} + \left(-\frac{2}{5}F_{461} + \frac{2}{5}F_{681} \right) \phi_2 \right) n_{x_2} = 0, \tag{119}$$

$\delta\phi_1 = 0 \quad \text{veya}$

$$
\begin{aligned}
&\Bigg(\left(\left(-2F_{470}-\frac{4}{5}F_{471}\right)\frac{\partial^3 u}{\partial x_1^3}+\left(-F_{470}-\frac{2}{3}F_{471}-\frac{1}{8}F_{472}\right)\frac{\partial^3 u}{\partial x_1\partial x_2^2}+F_{11}\frac{\partial u}{\partial x_1}\right.\\
&+\left(-2F_{470}+\frac{4}{15}F_{471}+\frac{1}{8}F_{472}\right)\frac{\partial^3 v}{\partial x_1^2\partial x_2}+\left(-F_{470}+\frac{2}{5}F_{471}\right)\frac{\partial^3 v}{\partial x_2^3}+F_{L11}\frac{\partial v}{\partial x_2}\\
&+\left(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1^4}+\left(-F_{440}+F_{480}+\frac{2}{5}F_{441}-\frac{2}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_2^4}\\
&+\left(-3F_{440}+3F_{480}-\frac{2}{5}F_{441}+\frac{2}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1^2\partial x_2^2}\\
&+\left(-F_{22}+F_{33}+2F_{550}-2F_{570}+\frac{2}{5}F_{461}+\frac{32}{15}F_{551}-\frac{16}{15}F_{571}+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\frac{\partial^2 w}{\partial x_1^2}\\
&+\left(-F_{L22}+F_{L33}+2F_{550}-2F_{570}+\frac{2}{15}F_{461}-\frac{8}{15}F_{551}+\frac{4}{15}F_{571}-\frac{1}{8}F_{462}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\right)\frac{\partial^2 w}{\partial x_2^2}\\
&+\left(-2F_{440}-\frac{4}{5}F_{441}\right)\frac{\partial^3 \phi_1}{\partial x_1^3}+\left(-F_{440}-\frac{2}{3}F_{441}-\frac{1}{8}F_{442}\right)\frac{\partial^3 \phi_1}{\partial x_1\partial x_2^2}\\
&+\left(F_{33}+2F_{550}+\frac{2}{5}F_{461}+\frac{32}{15}F_{551}+\frac{1}{4}F_{552}\right)\frac{\partial \phi_1}{\partial x_1}+\left(-F_{440}+\frac{2}{5}F_{441}\right)\frac{\partial^3 \phi_2}{\partial x_2^3}\\
&+\left(-2F_{440}+\frac{4}{15}F_{441}+\frac{1}{8}F_{442}\right)\frac{\partial^3 \phi_2}{\partial x_1^2\partial x_2}\\
&\left.+\left(F_{L33}+2F_{550}+\frac{2}{15}F_{461}-\frac{8}{15}F_{551}-\frac{1}{8}F_{462}-\frac{1}{4}F_{552}\right)\frac{\partial \phi_2}{\partial x_2}-F_{T11}\Delta T\right)n_{x_1}\\
&+\left(\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^3 u}{\partial x_2^3}+\left(-F_{470}-\frac{2}{3}F_{471}-\frac{1}{8}F_{472}\right)\frac{\partial^3 u}{\partial x_1^2\partial x_2}+F_{47}\frac{\partial u}{\partial x_2}\right.\\
&+\left(-\frac{8}{15}F_{471}+\frac{1}{8}F_{472}\right)\frac{\partial^3 v}{\partial x_1^3}+\left(-F_{470}-\frac{2}{3}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^3 v}{\partial x_1\partial x_2^2}+F_{47}\frac{\partial v}{\partial x_1}\\
&+\left(-F_{440}+F_{480}-\frac{6}{5}F_{441}+\frac{6}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1^3\partial x_2}+\left(-F_{440}+F_{480}-\frac{6}{5}F_{441}+\frac{6}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1\partial x_2^3}\\
&+\left(2F_{44}-2F_{48}+\frac{4}{15}F_{461}+\frac{8}{3}F_{551}-\frac{4}{3}F_{571}+\frac{1}{8}F_{462}+\frac{1}{2}F_{552}-F_{572}\right)\frac{\partial^2 w}{\partial x_1\partial x_2}\\
&+\left(-\frac{8}{15}F_{441}-\frac{1}{4}F_{442}\right)\frac{\partial^3 \phi_1}{\partial x_2^3}+\left(-F_{440}-\frac{2}{3}F_{441}-\frac{1}{8}F_{442}\right)\frac{\partial^3 \phi_1}{\partial x_1^2\partial x_2}\\
&+\left(F_{44}+\frac{2}{15}F_{461}+\frac{4}{3}F_{551}+\frac{1}{4}F_{462}+F_{552}\right)\frac{\partial \phi_1}{\partial x_2}+\left(-\frac{8}{15}F_{441}+\frac{1}{8}F_{442}\right)\frac{\partial^3 \phi_2}{\partial x_1^3}\\
&\left.+\left(-F_{440}-\frac{2}{3}F_{441}+\frac{1}{4}F_{442}\right)\frac{\partial^3 \phi_2}{\partial x_1\partial x_2^2}+\left(F_{44}+\frac{2}{15}F_{461}+\frac{4}{3}F_{551}-\frac{1}{8}F_{462}-\frac{1}{2}F_{552}\right)\frac{\partial \phi_2}{\partial x_1}\right)n_{x_2}=0,
\end{aligned}
$$

(120)

$$\delta \frac{\partial \phi_1}{\partial x_1} = 0 \quad \text{veya}$$

$$\Bigg(\left(\left(2F_{470} + \frac{4}{5}F_{471}\right)\frac{\partial^2 u}{\partial x_1^2} - \frac{2}{5}F_{471}\frac{\partial^2 u}{\partial x_2^2} + \left(2F_{470} - \frac{4}{5}F_{471}\right)\frac{\partial^2 v}{\partial x_1 \partial x_2}\right.$$
$$+\left(2F_{440} - 2F_{480} + \frac{4}{5}F_{441} - \frac{4}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_1^3} + \left(2F_{440} - 2F_{480} - \frac{6}{5}F_{441} + \frac{6}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_1 \partial x_2^2} - \frac{2}{5}F_{461}\frac{\partial w}{\partial x_1}$$
$$\left.+\left(2F_{440} + \frac{4}{5}F_{441}\right)\frac{\partial^2 \phi_1}{\partial x_1^2} - \frac{2}{5}F_{441}\frac{\partial^2 \phi_1}{\partial x_2^2} - \frac{2}{5}F_{461}\phi_1 + \left(2F_{440} - \frac{4}{5}F_{441}\right)\frac{\partial^2 \phi_2}{\partial x_1 \partial x_2}\right)n_{x_1}$$
$$+\left(\left(F_{470} + \frac{16}{15}F_{471} + \frac{1}{8}F_{472}\right)\frac{\partial^2 u}{\partial x_1 \partial x_2} + \left(\frac{8}{15}F_{471} - \frac{1}{8}F_{472}\right)\frac{\partial^2 v}{\partial x_1^2} + \left(F_{470} - \frac{2}{5}F_{471}\right)\frac{\partial^2 v}{\partial x_2^2}\right.$$
$$+\left(F_{440} - F_{480} - \frac{2}{5}F_{441} + \frac{2}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_2^3} + \left(F_{440} - F_{480} + \frac{8}{5}F_{441} - \frac{8}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_1^2 \partial x_2}$$
$$+\left(-\frac{2}{15}F_{461} + \frac{1}{8}F_{462}\right)\frac{\partial w}{\partial x_2} + \left(F_{440} + \frac{16}{15}F_{441} + \frac{1}{8}F_{442}\right)\frac{\partial^2 \phi_1}{\partial x_1 \partial x_2}$$
$$\left.+\left(\frac{8}{15}F_{441} - \frac{1}{8}F_{442}\right)\frac{\partial^2 \phi_2}{\partial x_1^2} + \left(F_{440} - \frac{2}{5}F_{441}\right)\frac{\partial^2 \phi_2}{\partial x_2^2} + \left(-\frac{2}{15}F_{461} + \frac{1}{8}F_{462}\right)\phi_2\right)n_{x_2} = 0, \tag{121}$$

$$\delta \frac{\partial \phi_1}{\partial x_2} = 0 \quad \text{veya}$$

$$\Bigg(\left(\left(F_{470} + \frac{16}{15}F_{471} + \frac{1}{8}F_{472}\right)\frac{\partial^2 u}{\partial x_1 \partial x_2} + \left(\frac{8}{15}F_{471} - \frac{1}{8}F_{472}\right)\frac{\partial^2 v}{\partial x_1^2} + \left(F_{470} - \frac{2}{5}F_{471}\right)\frac{\partial^2 v}{\partial x_2^2}\right.$$
$$+\left(F_{440} - F_{480} + \frac{8}{5}F_{441} - \frac{8}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_1^2 \partial x_2} + \left(F_{440} - F_{480} - \frac{2}{5}F_{441} + \frac{2}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_2^3}$$
$$+\left(-\frac{2}{15}F_{461} + \frac{1}{8}F_{462}\right)\frac{\partial w}{\partial x_2} + \left(F_{440} + \frac{16}{15}F_{441} + \frac{1}{8}F_{442}\right)\frac{\partial^2 \phi_1}{\partial x_1 \partial x_2}$$
$$\left.+\left(\frac{8}{15}F_{441} - \frac{1}{8}F_{442}\right)\frac{\partial^2 \phi_2}{\partial x_1^2} + \left(F_{440} - \frac{2}{5}F_{441}\right)\frac{\partial^2 \phi_2}{\partial x_2^2} + \left(-\frac{2}{15}F_{461} + \frac{1}{8}F_{462}\right)\phi_2\right)n_{x_1}$$
$$+\left(-\frac{2}{5}F_{471}\frac{\partial^2 u}{\partial x_1^2} + \left(\frac{8}{15}F_{471} + \frac{1}{4}F_{472}\right)\frac{\partial^2 u}{\partial x_2^2} + \left(\frac{16}{15}F_{471} - \frac{1}{4}F_{472}\right)\frac{\partial^2 v}{\partial x_1 \partial x_2}\right.$$
$$+\left(-\frac{2}{5}F_{441} + \frac{2}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_1^3} + \left(\frac{8}{5}F_{441} - \frac{8}{5}F_{481}\right)\frac{\partial^3 w}{\partial x_1 \partial x_2^2} + \left(-\frac{2}{15}F_{461} - \frac{1}{4}F_{462}\right)\frac{\partial w}{\partial x_1}$$
$$\left.-\frac{2}{5}F_{441}\frac{\partial^2 \phi_1}{\partial x_1^2} + \left(\frac{8}{15}F_{441} + \frac{1}{4}F_{442}\right)\frac{\partial^2 \phi_1}{\partial x_2^2} + \left(-\frac{2}{15}F_{461} - \frac{1}{4}F_{462}\right)\phi_1 + \left(\frac{16}{15}F_{441} - \frac{1}{4}F_{442}\right)\frac{\partial^2 \phi_2}{\partial x_1 \partial x_2}\right)n_{x_2} = 0, \tag{122}$$

$\delta\phi_2 = 0$ veya

$$
\begin{aligned}
&\Bigg(\bigg(\left(-\frac{8}{15}F_{471}+\frac{1}{8}F_{472}\right)\frac{\partial^3 u}{\partial x_2^3}+\left(-F_{470}-\frac{2}{3}F_{471}+\frac{1}{4}F_{472}\right)\frac{\partial^3 u}{\partial x_1^2\partial x_2}+F_{47}\frac{\partial u}{\partial x_2}\\
&+\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\frac{\partial^3 v}{\partial x_1^3}+\left(-F_{470}-\frac{2}{3}F_{471}-\frac{1}{8}F_{472}\right)\frac{\partial^3 v}{\partial x_1\partial x_2^2}+F_{47}\frac{\partial v}{\partial x_1}\\
&+\left(-F_{440}+F_{480}-\frac{6}{5}F_{441}+\frac{6}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1\partial x_2^3}+\left(-F_{440}+F_{480}-\frac{6}{5}F_{441}+\frac{6}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1^3\partial x_2}\\
&+\left(2F_{44}-2F_{48}+\frac{4}{15}F_{461}+\frac{8}{3}F_{551}-\frac{4}{3}F_{571}+\frac{1}{8}F_{462}+\frac{1}{2}F_{552}-F_{572}\right)\frac{\partial^2 w}{\partial x_1\partial x_2}\\
&+\left(-\frac{8}{15}F_{441}+\frac{1}{8}F_{442}\right)\frac{\partial^3\phi_1}{\partial x_2^3}+\left(-F_{440}-\frac{2}{3}F_{441}+\frac{1}{4}F_{442}\right)\frac{\partial^3\phi_1}{\partial x_1^2\partial x_2}\\
&+\left(F_{44}+\frac{2}{15}F_{461}+\frac{4}{3}F_{551}-\frac{1}{8}F_{462}-\frac{1}{2}F_{552}\right)\frac{\partial\phi_1}{\partial x_2}+\left(-\frac{8}{15}F_{441}-\frac{1}{4}F_{442}\right)\frac{\partial^3\phi_2}{\partial x_1^3}\\
&+\left(-F_{440}-\frac{2}{3}F_{441}-\frac{1}{8}F_{442}\right)\frac{\partial^3\phi_2}{\partial x_1\partial x_2^2}+\left(F_{44}+\frac{2}{15}F_{461}+\frac{4}{3}F_{551}+\frac{1}{4}F_{462}+F_{552}\right)\frac{\partial\phi_2}{\partial x_1}\bigg)n_{x_1}\\
&+\bigg(\left(-2F_{470}+\frac{4}{15}F_{471}+\frac{1}{8}F_{472}\right)\frac{\partial^3 u}{\partial x_1\partial x_2^2}+\left(-F_{470}+\frac{2}{5}F_{471}\right)\frac{\partial^3 u}{\partial x_1^3}+F_{L11}\frac{\partial u}{\partial x_1}\\
&+\left(-2F_{470}-\frac{4}{5}F_{471}\right)\frac{\partial^3 v}{\partial x_2^3}+\left(-F_{470}-\frac{2}{3}F_{471}-\frac{1}{8}F_{472}\right)\frac{\partial^3 v}{\partial x_1^2\partial x_2}+F_{11}\frac{\partial v}{\partial x_2}\\
&+\left(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_2^4}+\left(-3F_{440}+3F_{480}-\frac{2}{5}F_{441}+\frac{2}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1^2\partial x_2^2}\\
&+\left(-F_{440}+F_{480}+\frac{2}{5}F_{441}-\frac{2}{5}F_{481}\right)\frac{\partial^4 w}{\partial x_1^4}\\
&+\left(-F_{22}+F_{33}+2F_{550}-2F_{570}+\frac{2}{5}F_{461}+\frac{32}{15}F_{551}-\frac{16}{15}F_{571}+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\frac{\partial^2 w}{\partial x_2^2}\\
&+\left(-F_{L22}+F_{L33}+2F_{550}-2F_{570}+\frac{2}{15}F_{461}-\frac{8}{15}F_{551}+\frac{4}{15}F_{571}-\frac{1}{8}F_{462}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\right)\frac{\partial^2 w}{\partial x_1^2}\\
&+\left(-F_{440}+\frac{2}{5}F_{441}\right)\frac{\partial^3\phi_1}{\partial x_1^3}+\left(-2F_{440}+\frac{4}{15}F_{441}+\frac{1}{8}F_{442}\right)\frac{\partial^3\phi_1}{\partial x_1\partial x_2^2}\\
&+\left(F_{L33}+2F_{550}+\frac{2}{15}F_{461}-\frac{8}{15}F_{551}-\frac{1}{8}F_{462}-\frac{1}{4}F_{552}\right)\frac{\partial\phi_1}{\partial x_1}+\left(-2F_{440}-\frac{4}{5}F_{441}\right)\frac{\partial^3\phi_2}{\partial x_2^3}\\
&+\left(-F_{440}-\frac{2}{3}F_{441}-\frac{1}{8}F_{442}\right)\frac{\partial^3\phi_2}{\partial x_1^2\partial x_2}+\left(F_{33}+2F_{550}+\frac{2}{5}F_{461}+\frac{32}{15}F_{551}+\frac{1}{4}F_{552}\right)\frac{\partial\phi_2}{\partial x_2}-F_{T11}\Delta T\bigg)n_{x_2}=0,
\end{aligned}
$$

(123)

$$\delta \frac{\partial \phi_2}{\partial x_1} = 0 \quad \text{veya}$$

$$\left(\left(\frac{16}{15} F_{471} - \frac{1}{4} F_{472} \right) \frac{\partial^2 u}{\partial x_1 \partial x_2} + \left(\frac{8}{15} F_{471} + \frac{1}{4} F_{472} \right) \frac{\partial^2 v}{\partial x_1^2} - \frac{2}{5} F_{471} \frac{\partial^2 v}{\partial x_2^2} \right.$$

$$+ \left(-\frac{2}{5} F_{441} + \frac{2}{5} F_{481} \right) \frac{\partial^3 w}{\partial x_2^3} + \left(\frac{8}{5} F_{441} - \frac{8}{5} F_{481} \right) \frac{\partial^3 w}{\partial x_1^2 \partial x_2} + \left(-\frac{2}{15} F_{461} - \frac{1}{4} F_{462} \right) \frac{\partial w}{\partial x_2}$$

$$\left. + \left(\frac{16}{15} F_{441} - \frac{1}{4} F_{442} \right) \frac{\partial^2 \phi_1}{\partial x_1 \partial x_2} + \left(\frac{8}{15} F_{441} + \frac{1}{4} F_{442} \right) \frac{\partial^2 \phi_2}{\partial x_1^2} - \frac{2}{5} F_{441} \frac{\partial^2 \phi_2}{\partial x_2^2} + \left(-\frac{2}{15} F_{461} - \frac{1}{4} F_{462} \right) \phi_2 \right) n_{x_1}$$

$$+ \left(\left(F_{470} - \frac{2}{5} F_{471} \right) \frac{\partial^2 u}{\partial x_1^2} + \left(\frac{8}{15} F_{471} - \frac{1}{8} F_{472} \right) \frac{\partial^2 u}{\partial x_2^2} + \left(F_{470} + \frac{16}{15} F_{471} + \frac{1}{8} F_{472} \right) \frac{\partial^2 v}{\partial x_1 \partial x_2} \right.$$

$$+ \left(F_{440} - F_{480} - \frac{2}{5} F_{441} + \frac{2}{5} F_{481} \right) \frac{\partial^3 w}{\partial x_1^3} + \left(F_{440} - F_{480} + \frac{8}{5} F_{441} - \frac{8}{5} F_{481} \right) \frac{\partial^3 w}{\partial x_1 \partial x_2^2} + \left(-\frac{2}{15} F_{461} + \frac{1}{8} F_{462} \right) \frac{\partial w}{\partial x_1}$$

$$+ \left(F_{440} - \frac{2}{5} F_{441} \right) \frac{\partial^2 \phi_1}{\partial x_1^2} + \left(\frac{8}{15} F_{441} - \frac{1}{8} F_{442} \right) \frac{\partial^2 \phi_1}{\partial x_2^2} + \left(-\frac{2}{15} F_{461} + \frac{1}{8} F_{462} \right) \phi_1$$

$$\left. + \left(F_{440} + \frac{16}{15} F_{441} + \frac{1}{8} F_{442} \right) \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} \right) n_{x_2} = 0,$$

(124)

$$\delta \frac{\partial \phi_2}{\partial x_2} = 0 \quad \text{veya}$$

$$\left(\left(F_{470} - \frac{2}{5} F_{471} \right) \frac{\partial^2 u}{\partial x_1^2} + \left(\frac{8}{15} F_{471} - \frac{1}{8} F_{472} \right) \frac{\partial^2 u}{\partial x_2^2} + \left(F_{470} + \frac{16}{15} F_{471} + \frac{1}{8} F_{472} \right) \frac{\partial^2 v}{\partial x_1 \partial x_2} \right.$$

$$+ \left(F_{440} - F_{480} - \frac{2}{5} F_{441} + \frac{2}{5} F_{481} \right) \frac{\partial^3 w}{\partial x_1^3} + \left(F_{440} - F_{480} + \frac{8}{5} F_{441} - \frac{8}{5} F_{481} \right) \frac{\partial^3 w}{\partial x_1 \partial x_2^2}$$

$$+ \left(-\frac{2}{15} F_{461} + \frac{1}{8} F_{462} \right) \frac{\partial w}{\partial x_1} + \left(F_{440} - \frac{2}{5} F_{441} \right) \frac{\partial^2 \phi_1}{\partial x_1^2} + \left(\frac{8}{15} F_{441} - \frac{1}{8} F_{442} \right) \frac{\partial^2 \phi_1}{\partial x_2^2} + \left(-\frac{2}{15} F_{461} + \frac{1}{8} F_{462} \right) \phi_1$$

$$\left. + \left(F_{440} + \frac{16}{15} F_{441} + \frac{1}{8} F_{442} \right) \frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} \right) n_{x_1}$$

$$+ \left(\left(2F_{470} - \frac{4}{15} F_{471} \right) \frac{\partial^2 u}{\partial x_1 \partial x_2} - \frac{2}{5} F_{471} \frac{\partial^2 v}{\partial x_1^2} + \left(2F_{470} + \frac{4}{5} F_{471} \right) \frac{\partial^2 v}{\partial x_2^2} \right.$$

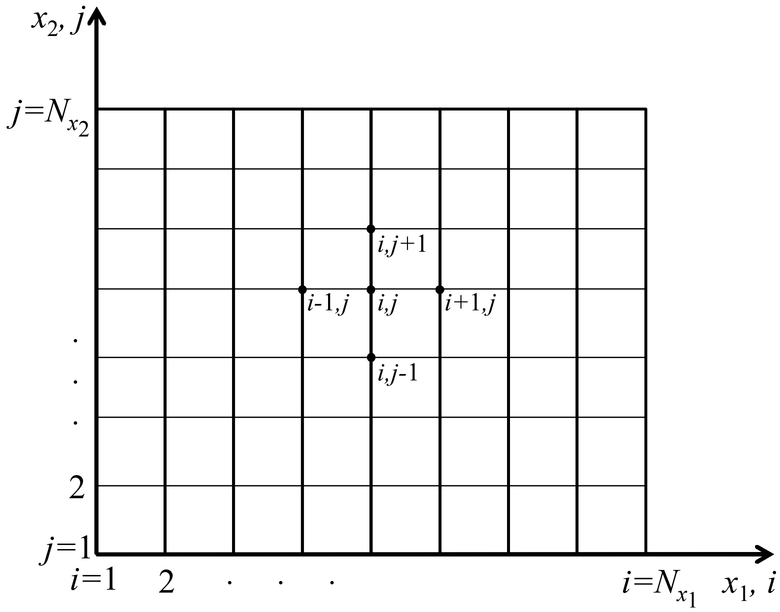
$$+ \left(2F_{440} - 2F_{480} + \frac{4}{5} F_{441} - \frac{4}{5} F_{481} \right) \frac{\partial^3 w}{\partial x_2^3} + \left(2F_{440} - 2F_{480} - \frac{6}{5} F_{441} + \frac{6}{5} F_{481} \right) \frac{\partial^3 w}{\partial x_1^2 \partial x_2} - \frac{2}{5} F_{461} \frac{\partial w}{\partial x_2}$$

$$\left. + \left(2F_{440} - \frac{4}{5} F_{441} \right) \frac{\partial^2 \phi_1}{\partial x_1 \partial x_2} - \frac{2}{5} F_{441} \frac{\partial^2 \phi_2}{\partial x_1^2} + \left(2F_{440} + \frac{4}{5} F_{441} \right) \frac{\partial^2 \phi_2}{\partial x_2^2} - \frac{2}{5} F_{461} \phi_2 \right) n_{x_2} = 0.$$

(125)

4. DİFERANSİYEL KARE YAPMA METODUNUN UYGULANMASI

Diferansiyel denklem sistemleri ve sınır koşullarından oluşan bağlaşık denklem sistemi diferansiyel kare yapma metodu aracılığı ile çözülmüştür. Bu metotda bir fonksiyonun herhangi bir noktadaki kısmi türevini hesaplamak için fonksiyonun çeşitli noktalardaki değerlerinin ağırlıklı lineer toplamı kullanılmaktadır. Diferansiyel kare yapma metodu uygulanırken, dikdörtgen bir mikro-plağın düzlemi Şekil 2'de gösterildiği gibi alt-alanlara bölünmektedir. Burada N_{x_1} ve N_{x_2} sırasıyla x_1 ve x_2 yönlerindeki nokta sayılarıdır.



Şekil 2. Plak yüzeyinin alt-alanlara bölümü.

Diferansiyel kare yapma yöntemine göre x_1 ve x_2'ye bağlı olan bir w fonksiyonun türevleri

$$\frac{\partial^n w}{\partial x_1^n} = \sum_{k=1}^{N_{x_1}} c_{ik}^{(n)} w_{k,j}, \tag{126}$$

$$\frac{\partial^n w}{\partial x_2^n} = \sum_{k=1}^{N_{x_2}} c_{jk}^{(n)} w_{i,k}, \tag{127}$$

$$\frac{\partial^{(n+m)} w}{\partial x_1^n \partial x_2^m} = \sum_{m=1}^{N_{x_2}} c_{jm}^{(m)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(n)} w_{k,m}, \tag{128}$$

şeklinde ifade edilmektedir. Burada $c_{ij}^{(m)}$, m dereceden türevin ağırlık katsayılarıdır ve Lagrange interpolasyon polinomları kullanılarak aşağıdaki gibi elde edilmiştir:

$$\begin{aligned}
c_{ij}^{(1)} &= \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)}, \quad i \neq j \text{ için}, \\
c_{ii}^{(1)} &= \frac{M^{(2)}(x_i)}{2M^{(1)}(x_i)}, \quad i = j \text{ için}, \\
c_{ij}^{(m)} &= m\left(c_{ii}^{(m-1)}c_{ij}^{(1)} - \frac{c_{ij}^{(m-1)}}{x_i - x_j}\right), \quad i \neq j \text{ için}, \quad m = 2, 3, ..., N, \quad i, j = 1, 2, ..., N, \\
c_{ii}^{(m)} &= -\sum_{j=1,\, j\neq i}^{N} c_{ij}^{(m)}, \quad \text{for } i = 1, 2, ..., N.
\end{aligned} \tag{129}$$

$M(x)$ ise

$$M(x) = \prod_{j=1}^{N}(x - x_j), \tag{130}$$

formunda ifade edilir. $M(x)$'in k dereceden türevi $M^{(k)}(x)$ ile gösterilmiştir. Yukarıda tarif edilen diferansiyel kare yapma metodu uygulanarak tüm denklemler sonlu seri formuna dönüştürülmüştür. Denklemlerin seri formları aşağıda sunulmaktadır:

$\delta u:$

$$\left(-2A_{550}-\frac{4}{5}A_{551}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(4)}u_{k,j}+\left(-\frac{8}{15}A_{551}-\frac{1}{4}A_{552}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(4)}u_{i,k}$$
$$+\left(-2A_{550}-\frac{4}{3}A_{551}-\frac{1}{4}A_{552}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}u_{k,m}$$
$$+A_{11}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}u_{k,j}+A_{55}\sum_{k=1}^{N_{x2}}c_{jk}^{(2)}u_{i,k}+\left(-2A_{550}-\frac{4}{15}A_{551}+\frac{1}{4}A_{552}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}v_{k,m}$$
$$+\left(-2A_{550}-\frac{4}{15}A_{551}+\frac{1}{4}A_{552}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(3)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}v_{k,m}+\left(A_{55}+A_{L11}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}v_{k,m}$$
$$+\left(-2F_{470}+2B_{550}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(5)}w_{k,j}$$
$$+\left(4B_{550}-4F_{470}+\frac{8}{5}B_{551}-\frac{8}{5}F_{471}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}w_{k,m}$$
$$+\left(-2F_{470}+2B_{550}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(4)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}w_{k,m}$$
$$+\left(F_{11}-B_{11}+\frac{2}{5}F_{671}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}w_{k,j}+\left(F_{11}-B_{11}+\frac{2}{5}F_{671}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}w_{k,m}$$
$$+\left(-2F_{470}-\frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(4)}\phi_{1k,j}+\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(4)}\phi_{1i,k}$$
$$+\left(-2F_{470}-\frac{4}{3}F_{471}-\frac{1}{4}F_{472}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}\phi_{1k,m}$$
$$+\left(F_{11}+\frac{2}{5}F_{671}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}\phi_{1k,j}+\left(F_{47}+\frac{2}{15}F_{671}+\frac{1}{4}F_{672}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(2)}\phi_{1i,k}$$
$$+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}\phi_{2k,m}$$
$$+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(3)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}\phi_{2k,m}$$
$$+\left(F_{L11}+F_{47}+\frac{4}{15}F_{671}-\frac{1}{4}F_{672}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}\phi_{2k,m}-A_{T11}\left(\frac{\partial\Delta T}{\partial x_1}\right)_{i,j}$$
$$=I_0\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j}+\left(I_3-I_1\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}w_{k,j}\left(\frac{\partial^2 w}{\partial t^2}\right)_{k,j}+I_3\left(\frac{\partial^2\phi_1}{\partial t^2}\right)_{i,j}, \qquad (131)$$

$\delta v:$

$$
\begin{aligned}
&\left(-2A_{550}-\frac{4}{15}A_{551}+\frac{1}{4}A_{552}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}u_{k,m}+\left(-2A_{550}-\frac{4}{15}A_{551}+\frac{1}{4}A_{552}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(3)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}u_{k,m}\\
&+\left(A_{55}+A_{L11}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}u_{k,m}\\
&+\left(-\frac{8}{15}A_{551}-\frac{1}{4}A_{552}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(4)}v_{k,j}+\left(-2A_{550}-\frac{4}{5}A_{551}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(4)}v_{i,k}\\
&+\left(-2A_{550}-\frac{4}{3}A_{551}-\frac{1}{4}A_{552}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}v_{k,m}+A_{55}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}v_{k,j}+A_{11}\sum_{k=1}^{N_{x2}}c_{jk}^{(2)}v_{i,k}\\
&+\left(-2F_{470}+2B_{550}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(5)}w_{i,k}\\
&+\left(2B_{550}+\frac{4}{5}B_{551}-2F_{470}-\frac{4}{5}F_{471}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(4)}w_{k,m}\\
&+\left(4B_{550}+\frac{8}{5}B_{551}-4F_{470}-\frac{8}{5}F_{471}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(3)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}w_{k,m}\\
&+\left(F_{11}-B_{11}+\frac{2}{5}F_{671}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(3)}w_{i,k}+\left(F_{11}-B_{11}+\frac{2}{5}F_{671}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}w_{k,m}\\
&+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}\phi_{1k,m}\\
&+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(3)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}\phi_{1k,m}\\
&+\left(+F_{L11}+F_{47}+\frac{4}{15}F_{671}-\frac{1}{4}F_{672}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}\phi_{1k,m}\\
&+\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(4)}\phi_{2k,j}+\left(-2F_{470}-\frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(4)}\phi_{2i,k}\\
&+\left(-2F_{470}-\frac{4}{3}F_{471}-\frac{1}{4}F_{472}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}\phi_{2k,m}\\
&+\left(F_{47}+\frac{2}{15}F_{671}+\frac{1}{4}F_{672}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}\phi_{2k,j}+\left(F_{11}+\frac{2}{5}F_{671}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(2)}\phi_{2i,k}\\
&-A_{T11}\left(\frac{\partial\Delta T}{\partial x_2}\right)_{i,j}=I_0\left(\frac{\partial^2 v}{\partial t^2}\right)_{i,j}+\left(I_3-I_1\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(1)}\left(\frac{\partial^2 w}{\partial t^2}\right)_{i,k}+I_3\left(\frac{\partial^2\phi_2}{\partial t^2}\right)_{i,j},
\end{aligned}
\tag{132}
$$

$\delta w:$

$$\left(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(5)}u_{k,j}+\left(-4B_{550}+4F_{470}-\frac{8}{5}B_{551}+\frac{8}{5}F_{471}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}u_{k,m}$$

$$+\left(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{5}F_{471}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(4)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}u_{k,m}+\left(B_{11}-F_{11}-\frac{2}{5}F_{671}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}u_{k,j}$$

$$+\left(B_{11}-F_{11}-\frac{2}{5}F_{671}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}u_{k,m}+\left(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(5)}v_{i,k}$$

$$+\left(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{5}F_{471}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(4)}v_{k,m}+\left(-4B_{550}+4F_{470}-\frac{8}{5}B_{551}\right.$$

$$\left.+\frac{8}{5}F_{471}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(3)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}v_{k,m}+\left(B_{11}-F_{11}-\frac{2}{5}F_{671}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(3)}v_{i,k}+\left(B_{11}-F_{11}-\frac{2}{5}F_{671}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}v_{k,m}$$

$$+\left(2D_{550}+2F_{440}-4F_{480}+\frac{4}{5}D_{551}+\frac{4}{5}F_{441}-\frac{8}{5}F_{481}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(6)}w_{k,j}$$

$$+\left(2D_{550}+2F_{440}-4F_{480}+\frac{4}{5}D_{551}+\frac{4}{5}F_{441}-\frac{8}{5}F_{481}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(6)}w_{i,k}$$

$$+\left(6D_{550}+6F_{440}-12F_{480}+\frac{12}{5}D_{551}+\frac{12}{5}F_{441}-\frac{24}{5}F_{481}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(4)}w_{k,m}$$

$$+\left(6D_{550}+6F_{440}-12F_{480}+\frac{12}{5}D_{551}+\frac{12}{5}F_{441}-\frac{24}{5}F_{481}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(4)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}w_{k,m}$$

$$+\left(-D_{11}+2F_{22}-F_{33}-2A_{550}-2F_{550}+4F_{570}-\frac{8}{15}A_{551}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{32}{15}F_{571}+\frac{4}{5}F_{681}\right.$$

$$\left.-A_{552}-\frac{1}{4}F_{552}+F_{572}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(4)}w_{k,j}+\left(-D_{11}+2F_{22}-F_{33}-2A_{550}-2F_{550}+4F_{570}\right.$$

$$\left.-\frac{8}{15}A_{551}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{32}{15}F_{571}+\frac{4}{5}F_{681}-A_{552}-\frac{1}{4}F_{552}+F_{572}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(4)}w_{i,k}$$

$$+\left(-2D_{11}+4F_{22}-2F_{33}-4A_{550}-4F_{550}+8F_{570}-\frac{16}{15}A_{551}-\frac{8}{5}F_{461}-\frac{64}{15}F_{551}+\frac{64}{15}F_{571}+\frac{8}{5}F_{681}\right.$$

$$\left.-2A_{552}-\frac{1}{2}F_{552}+2F_{572}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}w_{k,m}+\left(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}w_{k,j}$$

$$+\left(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(2)}w_{i,k}$$

$$+\left(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(5)}\phi_{1k,j}+\left(4F_{440}-4F_{480}+\frac{8}{5}F_{441}-\frac{8}{5}F_{481}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}\phi_{1k,m}$$

$$+\left(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(4)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}\phi_{1k,m}$$

$$+\left(F_{22}-F_{33}-2F_{550}+2F_{570}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{16}{15}F_{571}+\frac{2}{5}F_{681}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}\phi_{1k,j}$$

$$+\left(F_{22}-F_{33}-2F_{550}+2F_{570}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{16}{15}F_{571}+\frac{2}{5}F_{681}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\right)\sum_{m=1}^{N_{x_2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x_1}}c_{ik}^{(1)}\phi_{1k,m}$$

$$+\left(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\right)\sum_{k=1}^{N_{x_1}}c_{ik}^{(1)}\phi_{1k,j}+\left(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\right)\sum_{k=1}^{N_{x_2}}c_{jk}^{(5)}\phi_{2i,k}$$

$$+\left(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\right)\sum_{m=1}^{N_{x_2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x_1}}c_{ik}^{(4)}\phi_{2k,m}$$

$$+\left(4F_{440}-4F_{480}+\frac{8}{5}F_{441}-\frac{8}{5}F_{481}\right)\sum_{m=1}^{N_{x_2}}c_{jm}^{(3)}\sum_{k=1}^{N_{x_1}}c_{ik}^{(2)}\phi_{2k,m}$$

$$+\left(F_{22}-F_{33}-2F_{550}+2F_{570}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{16}{15}F_{571}+\frac{2}{5}F_{681}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\right)\sum_{k=1}^{N_{x_2}}c_{jk}^{(3)}\phi_{2i,k}$$

$$+\left(F_{22}-F_{33}-2F_{550}+2F_{570}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{16}{15}F_{571}+\frac{2}{5}F_{681}\right.$$

$$\left.-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\right)\sum_{m=1}^{N_{x_2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}}c_{ik}^{(2)}\phi_{2k,m}+\left(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\right)\sum_{k=1}^{N_{x_2}}c_{jk}^{(1)}\phi_{2i,k}$$

$$+\left(-B_{T11}+F_{T11}\right)\left(\frac{\partial^2\Delta T}{\partial x_1^2}\right)_{i,j}+\left(-B_{T11}+F_{T11}\right)\left(\frac{\partial^2\Delta T}{\partial x_2^2}\right)_{i,j}$$

$$+P_{x_1}^0\sum_{k=1}^{N_{x_1}}c_{ik}^{(2)}w_{k,j}+P_{x_2}^0\sum_{k=1}^{N_{x_2}}c_{jk}^{(2)}w_{i,k}+2P_{x_1x_2}^0\sum_{m=1}^{N_{x_2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}}c_{ik}^{(1)}w_{k,m}+q_{i,j}-P_{x_1}\sum_{k=1}^{N_{x_1}}c_{ik}^{(2)}w_{k,j}-P_{x_2}\sum_{k=1}^{N_{x_2}}c_{jk}^{(2)}w_{i,k}$$

$$=\left(I_1-I_3\right)\sum_{k=1}^{N_{x_1}}c_{ik}^{(1)}\left(\frac{\partial^2u}{\partial t^2}\right)_{k,j}+\left(I_1-I_3\right)\sum_{k=1}^{N_{x_2}}c_{jk}^{(1)}\left(\frac{\partial^2v}{\partial t^2}\right)_{i,k}+\left(2I_4-I_2-I_5\right)\sum_{k=1}^{N_{x_1}}c_{ik}^{(2)}\left(\frac{\partial^2w}{\partial t^2}\right)_{k,j}$$

$$+\left(2I_4-I_2-I_5\right)\sum_{k=1}^{N_{x_2}}c_{jk}^{(2)}\left(\frac{\partial^2w}{\partial t^2}\right)_{i,k}+I_0\left(\frac{\partial^2w}{\partial t^2}\right)_{i,j}$$

$$+\left(I_4-I_5\right)\sum_{k=1}^{N_{x_1}}c_{ik}^{(1)}\left(\frac{\partial^2\phi_1}{\partial t^2}\right)_{k,j}+\left(I_4-I_5\right)\sum_{k=1}^{N_{x_2}}c_{jk}^{(1)}\left(\frac{\partial^2\phi_2}{\partial t^2}\right)_{i,k},\qquad(133)$$

$\delta\phi_1$:

$$\left(-2F_{470}-\frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(4)}u_{k,j}+\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(4)}u_{i,k}$$
$$+\left(-2F_{470}-\frac{4}{3}F_{471}-\frac{1}{4}F_{472}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}u_{k,m}$$
$$+\left(F_{11}+\frac{2}{5}F_{671}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}u_{k,j}+\left(F_{47}+\frac{2}{15}F_{671}+\frac{1}{4}F_{672}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(2)}u_{i,k}$$
$$+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}v_{k,m}+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(3)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}v_{k,m}$$
$$+\left(F_{L11}+F_{47}+\frac{4}{15}F_{671}-\frac{1}{4}F_{672}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}v_{k,m}$$
$$+\left(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(5)}w_{k,j}+\left(-4F_{440}+4F_{480}-\frac{8}{5}F_{441}+\frac{8}{5}F_{481}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}w_{k,m}$$
$$+\left(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(4)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}w_{k,m}$$
$$+\left(-F_{22}+F_{33}+2F_{550}-2F_{570}+\frac{4}{5}F_{461}+\frac{32}{15}F_{551}-\frac{16}{15}F_{571}-\frac{2}{5}F_{681}+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}w_{k,j}$$
$$+\left(-F_{22}+F_{33}+2F_{550}-2F_{570}+\frac{4}{5}F_{461}+\frac{32}{15}F_{551}-\frac{16}{15}F_{571}-\frac{2}{5}F_{681}\right.$$
$$\left.+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}w_{k,m}+\left(-k_sF_{55}-\frac{8}{15}F_{661}-\frac{1}{4}F_{662}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}w_{k,j}$$
$$+\left(-2F_{440}-\frac{4}{5}F_{441}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(4)}\phi_{1k,j}+\left(-\frac{8}{15}F_{441}-\frac{1}{4}F_{442}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(4)}\phi_{1i,k}$$
$$+\left(-2F_{440}-\frac{4}{3}F_{441}-\frac{1}{4}F_{442}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}\phi_{1k,m}$$
$$+\left(F_{33}+2F_{550}+\frac{4}{5}F_{461}+\frac{32}{15}F_{551}+\frac{1}{4}F_{552}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}\phi_{1k,j}$$
$$+\left(F_{44}+\frac{4}{15}F_{461}+\frac{4}{3}F_{551}+\frac{1}{2}F_{462}+F_{552}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(2)}\phi_{1i,k}+\left(-k_sF_{55}-\frac{8}{15}F_{661}-\frac{1}{4}F_{662}\right)\phi_{1i,j}$$
$$+\left(-2F_{440}-\frac{4}{15}F_{441}+\frac{1}{4}F_{442}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}\phi_{2k,m}+\left(-2F_{440}-\frac{4}{15}F_{441}+\frac{1}{4}F_{442}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(3)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}\phi_{2k,m}$$
$$+\left(F_{L33}+F_{44}+2F_{550}+\frac{8}{15}F_{461}+\frac{4}{5}F_{551}-\frac{1}{2}F_{462}-\frac{3}{4}F_{552}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}\phi_{2k,m}$$
$$-F_{T11}\left(\frac{\partial\Delta T}{\partial x_1}\right)_{i,j}=I_3\left(\frac{\partial^2u}{\partial t^2}\right)_{i,j}+\left(I_5-I_4\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}\left(\frac{\partial^2w}{\partial t^2}\right)_{k,j}+I_5\left(\frac{\partial^2\phi_1}{\partial t^2}\right)_{i,j},$$

(134)

$\delta\phi_2$:

$$\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}u_{k,m}+\left(-2F_{470}-\frac{4}{15}F_{471}+\frac{1}{4}F_{472}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(3)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}u_{k,m}$$
$$+\left(F_{L11}+F_{47}+\frac{4}{15}F_{671}-\frac{1}{4}F_{672}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}u_{k,m}$$
$$+\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(4)}v_{k,j}+\left(-2F_{470}-\frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(4)}v_{i,k}$$
$$+\left(-2F_{470}-\frac{4}{3}F_{471}-\frac{1}{4}F_{472}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}v_{k,m}$$
$$+\left(F_{47}+\frac{2}{15}F_{671}+\frac{1}{4}F_{672}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}v_{k,j}+\left(F_{11}+\frac{2}{5}F_{671}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(2)}v_{i,k}$$
$$+\left(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(5)}w_{i,k}+\left(-4F_{440}+4F_{480}-\frac{8}{5}F_{441}+\frac{8}{5}F_{481}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(3)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}w_{k,m}$$
$$+\left(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(4)}w_{k,m}$$
$$+\left(-F_{22}+F_{33}+2F_{550}-2F_{570}+\frac{4}{5}F_{461}+\frac{32}{15}F_{551}-\frac{16}{15}F_{571}-\frac{2}{5}F_{681}+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(3)}w_{i,k}$$
$$+\left(-F_{22}+F_{33}+2F_{550}-2F_{570}+\frac{4}{5}F_{461}+\frac{32}{15}F_{551}-\frac{16}{15}F_{571}-\frac{2}{5}F_{681}\right.$$
$$\left.+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}w_{k,m}+\left(-k_sF_{55}-\frac{8}{15}F_{661}-\frac{1}{4}F_{662}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(1)}w_{i,k}$$
$$+\left(-2F_{440}-\frac{4}{15}F_{441}+\frac{1}{4}F_{442}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}\phi_{1k,m}+\left(-2F_{440}-\frac{4}{15}F_{441}+\frac{1}{4}F_{442}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(3)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}\phi_{1k,m}$$
$$+\left(F_{L33}+F_{44}+2F_{550}+\frac{8}{15}F_{461}+\frac{4}{5}F_{551}-\frac{1}{2}F_{462}-\frac{3}{4}F_{552}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}\phi_{1k,m}$$
$$+\left(-\frac{8}{15}F_{441}-\frac{1}{4}F_{442}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(4)}\phi_{2k,j}+\left(-2F_{440}-\frac{4}{5}F_{441}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(4)}\phi_{2i,k}$$
$$+\left(-2F_{440}-\frac{4}{3}F_{441}-\frac{1}{4}F_{442}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}\phi_{2k,m}+\left(F_{44}+\frac{4}{15}F_{461}+\frac{4}{3}F_{551}+\frac{1}{2}F_{462}+F_{552}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}\phi_{2k,j}$$
$$+\left(F_{33}+2F_{550}+\frac{4}{5}F_{461}+\frac{32}{15}F_{551}+\frac{1}{4}F_{552}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(2)}\phi_{2i,k}+\left(-k_sF_{55}-\frac{8}{15}F_{661}-\frac{1}{4}F_{662}\right)\phi_{2i,j}$$
$$-F_{T11}\left(\frac{\partial\Delta T}{\partial x_2}\right)_{i,j}=I_3\left(\frac{\partial^2 v}{\partial t^2}\right)_{i,j}+\left(I_5-I_4\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(1)}\left(\frac{\partial^2 w}{\partial t^2}\right)_{i,k}+I_5\left(\frac{\partial^2\phi_2}{\partial t^2}\right)_{i,j},$$

(135)

$u_{i,j}=0$ veya

$$
\begin{aligned}
&\Bigg(\Bigg(\left(-2A_{550}-\frac{4}{5}A_{551}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}u_{k,j}+\left(-A_{550}-\frac{2}{3}A_{551}-\frac{1}{8}A_{552}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}u_{k,m}+A_{11}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}u_{k,j}\\
&+\left(-A_{550}+\frac{2}{15}A_{551}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(3)}v_{i,k}+\left(-2A_{550}+\frac{4}{15}A_{551}+\frac{1}{8}A_{552}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}v_{k,m}+A_{L11}\sum_{k=1}^{N_{x2}}c_{jk}^{(1)}v_{i,k}\\
&+\left(-2F_{470}+2B_{550}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(4)}w_{k,j}+\left(-F_{470}+B_{550}-\frac{2}{5}B_{551}+\frac{2}{5}F_{471}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(4)}w_{i,k}\\
&+\left(3B_{550}-3F_{470}+\frac{2}{5}B_{551}-\frac{2}{5}F_{471}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}w_{k,m}\\
&+\left(F_{11}-B_{11}+\frac{2}{5}F_{671}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}w_{k,j}+\left(F_{L11}-B_{L11}+\frac{2}{15}F_{671}-\frac{1}{8}F_{672}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(2)}w_{i,k}\\
&+\left(-2F_{470}-\frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}\phi_{1k,j}+\left(-F_{470}-\frac{2}{3}F_{471}-\frac{1}{8}F_{472}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}\phi_{1k,m}+\left(F_{11}+\frac{2}{5}F_{671}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}\phi_{1k,j}\\
&+\left(-F_{470}+\frac{2}{5}F_{471}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(3)}\phi_{2i,k}+\left(-2F_{470}+\frac{4}{15}F_{471}+\frac{1}{8}F_{472}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}\phi_{2k,m}\\
&+\left(F_{L11}+\frac{2}{15}F_{671}-\frac{1}{8}F_{672}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(1)}\phi_{2i,k}-A_{T11}\Delta T_{i,j}\Bigg)n_{x_1}\\
&+\Bigg(\left(-\frac{8}{15}A_{551}-\frac{1}{4}A_{552}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(3)}u_{i,k}+\left(-A_{550}-\frac{2}{3}A_{551}-\frac{1}{8}A_{552}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}u_{k,m}+A_{55}\sum_{k=1}^{N_{x2}}c_{jk}^{(1)}u_{i,k}\\
&+\left(-\frac{8}{15}A_{551}+\frac{1}{8}A_{552}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}v_{k,j}+\left(-A_{550}-\frac{2}{3}A_{551}+\frac{1}{4}A_{552}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}v_{k,m}+A_{55}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}v_{k,j}\\
&+\left(B_{550}-F_{470}+\frac{6}{5}B_{551}-\frac{6}{5}F_{471}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}w_{k,m}+\left(-F_{470}+B_{550}+\frac{6}{5}B_{551}-\frac{6}{5}F_{471}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(3)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}w_{k,m}\\
&+\left(2F_{47}-2B_{55}+\frac{4}{15}F_{671}+\frac{1}{8}F_{672}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}w_{k,m}+\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(3)}\phi_{1i,k}\\
&+\left(-F_{470}-\frac{2}{3}F_{471}-\frac{1}{8}F_{472}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}\phi_{1k,m}+\left(F_{47}+\frac{2}{15}F_{671}+\frac{1}{4}F_{672}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(1)}\phi_{1i,k}\\
&+\left(-\frac{8}{15}F_{471}+\frac{1}{8}F_{472}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}\phi_{2k,j}+\left(-F_{470}-\frac{2}{3}F_{471}+\frac{1}{4}F_{472}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}\phi_{2k,m}\\
&+\left(F_{47}+\frac{2}{15}F_{671}-\frac{1}{8}F_{672}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}\phi_{2k,j}\Bigg)n_{x_2}=0,
\end{aligned}
$$

(136)

$$\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} u_{k,j} = 0 \quad \text{veya}$$

$$\Bigg(\Bigg(\left(2A_{550} + \frac{4}{5}A_{551}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} u_{k,j} - \frac{2}{5}A_{551}\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} u_{i,k} + \left(2A_{550} - \frac{4}{5}A_{551}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} v_{k,m}$$

$$+\left(2F_{470} - 2B_{550} - \frac{4}{5}B_{551} + \frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(3)} w_{k,j} + \left(-2B_{550} + 2F_{470} + \frac{6}{5}B_{551} - \frac{6}{5}F_{471}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(2)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} w_{k,m}$$

$$-\frac{2}{5}F_{671}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} w_{k,j} + \left(2F_{470} + \frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} \phi_{1k,j} - \frac{2}{5}F_{471}\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} \phi_{1i,k} - \frac{2}{5}F_{671}\phi_{1i,j}$$

$$+\left(2F_{470} - \frac{4}{5}F_{471}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{2k,m}\Bigg) n_{x_1} + \Bigg(\left(A_{550} + \frac{16}{15}A_{551} + \frac{1}{8}A_{552}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} u_{k,m}$$

$$+\left(\frac{8}{15}A_{551} - \frac{1}{8}A_{552}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} v_{k,j} + \left(A_{550} - \frac{2}{5}A_{551}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} v_{i,k} + \left(F_{470} - B_{550} + \frac{2}{5}B_{551} - \frac{2}{5}F_{471}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(3)} w_{i,k}$$

$$+\left(-B_{550} + F_{470} - \frac{8}{5}B_{551} + \frac{8}{5}F_{471}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} w_{k,m} + \left(-\frac{2}{15}F_{671} + \frac{1}{8}F_{672}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(1)} w_{i,k}$$

$$+\left(F_{470} + \frac{16}{15}F_{471} + \frac{1}{8}F_{472}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{1k,m} + \left(\frac{8}{15}F_{471} - \frac{1}{8}F_{472}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} \phi_{2k,j}$$

$$+\left(F_{470} - \frac{2}{5}F_{471}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} \phi_{2i,k} + \left(-\frac{2}{15}F_{671} + \frac{1}{8}F_{672}\right)\phi_{2i,j}\Bigg) n_{x_2} = 0, \tag{137}$$

$$\sum_{k=1}^{N_{x2}} c_{jk}^{(1)} u_{i,k} = 0 \quad \text{veya}$$

$$\Bigg(\Bigg(\left(A_{550} + \frac{16}{15}A_{551} + \frac{1}{8}A_{552}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x1}} c_{ik}^{(1)} u_{k,m} + \left(\frac{8}{15}A_{551} - \frac{1}{8}A_{552}\right)\sum_{k=1}^{N_{x1}} c_{ik}^{(2)} v_{k,j} + \left(A_{550} - \frac{2}{5}A_{551}\right)\sum_{k=1}^{N_{x2}} c_{jk}^{(2)} v_{i,k}$$

$$+\left(F_{470} - B_{550} + \frac{2}{5}B_{551} - \frac{2}{5}F_{471}\right)\sum_{k=1}^{N_{x2}} c_{jk}^{(3)} w_{i,k} + \left(-B_{550} + F_{470} - \frac{8}{5}B_{551} + \frac{8}{5}F_{471}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x1}} c_{ik}^{(2)} w_{k,m}$$

$$+\left(-\frac{2}{15}F_{671} + \frac{1}{8}F_{672}\right)\sum_{k=1}^{N_{x2}} c_{jk}^{(1)} w_{i,k} + \left(F_{470} + \frac{16}{15}F_{471} + \frac{1}{8}F_{472}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x1}} c_{ik}^{(1)} \phi_{1k,m}$$

$$+\left(\frac{8}{15}F_{471} - \frac{1}{8}F_{472}\right)\sum_{k=1}^{N_{x1}} c_{ik}^{(2)} \phi_{2k,j} + \left(F_{470} - \frac{2}{5}F_{471}\right)\sum_{k=1}^{N_{x2}} c_{jk}^{(2)} \phi_{2i,k} + \left(-\frac{2}{15}F_{671} + \frac{1}{8}F_{672}\right)\phi_{2i,j}\Bigg) n_{x_1}$$

$$+\Bigg(-\frac{2}{5}A_{551}\sum_{k=1}^{N_{x1}} c_{ik}^{(2)} u_{k,j} + \left(\frac{8}{15}A_{551} + \frac{1}{4}A_{552}\right)\sum_{k=1}^{N_{x2}} c_{jk}^{(2)} u_{i,k} + \left(\frac{16}{15}A_{551} - \frac{1}{4}A_{552}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x1}} c_{ik}^{(1)} v_{k,m}$$

$$+\left(\frac{2}{5}B_{551} - \frac{2}{5}F_{471}\right)\sum_{k=1}^{N_{x1}} c_{ik}^{(3)} w_{k,j} + \left(-\frac{8}{5}B_{551} + \frac{8}{5}F_{471}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x1}} c_{ik}^{(1)} w_{k,m} + \left(-\frac{2}{15}F_{671} - \frac{1}{4}F_{672}\right)\sum_{k=1}^{N_{x1}} c_{ik}^{(1)} w_{k,j}$$

$$-\frac{2}{5}F_{471}\sum_{k=1}^{N_{x1}} c_{ik}^{(2)} \phi_{1k,j} + \left(\frac{8}{15}F_{471} + \frac{1}{4}F_{472}\right)\sum_{k=1}^{N_{x2}} c_{jk}^{(2)} \phi_{1i,k} + \left(-\frac{2}{15}F_{671} - \frac{1}{4}F_{672}\right)\phi_{1i,j}$$

$$+\left(\frac{16}{15}F_{471} - \frac{1}{4}F_{472}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x1}} c_{ik}^{(1)} \phi_{2k,m}\Bigg) n_{x_2} = 0, \qquad \textit{(138)}$$

$v_{i,j}=0$ veya

$$\begin{aligned}
&\Bigg(\left(\left(-\frac{8}{15}A_{551}+\frac{1}{8}A_{552}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(3)}u_{i,k}+\left(-A_{550}-\frac{2}{3}A_{551}+\frac{1}{4}A_{552}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}u_{k,m}+A_{55}\sum_{k=1}^{N_{x2}}c_{jk}^{(1)}u_{i,k}\right.\\
&+\left(-A_{550}-\frac{2}{3}A_{551}-\frac{1}{8}A_{552}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}v_{k,m}+\left(-\frac{8}{15}A_{551}-\frac{1}{4}A_{552}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}v_{k,j}+A_{55}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}v_{k,j}\\
&+\left(B_{550}-F_{470}+\frac{6}{5}B_{551}-\frac{6}{5}F_{471}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}w_{k,m}+\left(B_{550}+\frac{6}{5}B_{551}-F_{470}-\frac{6}{5}F_{471}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(3)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}w_{k,m}\\
&+\left(2F_{47}-2B_{55}+\frac{4}{15}F_{671}+\frac{1}{8}F_{672}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}w_{k,m}+\left(-\frac{8}{15}F_{471}+\frac{1}{8}F_{472}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(3)}\phi_{1i,k}\\
&+\left(-F_{470}-\frac{2}{3}F_{471}+\frac{1}{4}F_{472}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}\phi_{1k,m}+\left(F_{47}+\frac{2}{15}F_{671}-\frac{1}{8}F_{672}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(1)}\phi_{1i,k}\\
&+\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}\phi_{2k,j}+\left(-F_{470}-\frac{2}{3}F_{471}-\frac{1}{8}F_{472}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}\phi_{2k,m}\\
&\left.+\left(F_{47}+\frac{2}{15}F_{671}+\frac{1}{4}F_{672}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}\phi_{2k,j}\right)n_{x_1}\\
&+\left(\left(-A_{550}+\frac{2}{5}A_{551}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}u_{k,j}+\left(-2A_{550}+\frac{4}{15}A_{551}+\frac{1}{8}A_{552}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}u_{k,m}+A_{L11}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}u_{k,j}\right.\\
&+\left(-2A_{550}-\frac{4}{5}A_{551}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(3)}v_{i,k}+\left(-A_{550}-\frac{2}{3}A_{551}-\frac{1}{8}A_{552}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}v_{k,m}+A_{11}\sum_{k=1}^{N_{x2}}c_{jk}^{(1)}v_{i,k}\\
&+\left(B_{550}-F_{470}-\frac{2}{5}B_{551}+\frac{2}{5}F_{471}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(4)}w_{k,j}+\left(-2F_{470}+2B_{550}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(4)}w_{i,k}\\
&+\left(3B_{550}+\frac{2}{5}B_{551}-3F_{470}-\frac{2}{5}F_{471}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}w_{k,m}\\
&+\left(F_{L11}-B_{L11}+\frac{2}{15}F_{671}-\frac{1}{8}F_{672}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}w_{k,j}+\left(F_{11}-B_{11}+\frac{2}{5}F_{671}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(2)}w_{i,k}\\
&+\left(-F_{470}+\frac{2}{5}F_{471}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}\phi_{1k,j}+\left(-2F_{470}+\frac{4}{15}F_{471}+\frac{1}{8}F_{472}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}\phi_{1k,m}\\
&+\left(F_{L11}+\frac{2}{15}F_{671}-\frac{1}{8}F_{672}\right)\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}\phi_{1k,j}+\left(-2F_{470}-\frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(3)}\phi_{2i,k}\\
&\left.+\left(-F_{470}-\frac{2}{3}F_{471}-\frac{1}{8}F_{472}\right)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}\phi_{2k,m}+\left(F_{11}+\frac{2}{5}F_{671}\right)\sum_{k=1}^{N_{x2}}c_{jk}^{(1)}\phi_{2i,k}-A_{T11}\Delta T_{i,j}\right)n_{x_2}=0,
\end{aligned}$$

(139)

$$\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} v_{k,j} = 0 \quad \text{veya}$$

$$\left(\left(\frac{16}{15} A_{551} - \frac{1}{4} A_{552} \right) \sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} u_{k,m} + \left(\frac{8}{15} A_{551} + \frac{1}{4} A_{552} \right) \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} v_{k,j} - \frac{2}{5} A_{551} \sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} v_{i,k} \right.$$

$$+ \left(\frac{2}{5} B_{551} - \frac{2}{5} F_{471} \right) \sum_{k=1}^{N_{x_2}} c_{jk}^{(3)} w_{i,k} + \left(-\frac{8}{5} B_{551} + \frac{8}{5} F_{471} \right) \sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} w_{k,m}$$

$$+ \left(-\frac{2}{15} F_{671} - \frac{1}{4} F_{672} \right) \sum_{k=1}^{N_{x_2}} c_{jk}^{(1)} w_{i,k} + \left(\frac{16}{15} F_{471} - \frac{1}{4} F_{472} \right) \sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{1k,m}$$

$$\left. + \left(\frac{8}{15} F_{471} + \frac{1}{4} F_{472} \right) \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} \phi_{2k,j} - \frac{2}{5} F_{471} \sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} \phi_{2i,k} + \left(-\frac{2}{15} F_{671} - \frac{1}{4} F_{672} \right) \phi_{2i,j} \right) n_{x_1}$$

$$+ \left(\left(A_{550} - \frac{2}{5} A_{551} \right) \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} u_{k,j} + \left(\frac{8}{15} A_{551} - \frac{1}{8} A_{552} \right) \sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} u_{i,k} + \left(A_{550} + \frac{16}{15} A_{551} + \frac{1}{8} A_{552} \right) \sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} v_{k,m} \right.$$

$$+ \left(-B_{550} + F_{470} + \frac{2}{5} B_{551} - \frac{2}{5} F_{471} \right) \sum_{k=1}^{N_{x_1}} c_{ik}^{(3)} w_{k,j} + \left(-B_{550} + F_{470} - \frac{8}{5} B_{551} + \frac{8}{5} F_{471} \right) \sum_{m=1}^{N_{x_2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} w_{k,m}$$

$$+ \left(-\frac{2}{15} F_{671} + \frac{1}{8} F_{672} \right) \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} w_{k,j} + \left(F_{470} - \frac{2}{5} F_{471} \right) \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} \phi_{1k,j} + \left(\frac{8}{15} F_{471} - \frac{1}{8} F_{472} \right) \sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} \phi_{1i,k}$$

$$\left. + \left(-\frac{2}{15} F_{671} + \frac{1}{8} F_{672} \right) \phi_{1i,j} + \left(F_{470} + \frac{16}{15} F_{471} + \frac{1}{8} F_{472} \right) \sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{2k,m} \right) n_{x_2} = 0, \qquad (140)$$

$$\sum_{k=1}^{N_{x2}} c_{jk}^{(1)} v_{i,k} = 0 \quad \text{veya}$$

$$\begin{aligned}
&\Bigg(\left(\left(A_{550}-\frac{2}{5}A_{551}\right)\sum_{k=1}^{N_{x1}} c_{ik}^{(2)} u_{k,j}+\left(\frac{8}{15}A_{551}-\frac{1}{8}A_{552}\right)\sum_{k=1}^{N_{x2}} c_{jk}^{(2)} u_{i,k}+\left(A_{550}+\frac{16}{15}A_{551}+\frac{1}{8}A_{552}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x1}} c_{ik}^{(1)} v_{k,m}\right.\\
&+\left(-B_{550}+F_{470}+\frac{2}{5}B_{551}-\frac{2}{5}F_{471}\right)\sum_{k=1}^{N_{x1}} c_{ik}^{(3)} w_{k,j}+\left(-B_{550}+F_{470}-\frac{8}{5}B_{551}+\frac{8}{5}F_{471}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(2)}\sum_{k=1}^{N_{x1}} c_{ik}^{(1)} w_{k,m}\\
&+\left(-\frac{2}{15}F_{671}+\frac{1}{8}F_{672}\right)\sum_{k=1}^{N_{x1}} c_{ik}^{(1)} w_{k,j}+\left(F_{470}-\frac{2}{5}F_{471}\right)\sum_{k=1}^{N_{x1}} c_{ik}^{(2)}\phi_{1k,j}+\left(\frac{8}{15}F_{471}-\frac{1}{8}F_{472}\right)\sum_{k=1}^{N_{x2}} c_{jk}^{(2)}\phi_{1i,k}\\
&\left.+\left(-\frac{2}{15}F_{671}+\frac{1}{8}F_{672}\right)\phi_{1i,j}+\left(F_{470}+\frac{16}{15}F_{471}+\frac{1}{8}F_{472}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x1}} c_{ik}^{(1)}\phi_{2k,m}\right)n_{x_1}\\
&+\left(\left(2A_{550}-\frac{4}{5}A_{551}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x1}} c_{ik}^{(1)} u_{k,m}-\frac{2}{5}A_{551}\sum_{k=1}^{N_{x1}} c_{ik}^{(2)} v_{k,j}+\left(2A_{550}+\frac{4}{5}A_{551}\right)\sum_{k=1}^{N_{x2}} c_{jk}^{(2)} v_{i,k}\right.\\
&+\left(-2B_{550}+2F_{470}+\frac{6}{5}B_{551}-\frac{6}{5}F_{471}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x1}} c_{ik}^{(2)} w_{k,m}+\left(2F_{470}-2B_{550}-\frac{4}{5}B_{551}+\frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x2}} c_{jk}^{(3)} w_{i,k}\\
&-\frac{2}{5}F_{671}\sum_{k=1}^{N_{x2}} c_{jk}^{(1)} w_{i,k}+\left(2F_{470}-\frac{4}{5}F_{471}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x1}} c_{ik}^{(1)}\phi_{1k,m}\\
&\left.-\frac{2}{5}F_{471}\sum_{k=1}^{N_{x1}} c_{ik}^{(2)}\phi_{2k,j}+\left(2F_{470}+\frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x2}} c_{jk}^{(2)}\phi_{2i,k}-\frac{2}{5}F_{671}\phi_{2i,j}\right)n_{x_2}=0, \qquad (141)
\end{aligned}$$

$w_{i,j} = 0$ veya

$$\Bigg(\Bigg(\left(-2B_{550} + 2F_{470} - \frac{4}{5}B_{551} + \frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x1}} c_{ik}^{(4)} u_{k,j} + \frac{8}{15}F_{471}\sum_{k=1}^{N_{x2}} c_{jk}^{(4)} u_{i,k} + \Big(-2B_{550} + 2F_{470} - \frac{14}{5}B_{551}$$

$$+\frac{4}{3}F_{471}\Big)\sum_{m=1}^{N_{x2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x1}} c_{ik}^{(2)} u_{k,m} + \left(B_{11} - F_{11} - \frac{2}{5}F_{671}\right)\sum_{k=1}^{N_{x1}} c_{ik}^{(2)} u_{k,j}$$

$$+\left(B_{55} - F_{47} - \frac{2}{15}F_{671} - \frac{1}{4}F_{672}\right)\sum_{k=1}^{N_{x2}} c_{jk}^{(2)} u_{i,k} + \left(-2B_{550} + 2F_{470} - \frac{4}{5}B_{551} + \frac{4}{15}F_{471}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x1}} c_{ik}^{(3)} v_{k,m}$$

$$+\left(-2B_{550} + 2F_{470} + \frac{6}{5}B_{551} + \frac{4}{15}F_{471}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(3)} \sum_{k=1}^{N_{x1}} c_{ik}^{(1)} v_{k,m} + \Big(B_{55} + B_{L11} - F_{47} - F_{L11} - \frac{4}{15}F_{671}$$

$$+\frac{1}{4}F_{672}\Big)\sum_{m=1}^{N_{x2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x1}} c_{ik}^{(1)} v_{k,m} + \left(2D_{550} + 2F_{440} - 4F_{480} + \frac{4}{5}D_{551} + \frac{4}{5}F_{441} - \frac{8}{5}F_{481}\right)\sum_{k=1}^{N_{x1}} c_{ik}^{(5)} w_{k,j}$$

$$+\left(4D_{550} + 4F_{440} - 8F_{480} + \frac{18}{5}D_{551} + \frac{8}{5}F_{441} - \frac{26}{5}F_{481}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x1}} c_{ik}^{(3)} w_{k,m}$$

$$+\left(2D_{550} + 2F_{440} - 4F_{480} - \frac{6}{5}D_{551} + \frac{4}{5}F_{441} + \frac{2}{5}F_{481}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(4)} \sum_{k=1}^{N_{x1}} c_{ik}^{(1)} w_{k,m}$$

$$+\Big(-D_{11} + 2F_{22} - F_{33} - 2A_{550} - 2F_{550} + 4F_{570} - \frac{8}{15}A_{551} - \frac{4}{5}F_{461} - \frac{32}{15}F_{551} + \frac{32}{15}F_{571} + \frac{4}{5}F_{681}$$

$$-A_{552} - \frac{1}{4}F_{552} + F_{572}\Big)\sum_{k=1}^{N_{x1}} c_{ik}^{(3)} w_{k,j} + \Big(-D_{11} + 2F_{22} - F_{33} - 2A_{550} - 2F_{550} + 4F_{570} - \frac{8}{15}A_{551} - \frac{4}{5}F_{461}$$

$$-\frac{32}{15}F_{551} + \frac{32}{15}F_{571} + \frac{4}{5}F_{681} - A_{552} - \frac{1}{4}F_{552} + F_{572}\Big)\sum_{m=1}^{N_{x2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x1}} c_{ik}^{(1)} w_{k,m} + \Big(k_s F_{55} + \frac{8}{15}F_{661}$$

$$+\frac{1}{4}F_{662}\Big)\sum_{k=1}^{N_{x1}} c_{ik}^{(1)} w_{k,j} + \left(2F_{440} - 2F_{480} + \frac{4}{5}F_{441} - \frac{4}{5}F_{481}\right)\sum_{k=1}^{N_{x1}} c_{ik}^{(4)} \phi_{1k,j} + \frac{8}{15}F_{441}\sum_{k=1}^{N_{x2}} c_{jk}^{(4)} \phi_{1i,k}$$

$$+\left(2F_{440} - 2F_{480} + \frac{4}{3}F_{441} - \frac{14}{5}F_{481}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x1}} c_{ik}^{(2)} \phi_{1k,m} + \Big(F_{22} - F_{33} - 2F_{550} + 2F_{570} - \frac{4}{5}F_{461}$$

$$-\frac{32}{15}F_{551} + \frac{16}{15}F_{571} + \frac{2}{5}F_{681} - \frac{1}{4}F_{552} + \frac{1}{2}F_{572}\Big)\sum_{k=1}^{N_{x1}} c_{ik}^{(2)} \phi_{1k,j} + \Big(-F_{44} + F_{48} - \frac{4}{15}F_{461} - \frac{4}{3}F_{551} + \frac{2}{3}F_{571}$$

$$-\frac{1}{4}F_{462} - \frac{1}{2}F_{552} + \frac{1}{2}F_{572}\Big)\sum_{k=1}^{N_{x2}} c_{jk}^{(2)} \phi_{1i,k} + \left(k_s F_{55} + \frac{8}{15}F_{661} + \frac{1}{4}F_{662}\right)\phi_{1i,j} + \Big(2F_{440} - 2F_{480} + \frac{4}{15}F_{441}$$

$$-\frac{4}{5}F_{481}\Big)\sum_{m=1}^{N_{x2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x1}} c_{ik}^{(3)} \phi_{2k,m} + \left(2F_{440} - 2F_{480} + \frac{4}{15}F_{441} + \frac{6}{5}F_{481}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(3)} \sum_{k=1}^{N_{x1}} c_{ik}^{(1)} \phi_{2k,m}$$

$$+\Big(F_{48} + F_{L22} - F_{44} - F_{L33} - 2F_{550} + 2F_{570} - \frac{8}{15}F_{461} - \frac{4}{5}F_{551} + \frac{2}{5}F_{571} + \frac{2}{5}F_{681} + \frac{1}{4}F_{462}$$

$$+\frac{1}{4}F_{552}\Big)\sum_{m=1}^{N_{x2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x1}} c_{ik}^{(1)} \phi_{2k,m} + \left(-B_{T11} + F_{T11}\right)\left(\frac{\partial \Delta T}{\partial x_1}\right)_{i,j} - P_{x_1}\sum_{k=1}^{N_{x1}} c_{ik}^{(1)} w_{k,j}\Bigg) n_{x_1}$$

$$+\Bigg(\bigg(-2B_{550}+2F_{470}+\frac{6}{5}B_{551}+\frac{4}{15}F_{471}\bigg)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}u_{k,m}$$

$$+\bigg(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{15}F_{471}\bigg)\sum_{m=1}^{N_{x2}}c_{jm}^{(3)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}u_{k,m}$$

$$+\bigg(B_{55}+B_{L11}-F_{47}-F_{L11}-\frac{4}{15}F_{671}+\frac{1}{4}F_{672}\bigg)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}u_{k,m}$$

$$+\frac{8}{15}F_{471}\sum_{k=1}^{N_{x1}}c_{ik}^{(4)}v_{k,j}+\bigg(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{5}F_{471}\bigg)\sum_{k=1}^{N_{x2}}c_{jk}^{(4)}v_{i,k}$$

$$+\bigg(-2B_{550}+2F_{470}-\frac{14}{5}B_{551}+\frac{4}{3}F_{471}\bigg)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}v_{k,m}+\bigg(B_{11}-F_{11}-\frac{2}{5}F_{671}\bigg)\sum_{k=1}^{N_{x2}}c_{jk}^{(2)}v_{i,k}$$

$$+\bigg(B_{55}-F_{47}-\frac{2}{15}F_{671}-\frac{1}{4}F_{672}\bigg)\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}v_{k,j}+\bigg(2D_{550}+2F_{440}-4F_{480}+\frac{4}{5}D_{551}+\frac{4}{5}F_{441}$$

$$-\frac{8}{5}F_{481}\bigg)\sum_{k=1}^{N_{x2}}c_{jk}^{(5)}w_{i,k}+\bigg(2D_{550}+2F_{440}-4F_{480}-\frac{6}{5}D_{551}+\frac{4}{5}F_{441}+\frac{2}{5}F_{481}\bigg)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(4)}w_{k,m}$$

$$+\bigg(4D_{550}+4F_{440}-8F_{480}+\frac{18}{5}D_{551}+\frac{8}{5}F_{441}-\frac{26}{5}F_{481}\bigg)\sum_{m=1}^{N_{x2}}c_{jm}^{(3)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}w_{k,m}$$

$$+\bigg(-D_{11}+2F_{22}-F_{33}-2A_{550}-2F_{550}+4F_{570}-\frac{8}{15}A_{551}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{32}{15}F_{571}+\frac{4}{5}F_{681}$$

$$-A_{552}-\frac{1}{4}F_{552}+F_{572}\bigg)\sum_{k=1}^{N_{x2}}c_{jk}^{(3)}w_{i,k}+\bigg(-D_{11}+2F_{22}-F_{33}-2A_{550}-2F_{550}+4F_{570}-\frac{8}{15}A_{551}-\frac{4}{5}F_{461}$$

$$-\frac{32}{15}F_{551}+\frac{32}{15}F_{571}+\frac{4}{5}F_{681}-A_{552}-\frac{1}{4}F_{552}+F_{572}\bigg)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}w_{k,m}$$

$$+\bigg(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\bigg)\sum_{k=1}^{N_{x2}}c_{jk}^{(1)}w_{i,k}+\bigg(2F_{440}-2F_{480}+\frac{4}{15}F_{441}+\frac{6}{5}F_{481}\bigg)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(3)}\phi_{1k,m}$$

$$+\bigg(2F_{440}-2F_{480}+\frac{4}{15}F_{441}-\frac{4}{5}F_{481}\bigg)\sum_{m=1}^{N_{x2}}c_{jm}^{(3)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}\phi_{1k,m}$$

$$+\bigg(F_{48}+F_{L22}-F_{44}-F_{L33}-2F_{550}+2F_{570}-\frac{8}{15}F_{461}-\frac{4}{5}F_{551}+\frac{2}{5}F_{571}+\frac{2}{5}F_{681}+\frac{1}{4}F_{462}$$

$$+\frac{1}{4}F_{552}\bigg)\sum_{m=1}^{N_{x2}}c_{jm}^{(1)}\sum_{k=1}^{N_{x1}}c_{ik}^{(1)}\phi_{1k,m}+\frac{8}{15}F_{441}\sum_{k=1}^{N_{x1}}c_{ik}^{(4)}\phi_{2k,j}+\bigg(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\bigg)\sum_{k=1}^{N_{x2}}c_{jk}^{(4)}\phi_{2i,k}$$

$$+\bigg(2F_{440}-2F_{480}+\frac{4}{3}F_{441}-\frac{14}{5}F_{481}\bigg)\sum_{m=1}^{N_{x2}}c_{jm}^{(2)}\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}\phi_{2k,m}+\bigg(-F_{44}+F_{48}-\frac{4}{15}F_{461}-\frac{4}{3}F_{551}+\frac{2}{3}F_{571}$$

$$-\frac{1}{4}F_{462}-\frac{1}{2}F_{552}+\frac{1}{2}F_{572}\bigg)\sum_{k=1}^{N_{x1}}c_{ik}^{(2)}\phi_{2k,j}+\bigg(F_{22}-F_{33}-2F_{550}+2F_{570}-\frac{4}{5}F_{461}-\frac{32}{15}F_{551}+\frac{16}{15}F_{571}$$

$$+\frac{2}{5}F_{681}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\bigg)\sum_{k=1}^{N_{x2}}c_{jk}^{(2)}\phi_{2i,k}+\bigg(k_sF_{55}+\frac{8}{15}F_{661}+\frac{1}{4}F_{662}\bigg)\phi_{2i,j}$$

$$+\left(-B_{T11}+F_{T11}\right)\left(\frac{\partial \Delta T}{\partial x_2}\right)_{i,j}-P_{x_2}\sum_{k=1}^{N_{x_2}}c_{jk}^{(1)}w_{i,k}\Bigg)n_{x_2}$$

$$=\left(\left(I_1-I_3\right)\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j}+\left(2I_4-I_2-I_5\right)\sum_{k=1}^{N_{x_1}}c_{ik}^{(1)}\left(\frac{\partial^2 w}{\partial t^2}\right)_{k,j}+\left(I_4-I_5\right)\left(\frac{\partial^2 \phi_1}{\partial t^2}\right)_{i,j}\right)n_{x_1}$$

$$+\left(\left(I_1-I_3\right)\left(\frac{\partial^2 v}{\partial t^2}\right)_{i,j}+\left(2I_4-I_2-I_5\right)\sum_{k=1}^{N_{x_2}}c_{jk}^{(1)}\left(\frac{\partial^2 w}{\partial t^2}\right)_{i,k}+\left(I_4-I_5\right)\left(\frac{\partial^2 \phi_2}{\partial t^2}\right)_{i,j}\right)n_{x_2}, \qquad (142)$$

$\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} w_{k,j} = 0$ veya

$$
\begin{aligned}
&\Bigg(\bigg(2B_{550} - 2F_{470} + \frac{4}{5}B_{551} - \frac{4}{5}F_{471}\bigg)\sum_{k=1}^{N_{x_1}} c_{ik}^{(3)} u_{k,j} + \bigg(2B_{550} - 2F_{470} + \frac{14}{5}B_{551} - \frac{14}{5}F_{471}\bigg)\sum_{m=1}^{N_{x_2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} u_{k,m} \\
&+\left(-B_{11} + F_{11}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} u_{k,j} + \bigg(2B_{550} - 2F_{470} - \frac{6}{5}B_{551} + \frac{6}{5}F_{471}\bigg)\sum_{k=1}^{N_{x_2}} c_{jk}^{(3)} v_{i,k} \\
&+\bigg(2B_{550} - 2F_{470} + \frac{4}{5}B_{551} - \frac{4}{5}F_{471}\bigg)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} v_{k,m} + \left(-B_{L11} + F_{L11}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(1)} v_{i,k} \\
&+\bigg(-2D_{550} - 2F_{440} + 4F_{480} - \frac{4}{5}D_{551} - \frac{4}{5}F_{441} + \frac{8}{5}F_{481}\bigg)\sum_{k=1}^{N_{x_1}} c_{ik}^{(4)} w_{k,j} \\
&+\bigg(-2D_{550} + 4F_{480} - 2F_{440} + \frac{6}{5}D_{551} + \frac{6}{5}F_{441} - \frac{12}{5}F_{481}\bigg)\sum_{k=1}^{N_{x_2}} c_{jk}^{(4)} w_{i,k} \\
&+\bigg(-4D_{550} + 8F_{480} - 4F_{440} - \frac{18}{5}D_{551} - \frac{18}{5}F_{441} + \frac{36}{5}F_{481}\bigg)\sum_{m=1}^{N_{x_2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} w_{k,m} \\
&+\bigg(D_{11} + F_{33} - 2F_{22} + 2A_{550} + 2F_{550} - 4F_{570} + \frac{8}{15}A_{551} + \frac{2}{5}F_{461} + \frac{32}{15}F_{551} - \frac{32}{15}F_{571} - \frac{2}{5}F_{681} \\
&+A_{552} + \frac{1}{4}F_{552} - F_{572}\bigg)\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} w_{k,j} + \big(D_{L11} - 2F_{L22} + F_{L33} + 2A_{550} + 2F_{550} - 4F_{570} \\
&-\frac{2}{15}A_{551} + \frac{2}{5}F_{461} - \frac{8}{15}F_{551} + \frac{8}{15}F_{571} - \frac{2}{5}F_{681} - A_{552} - \frac{1}{4}F_{552} + F_{572}\bigg)\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} w_{i,k} \\
&+\bigg(-2F_{440} + 2F_{480} - \frac{4}{5}F_{441} + \frac{4}{5}F_{481}\bigg)\sum_{k=1}^{N_{x_1}} c_{ik}^{(3)} \phi_{1k,j} + \bigg(-2F_{440} + 2F_{480} - \frac{14}{5}F_{441} + \frac{14}{5}F_{481}\bigg)\sum_{m=1}^{N_{x_2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{1k,m} \\
&+\bigg(-F_{22} + F_{33} + 2F_{550} - 2F_{570} + \frac{2}{5}F_{461} + \frac{32}{15}F_{551} - \frac{16}{15}F_{571} - \frac{2}{5}F_{681} + \frac{1}{4}F_{552} - \frac{1}{2}F_{572}\bigg)\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{1k,j} \\
&+\bigg(-2F_{440} + 2F_{480} + \frac{6}{5}F_{441} - \frac{6}{5}F_{481}\bigg)\sum_{k=1}^{N_{x_2}} c_{jk}^{(3)} \phi_{2i,k} + \bigg(-2F_{440} + 2F_{480} - \frac{4}{5}F_{441} + \frac{4}{5}F_{481}\bigg)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} \phi_{2k,m} \\
&+\bigg(-F_{L22} + F_{L33} + 2F_{550} - 2F_{570} + \frac{2}{5}F_{461} - \frac{8}{15}F_{551} + \frac{4}{15}F_{571} - \frac{2}{5}F_{681} - \frac{1}{4}F_{552} + \frac{1}{2}F_{572}\bigg)\sum_{k=1}^{N_{x_2}} c_{jk}^{(1)} \phi_{2i,k} \\
&+\left(B_{T11} - F_{T11}\right)\Delta T_{i,j}\Bigg) n_{x_1} + \Bigg(\left(-B_{55} + F_{47}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(1)} u_{i,k} + \left(-B_{55} + F_{47}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} v_{k,j} \\
&+\bigg(2D_{55} + 2F_{44} - 4F_{48} + \frac{2}{3}A_{551} + \frac{8}{3}F_{551} - \frac{8}{3}F_{571} + 2A_{552} + \frac{1}{2}F_{552} - 2F_{572}\bigg)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} w_{k,m} \\
&+\bigg(F_{44} - F_{48} + \frac{4}{3}F_{551} - \frac{2}{3}F_{571} + \frac{1}{4}F_{552} - \frac{1}{2}F_{572}\bigg)\sum_{k=1}^{N_{x_2}} c_{jk}^{(1)} \phi_{1i,k} \\
&+\bigg(F_{44} - F_{48} + \frac{4}{3}F_{551} - \frac{2}{3}F_{571} + \frac{1}{4}F_{552} - \frac{1}{2}F_{572}\bigg)\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{2k,j}\Bigg) n_{x_2} = 0, \qquad (143)
\end{aligned}
$$

$$\sum_{k=1}^{N_{x2}} c_{jk}^{(1)} w_{i,k} = 0 \quad \text{veya}$$

$$\Bigg(\left(-B_{55}+F_{47}\right)\sum_{k=1}^{N_{x2}} c_{jk}^{(1)} u_{i,k} + \left(-B_{55}+F_{47}\right)\sum_{k=1}^{N_{x1}} c_{ik}^{(1)} v_{k,j}$$

$$+\left(2D_{55}+2F_{44}-4F_{48}+\frac{2}{3}A_{551}+\frac{8}{3}F_{551}-\frac{8}{3}F_{571}+2A_{552}+\frac{1}{2}F_{552}-2F_{572}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x1}} c_{ik}^{(1)} w_{k,m}$$

$$+\left(F_{44}-F_{48}+\frac{4}{3}F_{551}-\frac{2}{3}F_{571}+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\sum_{k=1}^{N_{x2}} c_{jk}^{(1)}\phi_{1i,k}$$

$$+\left(F_{44}-F_{48}+\frac{4}{3}F_{551}-\frac{2}{3}F_{571}+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\sum_{k=1}^{N_{x1}} c_{ik}^{(1)}\phi_{2k,j}\Bigg) n_{x_1}$$

$$+\Bigg(\left(2B_{550}-2F_{470}-\frac{6}{5}B_{551}+\frac{6}{5}F_{471}\right)\sum_{k=1}^{N_{x1}} c_{ik}^{(3)} u_{k,j} + \left(2B_{550}-2F_{470}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(2)}\sum_{k=1}^{N_{x1}} c_{ik}^{(1)} u_{k,m}$$

$$+\left(-B_{L11}+F_{L11}\right)\sum_{k=1}^{N_{x1}} c_{ik}^{(1)} u_{k,j} + \left(2B_{550}-2F_{470}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x2}} c_{jk}^{(3)} v_{i,k}$$

$$+\left(2B_{550}-2F_{470}+\frac{14}{5}B_{551}-\frac{14}{5}F_{471}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x1}} c_{ik}^{(2)} v_{k,m} + \left(-B_{11}+F_{11}\right)\sum_{k=1}^{N_{x2}} c_{jk}^{(1)} v_{i,k}$$

$$+\left(-2D_{550}+4F_{480}-2F_{440}+\frac{6}{5}D_{551}+\frac{6}{5}F_{441}-\frac{12}{5}F_{481}\right)\sum_{k=1}^{N_{x1}} c_{ik}^{(4)} w_{k,j}$$

$$+\left(-2D_{550}-2F_{440}+4F_{480}-\frac{4}{5}D_{551}-\frac{4}{5}F_{441}+\frac{8}{5}F_{481}\right)\sum_{k=1}^{N_{x2}} c_{jk}^{(4)} w_{i,k}$$

$$+\left(-4D_{550}+8F_{480}-4F_{440}-\frac{18}{5}D_{551}-\frac{18}{5}F_{441}+\frac{36}{5}F_{481}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(2)}\sum_{k=1}^{N_{x1}} c_{ik}^{(2)} w_{k,m}$$

$$+\Bigg(D_{L11}-2F_{L22}+F_{L33}+2A_{550}+2F_{550}-4F_{570}-\frac{2}{15}A_{551}+\frac{2}{5}F_{461}-\frac{8}{15}F_{551}+\frac{8}{15}F_{571}-\frac{2}{5}F_{681}$$

$$-A_{552}-\frac{1}{4}F_{552}+F_{572}\Bigg)\sum_{k=1}^{N_{x1}} c_{ik}^{(2)} w_{k,j} + \Big(D_{11}+F_{33}-2F_{22}+2A_{550}+2F_{550}-4F_{570}$$

$$+\frac{8}{15}A_{551}+\frac{2}{5}F_{461}+\frac{32}{15}F_{551}-\frac{32}{15}F_{571}-\frac{2}{5}F_{681}+A_{552}+\frac{1}{4}F_{552}-F_{572}\Big)\sum_{k=1}^{N_{x2}} c_{jk}^{(2)} w_{i,k}$$

$$+\left(-2F_{440}+2F_{480}+\frac{6}{5}F_{441}-\frac{6}{5}F_{481}\right)\sum_{k=1}^{N_{x1}} c_{ik}^{(3)}\phi_{1k,j} + \left(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(2)}\sum_{k=1}^{N_{x1}} c_{ik}^{(1)}\phi_{1k,m}$$

$$+\left(-F_{L22}+F_{L33}+2F_{550}-2F_{570}+\frac{2}{5}F_{461}-\frac{8}{15}F_{551}+\frac{4}{15}F_{571}-\frac{2}{5}F_{681}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\right)\sum_{k=1}^{N_{x1}} c_{ik}^{(1)}\phi_{1k,j}$$

$$+\left(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\right)\sum_{k=1}^{N_{x2}} c_{jk}^{(3)}\phi_{2i,k} + \left(-2F_{440}+2F_{480}-\frac{14}{5}F_{441}+\frac{14}{5}F_{481}\right)\sum_{m=1}^{N_{x2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x1}} c_{ik}^{(2)}\phi_{2k,m}$$

$$+\left(-F_{22}+F_{33}+2F_{550}-2F_{570}+\frac{2}{5}F_{461}+\frac{32}{15}F_{551}-\frac{16}{15}F_{571}-\frac{2}{5}F_{681}+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\sum_{k=1}^{N_{x2}} c_{jk}^{(1)}\phi_{2i,k}$$

$$+\left(B_{T11}-F_{T11}\right)\Delta T_{i,j}\Big)n_{x_2}=0, \tag{144}$$

$$\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} w_{k,j}=0 \quad \text{veya}$$

$$\Bigg(\Bigg(\left(-2B_{550}+2F_{470}-\frac{4}{5}B_{551}+\frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} u_{k,j}+\left(+\frac{2}{5}B_{551}-\frac{2}{5}F_{471}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} u_{i,k}$$

$$+\left(-2B_{550}+2F_{470}+\frac{4}{5}B_{551}-\frac{4}{5}F_{471}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} v_{k,m}$$

$$+\left(2D_{550}+2F_{440}-4F_{480}+\frac{4}{5}D_{551}+\frac{4}{5}F_{441}-\frac{8}{5}F_{481}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(3)} w_{k,j}$$

$$+\left(2D_{550}+2F_{440}-4F_{480}-\frac{6}{5}D_{551}-\frac{6}{5}F_{441}+\frac{12}{5}F_{481}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(2)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} w_{k,m}$$

$$+\left(-\frac{2}{5}F_{461}+\frac{2}{5}F_{681}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} w_{k,j}$$

$$+\left(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} \phi_{1k,j}+\left(-\frac{2}{5}F_{441}+\frac{2}{5}F_{481}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} \phi_{1i,k}$$

$$+\left(-\frac{2}{5}F_{461}+\frac{2}{5}F_{681}\right)\phi_{1i,j}+\left(2F_{440}-2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{2k,m}\Bigg)n_{x_1}$$

$$+\Bigg(\Bigg(-2B_{550}+2F_{470}-\frac{16}{5}B_{551}+\frac{16}{5}F_{471}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} u_{k,m}$$

$$+\left(-\frac{8}{5}B_{551}+\frac{8}{5}F_{471}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} v_{k,j}+\left(-2B_{550}+2F_{470}+\frac{6}{5}B_{551}-\frac{6}{5}F_{471}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} v_{i,k}$$

$$+\left(2D_{550}+2F_{440}-4F_{480}-\frac{6}{5}D_{551}-\frac{6}{5}F_{441}+\frac{12}{5}F_{481}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(3)} w_{i,k}$$

$$+\left(2D_{550}+2F_{440}-4F_{480}+\frac{24}{5}D_{551}+\frac{24}{5}F_{441}-\frac{48}{5}F_{481}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} w_{k,m}$$

$$+\left(-\frac{2}{5}F_{461}+\frac{2}{5}F_{681}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(1)} w_{i,k}$$

$$+\left(2F_{440}-2F_{480}+\frac{16}{5}F_{441}-\frac{16}{5}F_{481}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{1k,m}+\left(\frac{8}{5}F_{441}-\frac{8}{5}F_{481}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} \phi_{2k,j}$$

$$+\left(2F_{440}-2F_{480}-\frac{6}{5}F_{441}+\frac{6}{5}F_{481}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} \phi_{2i,k}+\left(-\frac{2}{5}F_{461}+\frac{2}{5}F_{681}\right)\phi_{2i,j}\Bigg)n_{x_2}=0, \tag{145}$$

$$\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} w_{i,k} = 0 \quad \text{veya}$$

$$\left(\left(-2B_{550} + 2F_{470} + \frac{6}{5}B_{551} - \frac{6}{5}F_{471} \right) \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} u_{k,j} + \left(-\frac{8}{5}B_{551} + \frac{8}{5}F_{471} \right) \sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} u_{i,k} \right.$$

$$+ \left(-2B_{550} + 2F_{470} - \frac{16}{5}B_{551} + \frac{16}{5}F_{471} \right) \sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} v_{k,m}$$

$$+ \left(2D_{550} + 2F_{440} - 4F_{480} - \frac{6}{5}D_{551} - \frac{6}{5}F_{441} + \frac{12}{5}F_{481} \right) \sum_{k=1}^{N_{x_1}} c_{ik}^{(3)} w_{k,j}$$

$$+ \left(2D_{550} + 2F_{440} - 4F_{480} + \frac{24}{5}D_{551} + \frac{24}{5}F_{441} - \frac{48}{5}F_{481} \right) \sum_{m=1}^{N_{x_2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} w_{k,m}$$

$$+ \left(-\frac{2}{5}F_{461} + \frac{2}{5}F_{681} \right) \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} w_{k,j}$$

$$+ \left(2F_{440} - 2F_{480} - \frac{6}{5}F_{441} + \frac{6}{5}F_{481} \right) \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} \phi_{1k,j} + \left(\frac{8}{5}F_{441} - \frac{8}{5}F_{481} \right) \sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} \phi_{1i,k}$$

$$\left. + \left(-\frac{2}{5}F_{461} + \frac{2}{5}F_{681} \right) \phi_{1i,j} + \left(2F_{440} - 2F_{480} + \frac{16}{5}F_{441} - \frac{16}{5}F_{481} \right) \sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{2k,m} \right) n_{x_1}$$

$$+ \left(\left(-2B_{550} + 2F_{470} + \frac{4}{5}B_{551} - \frac{4}{5}F_{471} \right) \sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} u_{k,m} \right.$$

$$+ \left(\frac{2}{5}B_{551} - \frac{2}{5}F_{471} \right) \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} v_{k,j} + \left(-2B_{550} + 2F_{470} - \frac{4}{5}B_{551} + \frac{4}{5}F_{471} \right) \sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} v_{i,k}$$

$$+ \left(2D_{550} + 2F_{440} - 4F_{480} + \frac{4}{5}D_{551} + \frac{4}{5}F_{441} - \frac{8}{5}F_{481} \right) \sum_{k=1}^{N_{x_2}} c_{jk}^{(3)} w_{i,k}$$

$$+ \left(2D_{550} + 2F_{440} - 4F_{480} - \frac{6}{5}D_{551} - \frac{6}{5}F_{441} + \frac{12}{5}F_{481} \right) \sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} w_{k,m}$$

$$+ \left(-\frac{2}{5}F_{461} + \frac{2}{5}F_{681} \right) \sum_{k=1}^{N_{x_2}} c_{jk}^{(1)} w_{i,k}$$

$$+ \left(2F_{440} - 2F_{480} - \frac{4}{5}F_{441} + \frac{4}{5}F_{481} \right) \sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{1k,m} + \left(-\frac{2}{5}F_{441} + \frac{2}{5}F_{481} \right) \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} \phi_{2k,j}$$

$$\left. + \left(2F_{440} - 2F_{480} + \frac{4}{5}F_{441} - \frac{4}{5}F_{481} \right) \sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} \phi_{2i,k} + \left(-\frac{2}{5}F_{461} + \frac{2}{5}F_{681} \right) \phi_{2i,j} \right) n_{x_2} = 0, \tag{146}$$

$\phi_{1i,j} = 0$ veya

$$\Bigg(\bigg(\left(-2F_{470} - \frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(3)} u_{k,j} + \left(-F_{470} - \frac{2}{3}F_{471} - \frac{1}{8}F_{472}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} u_{k,m} + F_{11}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} u_{k,j}$$

$$+\left(-2F_{470} + \frac{4}{15}F_{471} + \frac{1}{8}F_{472}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} v_{k,m} + \left(-F_{470} + \frac{2}{5}F_{471}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(3)} v_{i,k} + F_{L11}\sum_{k=1}^{N_{x_2}} c_{jk}^{(1)} v_{i,k}$$

$$+\left(-2F_{440} + 2F_{480} - \frac{4}{5}F_{441} + \frac{4}{5}F_{481}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(4)} w_{k,j} + \left(-F_{440} + F_{480} + \frac{2}{5}F_{441} - \frac{2}{5}F_{481}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(4)} w_{i,k}$$

$$+\left(-3F_{440} + 3F_{480} - \frac{2}{5}F_{441} + \frac{2}{5}F_{481}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} w_{k,m}$$

$$+\left(-F_{22} + F_{33} + 2F_{550} - 2F_{570} + \frac{2}{5}F_{461} + \frac{32}{15}F_{551} - \frac{16}{15}F_{571} + \frac{1}{4}F_{552} - \frac{1}{2}F_{572}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} w_{k,j}$$

$$+\left(-F_{L22} + F_{L33} + 2F_{550} - 2F_{570} + \frac{2}{15}F_{461} - \frac{8}{15}F_{551} + \frac{4}{15}F_{571} - \frac{1}{8}F_{462} - \frac{1}{4}F_{552} + \frac{1}{2}F_{572}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} w_{i,k}$$

$$+\left(-2F_{440} - \frac{4}{5}F_{441}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(3)} \phi_{1k,j} + \left(-F_{440} - \frac{2}{3}F_{441} - \frac{1}{8}F_{442}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{1k,m}$$

$$+\left(F_{33} + 2F_{550} + \frac{2}{5}F_{461} + \frac{32}{15}F_{551} + \frac{1}{4}F_{552}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{1k,j}$$

$$+\left(-F_{440} + \frac{2}{5}F_{441}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(3)} \phi_{2i,k} + \left(-2F_{440} + \frac{4}{15}F_{441} + \frac{1}{8}F_{442}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} \phi_{2k,m}$$

$$+\left(F_{L33} + 2F_{550} + \frac{2}{15}F_{461} - \frac{8}{15}F_{551} - \frac{1}{8}F_{462} - \frac{1}{4}F_{552}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(1)} \phi_{2i,k} - F_{T11}\Delta T_{i,j}\bigg) n_{x_1}$$

$$+\bigg(\left(-\frac{8}{15}F_{471} - \frac{1}{4}F_{472}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(3)} u_{i,k} + \left(-F_{470} - \frac{2}{3}F_{471} - \frac{1}{8}F_{472}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} u_{k,m} + F_{47}\sum_{k=1}^{N_{x_2}} c_{jk}^{(1)} u_{i,k}$$

$$+\left(-\frac{8}{15}F_{471} + \frac{1}{8}F_{472}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(3)} v_{k,j} + \left(-F_{470} - \frac{2}{3}F_{471} + \frac{1}{4}F_{472}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} v_{k,m} + F_{47}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} v_{k,j}$$

$$+\left(-F_{440} + F_{480} - \frac{6}{5}F_{441} + \frac{6}{5}F_{481}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(3)} w_{k,m} + \left(-F_{440} + F_{480} - \frac{6}{5}F_{441} + \frac{6}{5}F_{481}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(3)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} w_{k,m}$$

$$+\left(2F_{44} - 2F_{48} + \frac{4}{15}F_{461} + \frac{8}{3}F_{551} - \frac{4}{3}F_{571} + \frac{1}{8}F_{462} + \frac{1}{2}F_{552} - F_{572}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} w_{k,m}$$

$$+\left(-\frac{8}{15}F_{441} - \frac{1}{4}F_{442}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(3)} \phi_{1i,k} + \left(-F_{440} - \frac{2}{3}F_{441} - \frac{1}{8}F_{442}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} \phi_{1k,m}$$

$$+\left(F_{44} + \frac{2}{15}F_{461} + \frac{4}{3}F_{551} + \frac{1}{4}F_{462} + F_{552}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(1)} \phi_{1i,k} + \left(-\frac{8}{15}F_{441} + \frac{1}{8}F_{442}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(3)} \phi_{2k,j}$$

$$+\left(-F_{440} - \frac{2}{3}F_{441} + \frac{1}{4}F_{442}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{2k,m} + \left(F_{44} + \frac{2}{15}F_{461} + \frac{4}{3}F_{551} - \frac{1}{8}F_{462} - \frac{1}{2}F_{552}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{2k,j}\bigg) n_{x_2} = 0, \tag{147}$$

$$\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)}\phi_{1k,j} = 0 \quad \text{veya}$$

$$\Bigg(\Bigg(\left(2F_{470}+\frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)}u_{k,j} - \frac{2}{5}F_{471}\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)}u_{i,k} + \left(2F_{470}-\frac{4}{5}F_{471}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)}v_{k,m}$$

$$+\left(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(3)}w_{k,j} + \left(2F_{440}-2F_{480}-\frac{6}{5}F_{441}+\frac{6}{5}F_{481}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(2)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)}w_{k,m}$$

$$-\frac{2}{5}F_{461}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)}w_{k,j} + \left(2F_{440}+\frac{4}{5}F_{441}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)}\phi_{1k,j} - \frac{2}{5}F_{441}\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)}\phi_{1i,k} - \frac{2}{5}F_{461}\phi_{1i,j}$$

$$+\left(2F_{440}-\frac{4}{5}F_{441}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)}\phi_{2k,m}\Bigg)n_{x_1}$$

$$+\Bigg(\left(F_{470}+\frac{16}{15}F_{471}+\frac{1}{8}F_{472}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)}u_{k,m} + \left(\frac{8}{15}F_{471}-\frac{1}{8}F_{472}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)}v_{k,j}$$

$$+\left(F_{470}-\frac{2}{5}F_{471}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)}v_{i,k} + \left(F_{440}-F_{480}-\frac{2}{5}F_{441}+\frac{2}{5}F_{481}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(3)}w_{i,k}$$

$$+\left(F_{440}-F_{480}+\frac{8}{5}F_{441}-\frac{8}{5}F_{481}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)}w_{k,m} + \left(-\frac{2}{15}F_{461}+\frac{1}{8}F_{462}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(1)}w_{i,k}$$

$$+\left(F_{440}+\frac{16}{15}F_{441}+\frac{1}{8}F_{442}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)}\phi_{1k,m} + \left(\frac{8}{15}F_{441}-\frac{1}{8}F_{442}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)}\phi_{2k,j}$$

$$+\left(F_{440}-\frac{2}{5}F_{441}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)}\phi_{2i,k} + \left(-\frac{2}{15}F_{461}+\frac{1}{8}F_{462}\right)\phi_{2i,j}\Bigg)n_{x_2} = 0, \qquad (148)$$

$$\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} \phi_{1i,k} = 0 \quad \text{veya}$$

$$\Bigg(\Bigg(F_{470} + \frac{16}{15}F_{471} + \frac{1}{8}F_{472}\Bigg)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} u_{k,m} + \Bigg(\frac{8}{15}F_{471} - \frac{1}{8}F_{472}\Bigg)\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} v_{k,j} + \Bigg(F_{470} - \frac{2}{5}F_{471}\Bigg)\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} v_{i,k}$$

$$+\Bigg(F_{440} - F_{480} + \frac{8}{5}F_{441} - \frac{8}{5}F_{481}\Bigg)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} w_{k,m} + \Bigg(F_{440} - F_{480} - \frac{2}{5}F_{441} + \frac{2}{5}F_{481}\Bigg)\sum_{k=1}^{N_{x_2}} c_{jk}^{(3)} w_{i,k}$$

$$+\Bigg(-\frac{2}{15}F_{461} + \frac{1}{8}F_{462}\Bigg)\sum_{k=1}^{N_{x_2}} c_{jk}^{(1)} w_{i,k} + \Bigg(F_{440} + \frac{16}{15}F_{441} + \frac{1}{8}F_{442}\Bigg)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{1k,m}$$

$$+\Bigg(\frac{8}{15}F_{441} - \frac{1}{8}F_{442}\Bigg)\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} \phi_{2k,j} + \Bigg(F_{440} - \frac{2}{5}F_{441}\Bigg)\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} \phi_{2i,k} + \Bigg(-\frac{2}{15}F_{461} + \frac{1}{8}F_{462}\Bigg)\phi_{2i,j}\Bigg) n_{x_1}$$

$$+\Bigg(-\frac{2}{5}F_{471}\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} u_{k,j} + \Bigg(\frac{8}{15}F_{471} + \frac{1}{4}F_{472}\Bigg)\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} u_{i,k} + \Bigg(\frac{16}{15}F_{471} - \frac{1}{4}F_{472}\Bigg)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} v_{k,m}$$

$$+\Bigg(-\frac{2}{5}F_{441} + \frac{2}{5}F_{481}\Bigg)\sum_{k=1}^{N_{x_1}} c_{ik}^{(3)} w_{k,j} + \Bigg(\frac{8}{5}F_{441} - \frac{8}{5}F_{481}\Bigg)\sum_{m=1}^{N_{x_2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} w_{k,m}$$

$$+\Bigg(-\frac{2}{15}F_{461} - \frac{1}{4}F_{462}\Bigg)\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} w_{k,j} - \frac{2}{5}F_{441}\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} \phi_{1k,j} + \Bigg(\frac{8}{15}F_{441} + \frac{1}{4}F_{442}\Bigg)\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} \phi_{1i,k}$$

$$+\Bigg(-\frac{2}{15}F_{461} - \frac{1}{4}F_{462}\Bigg)\phi_{1i,j}$$

$$+\Bigg(\frac{16}{15}F_{441} - \frac{1}{4}F_{442}\Bigg)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{2k,m}\Bigg) n_{x_2} = 0, \tag{149}$$

$\phi_{2i,j} = 0$ veya

$$\Bigg(\Bigg(\left(-\frac{8}{15}F_{471}+\frac{1}{8}F_{472}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(3)} u_{i,k} + \left(-F_{470}-\frac{2}{3}F_{471}+\frac{1}{4}F_{472}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} u_{k,m} + F_{47}\sum_{k=1}^{N_{x_2}} c_{jk}^{(1)} u_{i,k}$$

$$+\left(-\frac{8}{15}F_{471}-\frac{1}{4}F_{472}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(3)} v_{k,j} + \left(-F_{470}-\frac{2}{3}F_{471}-\frac{1}{8}F_{472}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} v_{k,m} + F_{47}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} v_{k,j}$$

$$+\left(-F_{440}+F_{480}-\frac{6}{5}F_{441}+\frac{6}{5}F_{481}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(3)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} w_{k,m} + \left(-F_{440}+F_{480}-\frac{6}{5}F_{441}+\frac{6}{5}F_{481}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(3)} w_{k,m}$$

$$+\left(2F_{44}-2F_{48}+\frac{4}{15}F_{461}+\frac{8}{3}F_{551}-\frac{4}{3}F_{571}+\frac{1}{8}F_{462}+\frac{1}{2}F_{552}-F_{572}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} w_{k,m}$$

$$+\left(-\frac{8}{15}F_{441}+\frac{1}{8}F_{442}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(3)} \phi_{1i,k} + \left(-F_{440}-\frac{2}{3}F_{441}+\frac{1}{4}F_{442}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} \phi_{1k,m}$$

$$+\left(F_{44}+\frac{2}{15}F_{461}+\frac{4}{3}F_{551}-\frac{1}{8}F_{462}-\frac{1}{2}F_{552}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(1)} \phi_{1i,k} + \left(-\frac{8}{15}F_{441}-\frac{1}{4}F_{442}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(3)} \phi_{2k,j}$$

$$+\left(-F_{440}-\frac{2}{3}F_{441}-\frac{1}{8}F_{442}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{2k,m} + \left(F_{44}+\frac{2}{15}F_{461}+\frac{4}{3}F_{551}+\frac{1}{4}F_{462}+F_{552}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{2k,j}\Bigg) n_{x_1}$$

$$+\Bigg(\left(-2F_{470}+\frac{4}{15}F_{471}+\frac{1}{8}F_{472}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} u_{k,m} + \left(-F_{470}+\frac{2}{5}F_{471}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(3)} u_{k,j} + F_{L11}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} u_{k,j}$$

$$+\left(-2F_{470}-\frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(3)} v_{i,k} + \left(-F_{470}-\frac{2}{3}F_{471}-\frac{1}{8}F_{472}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} v_{k,m} + F_{11}\sum_{k=1}^{N_{x_2}} c_{jk}^{(1)} v_{i,k}$$

$$+\left(-2F_{440}+2F_{480}-\frac{4}{5}F_{441}+\frac{4}{5}F_{481}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(4)} w_{i,k} + \left(-3F_{440}+3F_{480}-\frac{2}{5}F_{441}+\frac{2}{5}F_{481}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} w_{k,m}$$

$$+\left(-F_{440}+F_{480}+\frac{2}{5}F_{441}-\frac{2}{5}F_{481}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(4)} w_{k,j}$$

$$+\left(-F_{22}+F_{33}+2F_{550}-2F_{570}+\frac{2}{5}F_{461}+\frac{32}{15}F_{551}-\frac{16}{15}F_{571}+\frac{1}{4}F_{552}-\frac{1}{2}F_{572}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} w_{i,k}$$

$$+\left(-F_{L22}+F_{L33}+2F_{550}-2F_{570}+\frac{2}{15}F_{461}-\frac{8}{15}F_{551}+\frac{4}{15}F_{571}-\frac{1}{8}F_{462}-\frac{1}{4}F_{552}+\frac{1}{2}F_{572}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} w_{k,j}$$

$$+\left(-F_{440}+\frac{2}{5}F_{441}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(3)} \phi_{1k,j} + \left(-2F_{440}+\frac{4}{15}F_{441}+\frac{1}{8}F_{442}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{1k,m}$$

$$+\left(F_{L33}+2F_{550}+\frac{2}{15}F_{461}-\frac{8}{15}F_{551}-\frac{1}{8}F_{462}-\frac{1}{4}F_{552}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{1k,j}$$

$$+\left(-2F_{440}-\frac{4}{5}F_{441}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(3)} \phi_{2i,k} + \left(-F_{440}-\frac{2}{3}F_{441}-\frac{1}{8}F_{442}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} \phi_{2k,m}$$

$$+\left(F_{33}+2F_{550}+\frac{2}{5}F_{461}+\frac{32}{15}F_{551}+\frac{1}{4}F_{552}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(1)} \phi_{2i,k} - F_{T11}\Delta T_{i,j}\Bigg) n_{x_2} = 0, \qquad (150)$$

$$\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{2k,j} = 0 \quad \text{veya}$$

$$\left(\left(\left(\frac{16}{15} F_{471} - \frac{1}{4} F_{472} \right) \sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} u_{k,m} + \left(\frac{8}{15} F_{471} + \frac{1}{4} F_{472} \right) \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} v_{k,j} - \frac{2}{5} F_{471} \sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} v_{i,k} \right.\right.$$

$$+ \left(-\frac{2}{5} F_{441} + \frac{2}{5} F_{481} \right) \sum_{k=1}^{N_{x_2}} c_{jk}^{(3)} w_{i,k} + \left(\frac{8}{5} F_{441} - \frac{8}{5} F_{481} \right) \sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} w_{k,m} + \left(-\frac{2}{15} F_{461} - \frac{1}{4} F_{462} \right) \sum_{k=1}^{N_{x_2}} c_{jk}^{(1)} w_{i,k}$$

$$+ \left(\frac{16}{15} F_{441} - \frac{1}{4} F_{442} \right) \sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{1k,m}$$

$$\left. + \left(\frac{8}{15} F_{441} + \frac{1}{4} F_{442} \right) \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} \phi_{2k,j} - \frac{2}{5} F_{441} \sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} \phi_{2i,k} + \left(-\frac{2}{15} F_{461} - \frac{1}{4} F_{462} \right) \phi_{2i,j} \right) n_{x_1}$$

$$+ \left(\left(F_{470} - \frac{2}{5} F_{471} \right) \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} u_{k,j} + \left(\frac{8}{15} F_{471} - \frac{1}{8} F_{472} \right) \sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} u_{i,k} + \left(F_{470} + \frac{16}{15} F_{471} + \frac{1}{8} F_{472} \right) \sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} v_{k,m} \right.$$

$$+ \left(F_{440} - F_{480} - \frac{2}{5} F_{441} + \frac{2}{5} F_{481} \right) \sum_{k=1}^{N_{x_1}} c_{ik}^{(3)} w_{k,j} + \left(F_{440} - F_{480} + \frac{8}{5} F_{441} - \frac{8}{5} F_{481} \right) \sum_{m=1}^{N_{x_2}} c_{jm}^{(2)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} w_{k,m}$$

$$+ \left(-\frac{2}{15} F_{461} + \frac{1}{8} F_{462} \right) \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} w_{k,j} + \left(F_{440} - \frac{2}{5} F_{441} \right) \sum_{k=1}^{N_{x_1}} c_{ik}^{(2)} \phi_{1k,j} + \left(\frac{8}{15} F_{441} - \frac{1}{8} F_{442} \right) \sum_{k=1}^{N_{x_2}} c_{jk}^{(2)} \phi_{1i,k}$$

$$\left. + \left(-\frac{2}{15} F_{461} + \frac{1}{8} F_{462} \right) \phi_{1i,j} + \left(F_{440} + \frac{16}{15} F_{441} + \frac{1}{8} F_{442} \right) \sum_{m=1}^{N_{x_2}} c_{jm}^{(1)} \sum_{k=1}^{N_{x_1}} c_{ik}^{(1)} \phi_{2k,m} \right) n_{x_2} = 0, \tag{151}$$

$$\sum_{k=1}^{N_{x_2}} c_{jk}^{(1)}\phi_{2i,k} = 0 \quad \text{veya}$$

$$\Bigg(\Bigg(\left(F_{470}-\frac{2}{5}F_{471}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)}u_{k,j} + \left(\frac{8}{15}F_{471}-\frac{1}{8}F_{472}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)}u_{i,k} + \left(F_{470}+\frac{16}{15}F_{471}+\frac{1}{8}F_{472}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)}v_{k,m}$$

$$+\left(F_{440}-F_{480}-\frac{2}{5}F_{441}+\frac{2}{5}F_{481}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(3)}w_{k,j} + \left(F_{440}-F_{480}+\frac{8}{5}F_{441}-\frac{8}{5}F_{481}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(2)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)}w_{k,m}$$

$$+\left(-\frac{2}{15}F_{461}+\frac{1}{8}F_{462}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)}w_{k,j} + \left(F_{440}-\frac{2}{5}F_{441}\right)\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)}\phi_{1k,j} + \left(\frac{8}{15}F_{441}-\frac{1}{8}F_{442}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)}\phi_{1i,k}$$

$$+\left(-\frac{2}{15}F_{461}+\frac{1}{8}F_{462}\right)\phi_{1i,j} + \left(F_{440}+\frac{16}{15}F_{441}+\frac{1}{8}F_{442}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)}\phi_{2k,m}\Bigg)n_{x_1}$$

$$+\Bigg(\left(2F_{470}-\frac{4}{15}F_{471}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)}u_{k,m} - \frac{2}{5}F_{471}\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)}v_{k,j} + \left(2F_{470}+\frac{4}{5}F_{471}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)}v_{i,k}$$

$$+\left(2F_{440}-2F_{480}+\frac{4}{5}F_{441}-\frac{4}{5}F_{481}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(3)}w_{i,k} + \left(2F_{440}-2F_{480}-\frac{6}{5}F_{441}+\frac{6}{5}F_{481}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)}w_{k,m}$$

$$-\frac{2}{5}F_{461}\sum_{k=1}^{N_{x_2}} c_{jk}^{(1)}w_{i,k} + \left(2F_{440}-\frac{4}{5}F_{441}\right)\sum_{m=1}^{N_{x_2}} c_{jm}^{(1)}\sum_{k=1}^{N_{x_1}} c_{ik}^{(1)}\phi_{1k,m}$$

$$-\frac{2}{5}F_{441}\sum_{k=1}^{N_{x_1}} c_{ik}^{(2)}\phi_{2k,j} + \left(2F_{440}+\frac{4}{5}F_{441}\right)\sum_{k=1}^{N_{x_2}} c_{jk}^{(2)}\phi_{2i,k} - \frac{2}{5}F_{461}\phi_{2i,j}\Bigg)n_{x_2} = 0. \qquad (152)$$

5. SAYISAL ÇÖZÜM YÖNTEMLERİ

Diferansiyel kare yapma metodu uygulanırken, dikdörtgen mikro-plak düzlemi Şekil 2'de gösterildiği gibi alanlara bölünmektedir. Burada N_{x_1} ve N_{x_2} sırasıyla x_1 ve x_2 yönlerindeki nokta sayılarıdır. Şekilde (i,j) ile belirlenen genel bir *p* noktası $(i-1)\times N_{x_2}+j$ olarak numaralandırılmıştır. Toplam nokta sayısı $N_{x_1}\times N_{x_2}$ olarak ifade edilmektedir. Genel formülasyonda her düğümde 5 bilinmeyen bulunmaktadır. Bu bilinmeyenler u,v,w,ϕ_1,ϕ_2 ile gösterilmiştir. Bilinmeyen yerdeğiştirme vektörü **d** aşağıdaki gibi tanımlanmıştır:

$$\mathbf{d}=\left\{\{u_p\}^{\mathrm{T}},\ \{v_p\}^{\mathrm{T}},\ \{w_p\}^{\mathrm{T}},\ \{\phi_{1p}\}^{\mathrm{T}},\ \{\phi_{2p}\}^{\mathrm{T}}\right\}^{\mathrm{T}},\quad p=1,\ 2,\ ...,\ N_{x_1}\times N_{x_2}\ \text{için.} \tag{153}$$

$\{u_p\}$, $\{v_p\}$, $\{w_p\}$, $\{\phi_{1p}\}$, $\{\phi_{2p}\}$ sırasıyla u,v,w,ϕ_1,ϕ_2 bilinmeyenlerini içeren vektörlerdir. Örneğin $\{w_p\}$

$$\{w_p\}^{\mathrm{T}}=\begin{bmatrix} w_{1,1} \\ : \\ w_{1,N_{x2}} \\ w_{2,1} \\ : \\ w_{2,N_{x2}} \\ : \\ w_{N_{x1},1} \\ : \\ w_{N_{x1},N_{x2}} \end{bmatrix}=\begin{bmatrix} w_1 \\ : \\ w_{N_{x2}} \\ w_{N_{x2}+1} \\ : \\ w_{2N_{x2}} \\ : \\ w_{(N_{x1}-1)N_{x2}+1} \\ : \\ w_{N_{x1}N_{x2}} \end{bmatrix},\quad p=1,\ 2,\ ...,\ N_{x_1}\times N_{x_2}\ \text{için,} \tag{154}$$

formundadır. Yüksek derece plak ve Mindlin plak teorileri için denklem (153)'deki yerdeğiştirme vektörünün boyutu $5\times N_{x_1}\times N_{x_2}$'dir. Ancak Kirchhoff plak teorisinde ϕ_1 ve ϕ_2 mevcut değildir ve dolayısıyla denklem sayısı ve yerdeğiştirme vektörünün boyutu $3\times N_{x_1}\times N_{x_2}$'dir. Yukarıdaki vektörler ve numaralandırma metodu kullanılarak seri formunda olan hareket denklemleri ve sınır koşulları matris şeklinde ifade edilebilir. Sınırdaki noktalar için ($i=1$ veya $j=1$ veya $i=N_{x_1}$ veya $j=N_{x_2}$) türetilmiş olan sınır koşulları uygulanmalıdır. Bir değişken için birden fazla sınır koşulu varsa, sınırdan bir sonraki noktalar ($i=2$ veya $j=2$ veya $i=N_{x_1}-1$ veya $j=N_{x_2}-1$) için hareket denklemi yerine bu ekstra sınır şartının yazılması gerekmektedir.

5.1. Statik Eğilme

Statik eğilme probleminde, diferansiyel denklemlerde zamana bağlı kısmi türevler sıfır olarak alınmaktadır. Bu işlemden sonra, denklemlerin seri formu kullanılarak, hareket denklemleri aşağıdaki gibi ifade edilmiştir:

$$\mathbf{D_b d^e} + \mathbf{D_d d^i} + \mathbf{Q} = \mathbf{0}. \tag{155}$$

Burada $\mathbf{d^e}$ ve $\mathbf{d^i}$ sırasıyla kenar ve iç noktaların yerdeğiştirme vektörüdür. $\mathbf{D_b}$ ve $\mathbf{D_d}$ sırasıyla kenar ve iç noktaların katsayı matrisleridir; ve $\mathbf{Q}$ yayılı yük vektörünü temsil etmektedir. Sınır koşulları ise

$$\mathbf{B_b d^e} + \mathbf{B_d d^i} = \mathbf{0}, \tag{156}$$

formunda yazılmıştır. $\mathbf{B_b}$ ve $\mathbf{B_d}$ katsayı matrisleri sırasıyla kenar ve iç noktalara göre elde edilirler. Son olarak denklem (156) kullanılarak aşağıdaki lineer denklem sistemi elde edilmiştir:

$$\mathbf{K d^i} + \mathbf{Q} = \mathbf{0}. \tag{157}$$

Burada direngenlik matrisi $\mathbf{K}$

$$\mathbf{K} = -\mathbf{D_b B_b^{-1} B_d} + \mathbf{D_d}, \tag{158}$$

olarak yazılır.

5.2. Serbest Titreşim

Serbest titreşim probleminde mikro-plağa uygulanan tüm dış kuvvetler sıfırdır. Bu durumda dinamik yerdeğiştirme vektörü aşağıdaki gibi tanımlanabilir:

$$\mathbf{d} = \mathbf{d}^{*} \mathrm{e}^{\mathrm{i}\omega t}. \qquad (159)$$

Bu denklemde ω doğal frekansı temsil etmekte; $\mathbf{d}^{*}$ ise titreşim mod şeklini göstermektedir ve aşağıdaki gibi ifade edilir:

$$\mathbf{d}^{*} = \left\{ \left\{ u_p^{\ *} \right\}^{\mathrm{T}},\ \left\{ v_p^{\ *} \right\}^{\mathrm{T}},\ \left\{ w_p^{\ *} \right\}^{\mathrm{T}},\ \left\{ \phi_{1p}^{\ *} \right\}^{\mathrm{T}},\ \left\{ \phi_{2p}^{\ *} \right\}^{\mathrm{T}} \right\}^{\mathrm{T}},\ \ p = 1,\ 2,\ ...,\ N_{x_1} \times N_{x_2}\ \text{için.} \qquad (160)$$

Denklem (159)'daki dinamik yerdeğiştirme vektörü hareket denklemlerinde kullanılarak

$$\mathbf{D_b d^{*e}} + \mathbf{D_d d^{*i}} - \omega^2 \mathbf{M d^{*i}} = \mathbf{0}, \qquad (161)$$

formunda bir denklem sistemi bulunmuştur. Bu denklemde mod şekli $\mathbf{d}^{*}$ kenar ve iç noktalar için sırasıyla $\mathbf{d}^{*\mathbf{e}}$ ve $\mathbf{d}^{*\mathbf{i}}$ alt vektörlerine ayrılmıştır. $\mathbf{D_b}$ ve $\mathbf{D_d}$ kenar ve iç noktaların katsayı matrisleridir; ve $\mathbf{M}$ atalet terimlerinden türetilen kütle matrisidir. Benzer biçimde sınır koşulları

$$\mathbf{B_b d^{*e}} + \mathbf{B_d d^{*i}} = \mathbf{0}, \qquad (162)$$

şeklinde ifade edilir. $\mathbf{B_b}$ ve $\mathbf{B_d}$ statik eğilme problemindeki gibi tanımlanmıştır. (161) ve (162) numaralı denklemler kullanılarak mikro-plağın serbest titreşimini temsil eden özdeğer problemi

$$\left\{ \mathbf{K} - \omega^2 \mathbf{M} \right\} \mathbf{d}^{*\mathbf{i}} = \mathbf{0}, \qquad (163)$$

formunda türetilmiştir. Direngenlik matrisi $\mathbf{K}$ denklem (158) kullanılarak hesaplanmaktadır. Doğal frekanslar ve mod şekilleri denklem (163) çözülerek bulunmaktadır. Bu çözümden elde edilen frekans beş yerdeğiştirme modundan herhangi birine ait olabilir. Bu titreşim biçimleri u ve v için eksenel yerdeğiştirme, w için enine yerdeğiştirme ve ϕ_1 ve ϕ_2 için dönme olarak tanımlanmaktadır. Hesaplanan doğal frekansların yerdeğiştirme modları, mod şekilleri incelenerek belirlenmiştir.

5.3. Burkulma

Burkulma probleminde diferansiyel denklemlerde zaman cinsinden kısmi türevler sıfırdır ve dış yük olarak sadece x_1 ve x_2 yönlerindeki düzlem içi basma kuvvetleri P_{x_1} ve P_{x_2} uygulanır. Burkulma problemini matris formunda ifade ederken serbest titreşim probleminde uygulanan prosedüre benzer bir prosedür takip edilmiştir. Kütle matrisi $\mathbf{M}$ yerine $\frac{\partial^2 w}{\partial x_1^2}$ ve $\frac{\partial^2 w}{\partial x_2^2}$ terimlerinden elde edilen katsayı matrisi $\mathbf{X}$ kullanılmıştır. Böylece mikro-plağın burkulmasını temsil eden özdeğer problemi şu formda elde edilmiştir:

$$\{\mathbf{K} - P\mathbf{X}\}\ \mathbf{d}_{\mathbf{b}}^{*\mathbf{i}} = \mathbf{0}. \qquad (164)$$

Bu eşitlikte *P* kritik burkulma kuvvetini, $\mathbf{d}_{\mathbf{b}}^{*\mathbf{i}}$ ise mod şeklini simgelemektedir.

5.4. Termal Etkiler

Micro-plakların termal yük altında mekanik davranışı mesnet türüne göre iki farklı grupta incelenebilir: Yanal yönde kısıtlı olmayan (hareket eden) ve yanal yönde kısıtlı olan (hareket etmeyen) mikro-plaklar. Hesaplamalarda sıcaklığın sadece kalınlık koordinatı boyunca değiştiği varsayılmıştır. Termal yükler altında olan bir mikro-plağın serbest titreşim analizinden önce sıcaklık değişiminden dolayı oluşan başlangıç düzlem içi termal yüklerin ($P^0_{x_1}$, $P^0_{x_2}$ ve $P^0_{x_1x_2}$) belirlenmesi gerekmektedir. Bu kuvvetler statik termal eğilme probleminden elde edilir. Statik termal probleminin çözümünde atalet terimleri ve düzlem içi termal ve mekanik yükler gibi kuvvetler göz önüne alınmaz. Bu problemden hesaplanan yerdeğiştirmeler, M^0_{11}, M^0_{22} ve M^0_{12} ifadelerinde yerlerine konulduklarında sırasıyla $P^0_{x_1}$, $P^0_{x_2}$ ve $P^0_{x_1x_2}$ 'i vermektedir.

Yanal kısıtlaması olmayan mikro-plaklarda kenarların serbestçe hareket edebilmesinden dolayı termal yüklerin bir kısmı termal gerinimin oluşmasına yol açmaktadır. Yanal hareketleri kısıtlı olan mikro-plaklarda ise, sıcaklık değişiminden dolayı başlangıç yerdeğiştirme oluşmamaktadır. Dolayısıyla düzlem içi kuvvetler aşağıdaki gibi elde edilir (Ansari vd., 2013; Javaheri ve Eslami, 2002):

$$P_{x_1}^0 = P_{x_2}^0 = -\int_{-h/2}^{h/2} \frac{E(x_3)\alpha(x_3)\Delta T(x_3)}{1-\nu(x_3)} dx_3, \quad P_{x_1x_2}^0 = 0. \tag{165}$$

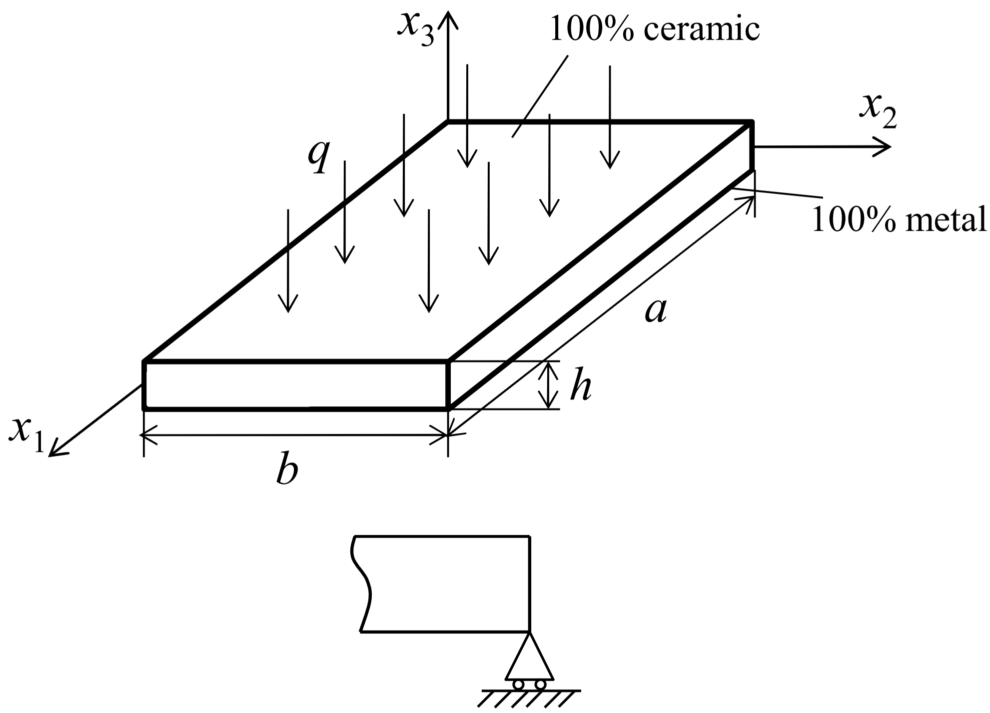
Düzgün sıcaklık değişimi uygulanması durumunda ΔT sabittir. Isı iletimi göz önüne alındığı takdirde is sıcaklık farkı

$$\Delta T(x_3) = \frac{(\Delta T_c - \Delta T_m)}{\int_{-h/2}^{h/2} \frac{dx_3}{K(x_3)}} \int_{-h/2}^{x_3} \frac{dx_3}{K(x_3)} + \Delta T_m. \tag{166}$$

integrali ile hesaplanır. Bu denklemde K malzemenin ısıl iletkenliğini simgelemektedir. *c* ve *m* alt indisleri sırasıyla seramik ve metal fazlarını göstermektedir. Düzlem içi termal yükler diğer düzlem içi kuvvetler, P_{x_1} ve P_{x_2}, ile toplanarak formülasyona girilmektedir. Bu durumda kritik burkulma kuvveti, serbest titreşim doğal frekansları, ve düzgün yayılı kuvvetin altında statik eğilme sonuçları etkilenmektedir.

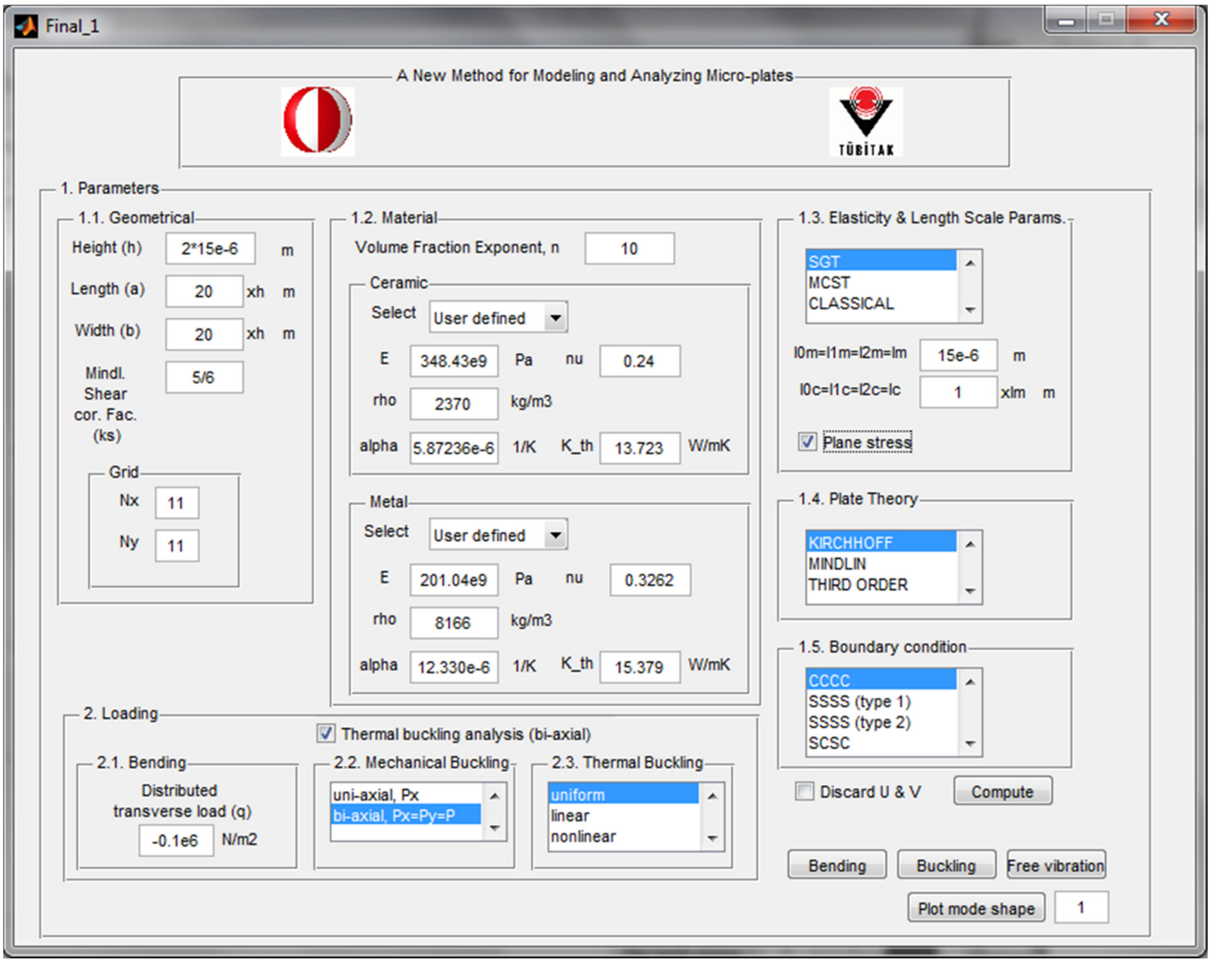
6. SAYISAL SONUÇLAR

Sayısal analizlerde göz önüne alınan dikdörtgen FDM mikro-plak konfigürasyonu ve basit mesnet geometrisi Şekil 3'de gösterilmektedir. Mikro-plağın dört kenarının da basit mesnetle desteklendiği varsayılmıştır. Statik eğilme probleminde plak şekilde *q* ile gösterilen düzgün yayılı yük ile yüklenmektedir. Serbest titreşim ve burkulma analizlerinde bu tür bir yükleme bulunmamaktadır.



Şekil 3. FDM dikdörtgen mikro-plak ve basit mesnet geometrisi.

Geliştirilen sayısal çözüm teknikleri MATLAB adlı yazılım içersine entegre edilmiş ve sayısal sonuçları hesaplamakta kullanılan genel bir bilgisayar programı geliştirilmiştir. Geliştirilen MATLAB programı kullanıcı arayüzü Şekil 4'de gösterilmektedir. Bu arayüz ile kullanıcı yapılacak olan analiz ile ilgili olarak istenilen seçimleri yapabilmektedir.



Şekil 4. MATLAB ile hazırlanan grafiksel kullanıcı arayüzü.

Yürüttüğümüz sayısal parametrik analizler iki ana bölümden oluşmaktadır. Öncelikle geliştirilen yöntemlerin doğrulamasını yapabilmek için literatürde bulunan verilerle MATLAB programı aracılığı ile hesapladığımız sonuçlar arasında karşılaştırmalar yapılmıştır. Bu karşılaştırmalar 6.1 numaralı bölümde sunulmaktadır. 6.2 numaralı bölümde ise çalışmamız kapsamında elde ettiğimiz yeni sayısal sonuçlara yer verilmektedir.

6.1. Model Doğrulama Çalışmaları

6.1.1. Statik Eğilme

Statik eğilme için yaptığımız model doğrulama çalışmasında Thai ve Choi (2013) tarafından verilen sonuçlar baz alınmıştır. Bu makaledeki sonuçlar modifiye edilmiş kuvvet çifti teorisi kullanılarak üretilmiştir. Bu durumda raporda da değinildiği gibi malzemenin tek bir uzunluk ölçeği parametresi bulunmaktadır. Bu parametre *l* ile gösterilmektedir. Çalışmamız kapsamında FDM bir mikro-plak için düzgün yükleme altında elde edilen boyutsuz orta nokta yerdeğiştirmeleri ile Thai ve Choi (2013) tarafından sunulan sonuçların karşılaştırmaları Tablo 1 ve 2'de sunulmuştur. Bu tablolarda verilen sonuçlar üretilirken Şekil 3'de gösterilen mesnet tipi kullanılmıştır. Tablo 1 değerleri Kirchhoff plak teorisine göre, Tablo 2'deki değerler ise Mindlin plak teorisine göre bulunmuştur.

Tablo 1. Boyutsuz orta nokta yerdeğiştirmesi $\bar{w}$, 13×13 ağ boyutu.

a/h	*l/h*	*n*=0		*n*=10	
		Thai ve Choi (2013)	Bu çalışma	Thai ve Choi (2013)	Bu çalışma
5	0	0.4171	0.4171	2.0905	2.0904
	0.2	0.3631	0.3631	1.8409	1.8408
	0.4	0.2615	0.2615	1.3553	1.3553
	0.6	0.1783	0.1783	0.9415	0.9415
	0.8	0.1234	0.1234	0.6595	0.6595
	1	0.0884	0.0884	0.4762	0.4762
10	0	0.4171	0.4171	2.0905	2.0904
	0.2	0.3631	0.3631	1.8409	1.8408
	0.4	0.2615	0.2615	1.3553	1.3553
	0.6	0.1783	0.1783	0.9415	0.9415
	0.8	0.1234	0.1234	0.6595	0.6595
	1	0.0884	0.0884	0.4762	0.4762

Tablo 2. Boyutsuz orta nokta yerdeğiştirmesi $\bar{w}$, 23×23 ağ boyutu.

a/h	*l/h*	*p*=0		*p*=10	
		Thai ve Choi (2013)	Bu çalışma	Thai ve Choi (2013)	Bu çalışma
5	0	0.5147	0.5147	2.6273	2.6273
	0.2	0.4479	0.4479	2.3127	2.3127
	0.4	0.3250	0.3250	1.7138	1.7138
	0.6	0.2268	0.2268	1.2163	1.2163
	0.8	0.1631	0.1631	0.8841	0.8841
	1	0.1230	0.1230	0.6710	0.6710
10	0	0.4415	0.4415	2.2247	2.2247
	0.2	0.3844	0.3843	1.9593	1.9591
	0.4	0.2775	0.2775	1.4461	1.4461
	0.6	0.1907	0.1907	1.0116	1.0116
	0.8	0.1335	0.1335	0.7171	0.7171
	1	0.0972	0.0972	0.5263	0.5263

Boyutsuz orta nokta yerdeğiştirmesi şu şekilde tanımlanmıştır

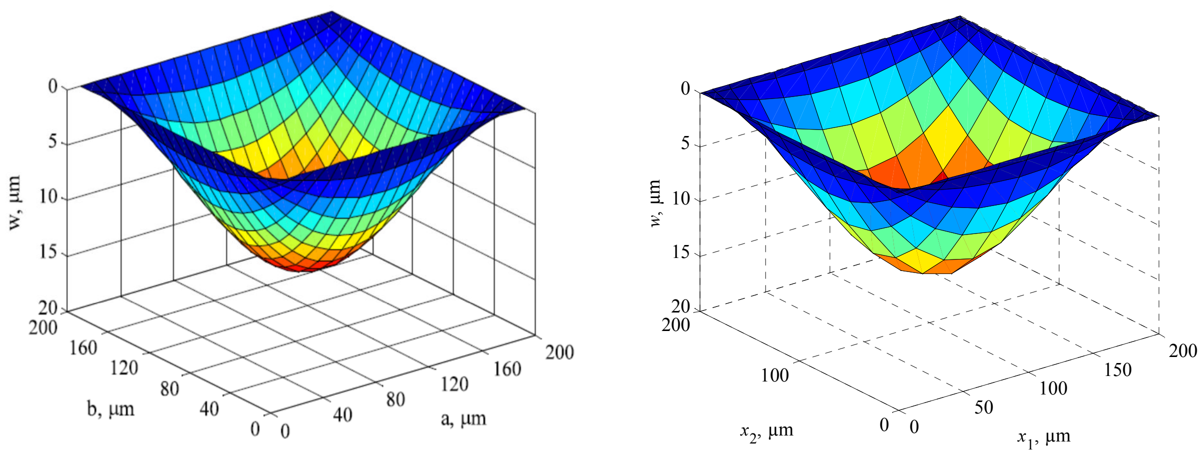
$$\bar{w} = w_{\max} \frac{100 E_2 h^3}{q a^4}. \qquad (167)$$

Bu sayısal sonuçlar üretilirken $h = 17.6 \times 10^{-6}$ m, $b = a$, $q = 1.0$ N/m^2 olarak alınmıştır. Kalınlık koordinatı boyunca FDM elastisite modülünün şu şekilde değiştiği varsayılmıştır:

$$E(z) = E_2 + (E_1 - E_2)\left(\frac{1}{2} + \frac{x_3}{h}\right)^n, \quad E_1 = 14.4 \text{ GPa}, \quad E_2 = 1.44 \text{ GPa}. \qquad (168)$$

Poisson oranının sabit olduğu kabul edilmiştir ve $\nu = 0.38$ olarak tanımlanmıştır. Tablo 1 ve 2'de verilen sonuçlar incelendiğinde araştırma projemiz kapsamında yapılan çalışmalarda elde edilen statik eğilme sonuçları ile Thai ve Choi (2013) tarafından verilen sonuçların çok iyi bir uyum içinde olduğu görülmektedir.

Statik eğilme için ikinci doğrulama çalışmasında Akgöz ve Civalek (2013) tarafından verilen sonuçlar kullanılmıştır. Bu makalede yükleme özellikleri ve geometrik özellikler $h = l$, $a / h = 50$, $b = a = 200$ µm, $q = 0.1$ N/m^2 şeklinde alınmıştır. Sonuçlar homojen bir mikro-plak için basit mesnet tipi göz önüne alınarak hesaplanmıştır. Malzeme olarak ise homojen epoksi ele alınmıştır. Plak teorisi olarak Kirchhoff plak teorisi kullanılmıştır. Şekil 5'de araştırma projemiz kapsamında hesaplanan mikro-plak deforme şekli Akgöz ve Civalek (2013) tarafından verilen sonuçla karşılaştırılmıştır. Bu karşılaştırmada da aradaki uyumun çok iyi olduğu görülmektedir.



(a) Akgöz ve Civalek (2013)   (b) Bu çalışma

Şekil 5. Deforme olmuş mikro-plak: (a) Akgöz ve Civalek, (2013); (b) bu çalışma.

6.1.2. Serbest Titreşim

Serbest titreşim için yapılan doğrulamalarda Thai ve Choi (2013) tarafından verilen sonuçlar kullanılmıştır. Bu kapsamda kullanılan malzeme özellikleri ve geometrik özellikler 6.1.1 numaralı bölümde sunulan özellikler ile aynıdır. Bu özelliklere ek olarak FDM mikro-plak yoğunluk değişimi

$$\rho(x_3) = \rho_2 + (\rho_1 - \rho_2)\left(\frac{1}{2} + \frac{x_3}{h}\right)^n, \quad \rho_1 = 12200 \text{ kg/m}^3, \quad \rho_2 = 12200 \text{ kg/m}^3, \qquad (169)$$

eşitlikleri ile tanımlanmıştır. Boyutsuz birinci ve ikinci doğal frekansların karşılaştırmaları Tablo 3 ve 4'de sunulmaktadır. Boyutsuz doğal frekans tanımı

$$\bar{\omega} = \omega \frac{a^2}{h}\sqrt{\frac{\rho_2}{E_2}}, \qquad (170)$$

şeklindedir. Tablo 3'de verilen sonuçların hesabında Kirchhoff plak teorisi, Tablo 4'de ise Mindlin plak teorisi kullanılmıştır. Formülasyonda modifiye edilmiş kuvvet çifti gerilmesi teorisi baz alınmıştır.

Tablo 3. Boyutsuz ilk iki doğal frekans $\bar{\omega}$, 17×17 ağ boyutu.

	l/h	*n*=0		*n*=10	
		Thai ve Choi (2013)	Bu çalışma	Thai ve Choi (2013)	Bu çalışma
$\bar{\omega}_1$	0	6.1103	6.1103	6.3958	6.3958
	0.2	6.5491	6.5491	6.8156	6.8156
	0.4	7.7174	7.7174	7.9431	7.9431
	0.6	9.3453	9.3453	9.5303	9.5303
	0.8	11.2349	11.2349	11.3866	11.3866
	1	13.2749	13.2749	13.4006	13.4006
$\bar{\omega}_2$	0	15.0936	15.0936	15.7809	15.7810
	0.2	16.1776	16.1776	16.8169	16.8169
	0.4	19.0634	19.0634*	19.5989	19.5989
	0.6	23.0848	23.0848**	23.5151	23.5151
	0.8	27.7525	27.7525	28.0953	28.0953
	1	32.7917	32.7917	33.0646	33.0646

*19×19 ağ kullanıldı

**21×21 ağ kullanıldı

Tablo 4. Boyutsuz ilk iki doğal frekans $\bar{\omega}$, 15×15 ağ boyutu.

	l/h	*n*=0		*n*=10	
		Thai ve Choi (2013)	Bu çalışma	Thai ve Choi (2013)	Bu çalışma
$\bar{\omega}_1$	0	5.9301	5.9301	6.1903	6.1903
	0.2	6.3559	6.3559	6.5967	6.5967
	0.4	7.4807	7.4807	7.6797	7.6797
	0.6	9.0261	9.0261	9.1829	9.1829
	0.8	10.7848	10.7848	10.9066	10.9066
	1	12.6360	12.6360	12.7303	12.7303
$\bar{\omega}_2$	0	14.0893	14.0893	14.6464	14.6464
	0.2	15.1064	15.1064	15.6144	15.6144
	0.4	17.7680	17.7680	18.1705	18.1705
	0.6	21.3648	21.3648	21.6607	21.6607
	0.8	25.3657	25.3657	25.5744	25.5744
	1	29.4588	29.4588	29.6009	29.6009

Araştırma projemiz kapsamında hesaplanan doğal frekansların Thai ve Choi (2013) tarafından verilen sonuçlarla çok bir iyi uyum içinde olduğu görülmektedir. Bu da serbest titreşim formülasyon ve modelleme yöntemlerimizin doğruluğunu kanıtlamaktadır.

6.1.3. Burkulma

Burkulma konusunda yaptığımız doğrulama çalışmasında Thai ve Choi (2013) tarafından verilen sayısal sonuçlar kullanılmıştır. Kritik burkulma kuvvetleri modifiye edilmiş kuvvet çifti gerilmesi teorisine göre hesaplanmıştır. Hesaplamalarda kullanılan malzeme özellikleri ve geometrik özellikler 6.1.1'de verilen özellikler ile aynıdır. Bu özelliklere ek olarak $a / h = 10$ olarak tanımlanmıştır. FDM mikro-plak için boyutsuz burkulma kuvveti,

$$\overline{P}=\frac{Pa^2}{E_2h^3}, \quad (171)$$

formundadır. Boyutsuz burkulma kuvveti ile ilgili karşılaştırmalar Tablo 5 ve 6'da sunulmaktadır. Tablo 5'de verilen sonuçlar Kirchhoff plak teorisi ile, Tablo 6 sonuçları ise Mindlin plak teorisi ile hesaplanmıştır. Sunulan sonuçlar farklı *n* ve *l/h* değerleri için aradaki uyumun çok iyi olduğunu göstermektedir. Dolayısıyla, bu karşılaştırmalar burkulma üzerine yaptığımız formülasyon ve sayısal çözüm çalışmalarımızı doğrulamaktadır.

Tablo 5. Boyutsuz burkulma kuvveti $\overline{P}$, 15×15 ağ boyutu.

Burkulma ekseni	*l/h*	*n*=0		*n*=10	
		Thai ve Choi (2013)	Bu çalışma	Thai ve Choi (2013)	Bu çalışma
Tek eksenli burkulma	0	38.4510	38.4511	7.6717	7.6718
	0.2	44.1725	44.1726	8.7120	8.7121
	0.4	61.3370	61.3371	11.8328	11.8329
	0.6	89.9446	89.9447	17.0342	17.0343
	0.8	129.9952	129.9952	24.3161	24.3162
	1	181.4888	181.4888	33.6786	33.6787
İki eksenli burkulma	0	19.2255	19.2255	3.8359	3.8359
	0.2	22.0863	22.0863	4.3560	4.3560
	0.4	30.6685	30.6686	5.9164	5.9164
	0.6	44.9723	44.9723	8.5171	8.5171
	0.8	64.9976	64.9976	12.1581	12.1581
	1	90.7444	90.7444	16.8393	16.8394

*19×19 ağ kullanıldı

**21×21 ağ kullanıldı

Tablo 6. Boyutsuz burkulma kuvveti $\overline{P}$, 13×13 ağ boyutu.

Burkulma ekseni	*l/h*	*n*=0		*n*=10	
		Thai ve Choi (2013)	Bu çalışma	Thai ve Choi (2013)	Bu çalışma
Tek eksenli burkulma	0	36.1492	36.1493	7.1707	7.1707
	0.2	41.5214	41.5214	8.1420	8.1420
	0.4	57.4956	57.4956	11.0303	11.0303
	0.6	83.6543	83.6543	15.7605	15.7606
	0.8	119.3314	119.3315	22.2129	22.2130
	1	163.6539	163.6539	30.2304	30.2305
İki eksenli burkulma	0	18.0746	18.0746	3.5854	3.5854
	0.2	20.7607	20.7607	4.0710	4.0710
	0.4	28.7478	28.7478	5.5151	5.5152
	0.6	41.8271	41.8272	7.8802	7.8803
	0.8	59.6657	59.6657	11.1065	11.1065
	1	81.8269	81.8270	15.1152	15.1153

6.2. Parametrik Analizler

Yürütülen sayısal analizlerde Şekil 3'de gösterilen mikro-plak konfigürasyonu göz önüne alınmıştır. Analizler en genel mikro-ölçek sürekli ortam teorisi olan gerinim gradyanı elastisite teorisine göre yürütülmüştür. Mikro-plak için malzeme özellikleri şu şekilde ifade edilmiştir:

$$E(x_3) = E_2 + (E_1 - E_2)\left(\frac{1}{2} + \frac{x_3}{h}\right)^n, \tag{172}$$

$$\rho(x_3) = \rho_2 + (\rho_1 - \rho_2)\left(\frac{1}{2} + \frac{x_3}{h}\right)^n, \tag{173}$$

$$\nu(x_3) = \nu_2 + (\nu_1 - \nu_2)\left(\frac{1}{2} + \frac{x_3}{h}\right)^n, \tag{174}$$

$$l_i(x_3) = l_{i_2} + (l_{i_1} - l_{i_2})\left(\frac{1}{2} + \frac{x_3}{h}\right)^n, \quad i = 0,1,2, \tag{175}$$

$$\alpha(x_3) = \alpha_2 + (\alpha_1 - \alpha_2)\left(\frac{1}{2} + \frac{x_3}{h}\right)^n. \tag{176}$$

Görülebileceği gibi uzunluk ölçeği parametreleri $l_i, \ i = 0,1,2,$ de dahil olmak üzere tüm malzeme özellikleri kalınlık koordinatı olan x_3'ün fonksiyonları olarak alınmıştır. Mikro-plağın $x_3 = -h/2$'de %100 metal $x_3 = h/2$'de ise %100 seramik olduğu varsayılmıştır. Seramik özelliklerini 1 numaralı alt indis, metal özelliklerini ise 2 numaralı alt indis temsil etmektedir. Metal bileşen olarak aluminyum (Al) seramik bileşen olarak ise silisyum karbür (SiC) kullanılmıştır. Bu malzemelerin özellikleri şu şekildedir (Eshraghi vd., 2016):

$$E_1 = 427 \text{ GPa}, \qquad E_2 = 70 \text{ GPa}, \tag{177}$$

$$\nu_1 = 0.17, \qquad \nu_2 = 0.3, \tag{178}$$

$$\rho_1 = 3100 \text{ kg/m}^3, \qquad \rho_2 = 2702 \text{ kg/m}^3, \tag{179}$$

$$\alpha_1 = 4.3(10)^{-6} \text{ 1/K}, \qquad \alpha_2 = 23(10)^{-6} \text{ 1/K}. \tag{180}$$

Metal ve seramiklerin uzunluk ölçeği parametreleri ile ilgili literatürde yeterli veri olmadığından, bu özellikler için yaklaşık değerler kullanılmıştır. Metal bileşenin uzunluk ölçeği parametreleri 10 μm olarak aşağıdaki gibi ifade edilmiştir:

$$l_{0_2} = l_{1_2} = l_{2_2} = l = 10 \ \mu\text{m}. \tag{181}$$

Bu değer literatürdeki çeşitli çalışmalarda kullanılmış olan bir referans değeridir (Akgöz ve Civalek, 2013). Seramik bileşenin uzunluk ölçeği parametreleri ise parametrik olarak

$$l_{0_1} = l_{1_1} = l_{2_1} = \beta l, \tag{182}$$

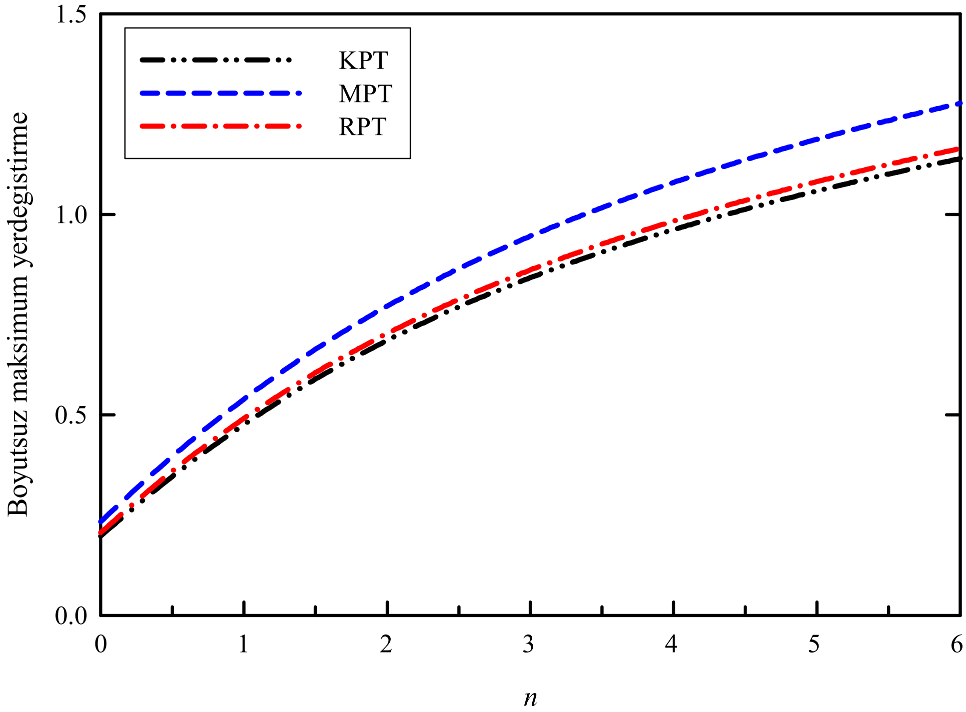
şeklinde ifade edilmiştir. Bu tanımdaki β faktörü aracılığı ile seramik bileşenin uzunluk ölçeği parametrelerini değiştirmek mümkün olmaktadır.

Sayısal sonuçlar statik eğilme, serbest titreşim, ve burkulma problemleri için üretilmiştir. Çeşitli analizlerde termal etkiler de göz önüne alınmıştır. Bu kapsamda bulduğumuz sonuçlar aşağıda sunulmaktadır.

6.2.1. Statik Eğilme

Statik eğilme kapsamında mekanik ve termal yüklemeler altında olan mikro-plaklar için analizler yürütülmüştür. Şekil 6'da q = 1 N/m^2 düzgün yayılı yükleme uygulanan bir mikro-plak için mikro-plağın orta noktasındaki boyutsuz maksimum yerdeğiştirme, malzeme özelliklerini tanımlayan üstel fonksiyonların üssü olan *n*'ye göre çizilmiştir. Buradaki *n* malzeme değişim profilini belirlemektedir. Örneğin *n* değerinin 0 ile 1 arasında olması durumunda malzeme değişim profili seramik-ağırlıklı, 1 ile sonsuz arasında olması durumunda ise metal-ağırlıklı olmaktadır. Sonuçlar üç farklı plak teorisi KPT (Kirchhoff plak teorisi), MPT (Mindlin plak teorisi), ve RPT (Reddy plak teorisi veya üçüncü derece plak teorisi) için hesaplanmıştır. Orta noktanın boyutsuz yerdeğiştirmesi şu şekilde tanımlanmıştır:

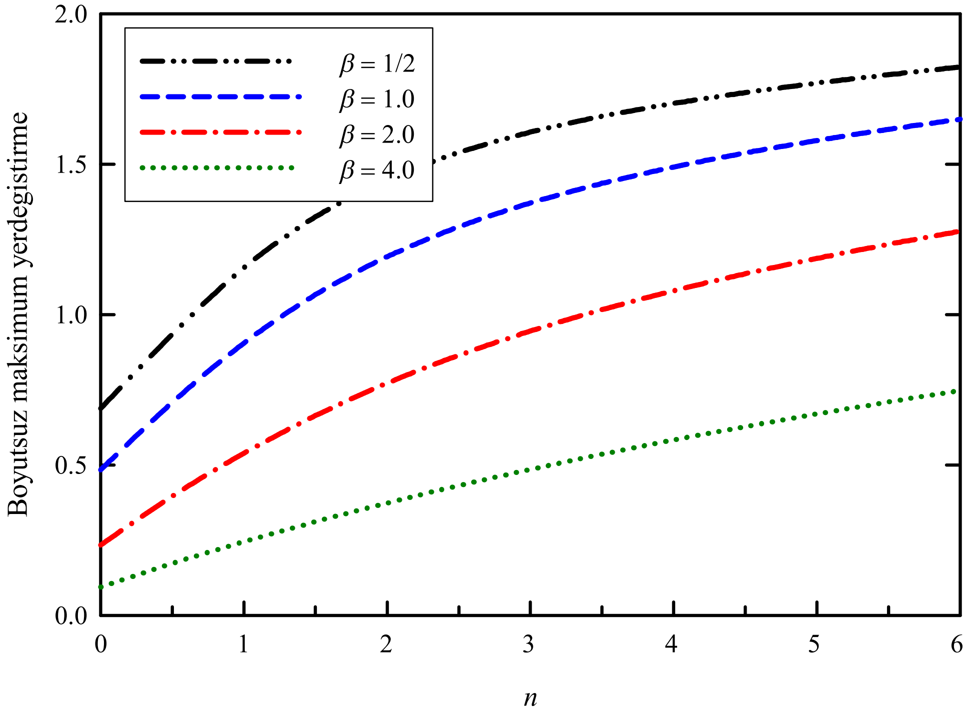
$$\bar{w} = \frac{100 E_2 h^3}{q a^4} w_{\max}. \tag{183}$$



Şekil 6. Boyutsuz orta nokta yerdeğiştirmesi, $\bar{w}$, l=10 µm, l/h=0.2, a/h=10, b/a=1.0, β=2.0, q=1 N/m^2, ΔT=0 K.

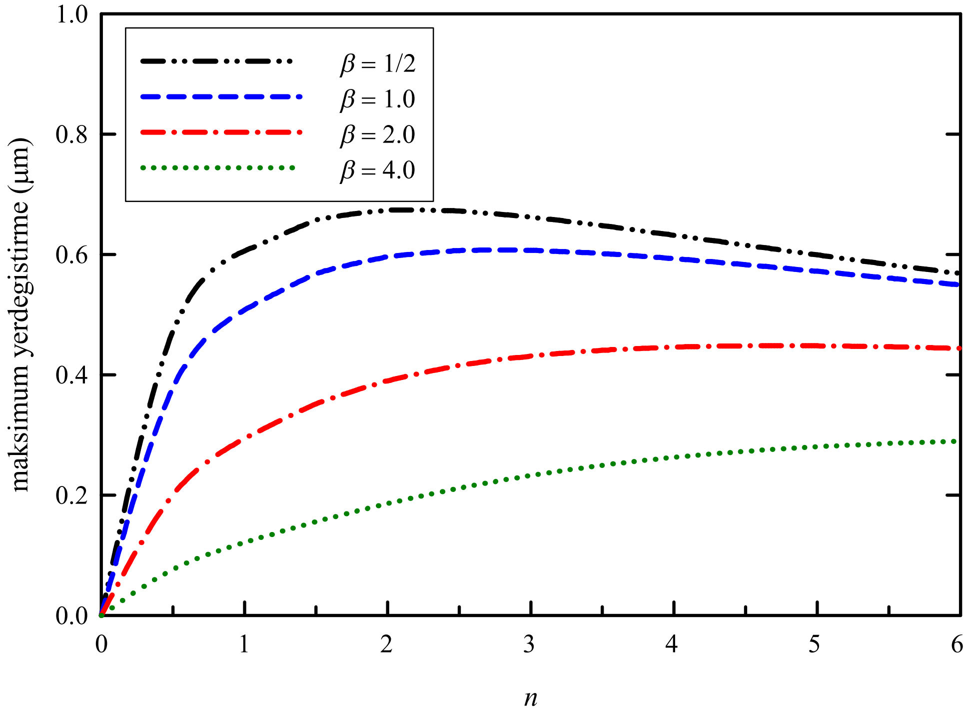
Üç farklı plak teorisi ile elde edilen sonuçlar genel olarak birbirine yakındır. Boyutsuz maksimum yerdeğiştirme n üssündeki artışla beraber artmaktadır. Bu durumda seramik-ağırlıklı mikro-plaklardaki boyutsuz yerdeğiştirme metal ağırlıklı mikro-plaklardakine göre daha düşük olacaktır.

Aynı mekanik yükleme için uzunluk ölçeği parametrelerindeki değişimin etkisi Şekil 7'de incelenmiştir. Bu şekilde boyutsuz maksimum yerdeğiştirme dört farklı β değeri için çizilmiştir. Hesaplamalarda Mindlin plak teorisi kullanılmıştır (MPT). β faktörü seramik uzunluğu ölçeği parametreleri ile metal uzunluk ölçeği parametreleri arasındaki oranı vermektedir. β değeri arttıkça boyutsuz yerdeğiştirmenin azalmakta olduğu görülmektedir. $\beta = 1$ olması durumunda mikro-plak içerisinde uzunluk ölçeği parametresi sabit olmaktadır. Ancak β'nın 1'den farklı olduğu durumlar için elde edilen sonuçlardaki farklılık, yeterli doğrulukta sonuç üretebilmek için uzunluk ölçeği parametrelerindeki değişimleri göz önüne almak gerektiğini kanıtlamaktadır. Bu araştırma projesi kapsamında ortaya koyduğumuz yöntemler ile uzunluk ölçeği parametrelerindeki uzaysal değişimler herhangi bir fonksiyon için göz önüne alınabilmektedir.

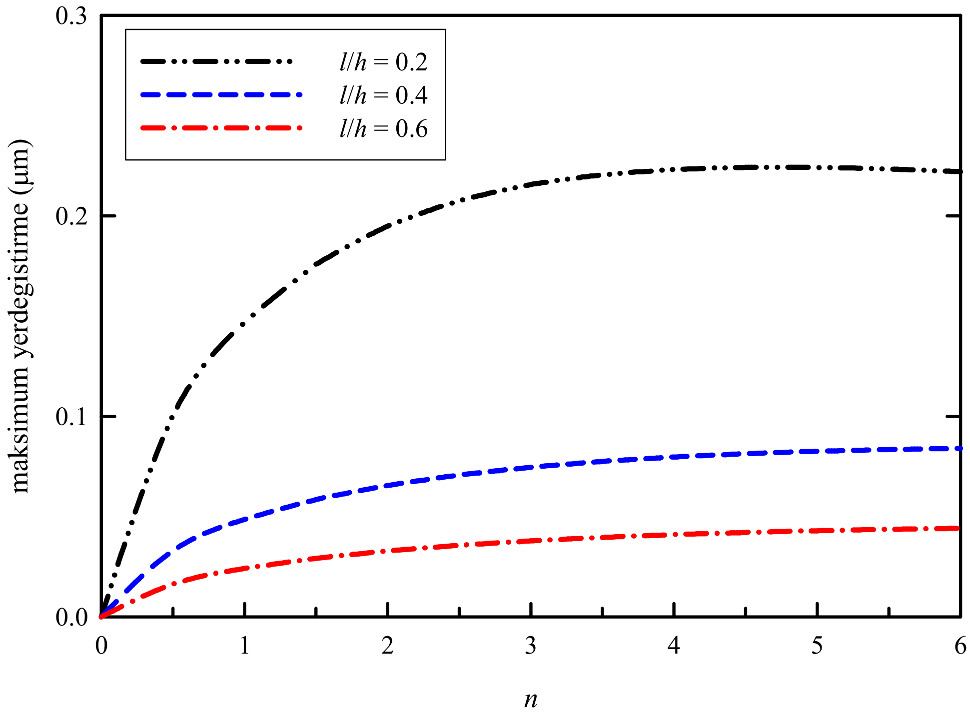


Şekil 7. Boyutsuz orta nokta yerdeğiştirmesi, $\bar{w}$, *l*=10 μm, *l*/*h*=0.2, *a*/*h*=10, *b*/*a*=1.0, *q*=1 N/m^2, Δ*T*=0 K.

Termal etki altında olan mikro-plaklar için orta nokta yerdeğiştirmeleri Şekil 8 ve 9'da sunulmaktadır. Bu şekillerde verilen sonuçlar Mindlin plak teorisine göre hesaplanmıştır, ve $\Delta T = 100$ K, $q = 0$ olarak tanımlanmıştır. Termal yükleme altında orta nokta deplasmanının *n*'deki artış ile birlikte önce artış gösterdiği ve daha sonra büyük *n* değerleri için sabit bir değere yakınsadığı görünmektedir. Şekil 8'de termal orta nokta yerdeğiştirmesi dört farklı β değeri için çizilmiştir. Uzunluk ölçeği parametresi oranı olan β arttıkça yerdeğiştirmede azalma meydana gelmektedir. 9. Şekil'de ise orta nokta deplasmanı üç farklı l/h değeri için sunulmuştur. l/h, uzunluk ölçeği parametresi *l*'nin plak kalınlığı *h*'ye oranıdır. Bu oran arttıkça, orta nokta deplasmanı azalmaktadır.



Şekil 8. Orta nokta yerdeğiştirmesi w, l=10 μm, l/h=0.2, a/h=10, b/a=1.0, q=0 N/m^2, ΔT=100 K.



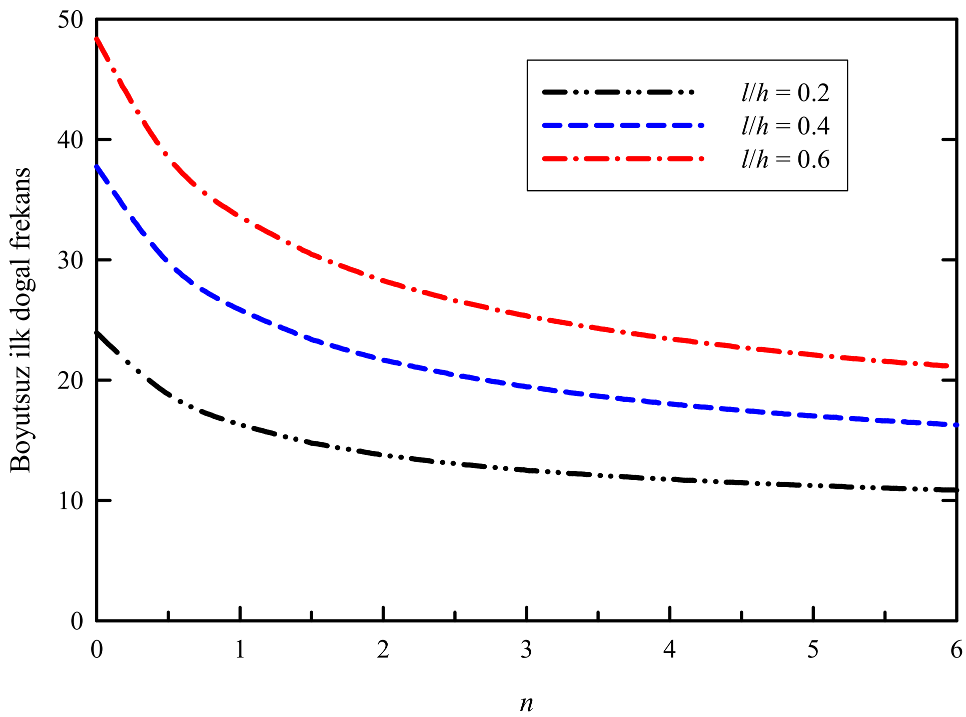
Şekil 9. Orta nokta yerdeğiştirmesi w, l=10 μm, a/h=10, b/a=1.0, β=2.0, q=0 N/m^2, ΔT=100 K.

6.2.2. Serbest titreşim

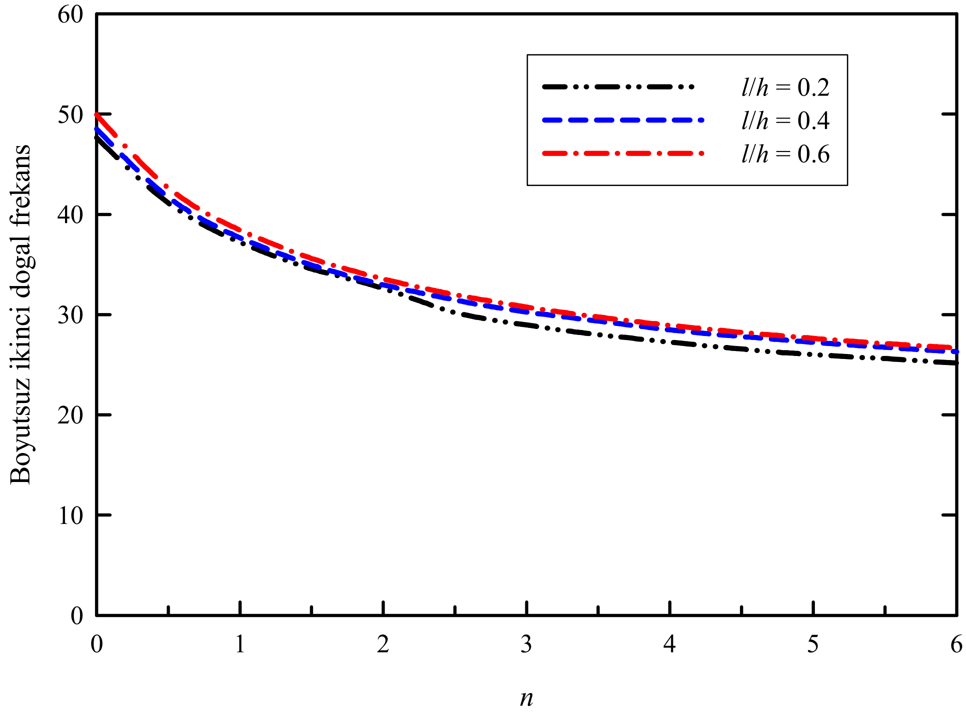
Gerinim gradyanı elastisite teorisini baz alarak, mikro-plakların serbest titreşimi üzerine elde ettiğimiz sonuçlar Şekil 10-13'de gösterilmektedir. Bu sonuçlar elde edilirken Mindlin plak teorisi kullanılmıştır. Şekillerde boyutsuz birinci ve ikinci serbest titreşim frekansları sunulmuştur. Boyutsuz frekanslar

$$\bar{\omega} = \frac{a^2}{h}\sqrt{\frac{\rho_2}{E_2}}\omega, \qquad (184)$$

şeklinde tanımlanmıştır. 10. ve 11. Figürlerde termal yüklemenin olmadığı durum için boyutsuz birinci ve ikinci doğal frekansların, üstel fonksiyonların üssü *n* ile boyutsuz uzunluk ölçeği parametresi *l* / *h*'ye göre değişimleri sunulmaktadır. Her iki boyutsuz frekansda da *n* arttıkça azalma görülmektedir. Bu durumda seramik-yoğun FDM mikro-plaklar metal-yoğun mikro-plaklara göre daha yüksek doğal frekanslara sahip olmaktadır. Farklı *l* / *h* oranları için boyutsuz ikinci frekanslar birbirine yakın olmakla birlikte, boyutsuz birinci frekansda *l* / *h* arttıkça artma gözlemlenmektedir.

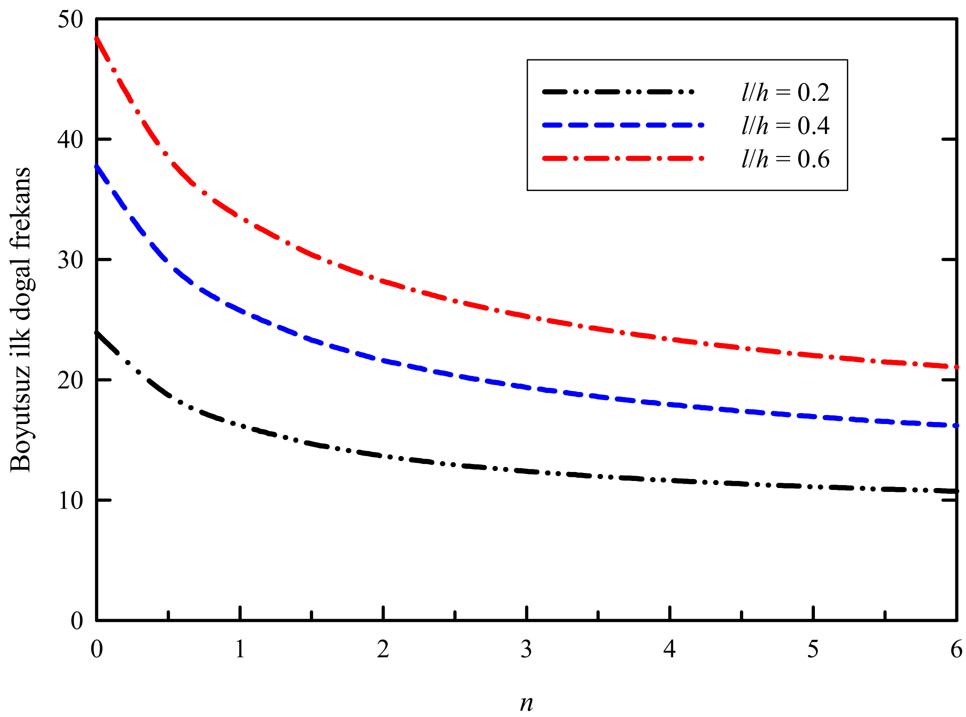


Şekil 10. Boyutsuz ilk doğal frekans $\bar{\omega}_1$, *h*=25 μm, *a*/*h*=10, *b*/*a*=1.0, *β*=2.0, Δ*T*=0 K.

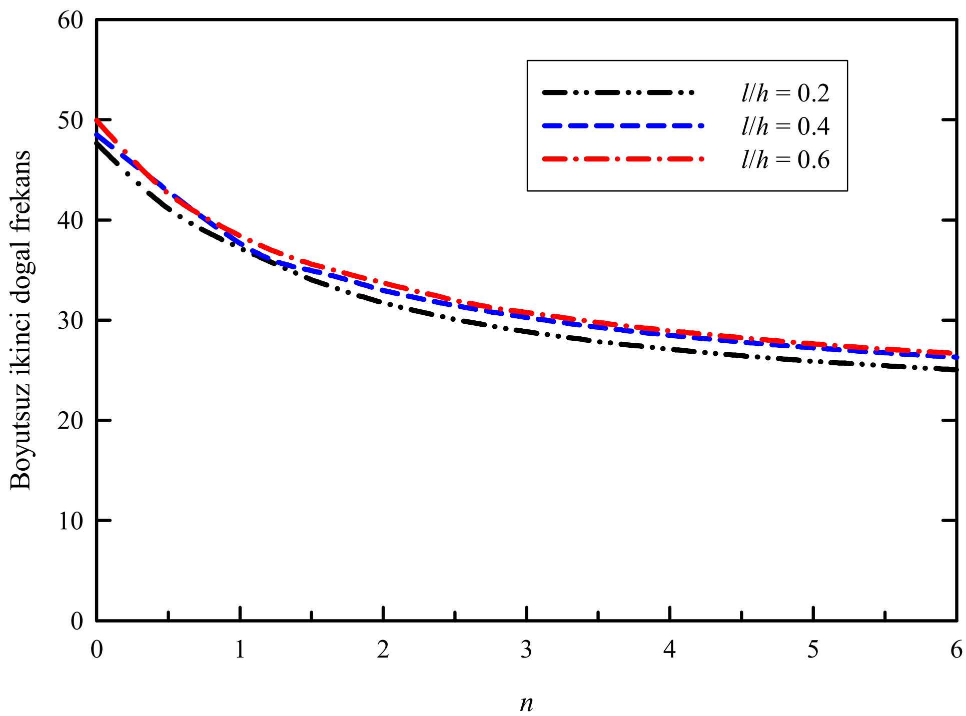


Şekil 11. Boyutsuz ikinci doğal frekans $\bar{\omega}_2$, *h*=25 μm, *a*/*h*=10, *b*/*a*=1.0, *β*=2.0, Δ*T*=0 K.

Termal etki altında olan mikro-plakların serbest titreşimi üzerine sonuçlar 12. ve 13. figürlerde sunulmaktadır. Bu şekillerde verilen sonuçlar elde edilirken ΔT = 100 K olarak uygulanmıştır. Bu sıcaklık farkı için elde edilen boyutsuz doğal frekans değerlerinin sıcaklık farkının göz önüne alınmadığı durum için hesaplanan Şekil 10 ve 11'deki sonuçlara genel olarak çok yakın olduğu bulunmuştur. Dolayısıyla n ve $l\ /\ h$ parametrelerine bağlı olarak termal etkinin olmadığı durumda gözlemlenen değişimler termal etki altında da geçerlidir. Yaptığımız diğer sayısal analizlerde sıcaklık farkı ΔT'nin daha yüksek olduğu durumlarda termal etki göz önüne alınmadan bulunan serbest titreşim sonuçlarıyla, göz önüne alınarak bulunan serbest titreşim sonuçları arasındaki farkın arttığı görülmüştür. Buna ek olarak sabit mesnet, ya da ankastre plak gibi problemlerde de sıcaklık farkı etkisi daha yüksek düzeyde gözlemlenecektir.



Şekil 12. Boyutsuz ilk doğal frekans $\bar{\omega}_1$, h=25 μm, a/h=10, b/a=1.0, β=2.0, ΔT=100 K.



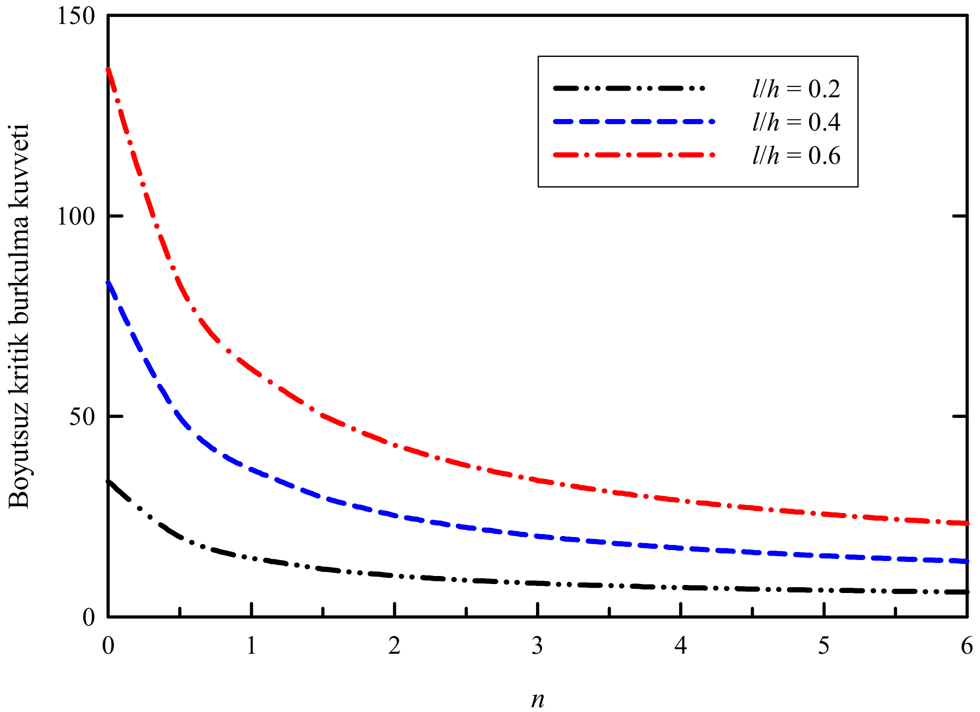
Şekil 13. Boyutsuz ikinci doğal frekans $\bar{\omega}_2$, h=25 μm, a/h=10, b/a=1.0, β=2.0, ΔT=100 K.

6.2.3. Burkulma

Gerinim gradyanı elastisite teorisi baz alınarak burkulma ile ilgili elde ettiğimiz sonuçlar 14. ve 15. Şekillerde sunulmuştur. Bu figürlerde verilen sayısal sonuçlar hesaplanırken Mindlin plak teorisi kullanılmıştır. Boyutsuz kritik burkulma kuvveti

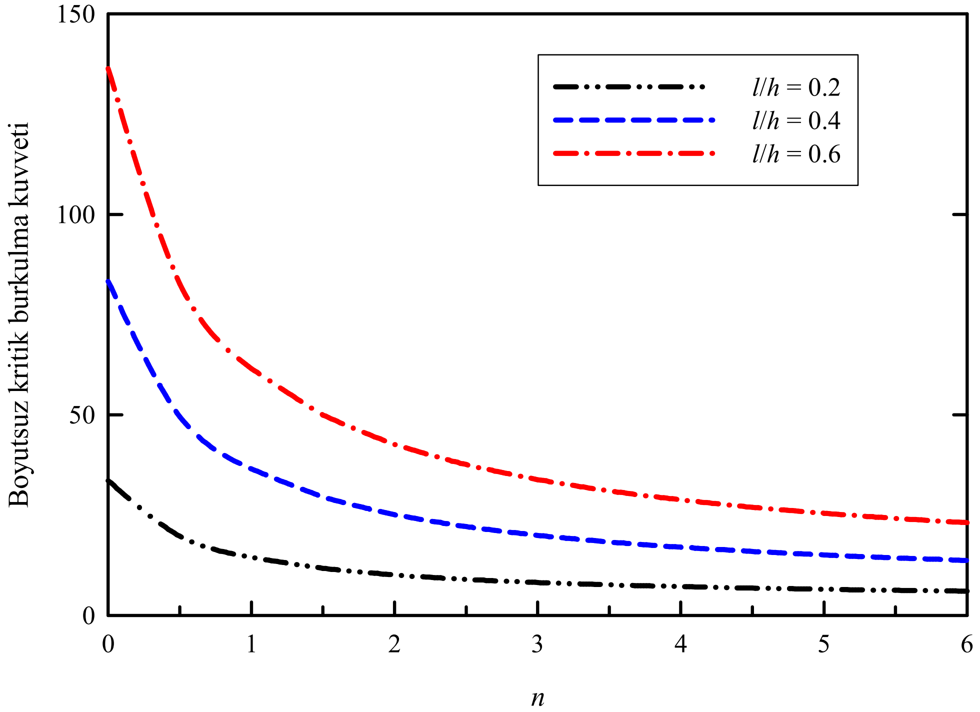
$$\bar{P} = \frac{Pa^2}{E_2 h^3}, \tag{185}$$

formunda tanımlanmıştır. Şekil 14'de boyutsuz burkulma kuvveti *n* üssü ve *l* / *h* oranına bağlı olarak çizilmiştir. *n* arttıkça boyutsuz burkulma kuvvetinde azalma olduğu görülmektedir. Bu durumda kritik burkulma kuvveti seramik-yoğun FDM mikro-plaklar için, metal-yoğun FDM mikro-plaklara göre daha yüksek seviyede olmaktadır. Boyutsuz uzunluk ölçeği parametresi *l* / *h*, 0.2'den 0.6'ya arttıkça, boyutsuz burkulma kuvvetinde de artma meydana gelmektedir.



Şekil 14. Boyutsuz burkulma kuvveti $\bar{P}$, *h*=25 μm, *a*/*h*=10, *b*/*a*=1.0, *β*=2.0, Δ*T*=0 K.

Termal etki altında olan FDM mikro-plaklar için hesaplanan kritik burkulma kuvveti ise Şekil 15'de sunulmaktadır. Δ*T*=100 K büyüklüğünde düzgün sıcaklık farkı altında elde edilen bu sonuçlar, Şekil 14'de verilen sonuçlara genel olarak yakındır. Fakat, farklı malzemeler ve geometrik konfigürasyonlar için sıcaklık etkisinin daha yüksek düzeyde olması beklenmelidir.



Şekil 15. Boyutsuz burkulma kuvveti $\bar{P}$, h=25 μm, a/h=10, b/a=1.0, β=2.0, ΔT=100 K.

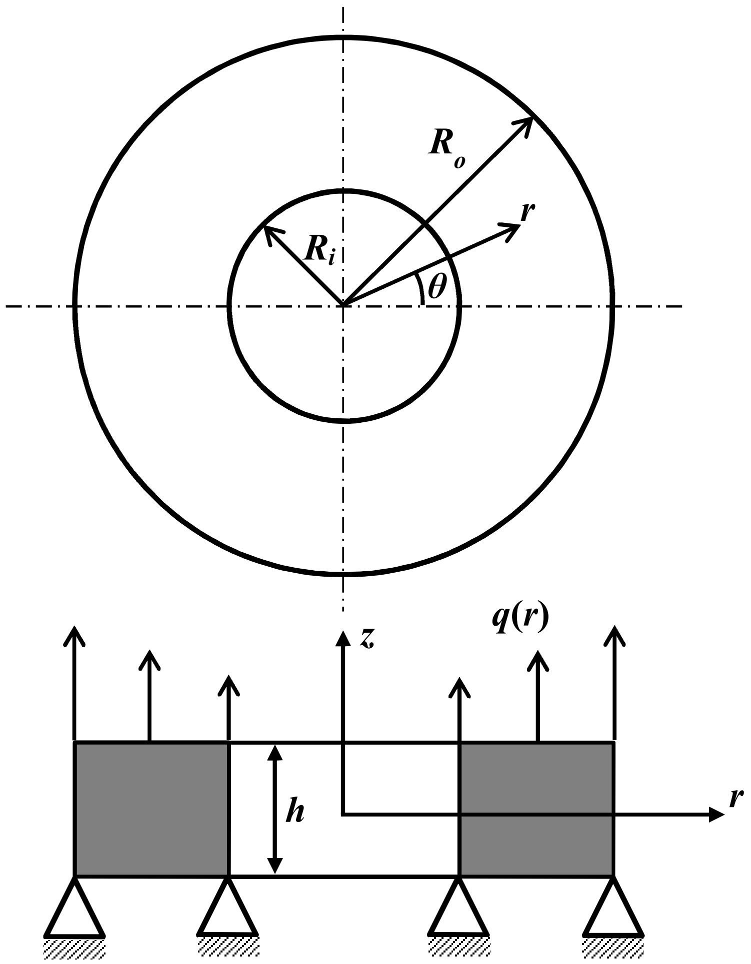
7. EK ÇALIŞMALAR

Önceki bölümlerde detaylandırdığımız çalışmalar proje önerimizde tanımlanmıştı. Öneride tanımlanan bu çalışmalara ek olarak proje ekibimizce mikro- ve nano-plakların modellenmesi ve analizi üzerine üç farklı çalışma yürütülmüştür. Bu kapsamda yapılan çalışmalar ile ilgili iki makalemiz yayımlanmıştır, bir makalemiz de şu an değerlendirmededir. Bu makaleler Ekler bölümünde sunulmuştur. Bu bölümde ek çalışmalarımızda ele alınan problemler kısaca özetlenmiş ve hazırlanmış olan makaleler ile ilgili bilgiler verilmiştir.

7.1. Fonksiyonel Derecelendirilmiş Halka Şeklinde ve Dairesel Mikro-Plakların Statik ve Dinamik Analizleri

Bu ek çalışma proje yürütücüsü Prof. Dr. Serkan Dağ'ın University of Tehran'da görevli Prof. Dr. Nassel Soltani ve doktora öğrencisi Iman Eshraghi ile yapmış olduğu işbirliği neticesinde gerçekleşmiştir. Çalışmada Şekil 16'da gösterilen mikro-plak konfigürasyonu ele alınmıştır. Bu şekilde iç ve dış yarıçapları sırasıyla R_i ve R_o; ve kalınlığı *h* olan halka şeklinde bir mikro-plak gösterilmektedir. Mikro-plak üzerine *q*(*r*) yüklemesi uygulanmıştır.

Yaptığımız çalışmada mikro-plağın statik eğilme ve serbest titreşim davranışları incelenmiştir. Bağlaşık kısmi diferansiyel denklemler ile sınır koşulları türetilirken modifiye edilmiş kuvvet çifti gerilmesi teorisi kullanılmıştır. Bu formülasyonda uzunluk ölçeği parametresi de dahil olmak üzere tüm malzeme özellikleri kalınlık koordinatının fonksiyonları olarak alınmıştır. Türetilen denklemler sayısal olarak diferansiyel kare yapma metodu ile çözülmüştür. Yürütülen sayısal analizler ile geometri ve malzeme özelliklerinin statik deformasyon ile serbest titreşim frekansları üzerindeki etkileri açığa çıkarılmıştır.



Şekil 16. Halka şeklinde FDM mikro-plak.

Bu çalışmamız Composites Part B dergisinde yayımlanmıştır. Makale bilgileri şu şekildedir (Eshraghi vd., 2015):

ESHRAGHI I., Dag S., Soltani N., Consideration of spatial variation of the length scale parameter in static and dynamic analyses of functionally graded annular and circular micro-plates, *Composites Part B: Engineering* **78**, 338-348, (2015).

Makalemizin tümüne Ekler bölümünde yer verilmiştir.

7.2. Termal Yükleme Altındaki Fonksiyonel Derecelendirilmiş Halka Şeklinde ve Dairesel Mikro-Plakların Statik ve Dinamik Analizleri

Bu çalışmada termal yükleme etkisi altında olan fonksiyonel derecelendirilmiş halka şeklinde ve dairesel mikro-plakların statik eğilme ve serbest titreşim analizleri yapılmıştır. Çalışma proje yürütücüsü Prof. Dr. Serkan Dağ, Tahran Üniversitesi'nde görevli Prof. Dr. Nasser Soltani, ve doktora öğrencisi Iman Eshraghi tarafından yürütülmüştür. Bu araştırmada göz önüne alınan mikro-plak konfigürasyonu Şekil 16'da gösterilen konfigürasyon ile aynıdır. Ancak, Bölüm 7.1'de tanımlanan problemden farklı olarak mikro-plağın termal gerilme etkisi altında olduğu varsayılmıştır.

Bu problem için gerekli olan diferansiyel denklemler ile sınır koşulları varyasyonel yöntem uygulanarak türetilmiştir. Formülasyonda modifiye edilmiş kuvvet çifti gerilmesi teorisi kullanılmıştır. Elde edilen denklem sistemleri, diferansiyel kare yapma metodu ile çözülmüştür. Yürütülen parametrik analizlerle malzeme ve geometri özelliklerinin termal yükleme altında olan mikro-plaklardaki statik eğilme deformasyonları ile serbest titreşim frekansları üzerlerindeki etkileri araştırılmıştır.

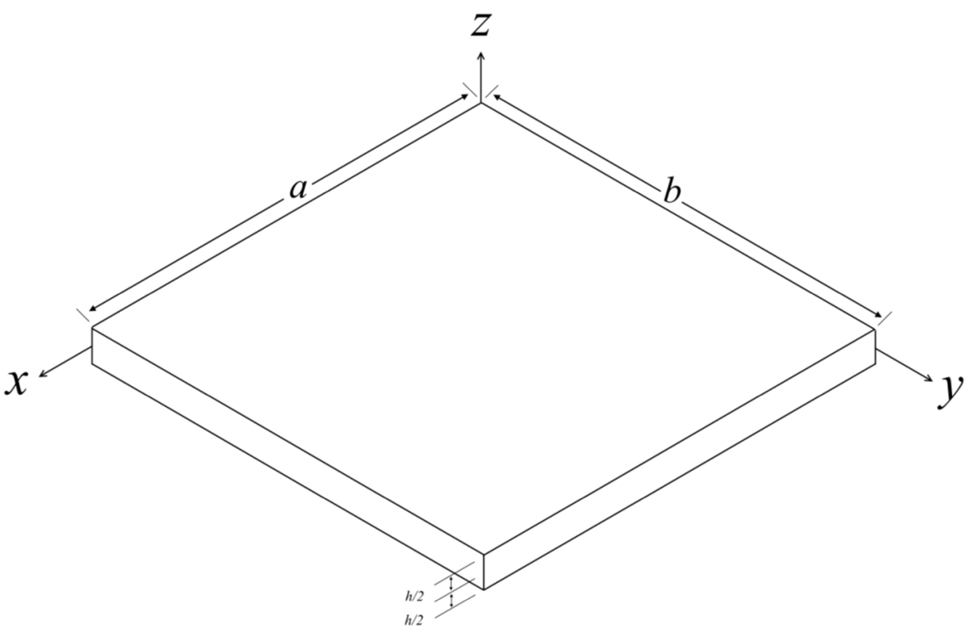
Bu kapsamda yürüttüğümüz çalışma Composite Structures adlı dergide yayımlanmıştır. Makale bilgileri şu şekildedir (Eshraghi vd., 2016):

ESHRAGHI I., Dag S., Soltani N., Bending and free vibrations of functionally graded annular and circular micro-plates under thermal loading, *Composite Structures*, **137**, 196–207, (2016).

Makalemiz Ekler bölümünde sunulmuştur.

7.3. Fonksiyonel Derecelendirilmiş Nano-Plakların Lokal Olmama Parametresinin Uzaysal Değişimi Göz Önüne Alınarak Serbest Titreşim Analizi

Bu çalışma proje yürütücüsü Prof. Dr. Serkan Dağ, araştırmacı Doç. Dr. Ender Ciğeroğlu, ve bursiyer Ata Alipour Ghassabi tarafından yapılmıştır. Ele alınan fonksiyonel derecelendirilmiş nano-plak geometrisi Şekil 17'de gösterilmektedir.



Şekil 17. FDM dikdörtgen nano-plak geometrisi.

Çalışma kapsamında lokal olmayan elastisite teorisi kullanılarak serbest titreşim davranışını belirleyen diferansiyel denklemler ile sınır koşulları türetilmiştir. Formülasyonda lokal olmama parametresi de dahil olmak üzere, tüm malzeme özelliklerinin *z*-koordinatı boyunca değiştiği varsayılmıştır. Türetilen denklem sistemi diferansiyel kare yapma metodu kullanılarak sayısal olarak çözülmüştür. Yürütülen sayısal analizler ile çeşitli malzeme ve geometri parametrelerinin, serbest titreşim frekansları üzerlerindeki etkileri ortaya konulmuştur. Bu kapsamda bir makale ilgili bir dergiye gönderilmek üzere hemen hemen hazır duruma getirilmiştir. Bazı ek sonuçlar da bulunduktan sonra konu ile ilgili uygun bir dergiye gönderilecektir. Makalemizin şu aşamadaki başlığı şu şekildedir:

ALIPOUR GHASSABI A., Dag S., Cigeroglu E., Free vibration analysis of functionally graded rectangular nano-plates considering spatial variation of the nonlocal parameter, konu ile ilgili bir dergiye gönderilecek.

Bu makale çalışmamız da Ekler bölümünde sunulmuştur.

8. SONUÇ

Yürüttüğümüz araştırma projesi kapsamında mikro-plakların analizi ve modellenmesi için yeni yöntemler geliştirilmiştir. Proje önerisinde detaylandırdığımız dikdörtgen FDM mikro-plakların gerinim gradyanı elastisite teorisi ile analizi dışında, halka şeklinde ve dairesel FDM mikro-plaklar ile lokal olmayan elastisite teorisine göre davranan dikdörtgen FDM mikro-plaklar için de analiz ve modelleme çalışmaları yürütülmüştür. Her bir çalışma ile ilgili çok sayıda parametrik analiz yapılarak, literatürde bulunmayan orijinal sonuçlar üretilmiştir. Şu ana kadar, yaptığımız çalışmalarla ilgili üç dergi makalesi hazırlanmış, ve bunlardan iki tanesi yayımlanmıştır. Bir makalemiz de konu ile ilgili uygun bir dergiye gönderilecektir. Hazırlamış olduğumuz bu makalelere ek olarak, hem dikdörtgen hem de halka şeklinde ve dairesel FDM mikro-plaklar ile ilgili yeni makaleler de hazırlanacaktır. Dikdörtgen mikro-plaklar üzerine hazırlayacağımız makalelerde gerinim gradyanı elastisite bazlı formülasyona dayalı çözümler sunulacak; halka şeklinde ve dairesel mikro-plaklar ile ilgili makalelerde ise termal burkulma problemleri incelenecektir.

Gerinim gradyanı elastisite teorisini baz alarak dikdörtgen FDM mikro-plaklar üzerine yaptığımız çalışmalarda formülasyon varyasyonel yöntem esas alınarak geliştirilmiştir. Uzunluk ölçeği parametrelerinde gerekli sadeleştirmeler yapıldığında bu formülasyon modifiye edilmiş kuvvet çifti teorisi için de geçerli olmaktadır. 6. Bölüm'de hem modifiye edilmiş kuvvet çifti gerilmesi teorisine hem de gerinim gradyanı elastisite teorisine dayalı sayısal sonuçlar sunulmuştur. Mekanik ve termal yüklemeler göz önüne alınmıştır. Elde edilen diferansiyel denklem sistemleri, diferansiyel kare yapma metodu kullanılarak sayısal olarak çözülmüştür. Sayısal çözüm tekniği MATLAB adlı yazılıma entegre edilmiştir. Geliştirilen program ile statik eğilme, serbest titreşim, ve burkulma problemleri için hem mekanik hem de termal yükleme altında sayısal çözümler üretmek mümkün olmaktadır. Ortaya koyduğumuz yöntemler literatürde bulunan sayısal sonuçlar ile karşılaştırmalar yapılarak doğrulanmıştır. Yaptığımız parametrik analizlerle mikro-plak teorisi, uzunluk ölçeği parametresi değişimi, uzunluk ölçeği parametresinin mikro-plak kalınlığına oranı, FDM malzeme değişim profili, ve termal yükleme gibi faktörlerin statik eğilme deformasyonu, serbest titreşim frekansları, ve kritik burkulma yükleri üzerlerindeki etkileri belirlenmiştir.

Proje önerisinde bulunan kapsama ek olarak, halka şeklinde ve dairesel mikro-plaklar için modifiye edilmiş kuvvet çifti teorisi bazlı; ve dikdörtgen nano-plaklar için lokal olmayan elastisite teorisi bazlı çalışmalar yürütülmüştür. Halka şeklinde ve dairesel FDM mikro-plaklar için yapılan çalışmalarda mekanik ve termal etkiler altında statik eğilme ile serbest titreşim problemleri incelenmiştir. Bu çalışmalar sonucunda plak teorisi, uzunluk ölçeği parametresi değişimi, geometrik özellikler, ve malzeme özelliklerinin statik eğilme davranışı ile serbest titreşim frekansları üzerindeki etkileri ortaya çıkarılmıştır. Dikdörtgen FDM nano-plaklar için yürüttüğümüz lokal olmayan elastisite teorisi bazlı çalışmamızda serbest titreşim problemi ele alınmış ve malzeme özellikleri ile geometrik özelliklerin titreşim davranışını nasıl etkileyeceği araştırılmıştır.

Ortaya koyduğumuz yeni yöntemler kullanılarak mikro-plaklar üzerine detaylı analiz, modelleme, ve tasarım çalışmaları yapmak mümkündür. Bu yöntemler farklı plak teorileri için sonuç üretilmesine, uzunluk ölçeği parametrelerindeki uzaysal değişimlerin göz önüne alınmasına, ve termal etki altında olan mikro-plakların modellenmesine olanak sağlamaktadır. Uzunluk ölçeği parametrelerindeki değişimlerin, dikdörtgen, halka şeklinde, ve dairesel mikro-plakların termomekanik davranışını önemli ölçüde etkilediği bu çalışmamızda kanıtlanmıştır. Bu nedenle bu değişimler göz önüne alınmadan yürütülecek analizler ile gerekli doğruluk derecesinde sonuç üretmek mümkün olmayacaktır. Araştırma projemiz kapsamında yapılan çalışmalar ile teknik literatürde FDM mikro-plaklar için uzunluk ölçeği parametrelerindeki değişimler ilk kez göz önüne alınmıştır.

Geliştirdiğimiz analiz ve modelleme yöntemleri ile dikdörtgen, halka şeklinde ve dairesel mikro-plaklarin statik deformasyon, serbest titreşim, ve burkulma davranışları yüksek doğrulukta ve termomekanik etkiler altında incelenebilmektedir. Bu nedenle bu analiz yöntemlerinin konu üzerinde çalışan araştırmacılar ve mühendisler tarafından mikro-plak analizi, tasarımı, ve optimizasyonu gibi çalışmalarda kullanılması mümkün olacaktır.

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Consideration of spatial variation of the length scale parameter in static and dynamic analyses of functionally graded annular and circular micro-plates

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ABSTRACT

This article introduces new methods for static and free vibration analyses of functionally graded annular and circular micro-plates, which can take into account spatial variation of the length scale parameter. The underlying higher order continuum theory behind the proposed approaches is the modified couple stress theory. A unified way of expressing the displacement field is adopted so as to produce numerical results for three different plate theories, which are Kirchhoff plate theory (KPT), Mindlin plate theory (MPT), and third-order shear deformation theory (TSDT). Governing partial differential equations and corresponding boundary conditions are obtained following the variational approach and the Hamilton's principle. Derived systems of differential equations are solved numerically by utilizing the differential quadrature method (DQM). Comparisons to the results available in the literature demonstrate the high level of accuracy of the numerical results generated through the developed methods. Extensive analyses are presented in order to illustrate the influences of various geometric and material parameters upon static deformation profiles, stresses, and natural vibration frequencies. In particular, the length scale parameter ratio -which defines the length scale parameter variation profile-is shown to possess a profound impact on both static and dynamic behaviors of functionally graded annular and circular micro-plates.

1. Introduction

Functionally graded materials (FGMs) belong to a special class of composites, which possess smooth spatial variations in the volume fractions of the constituent phases. These variations facilitate design of structures with customized physical properties and improved performance. Previous work show that property gradation enhances mechanical response of components utilized in a number of technological applications including thermal barrier coatings [1], wear resistant surfaces [2], cutting tools [3], solid oxide fuel cells [4], and biomaterials [5]. Recently, with the advent of micro-scale FGM component fabrication techniques such as magnetron sputtering [6], chemical vapor deposition and plasma-enhanced chemical vapor deposition [7], and modified soft lithography [8]; research focus has been placed on behavior of functionally graded micro-scale structures.

The present study aims at putting forward new analysis techniques for micro-scale annular and circular plates built from functionally graded materials. Annular and circular micro-plates have been employed as components in a wide variety of micro-electro-mechanical-systems (MEMS). Annular micro-plates are utilized in MEMS such as gear pumps [9], stiction valves [10], and resonators [11,12]; whereas circular micro-plates find applications in pressure sensors [13], acoustic energy harvesters [14], and optical MEMS sensors [15]. Functionally graded annular and circular plates possess the intrinsic advantages that come along with spatial variations in physical properties. Depending on the form of these variations, in graded annular and circular plates deflections and tensile static stresses could be lower [16] and critical buckling loads could be higher [17] compared to those evaluated for non-graded counterparts. Furthermore, natural frequencies of free vibrations of a graded plate are strongly dependent upon property distribution profiles and can be optimized by devising a suitable composition architecture [18]. Thus, it is of significance to employ solution methodologies delivering sufficiently accurate results regarding

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both static and dynamic behaviors of graded annular and circular micro-plates.

Conventional continuum theories, such as classical elasticity, can not accurately predict the response of micro-components because of the size effect prevailing at the micro-scale. Analysis of micro-scale structures needs to be based on a higher order continuum theory, examples of which are nonlocal theory of Eringen [19–21], modified couple stress theory [22–24], strain gradient elasticity [25,26], and finite deformation gradient elasticity [27]. Modified couple stress theory and strain gradient elasticity are the most commonly used higher-order theories in the analyses of micro-scale functionally graded components.

There are several articles in the technical literature that present higher order continuum theory based analysis techniques for micro-scale FGM plates. Adopting modified couple stress theory, Kim and Reddy [28] and Thai and Kim [29] developed solutions for rectangular graded micro-plates; and Asghari and Taati [30] treated the arbitrarily shaped micro-plate problem. Micro-scale axisymmetric functionally graded plates were considered by Ansari et al. [17] and Ke et al. [18]. Ansari et al. [17] studied bending, buckling, and free vibrations of annular and circular micro-plates using modified strain gradient elasticity, whereas Ke et al. [18] examined free vibrations of annular plates by employing modified couple stress theory.

Higher order stresses and strain gradient measures in higher order continuum theories are related through length scale parameters. The single length scale parameter in modified couple stress theory for instance is defined as the ratio of the modulus of curvature to the shear modulus [31,32], and thus within this context it is essentially an elastic material property. Since all elastic properties of a functionally graded structure are expected to possess spatial variations, the length scale parameter is in general also a function of the spatial coordinates. Hence, the employed solution methodology should be able to account for the spatial variation of the length scale parameter. However, in all studies mentioned in the foregoing paragraph, length scale parameter is assumed to be a constant quantity. The only study presenting a systematic approach in the consideration of the spatial variation of the length scale parameter is that by Aghazadeh et al. [33], which treats problems involving functionally graded micro-beams. Yet, spatial variation of the length scale parameter has not been incorporated into the analysis of FGM micro-plates.

The main objective of the present study is to put forth modified couple stress theory-based modeling and analysis techniques for functionally graded annular and circular micro-plates, which take into account the *spatial variation of the length scale parameter*. Both static bending and free vibration problems of FGM micro-plates are studied to develop the proposed methods. In the formulation, displacement field is expressed in a certain unified form so as to generate results for three different plate theories, which are Kirchhoff plate theory (KPT), Mindlin plate theory (MPT), and third-order shear deformation theory (TSDT). Governing partial differential equations and boundary conditions are derived by applying Hamilton's principle in accordance with modified couple stress theory. All material properties including the length scale parameter are assumed to be functions of the thickness coordinate in the derivations. The equations are solved numerically by means of the differential quadrature method (DQM). Comparisons of the generated numerical results to those available in the literature illustrate the high degree of accuracy attained by the application of the developed procedures. Further parametric analyses are carried out to shed light upon the influences of material and geometric parameters on static deflections and natural frequencies of annular and circular FGM micro-plates. Numerical results unequivocally demonstrate that in modeling and analysis of graded micro-structures, it is necessary to take into consideration the spatial variation of the length scale parameter.

2. Formulation

In this study, we examine static and dynamic behaviors of functionally graded annular and circular micro-plates. The geometry of the annular micro-plate is depicted in Fig. 1. Inner and outer radii are respectively denoted by R_i and R_o. Circular micro-plate has an identical geometry except for the fact that $R_i = 0$. The loading function and the boundary conditions are assumed to be independent of θ, hence the underlying problems become axisymmetric. Material properties vary continuously along the z-direction. Both the annular and the circular micro-plates are 100% metallic at $z = -h/2$ and 100% ceramic at $z = h/2$. According to the modified couple stress theory [22], strain energy of the plates under consideration reads:

$$U = \frac{1}{2}\iiint_V (\sigma_{ij}\varepsilon_{ij} + m_{ij}\chi_{ij})dV, \tag{1}$$

where V designates volume; σ_{ij} represents Cauchy stress; ε_{ij} is strain; m_{ij} stands for the deviatoric part of the couple stress tensor; and χ_{ij} is the symmetric curvature tensor. The tensorial quantities are expressed in the following form:

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk}, \tag{2a}$$

$$m_{ij} = 2\mu l^2\chi_{ij}, \tag{2b}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}\right], \tag{2c}$$

$$\boldsymbol{\chi} = \frac{1}{2}\left[\nabla\boldsymbol{\omega} + (\nabla\boldsymbol{\omega})^{\mathrm{T}}\right]. \tag{2d}$$

In Eq. (2), μ and λ are Lame parameters, δ_{ij} is Kronecker delta, l denotes the length scale parameter, $\mathbf{u}$ symbolizes displacement vector, and $\boldsymbol{\omega}$ is the rotation vector. μ, λ, and the rotation vector are defined by

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \tag{3a}$$

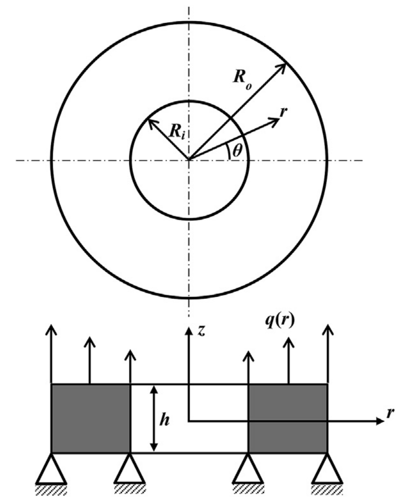

Fig. 1. A functionally graded annular micro-plate under the effect of distributed loading $q(r)$.

$$\boldsymbol{\omega} = \frac{1}{2}\mathrm{curl}(\mathbf{u}), \tag{3b}$$

where E and ν are respectively modulus of elasticity and Poisson's ratio. All of the material parameters, including the length scale parameter l, are assumed to be functions of the thickness coordinate z.

In order to be able to examine the behavior of micro-plates according to three different plate theories, namely Kirchhoff plate theory, Mindlin plate theory, and third-order shear deformation theory, the displacement field is expressed in a general form in the following way:

$$u_r(r,z,t) = u(r,t) - z\frac{\partial w}{\partial r} + f(z)\gamma(r,t), \tag{4a}$$

$$u_\theta(r,z,t) = 0, \tag{4b}$$

$$u_z(r,z,t) = w(r,t), \tag{4c}$$

where

$$\gamma(r,t) = \frac{\partial w}{\partial r} - \phi(r,t), \tag{5a}$$

$$f(z) = \begin{cases} 0, & \text{for Kirchhoff plate theory,} \\ z, & \text{for Mindlin plate theory,} \\ z\left(1 - \dfrac{4z^2}{3h^2}\right), & \text{for third} - \text{order shear deformation theory.} \end{cases} \tag{5b}$$

ϕ here is the rotation at the mid-plane. Using Eqs. (2) and (4), nonzero components of strain and symmetric curvature tensors are written as

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} - z\frac{\partial^2 w}{\partial r^2} + f\frac{\partial \gamma}{\partial r}, \tag{6a}$$

$$\varepsilon_{\theta\theta} = \frac{u}{r} - \frac{z}{r}\frac{\partial w}{\partial r} + f\frac{\gamma}{r}, \tag{6b}$$

$$\varepsilon_{rz} = \frac{1}{2}f'\gamma, \tag{6c}$$

$$\chi_{r\theta} = \frac{1}{2}\left\{\frac{1}{r}\frac{\partial w}{\partial r} - \frac{\partial^2 w}{\partial r^2} + \frac{1}{2}\left(\frac{\partial \gamma}{\partial r} - \frac{\gamma}{r}\right)f'\right\}, \tag{6d}$$

$$\chi_{\theta z} = \frac{1}{4}f''\gamma. \tag{6e}$$

Hamilton's principle is used to derive the governing partial differential equations and the corresponding boundary conditions. This principle postulates that

$$\delta\int_{t_1}^{t_2}(K - U + W)dt = 0, \tag{7}$$

where K is total kinetic energy, U is strain energy, and W is the work done by external forces. These terms are as follows:

$$U = \frac{1}{2}\int_{-h/2}^{h/2}\int_{R_i}^{R_o}\{\sigma_{rr}\varepsilon_{rr} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + 2(\sigma_{rz}\varepsilon_{rz} + m_{r\theta}\chi_{r\theta} + m_{z\theta}\chi_{z\theta})\} \times 2\pi r dr dz, \tag{8a}$$

$$K = \frac{1}{2}\int_{-h/2}^{h/2}\int_{R_i}^{R_o}\rho(z)\left(\dot{u}_r^2 + \dot{u}_z^2\right)2\pi r dr, \tag{8b}$$

$$W = \int_{R_i}^{R_o} q(r)w2\pi r dr. \tag{8c}$$

ρ in Eq. (8b) is the mass density, and $q(r)$ in Eq. (8c) is the axisymmetrical distributed loading applied to the plate surface. Note that R_i is to be taken as zero for the circular plate. Utilizing Eqs. (7) and (8), and variational principles, governing partial differential equations are derived in the following form:

$$(F_{11} - B_{11})\left\{\frac{\partial^3 w}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r}\right\} + A_{11}\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right\} - F_{11}\left\{\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} - \frac{\phi}{r^2}\right\} = I_1\frac{\partial^2 u}{\partial t^2} + (I_4 - I_2)\frac{\partial^3 w}{\partial r \partial t^2} - I_4\frac{\partial^2 \phi}{\partial t^2}, \tag{9a}$$

$$\begin{aligned}&\left\{2F_{22} - D_{11} - F_{33} - A_{552} + F_{572} - \frac{F_{552}}{4}\right\}\left\{r\frac{\partial^4 w}{\partial r^4} + 2\frac{\partial^3 w}{\partial r^3} - \frac{1}{r}\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial w}{\partial r}\right\} + \left\{\frac{F_{662}}{4} + k_s F_{55}\right\}\left\{r\frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{\partial r}\right\} \\ &+ (B_{11} - F_{11})\left\{r\frac{\partial^3 u}{\partial r^3} + 2\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{u}{r^2}\right\} + \left\{F_{33} - F_{22} - \frac{F_{572}}{2} + \frac{F_{552}}{4}\right\}\left\{r\frac{\partial^3 \phi}{\partial r^3} + 2\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\phi}{r^2}\right\} - \left\{k_s F_{55} + \frac{F_{662}}{4}\right\}\left\{r\frac{\partial \phi}{\partial r} + \phi\right\} + rq \\ &= I_1 r\frac{\partial^2 w}{\partial t^2} + (2I_5 - I_3 - I_6)\left\{\frac{\partial^3 w}{\partial r \partial t^2} + r\frac{\partial^4 w}{\partial r^2 \partial t^2}\right\} + (I_2 - I_4)\left\{\frac{\partial^2 u}{\partial t^2} + r\frac{\partial^3 u}{\partial r \partial t^2}\right\} + (I_6 - I_5)\left\{\frac{\partial^2 \phi}{\partial t^2} + r\frac{\partial^3 \phi}{\partial r \partial t^2}\right\},\end{aligned} \tag{9b}$$

$$\left\{F_{33}+\frac{F_{552}}{4}\right\}\left\{\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}-\frac{\phi}{r^2}\right\}+\left\{F_{22}-F_{33}+\frac{F_{572}}{2}-\frac{F_{552}}{4}\right\}\left\{\frac{\partial^3 w}{\partial r^3}+\frac{1}{r}\frac{\partial^2 w}{\partial r^2}-\frac{1}{r^2}\frac{\partial w}{\partial r}\right\}+\left\{k_sF_{55}+\frac{F_{662}}{4}\right\}\left\{\frac{\partial w}{\partial r}-\phi\right\}-F_{11}\left\{\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right\}$$
$$=I_6\frac{\partial^2\phi}{\partial t^2}+(I_5-I_6)\frac{\partial^3 w}{\partial r\partial t^2}-I_4\frac{\partial^2 u}{\partial t^2}. \tag{9c}$$

Boundary conditions at $r=R_i$, R_o are obtained as:

$$\delta u=0,\quad \text{or}\quad (F_{11}-B_{11})r\frac{\partial^2 w}{\partial r^2}+\left(F^*_{11}-B^*_{11}\right)\frac{\partial w}{\partial r}+A_{11}r\frac{\partial u}{\partial r}+A^*_{11}u-F_{11}r\frac{\partial\phi}{\partial r}-F^*_{11}\phi=0, \tag{10a}$$

$$\begin{aligned}\delta w=0,\quad \text{or}\quad &\left\{2F_{22}-D_{11}-F_{33}-A_{552}+F_{572}-\frac{F_{552}}{4}\right\}\\ &\times\left\{\frac{\partial^3 w}{\partial r^3}+\frac{1}{r}\frac{\partial^2 w}{\partial r^2}-\frac{1}{r^2}\frac{\partial w}{\partial r}\right\}+\left\{\frac{F_{662}}{4}+k_sF_{55}\right\}\frac{\partial w}{\partial r}\\ &+(B_{11}-F_{11})\left\{\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right\}\\ &+\left\{F_{33}-F_{22}-\frac{F_{572}}{2}+\frac{F_{552}}{4}\right\}\left\{\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}-\frac{\phi}{r^2}\right\}\\ &-\left\{\frac{F_{662}}{4}+k_sF_{55}\right\}\phi=(2I_5-I_3-I_6)\frac{\partial^3 w}{\partial r\partial t^2}\\ &+(I_2-I_4)\frac{\partial^2 u}{\partial t^2},\end{aligned} \tag{10b}$$

$$\begin{aligned}\delta\left(\frac{\partial w}{\partial r}\right)=0,\quad \text{or}\quad &\left\{D_{11}-2F_{22}+F_{33}+A_{552}-F_{572}+\frac{F_{552}}{4}\right\}r\frac{\partial^2 w}{\partial r^2}\\ &+\left\{D^*_{11}-2F^*_{22}+F^*_{33}-A_{552}+F_{572}-\frac{F_{552}}{4}\right\}\frac{\partial w}{\partial r}\\ &+(F_{11}-B_{11})r\frac{\partial u}{\partial r}+\left(F^*_{11}-B^*_{11}\right)u\\ &+\left\{F_{22}-F_{33}+\frac{F_{572}}{2}-\frac{F_{552}}{4}\right\}r\frac{\partial\phi}{\partial r}\\ &+\left\{F^*_{22}-F^*_{33}-\frac{F_{572}}{2}+\frac{F_{552}}{4}\right\}\phi=0,\end{aligned} \tag{10c}$$

$$\begin{aligned}\delta\phi=0,\quad \text{or}\quad &\left\{F_{22}-F_{33}+\frac{F_{572}}{2}-\frac{F_{552}}{4}\right\}r\frac{\partial^2 w}{\partial r^2}\\ &+\left\{F^*_{22}-F^*_{33}-\frac{F_{572}}{2}+\frac{F_{552}}{4}\right\}\frac{\partial w}{\partial r}\\ -F_{11}r\frac{\partial u}{\partial r}-F^*_{11}u&+\left\{F_{33}+\frac{F_{552}}{4}\right\}r\frac{\partial\phi}{\partial r}+\left\{F^*_{33}-\frac{F_{552}}{4}\right\}\phi=0.\end{aligned} \tag{10d}$$

The coefficient terms in Eqs. (9) and (10) are given by

$$\{A_{11},B_{11},D_{11},F_{11},F_{22},F_{33}\}=\int_{-h/2}^{h/2}\frac{E(z)}{1-\nu^2(z)}\left\{1,z,z^2,f,zf,f^2\right\}dz, \tag{11a}$$

$$\{A_{552},F_{552},F_{572},F_{662}\}=\int_{-h/2}^{h/2}\frac{E(z)l^2(z)}{2(1+\nu(z))}\left\{1,f'^2,f',f''^2\right\}dz, \tag{11b}$$

$$F_{55}=\int_{-h/2}^{h/2}\frac{E(z)}{2(1+\nu(z))}f'^2dz, \tag{11c}$$

$$\left\{A^*_{11},B^*_{11},D^*_{11},F^*_{11},F^*_{22},F^*_{33}\right\}=\int_{-h/2}^{h/2}\frac{E(z)\nu(z)}{1-\nu^2(z)}\left\{1,z,z^2,f,zf,f^2\right\}dz, \tag{11d}$$

$$\{I_1,I_2,I_3,I_4,I_5,I_6\}=\int_{-h/2}^{h/2}\rho(z)\left\{1,z,z^2,f,zf,f^2\right\}dz. \tag{11e}$$

Moreover, k_s in Eqs. (9) and (10) is the shear correction factor, which is equal to $\pi^2/12$ in Mindlin plate theory [18] and taken as unity in the third-order shear deformation theory [34]. The results are independent of k_s in Kirchhoff plate theory. Note that the partial differential equations are valid for both functionally graded annular and circular micro-plates. However, the specifications regarding the boundary conditions depend on the type of the plate as will be elucidated in Section 3. Both static bending and free vibrations of circular micro-plates can be examined through the use of the governing equations. In the case of static loading, there are no time-derivatives and right-hand-sides have to be taken as zero. External loading q has to be equated to zero in free vibration analysis.

In order to write the governing equations and the boundary conditions in normalized form, we define the following quantities:

$$\xi=\frac{r-R_i}{R_o-R_i},\quad \gamma=\frac{R_i}{R_o-R_i},\quad \eta=\frac{R_o-R_i}{h},\quad \chi=\xi+\gamma, \tag{12a}$$

$$q_0=\frac{R_o-R_i}{A_{110}}q,\quad \tau=\frac{1}{R_o-R_i}\sqrt{\frac{A_{110}}{I_{10}}}t, \tag{12b}$$

$$\{\overline{u},\overline{w}\}=\frac{\{u,w\}}{h},\quad \varphi=\phi, \tag{12c}$$

$$\left\{\overline{A}_{11},\overline{B}_{11},\overline{D}_{11},\overline{F}_{11},\overline{F}_{22},\overline{F}_{33},\overline{F}_{55}\right\}=\left\{\frac{A_{11}}{A_{110}},\frac{B_{11}}{hA_{110}},\frac{D_{11}}{h^2A_{110}},\frac{F_{11}}{hA_{110}},\frac{F_{22}}{h^2A_{110}},\frac{F_{33}}{h^2A_{110}},\frac{F_{55}}{A_{110}}\right\}, \tag{12d}$$

$$\left\{\overline{A}_{552},\overline{F}_{552},\overline{F}_{572},\overline{F}_{662}\right\}=\left\{\frac{A_{552}}{h^2A_{110}},\frac{F_{552}}{h^2A_{110}},\frac{F_{572}}{h^2A_{110}},\frac{F_{662}}{A_{110}}\right\}, \tag{12e}$$

$$\left\{\overline{A}^*_{11},\overline{B}^*_{11},\overline{D}^*_{11},\overline{F}^*_{11},\overline{F}^*_{22},\overline{F}^*_{33}\right\} = \left\{\frac{A^*_{11}}{A_{110}},\frac{B^*_{11}}{hA_{110}},\frac{D^*_{11}}{h^2A_{110}},\frac{F^*_{11}}{hA_{110}},\frac{F^*_{22}}{h^2A_{110}},\frac{F^*_{33}}{h^2A_{110}}\right\}, \tag{12f}$$

$$\{\overline{I}_1,\overline{I}_2,\overline{I}_3,\overline{I}_4,\overline{I}_5,\overline{I}_6\} = \left\{\frac{I_1}{I_{10}},\frac{I_2}{hI_{10}},\frac{I_3}{h^2I_{10}},\frac{I_4}{hI_{10}},\frac{I_5}{h^2I_{10}},\frac{I_6}{h^2I_{10}}\right\}. \tag{12g}$$

A_{110} and I_{10} in the above equations are respectively the values of A_{11} and I_1 computed by considering a homogeneous plate with properties same as those of the graded plate at $z=-h/2$.

3. Numerical solution

The system comprising the partial differential equations and the boundary conditions is solved by using the differential quadrature method. According to this technique, an m-th order differential operator is represented as a finite series as follows:

$$\left.\frac{\partial^m u(r,t)}{\partial r^m}\right|_{r=r_i} = \sum_{k=1}^{N} C_{ik}^{(m)} u(r_k,t), \quad i=1,\ldots,N, \tag{13}$$

where N is the number of nodes, and $C_{ik}^{(m)}$ are the weighting coefficients for the m-th derivative [35]. Shifted Chebyshev-Gauss-Lobatto points are utilized as nodal points, which are computed according to the relation [36].

$$r_k = \frac{1}{2}\left\{1-\cos\left(\frac{\pi(k-1)}{N-1}\right)\right\}, \quad k=1,\ldots,N. \tag{14}$$

Considering the normalizations given by Eq. (12) and finite series representations of the unknown functions, governing partial differential equations are recast into the following form:

$$\begin{aligned}&\frac{1}{\eta}\left(\overline{F}_{11}-\overline{B}_{11}\right)\left\{\sum_{k=1}^{N}C_{ik}^{(3)}\overline{w}_k+\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(2)}\overline{w}_k-\frac{1}{\chi_i^2}\sum_{k=1}^{N}C_{ik}^{(1)}\overline{w}_k\right\}+\overline{A}_{11}\left\{\sum_{k=1}^{N}C_{ik}^{(2)}\overline{u}_k+\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(1)}\overline{u}_k-\frac{\overline{u}_i}{\chi_i^2}\right\}\\&-\overline{F}_{11}\left\{\sum_{k=1}^{N}C_{ik}^{(2)}\varphi_k+\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(1)}\varphi_k-\frac{\varphi_i}{\chi_i^2}\right\}\\&=\overline{I}_1\ddot{\overline{u}}_i+\frac{1}{\eta}\left(\overline{I}_4-\overline{I}_2\right)\sum_{k=1}^{N}C_{ik}^{(1)}\ddot{\overline{w}}_k-\overline{I}_4\ddot{\varphi}_i,\end{aligned} \tag{15a}$$

$$\begin{aligned}&\frac{1}{\eta}\left\{2\overline{F}_{22}-\overline{D}_{11}-\overline{F}_{33}-\overline{A}_{552}+\overline{F}_{572}-\frac{\overline{F}_{552}}{4}\right\}\left\{\chi_i\sum_{k=1}^{N}C_{ik}^{(4)}\overline{w}_k+2\sum_{k=1}^{N}C_{ik}^{(3)}\overline{w}_k-\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(2)}\overline{w}_k+\frac{1}{\chi_i^2}\sum_{k=1}^{N}C_{ik}^{(1)}\overline{w}_k\right\}\\&+\eta\left\{\frac{\overline{F}_{662}}{4}+k_s\overline{F}_{55}\right\}\left\{\chi_i\sum_{k=1}^{N}C_{ik}^{(2)}\overline{w}_k+\sum_{k=1}^{N}C_{ik}^{(1)}\overline{w}_k\right\}+\left(\overline{B}_{11}-\overline{F}_{11}\right)\left\{\chi_i\sum_{k=1}^{N}C_{ik}^{(3)}\overline{u}_k+2\sum_{k=1}^{N}C_{ik}^{(2)}\overline{u}_k-\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(1)}\overline{u}_k+\frac{\overline{u}_i}{\chi_i^2}\right\}\\&+\left\{\overline{F}_{33}-\overline{F}_{22}-\frac{\overline{F}_{572}}{2}+\frac{\overline{F}_{552}}{4}\right\}\left\{\chi_i\sum_{k=1}^{N}C_{ik}^{(3)}\varphi_k+2\sum_{k=1}^{N}C_{ik}^{(2)}\varphi_k-\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(1)}\varphi_k+\frac{\varphi_i}{\chi_i^2}\right\}-\eta^2\left\{k_s\overline{F}_{55}+\frac{\overline{F}_{662}}{4}\right\}\left\{\chi_i\sum_{k=1}^{N}C_{ik}^{(1)}\varphi_k+\varphi_i\right\}+\eta^2\chi_i q_{oi}\\&=\eta\chi_i\overline{I}_1\ddot{\overline{w}}_i+\frac{1}{\eta}\left(2\overline{I}_5-\overline{I}_3-\overline{I}_6\right)\left\{\sum_{k=1}^{N}C_{ik}^{(1)}\ddot{\overline{w}}_k+\chi_i\sum_{k=1}^{N}C_{ik}^{(2)}\ddot{\overline{w}}_k\right\}+\left(\overline{I}_2-\overline{I}_4\right)\left\{\ddot{\overline{u}}_i+\chi_i\sum_{k=1}^{N}C_{ik}^{(1)}\ddot{\overline{u}}_k\right\}+\left(\overline{I}_6-\overline{I}_5\right)\left\{\ddot{\varphi}_i+\chi_i\sum_{k=1}^{N}C_{ik}^{(1)}\ddot{\varphi}_k\right\},\end{aligned} \tag{15b}$$

$$\begin{aligned}&\left\{\overline{F}_{33}+\frac{\overline{F}_{552}}{4}\right\}\left\{\sum_{k=1}^{N}C_{ik}^{(2)}\varphi_k+\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(1)}\varphi_k-\frac{\varphi_i}{\chi_i^2}\right\}\\&+\frac{1}{\eta}\left\{\overline{F}_{22}-\overline{F}_{33}+\frac{\overline{F}_{572}}{2}-\frac{\overline{F}_{552}}{4}\right\}\left\{\sum_{k=1}^{N}C_{ik}^{(3)}\overline{w}_k+\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(2)}\overline{w}_k\right.\\&\left.-\frac{1}{\chi_i^2}\sum_{k=1}^{N}C_{ik}^{(1)}\overline{w}_k\right\}+\eta\left\{k_s\overline{F}_{55}+\frac{\overline{F}_{662}}{4}\right\}\left\{\sum_{k=1}^{N}C_{ik}^{(1)}\overline{w}_k-\eta\varphi_i\right\}\\&-\overline{F}_{11}\left\{\sum_{k=1}^{N}C_{ik}^{(2)}\overline{u}_k+\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(1)}\overline{u}_k-\frac{\overline{u}_i}{\chi_i^2}\right\}\\&=\overline{I}_6\ddot{\varphi}_i+\frac{1}{\eta}\left(\overline{I}_5-\overline{I}_6\right)\sum_{k=1}^{N}C_{ik}^{(1)}\ddot{\overline{w}}_k-\overline{I}_4\ddot{\overline{u}}_i.\end{aligned} \tag{15c}$$

In all three governing equations, $i=1,\ldots,N$; and the dot stands for differentiation with respect to time.

Two different types of micro-plate configurations are considered in the numerical analyses. The first one is that of an annular micro-plate simply supported along its boundaries at $r=R_i$ and $r=R_o$. The second one is the problem of a circular micro-plate simply supported over its periphery $r=R_o$. In terms of the finite series, boundary conditions for the annular micro-plate problem read

$$\overline{u}_1 = \overline{u}_N = 0, \tag{16a}$$

$$\overline{w}_1 = \overline{w}_N = 0, \tag{16b}$$

$$\begin{aligned}&\left\{\overline{D}_{11}-2\overline{F}_{22}+\overline{F}_{33}+\overline{A}_{552}-\overline{F}_{572}+\frac{\overline{F}_{552}}{4}\right\}\gamma\sum_{k=1}^{N}C_{1k}^{(2)}\overline{w}_k+\left\{\overline{D}_{11}^{*}-2\overline{F}_{22}^{*}+\overline{F}_{33}^{*}-\overline{A}_{552}+\overline{F}_{572}-\frac{\overline{F}_{552}}{4}\right\}\sum_{k=1}^{N}C_{1k}^{(1)}\overline{w}_k\\&+\eta\left(\overline{F}_{11}-\overline{B}_{11}\right)\gamma\sum_{k=1}^{N}C_{1k}^{(1)}\overline{u}_k+\eta\left\{\overline{F}_{22}-\overline{F}_{33}+\frac{\overline{F}_{572}}{2}-\frac{\overline{F}_{552}}{4}\right\}\gamma\sum_{k=1}^{N}C_{1k}^{(1)}\varphi_k+\eta\left\{\overline{F}_{22}^{*}-\overline{F}_{33}^{*}-\frac{\overline{F}_{572}}{2}+\frac{\overline{F}_{552}}{4}\right\}\varphi_1\\&=\left\{\overline{D}_{11}-2\overline{F}_{22}+\overline{F}_{33}+\overline{A}_{552}-\overline{F}_{572}+\frac{\overline{F}_{552}}{4}\right\}(1+\gamma)\sum_{k=1}^{N}C_{Nk}^{(2)}\overline{w}_k+\left\{\overline{D}_{11}^{*}-2\overline{F}_{22}^{*}+\overline{F}_{33}^{*}-\overline{A}_{552}+\overline{F}_{572}-\frac{\overline{F}_{552}}{4}\right\}\sum_{k=1}^{N}C_{Nk}^{(1)}\overline{w}_k\\&+\eta\left(\overline{F}_{11}-\overline{B}_{11}\right)(1+\gamma)\sum_{k=1}^{N}C_{Nk}^{(1)}\overline{u}_k+\eta\left\{\overline{F}_{22}-\overline{F}_{33}+\frac{\overline{F}_{572}}{2}-\frac{\overline{F}_{552}}{4}\right\}(1+\gamma)\sum_{k=1}^{N}C_{Nk}^{(1)}\varphi_k+\eta\left\{\overline{F}_{22}^{*}-\overline{F}_{33}^{*}-\frac{\overline{F}_{572}}{2}+\frac{\overline{F}_{552}}{4}\right\}\varphi_N=0,\end{aligned}\tag{16c}$$

$$\begin{aligned}&\frac{1}{\eta}\left\{\overline{F}_{22}-\overline{F}_{33}+\frac{\overline{F}_{572}}{2}-\frac{\overline{F}_{552}}{4}\right\}\gamma\sum_{k=1}^{N}C_{1k}^{(2)}\overline{w}_k+\frac{1}{\eta}\left\{\overline{F}_{22}^{*}-\overline{F}_{33}^{*}-\frac{\overline{F}_{572}}{2}+\frac{\overline{F}_{552}}{4}\right\}\sum_{k=1}^{N}C_{1k}^{(1)}\overline{w}_k-\overline{F}_{11}\gamma\sum_{k=1}^{N}C_{1k}^{(1)}\overline{u}_k\\&+\left\{\overline{F}_{33}+\frac{\overline{F}_{552}}{4}\right\}\gamma\sum_{k=1}^{N}C_{1k}^{(1)}\varphi_k+\left\{\overline{F}_{33}^{*}-\frac{\overline{F}_{552}}{4}\right\}\varphi_1\\&=\frac{1}{\eta}\left\{\overline{F}_{22}-\overline{F}_{33}+\frac{\overline{F}_{572}}{2}-\frac{\overline{F}_{552}}{4}\right\}(1+\gamma)\sum_{k=1}^{N}C_{Nk}^{(2)}\overline{w}_k+\frac{1}{\eta}\left\{\overline{F}_{22}^{*}-\overline{F}_{33}^{*}-\frac{\overline{F}_{572}}{2}+\frac{\overline{F}_{552}}{4}\right\}\sum_{k=1}^{N}C_{Nk}^{(1)}\overline{w}_k-\overline{F}_{11}(1+\gamma)\sum_{k=1}^{N}C_{Nk}^{(1)}\overline{u}_k\\&+\left\{\overline{F}_{33}+\frac{\overline{F}_{552}}{4}\right\}(1+\gamma)\sum_{k=1}^{N}C_{Nk}^{(1)}\varphi_k+\left\{\overline{F}_{33}^{*}-\frac{\overline{F}_{552}}{4}\right\}\varphi_N=0.\end{aligned}\tag{16d}$$

The circular micro-plate is approximated as an annular micro-plate with a very small inner radius R_i in the numerical solution. Thus, the conditions at $r = R_o$ given above are also valid for the simply-supported circular micro-plate whereas the conditions at $r = R_i$ have to be replaced by the following equalities:

$$\overline{u}_1=0,\tag{17a}$$

$$\begin{aligned}&\left\{2\overline{F}_{22}-\overline{D}_{11}-\overline{F}_{33}-\overline{A}_{552}+\overline{F}_{572}-\frac{\overline{F}_{552}}{4}\right\}\left\{\sum_{k=1}^{N}C_{1k}^{(3)}\overline{w}_k\right.\\&\left.+\frac{1}{\gamma_\varepsilon}\sum_{k=1}^{N}C_{1k}^{(2)}\overline{w}_k\right\}+\eta\left(\overline{B}_{11}-\overline{F}_{11}\right)\left\{\sum_{k=1}^{N}C_{1k}^{(2)}\overline{u}_k+\frac{1}{\gamma_\varepsilon}\sum_{k=1}^{N}C_{1k}^{(1)}\overline{u}_k\right\}\\&+\eta\left\{\overline{F}_{33}-\overline{F}_{22}-\frac{\overline{F}_{572}}{2}+\frac{\overline{F}_{552}}{4}\right\}\left\{\sum_{k=1}^{N}C_{1k}^{(2)}\varphi_k+\frac{1}{\gamma_\varepsilon}\sum_{k=1}^{N}C_{1k}^{(1)}\varphi_k\right\}=0,\end{aligned}\tag{17b}$$

$$\sum_{k=1}^{N}C_{1k}^{(1)}\overline{w}_k=0,\tag{17c}$$

$$\varphi_1=0.\tag{17d}$$

γ_ε in Eq. (17b) is to be taken as a sufficiently small number.

In the case of static loading, for both annular and circular plate problems final matrix form of the system comprising partial differential equations and boundary conditions is derived as follows:

$$\mathbf{KX}+\mathbf{Q}=\mathbf{0}.\tag{18}$$

where **K** is the stiffness matrix, **Q** is the generalized distributed load vector, and **X** is the vector of nodal displacements written as

$$\mathbf{X}=\left\{\{\overline{u}_i\}^{\mathrm{T}},\{\overline{w}_i\}^{\mathrm{T}},\{\varphi_i\}^{\mathrm{T}}\right\}^{\mathrm{T}},\quad i=1,\,2,\ldots,N.\tag{19}$$

In a similar way, matrix form of the equations for the free vibration problems is obtained in the form:

$$\mathbf{KX}+\mathbf{M}\ddot{\mathbf{X}}=\mathbf{0}.\tag{20}$$

M here is the mass matrix. Assuming a solution of the form

$$\mathbf{X}=\mathbf{X}^{*}e^{i\omega\tau},\tag{21}$$

Eq. (20) is reduced to

$$\left(\mathbf{K}-\omega^2\mathbf{M}\right)\mathbf{X}^{*}=\mathbf{0}.\tag{22}$$

$\mathbf{X}^{*}$ in this equation is the free vibration mode shape vector, and ω designates dimensionless natural frequency of vibration, which are defined by

$$\mathbf{X}^{*}=\left\{\left\{\overline{u}_i^{*}\right\}^{\mathrm{T}},\left\{\overline{w}_i^{*}\right\}^{\mathrm{T}},\left\{\varphi_i^{*}\right\}^{\mathrm{T}}\right\}^{\mathrm{T}},\quad i=1,\,2,\ldots,N.\tag{23a}$$

$$\omega=\sqrt{\frac{I_{10}}{A_{110}}}(R_o-R_i)\Omega,\tag{23b}$$

where Ω stands for the natural frequency of free vibrations. Solution of Eq. (22) yields the dimensionless natural frequencies and the corresponding mode shape vectors. Notice that there are three possible deformation modes in the examined micro-plate problems, namely radial, transverse, and rotational deformation modes; which respectively correspond to the unknown functions u, w, and ϕ. In numerical analyses, the deformation mode to which a computed natural frequency corresponds is ascertained by examining the mode shape vector.

4. Numerical results

Graded ceramic-metal annular and circular micro-plates are considered in the parametric analyses. Referring to Fig. 1, volume fractions of the constituent phases are expressed as follows

$$V_c(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^{\lambda}, \tag{24a}$$

$$V_m(z) = 1 - V_c(z), \tag{24b}$$

where the subscripts c and m respectively represent the ceramic and the metallic components; and the exponent λ is an inhomogeneity constant defining the property variation. The composition is 100% metallic at $z = -h/2$, and 100% ceramic at $z = h/2$. Note that for $\lambda>1$ the plate considered is metal-rich whereas for $\lambda<1$ it has a ceramic-rich profile. Elastic properties are computed by means of the Mori-Tanaka technique [37], according to which effective bulk modulus B_e, and effective shear modulus μ_e are evaluated from

$$B_e = \frac{V_c(B_c - B_m)}{1 + \dfrac{(B_c - B_m)V_m}{\dfrac{4\mu_m}{3} + B_m}} + B_m, \tag{25a}$$

$$\mu_e = \frac{V_c(\mu_c - \mu_m)}{1 + \dfrac{(\mu_c - \mu_m)V_m}{\left\{\mu_m + \dfrac{(9B_m + 8\mu_m)\mu_m}{6(B_m + 2\mu_m)}\right\}}} + \mu_m. \tag{25b}$$

Once B_e and μ_e are computed at a given location z across the thickness, modulus of elasticity E and Poisson's ratio ν can be calculated through

$$E(z) = \frac{9B_e(z)\mu_e(z)}{3B_e(z) + \mu_e(z)}, \tag{26a}$$

$$\nu(z) = \frac{3B_e(z) - 2\mu_e(z)}{6B_e(z) + 2\mu_e(z)}. \tag{26b}$$

Mass density ρ, and the length scale parameter l are evaluated by employing the rule of mixtures. These two properties are expressed in the following form:

$$\rho(z) = \rho_c V_c(z) + \rho_m V_m(z), \tag{27a}$$

$$l(z) = l_c V_c(z) + l_m V_m(z). \tag{27b}$$

The particular ceramic and metallic phases considered in the parametric analyses are silicon carbide (SiC) and aluminum (Al), whose material properties are given as:

$$E_c = 427\ \text{GPa}, \quad \nu_c = 0.17, \quad \rho_c = 3100\ \text{kg}\Big/\text{m}^3, \tag{28a}$$

$$E_m = 70\ \text{GPa}, \quad \nu_m = 0.3, \quad \rho_m = 2702\ \text{kg}\Big/\text{m}^3. \tag{28b}$$

Table 2
Comparisons of the natural frequency $\Omega/(2\pi)$ (in kHz) generated by considering a homogeneous circular micro-plate hinged along its boundary. $E = 1.44$ GPa, ν=0.38, ρ=1220 kg/m^3, h = 100 μm, $R_o/h = 50$.

h/l	1			2		
Mode	First	Second	Third	First	Second	Third
Wang et al. [38]	1.2051	13.3250	34.1840	1.1565	8.6931	22.0114
Present study	1.2051	13.3263	34.1874	1.1566	8.6939	22.0136

The length scale parameter of the metallic component l_m is taken as 15 μm, which is a reference value commonly used in the literature [18,23]. Ceramic component's length scale parameter l_c on the other hand is set as 22.5 μm in a number of parametric analyses. In the remaining cases, l_c is varied to be able to illustrate the influence of the length scale parameter variation.

In order to exhibit the accuracy of the numerical results generated by means of the developed procedures, two sets of comparisons are given as presented by Tables 1 and 2. Table 1 tabulates dimensionless first natural frequencies obtained for a functionally graded annular micro-plate hinged along its boundaries. The results computed in the present study are seen to be in very good agreement with those provided by Ke et al. [18]. Note that in the work of Ke et al. [18], Mindlin plate theory is used and the length scale parameter l is assumed to have a constant value of 15 μm. Table 2 presents comparisons of the first three natural frequencies calculated by considering a simply-supported homogeneous circular micro-plate. Our results are in excellent agreement with those of Wang et al. [38], who adopted Kirchhoff plate theory in their analysis. The comparisons provided are indicative of the high degree of accuracy achieved through the use of developed methods. All frequencies tabulated in Tables 1 and 2 correspond to the transverse deformation mode.

In Figs. 2–9 and Tables 3 and 4, we present the results of parametric analyses carried out to examine the influences of various factors on the behaviors of graded annular and circular micro-plates. The results corresponding to static loading of micro-plates are provided in Figs. 2–5. In all analyses, annular and circular micro-plates are assumed to be simply-supported at their boundaries. Fig. 2 depicts static deflection profiles of a graded annular micro-plate that are generated through the use of three different plate theories. Kirchhoff and third-order shear deformation theories are seen to lead to almost identical profiles whereas Mindlin plate theory slightly overestimates the micro-plate deflection. Static loading results provided in Figs. 3–5 are computed by means of the third-order shear deformation theory. Fig. 3 demonstrates the effect of the spatial variation of the length scale parameter upon the static deflection of a functionally graded annular micro-plate. The deflection profiles are given for four different values of the length scale parameter ratio l_c/l_m. According to Eq. (27b), the length scale parameter varies within the plate when $l_c/l_m \neq 1$, and it is constant when $l_c/l_m = 1$. Fig. 3 indicates that the impact of the length scale parameter variation on the static deflection profile is rather significant. Static deflection drops as the length scale parameter ratio l_c/l_m is increased from 1/3 to 2. A similar set of results illustrating the influence of the length scale parameter ratio

Table 1
Comparisons of the first dimensionless natural frequency $R_o\Omega\sqrt{I_{10}/A_{110}}$ generated by considering an FGM annular micro-plate hinged along the boundaries. λ=1.2, $R_o/h = 10$, $R_o/R_i = 5$, $l = 15$ μm.

h/l	1	1.5	2	3	6	10	16	Classical
Ke et al. [18]	1.5711	1.1996	1.0252	0.8718	0.7582	0.7301	0.7201	0.7141
Present study	1.5710	1.1996	1.0252	0.8720	0.7586	0.7306	0.7206	0.7141

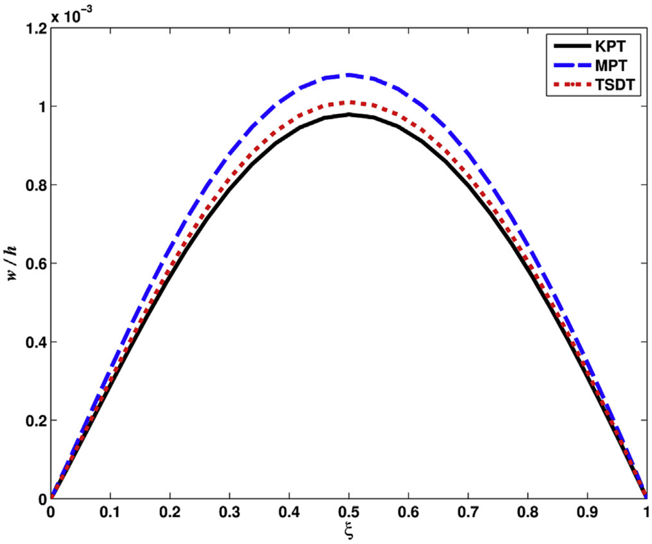

Fig. 2. Static deflection profiles of a graded annular micro-plate generated by means of three different plate theories. $q = 1$ MPa, $R_o/h = 10$, $l_m = 15$ μm, $h/l_m = 2$, $l_c/l_m = 3/2$, λ=2, $R_o/R_i = 4$.

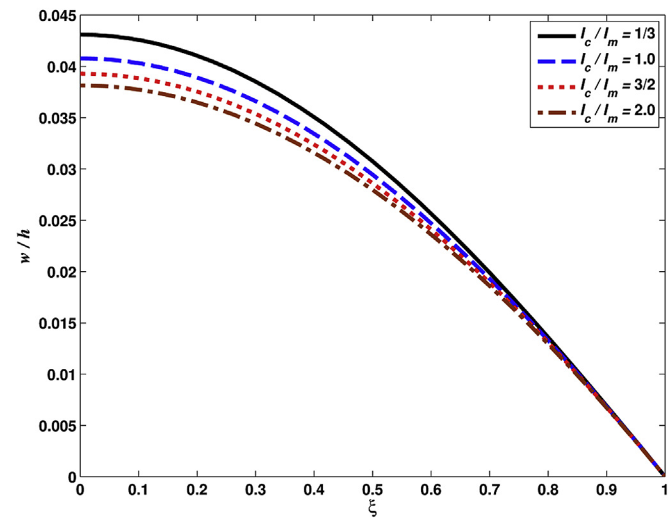

Fig. 4. Static deflection profiles of a graded circular micro-plate generated by considering four different values of l_c/l_m. $q = 1$ MPa, $R_o/h = 10$, $l_m = 15$ μm, $h/l_m = 2$, λ=2.

is provided in Fig. 4. These results are calculated by considering a functionally graded circular micro-plate. Again, the influence of the length scale parameter ratio is found to be notable. Fig. 5 shows through-the-thickness distributions of the normalized normal stress σ_{rr}/q computed at $r = (R_i+R_o)/2$ in an annular micro-plate. The tensile stress evaluated at $z = h/2$ is larger in magnitude compared to the compressive stress calculated at $z = -h/2$. As the length scale parameter ratio l_c/l_m is increased, magnitude of the normalized normal stress decreases. This figure demonstrates the fact that under static loading condition, length scale parameter variation is influential also on the stress distribution.

Numerical results regarding free vibration behaviors of annular and circular micro-plates are provided in Tables 3 and 4, and Figs. 6–9. Table 3 presents first three dimensionless natural frequencies corresponding to the transverse deformation mode, that are computed by considering a graded annular micro-plate. Frequencies are given as functions of the inhomogeneity exponent λ and the length scale parameter ratio l_c/l_m. As the inhomogeneity exponent λ increases from 0.5 to 5, i.e. as the micro-plate becomes metal-rich, frequency of each mode gets smaller. The rise in l_c/l_m on the other hand leads to increases in the natural frequencies. As can be observed from Table 4, general free vibration behavior of a functionally graded circular micro-plate is similar to that of the annular micro-plate.

Figs. 6–9 depict variations of the first dimensionless natural frequency ω_1. In all cases ω_1 corresponds to the transverse deformation mode. Fig. 6 provides ω_1 vs h/l_m curves for the three different plate theories. ω_1 values are extracted by considering a functionally graded annular micro-plate. The curves obtained for the different plate theories are seen to lie in close proximity of each other. In each case, natural frequency tends to a constant value as h/l_m gets larger. However, as h/l_m becomes smaller a sharp rise is

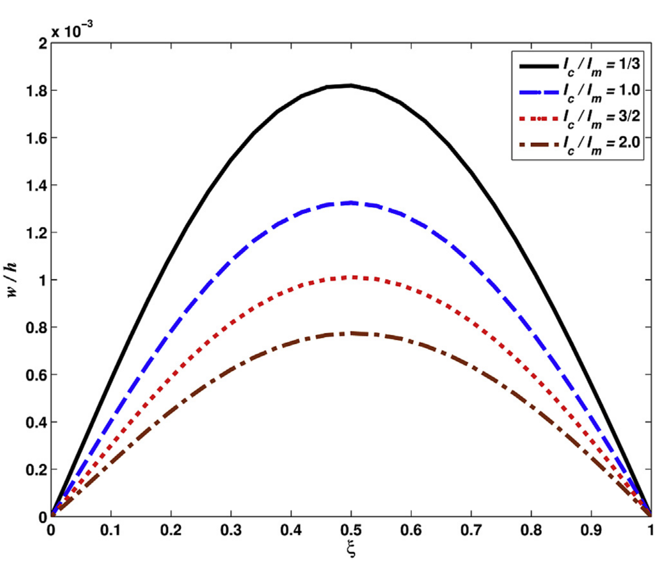

Fig. 3. Static deflection profiles of a graded annular micro-plate generated by considering four different values of l_c/l_m. $q = 1$ MPa, $R_o/h = 10$, $l_m = 15$ μm, $h/l_m = 2$, λ=2, $R_o/R_i = 4$.

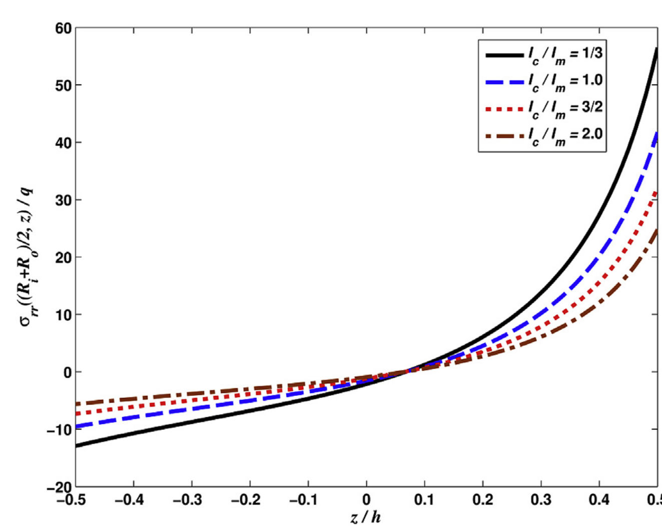

Fig. 5. Through-the-thickness distributions of normal stress in a graded annular micro-plate generated by considering four different values of l_c/l_m. $q = 1$ MPa, $R_o/h = 10$, $l_m = 15$ μm, $h/l_m = 2$, λ=2, $R_o/R_i = 4$.

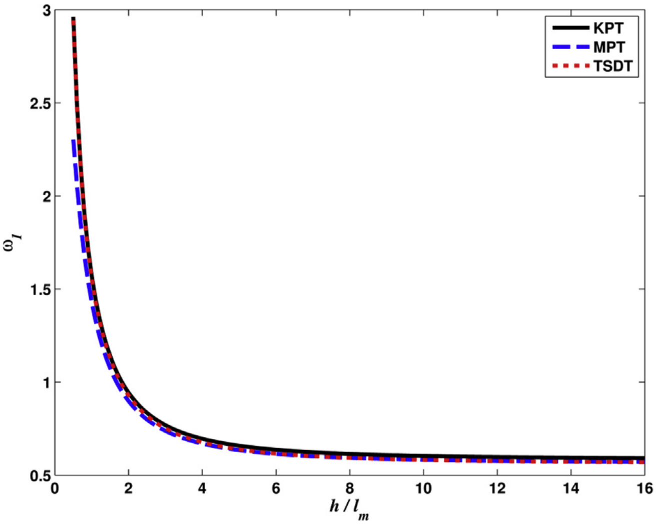

Fig. 6. Variations of the first dimensionless natural frequency with respect to h/l_m generated by considering a graded annular micro-plate and three different plate theories. $R_o/h = 10$, $l_m = 15$ μm, $l_c/l_m = 3/2$, λ=2, $R_o/R_i = 4$.

induced in ω_1, which is an indication of the size effect at the micro-scale. When thickness h is at the order of the length scale parameter l_m, the first natural frequency is significantly larger due to the stiffening of the micro-plate. The results displayed in Figs. 7–9 are calculated using third-order shear deformation theory. In Fig. 7, we give natural frequency variations of an annular micro-plate for four different values of the inhomogeneity exponent λ. ω_1 decreases continuously as the inhomogeneity parameter is increased from 0.5 to 5. As a result, we can deduce that a ceramic-rich annular micro-plate is stiffer and possesses higher natural frequencies compared to a metal-rich micro-plate. Fig. 8 shows first dimensionless natural frequency of an annular micro-plate as functions of h/l_m and the length scale parameter ratio l_c/l_m. The impact of l_c/l_m is considerable especially when h is at the order of l_m. An increase in the length scale parameter ratio causes a corresponding increase in ω_1. Finally, in Fig. 9 we present ω_1 variations illustrating the influence of the length scale parameter ratio for a graded circular micro-plate.

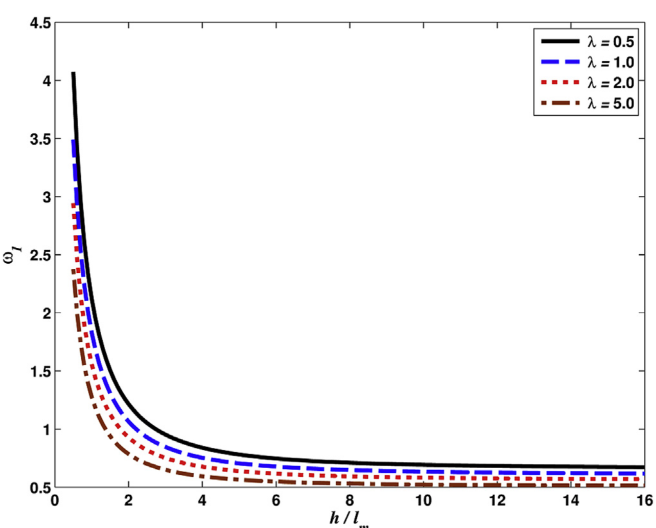

Fig. 7. Variations of the first dimensionless natural frequency with respect to h/l_m generated by considering a graded annular micro-plate and four different values of λ. $R_o/h = 10$, $l_m = 15$ μm, $l_c/l_m = 3/2$, $R_o/R_i = 4$.

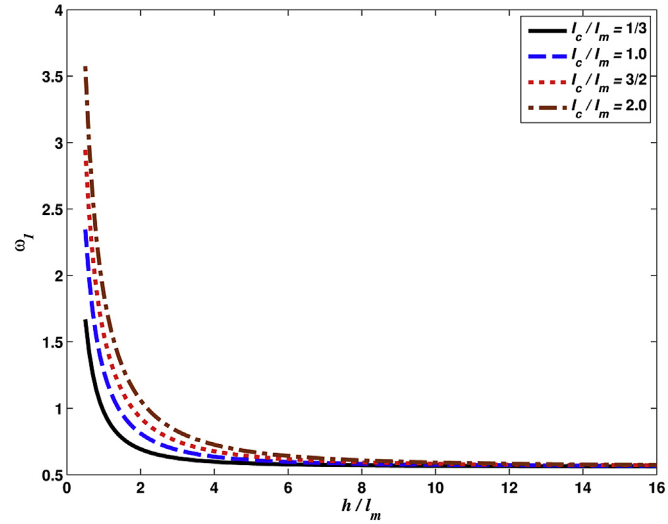

Fig. 8. Variations of the first dimensionless natural frequency with respect to h/l_m generated by considering a graded annular micro-plate and four different values of l_c/l_m. $R_o/h = 10$, $l_m = 15$ μm, λ=2, $R_o/R_i = 4$.

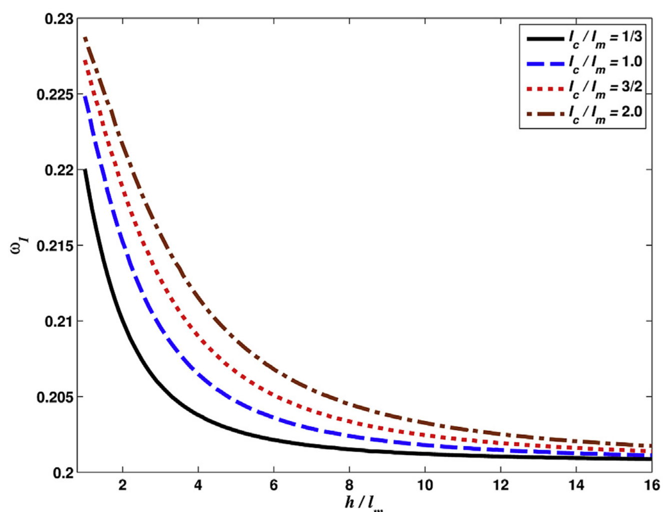

Fig. 9. Variations of the first dimensionless natural frequency with respect to h/l_m generated by considering a graded circular micro-plate and four different values of l_c/l_m. $R_o/h = 10$, $l_m = 15$ μm, λ=2.

Table 3
Dimensionless natural frequencies corresponding to the transverse deformation mode generated by considering a graded annular micro-plate. $R_o/h = 10$, $l_m = 15$ μm, $h/l_m = 2$, $R_o/R_i = 4$.

Mode	λ	l_c/l_m			
		1/3	1.0	3/2	2.0
First	0.5	0.7635	0.9928	1.2150	1.4590
	1.0	0.7248	0.8968	1.0645	1.2506
	2.0	0.6926	0.8113	0.9285	1.0606
	5.0	0.6556	0.7202	0.7854	0.8610
Second	0.5	2.6246	3.5248	4.3705	5.1623
	1.0	2.4536	3.1366	3.7758	4.3993
	2.0	2.3420	2.8206	3.2731	3.7378
	5.0	2.2660	2.5366	2.8001	3.0908
Third	0.5	5.3634	7.3193	9.0952	11.7226
	1.0	5.0527	6.5193	7.8695	10.2989
	2.0	4.7903	5.8567	6.8301	7.7069
	5.0	4.6137	5.2470	5.8417	6.4567

Table 4
Dimensionless natural frequencies corresponding to the transverse deformation mode generated by considering a graded circular micro-plate. $R_o/h = 10$, $l_m = 15$ μm, $h/l_m = 2$.

Mode	λ	l_c/l_m			
		1/3	1.0	3/2	2.0
First	0.5	0.2389	0.2483	0.2545	0.2645
	1.0	0.2238	0.2311	0.2362	0.2394
	2.0	0.2103	0.2158	0.2188	0.2211
	5.0	0.1918	0.1946	0.1971	0.1981
Second	0.5	1.5000	1.9505	2.3942	2.8819
	1.0	1.4044	1.7456	2.0810	2.4547
	2.0	1.3426	1.5777	1.8114	2.0753
	5.0	1.2930	1.4197	1.5482	1.6977
Third	0.5	3.5711	4.7421	5.8497	7.2054
	1.0	3.3361	4.2290	5.0711	6.1920
	2.0	3.1829	3.8097	4.4063	4.9419
	5.0	3.0736	3.4258	3.7703	4.1435

Length scale parameter ratio effect is again found to be significant. Thus, as is the case in static bending solution, it is necessary to incorporate the variation of the length scale parameter into the formulation in dynamic analysis.

5. Closure

In this article, we propose new methods that can take into account spatial variation of the length scale parameter in static and free vibration analyses of functionally graded annular and circular micro-plates. The developments are based upon the modified couple stress theory. Displacement field is expressed in a unified way to be able to generate numerical results in accordance with three different beam theories, namely: Kirchhoff, Mindlin, and third-order shear deformation theories. Partial differential equations are derived by employing the variational approach and Hamilton's principle. These equations are solved numerically by means of the differential quadrature method. Comparisons to the results available in the literature verify the developed procedures. Numerical analyses are carried out to study the influences of various problem parameters on static deformation profiles, stresses, and natural vibration frequencies.

The effect of the length scale parameter variation is examined through the use of the length scale parameter ratio l_c/l_m. When this ratio is equal to one, length scale parameter is constant across the micro-plate thickness. Non-unity values of l_c/l_m implies a variation in the length scale parameter. The numerical results presented point out that length scale parameter ratio significantly affects both static and the dynamic responses of annular and circular micro-plates. An increase in l_c/l_m is shown to lead to drops in static deflection and stress magnitude; and a rise in the first dimensionless natural frequency. Thus, the methods presented in this article are capable of producing more realistic results regarding the behavior of FGM micro-plates possessing circular and annular profiles.

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Bending and free vibrations of functionally graded annular and circular micro-plates under thermal loading

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ABSTRACT

We introduce solution methods capable of treating static bending and free vibration problems involving thermally loaded functionally graded annular and circular micro-plates. Formulation is based on modified couple stress theory; and related governing partial differential equations and boundary conditions are derived by means of Hamilton's principle. Displacement field is expressed in a unified way so as to produce numerical results in accordance with Kirchhoff, Mindlin, and third-order shear deformation theories. All material properties, including the length scale parameter, are assumed to be functions of the thickness coordinate. The static and dynamic problems are solved by means of differential quadrature method. Proposed procedures are verified through comparisons made to the findings available in the technical literature on thermally stressed axisymmetric plates. Detailed numerical results are presented in order to demonstrate the influences of thermal loading magnitude, and material and geometric parameters upon static deformation profiles, stresses, and natural vibration frequencies.

1. Introduction

Functionally graded materials (FGMs) are advanced composites, that possess smooth spatial variations in the volume fractions of the constituent phases. These variations are additional degrees of freedom in materials design and allow customization of physical properties. The characteristic feature of FGMs is inhomogeneity at both micro- and macro-scale. The inhomogeneity and continuous spatial variations of the physical properties need to be accounted for in theoretical and computational studies so as to produce realistic results regarding behavior of graded structures.

Since the inception of the concept, FGMs have been proposed to be employed in a number of technological applications such as thermal barrier and tribological coatings [1,2], biomaterials [3,4], and solid oxide fuel cells [5]. In recent years, use of functionally graded structures in micro-electro-mechanical-systems (MEMSs) also became feasible with the introduction of micro-scale FGM component production techniques like magnetron sputtering [6], plasma-enhanced chemical vapor deposition [7], and modified soft lithography [8]. As a result, there have been extensive research efforts directed towards examining mechanical behavior of micro-scale functionally graded structures. An important group of such structural members is comprised of micro-plates. Annular, circular, and rectangular micro-plates find applications in MEMS including micro-scale resonators, optical and pressure sensors, and gear pumps [9–12]. Our main objective in the present study is to develop methods of analysis for functionally graded *annular and circular* micro-plates, that are under the influence of *thermal loading.*

Modeling and analysis of micro-scale structures require adoption of an higher order continuum theory, that takes into account the size effect. The most commonly used higher order continuum theory in the analysis of annular and circular micro-plates is modified couple stress theory [13]. A single length scale parameter is needed in this theory to describe material response at the micro-scale. Modified couple stress theory based previous work on annular and circular micro-plates encompass both homogeneous and functionally graded structures. Wang et al. [14] and Zhou and Gao [15] presented procedures capable of resolving static bending problems involving homogeneous circular micro-plates. Results regarding large amplitude free vibrations of homogeneous circular plates are provided by Wang et al. [16]. In articles on functionally graded annular and circular micro-plates, examined problems include bending [17–19], free vibrations [17,18,20], buckling [18], and post-buckling [21].

The articles mentioned in the foregoing paragraph do not take thermal effects into consideration. Thermal loads on micro-scale

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structures could be induced due to environmental or electrical effects. Depending on the constraints imposed on the micro-structure, these loads may lead to severe thermal stresses that could jeopardize structural integrity. Thus, in analysis and design, it is important to possess tools capable of considering thermal loads acting on micro-scale structures. In the present article, we put forward modified couple stress theory based solutions for *thermally loaded* functionally graded annular and circular micro-plates. Presented techniques are capable of resolving static bending and free vibration problems by taking into account through-the-thickness temperature variation and thermal strains.

The main advantage of modified couple stress theory over other higher order continuum theories is that, it requires a single material property for micro-scale material characterization. This property is the length scale parameter, which can be determined through experimental procedures as described by Lam et al. [22]. Strain gradient elasticity theory, which is another commonly used higher order continuum theory, requires three additional material properties in the characterization [22]. Hence, modified couple stress theory is simpler to implement, and capable of successfully capturing the size effect prevailing at the micro-scale.

The posed static and dynamic problems are formulated in terms of partial differential equations. Two sets of governing partial differential equations are derived for *thermally loaded* FGM annular and circular micro-plates. One group of equations is valid for micro-plates in static bending and the other is applicable for micro-plates undergoing free vibrations. A unified formulation is established to be able generate results for three different plate theories, namely Kirchhoff plate theory (KPT), Mindlin plate theory (MPT), and third-order shear deformation plate theory (TSDT). The systems comprising governing equations and boundary conditions are solved numerically by means of differential quadrature method (DQM). Proposed techniques are verified by making comparisons to the findings available in the literature. Presented numerical results shed light on the influences of thermal loading, and material and geometric parameters upon static deformation profiles, stresses, and natural vibration frequencies.

2. Formulation

Fig. 1 depicts a functionally graded annular micro-plate, that is assumed to be under the influence of thermal loading. Inner and outer radii, and thickness of the plate are respectively denoted by R_i, R_o, and h. Circular micro-plate possesses exactly the same geometric features except for the fact that $R_i = 0$. All material properties and temperature are functions of only the thickness coordinate, i.e. z. As a consequence, temperature and deformation fields are both axisymmetric.

Ceramic–metal functionally graded annular and circular micro-plates are considered in the parametric analyses. All plates are 100% metallic at $z = -h/2$ and 100% ceramic at $z = h/2$. In the computation of elastic properties modulus of elasticity and Poisson's ratio, we utilize Mori–Tanaka method [23]. These two properties are of the forms

$$E(z) = \frac{9B_e(z)\mu_e(z)}{3B_e(z) + \mu_e(z)}, \tag{1a}$$

$$\nu(z) = \frac{3B_e(z) - 2\mu_e(z)}{6B_e(z) + 2\mu_e(z)}, \tag{1b}$$

$$B_e = \frac{V_c(B_c - B_m)}{1 + \frac{(B_c - B_m)V_m}{\frac{4\mu_m}{3} + B_m}} + B_m, \quad \mu_e = \frac{V_c(\mu_c - \mu_m)}{1 + \frac{(\mu_c - \mu_m)V_m}{\left\{\mu_m + \frac{(9B_m + 8\mu_m)\mu_m}{6(B_m + 2\mu_m)}\right\}}} + \mu_m. \tag{1c}$$

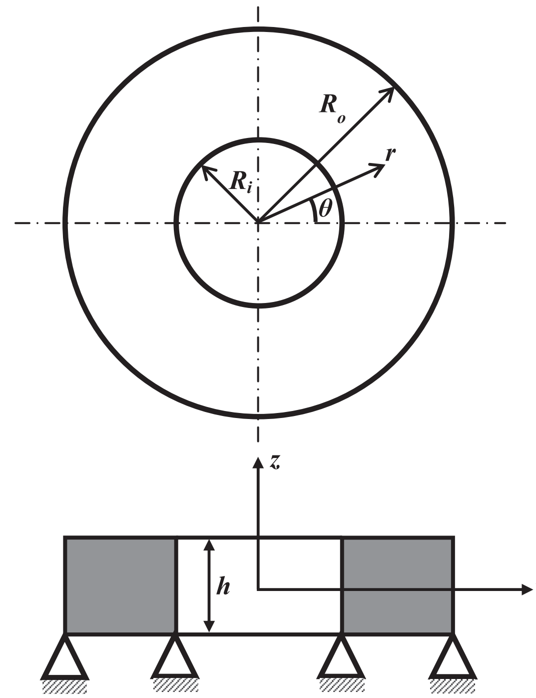

Fig. 1. A functionally graded simply-supported annular micro-plate.

E, ν, B_e, μ_e, and V here respectively designate modulus of elasticity, Poisson's ratio, effective bulk modulus, effective shear modulus, and volume fraction. The subscripts c and m stand for ceramic and metallic phases. Volume fractions are written in terms of a power function as follows:

$$V_c(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^{\lambda}, \tag{2a}$$

$$V_m(z) = 1 - V_c(z), \tag{2b}$$

where λ is an inhomogeneity parameter. Mass density, length scale parameter, thermal conductivity, and thermal expansion coefficient variations are evaluated employing rule of mixtures and respectively expressed as:

$$\rho(z) = \rho_c V_c(z) + \rho_m V_m(z), \tag{3a}$$

$$l(z) = l_c V_c(z) + l_m V_m(z), \tag{3b}$$

$$k(z) = k_c V_c(z) + k_m V_m(z), \tag{3c}$$

$$\alpha(z) = \alpha_c V_c(z) + \alpha_m V_m(z). \tag{3d}$$

Computation of spatial variation of a physical property of a functionally graded material requires the use of either a micromechanics model or experimental characterization data. In the absence of such theoretical and experimental results, approximate representations are utilized to evaluate material properties. Exponential [24,25] and power functions [26,27] are commonly employed to directly represent property variations. Another commonly used approximation involves expressing material properties by means of rule of mixtures. Variations of elastic properties such as modulus of elasticity, Poisson's ratio, and Lame's parameters; and thermal properties like thermal expansion coefficient, and thermal conductivity were evaluated by rule of mixtures in previous studies [28–32]. Due to lack of micromechanics formulations and experimental data, we employed the rule of mixtures representations given in Eq. (3) to approximate the spatial variations.

In the analysis of both static bending and free vibration problems, we suppose that the micro-plate shown in Fig. 1 is subjected to a temperature difference defined by

$$\theta(z) = T(z) - T_o, \tag{4}$$

in which T is final temperature, and T_o is temperature of the stress-free state. Solution of the one-dimensional heat equation for the FGM micro-plate results in the following expression for the temperature distribution

$$\theta(z) = \theta_L + \frac{\theta_U - \theta_L}{\int_{-h/2}^{h/2} \frac{dz}{k(z)}} \int_{-h/2}^{z} \frac{dz}{k(z)}, \tag{5}$$

where $\theta_U = T(h/2) - T_o$, $\theta_L = T(-h/2) - T_o$.

2.1. Formulation for static bending

According to modified couple stress theory [13], constitutive relations are given by:

$$\sigma_{ij} = 2\mu(\varepsilon_{ij} - \alpha\theta\delta_{ij}) + \lambda\delta_{ij}(\varepsilon_{kk} - 3\alpha\theta), \tag{6a}$$

$$m_{ij} = 2\mu l^2 \chi_{ij}, \tag{6b}$$

where σ_{ij} is Cauchy stress, ε_{ij} is total strain, μ and λ are Lame parameters, m_{ij} denotes deviatoric part of the couple stress tensor, and χ_{ij} is symmetric curvature tensor. The material properties μ, λ, α, and l are all functions of the thickness coordinate z. ε_{ij} and χ_{ij} are of the forms

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}\right], \tag{7a}$$

$$\boldsymbol{\chi} = \frac{1}{2}\left[\nabla\boldsymbol{\omega} + (\nabla\boldsymbol{\omega})^{\mathrm{T}}\right]. \tag{7b}$$

$\mathbf{u}$ and $\boldsymbol{\omega}$ here are respectively displacement and rotation vectors. Displacement vector components are expressed in the following way:

$$u_r(r,z,t) = u(r,t) - z\frac{\partial w}{\partial r} + f(z)\gamma(r,t), \tag{8a}$$

$$u_\theta(r,z,t) = 0, \tag{8b}$$

$$u_z(r,z,t) = w(r,t), \tag{8c}$$

where

$$\gamma(r,t) = \frac{\partial w}{\partial r} - \phi(r,t), \tag{9}$$

ϕ being the rotation at the mid-plane. The function f in Eq. (8a) depends on the plate theory used to represent plate deformation and is defined as:

$$f(z) = \begin{cases} 0, & \text{for Kirchhoff plate theory (KPT)}, \\ z, & \text{for Mindlin plate theory (MPT)}, \\ z\left(1 - \frac{4z^2}{3h^2}\right), & \text{for third-order shear deformation theory (TSDT)}. \end{cases} \tag{10}$$

Governing equations and corresponding boundary conditions are derived by means of Hamilton's principle, which for a static problem postulates that

$$\delta U = 0, \tag{11}$$

where U is strain energy evaluated from

$$U = \frac{1}{2}\int_{-h/2}^{h/2}\int_{R_i}^{R_o} \{\sigma_{rr}(\varepsilon_{rr} - \alpha\theta) + \sigma_{\theta\theta}(\varepsilon_{\theta\theta} - \alpha\theta) + 2(\sigma_{rz}\varepsilon_{rz} + m_{r\theta}\chi_{r\theta} + m_{z\theta}\chi_{z\theta})\}2\pi r dr dz. \tag{12}$$

Using Eqs. (6)–(10) in conjunction with Eqs. (11) and (12), we derive governing partial differential equations as follows:

$$(F_{11} - B_{11})\left\{\frac{\partial^3 w}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r}\right\} + A_{11}\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right\} - F_{11}\left\{\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} - \frac{\phi}{r^2}\right\} = 0, \tag{13a}$$

$$\begin{aligned}&\left\{2F_{22} - D_{11} - F_{33} - A_{552} + F_{572} - \frac{F_{552}}{4}\right\} \times \left\{r\frac{\partial^4 w}{\partial r^4} + 2\frac{\partial^3 w}{\partial r^3} - \frac{1}{r}\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial w}{\partial r}\right\} + \left\{\frac{F_{662}}{4} + k_s F_{55}\right\}\left\{r\frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{\partial r}\right\} \\ &+ (B_{11} - F_{11})\left\{r\frac{\partial^3 u}{\partial r^3} + 2\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{u}{r^2}\right\} + \left\{F_{33} - F_{22} - \frac{F_{572}}{2} + \frac{F_{552}}{4}\right\}\left\{r\frac{\partial^3 \phi}{\partial r^3} + 2\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\phi}{r^2}\right\} \\ &- \left\{k_s F_{55} + \frac{F_{662}}{4}\right\}\left\{r\frac{\partial \varphi}{\partial r} + \varphi\right\} = 0,\end{aligned} \tag{13b}$$

$$\begin{aligned}&\left\{F_{33} + \frac{F_{552}}{4}\right\}\left\{\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} - \frac{\phi}{r^2}\right\} + \left\{F_{22} - F_{33} + \frac{F_{572}}{2} - \frac{F_{552}}{4}\right\}\left\{\frac{\partial^3 w}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r}\right\} \\ &+ \left\{k_s F_{55} + \frac{F_{662}}{4}\right\}\left\{\frac{\partial w}{\partial r} - \varphi\right\} - F_{11}\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right\} = 0.\end{aligned} \tag{13c}$$

And, boundary conditions at $r = R_i$ and R_o read

$$\delta u = 0, \text{ or } (F_{11} - B_{11})r\frac{\partial^2 w}{\partial r^2} + (F_{11}^* - B_{11}^*)\frac{\partial w}{\partial r} + A_{11} r\frac{\partial u}{\partial r} + A_{11}^* u - F_{11} r\frac{\partial \phi}{\partial r} - F_{11}^*\phi - rN^T = 0, \tag{14a}$$

$$\begin{aligned}\delta w = 0, \text{ or } &\left\{2F_{22} - D_{11} - F_{33} - A_{552} + F_{572} - \frac{F_{552}}{4}\right\} \times \left\{\frac{\partial^3 w}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r}\right\} + \left\{\frac{F_{662}}{4} + k_s F_{55}\right\}\frac{\partial w}{\partial r} \\ &+ (B_{11} - F_{11})\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right\} + \left\{F_{33} - F_{22} - \frac{F_{572}}{2} + \frac{F_{552}}{4}\right\}\left\{\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} - \frac{\phi}{r^2}\right\} \\ &- \left\{\frac{F_{662}}{4} + k_s F_{55}\right\}\phi = 0,\end{aligned} \tag{14b}$$

$$\begin{aligned}\delta\left(\frac{\partial w}{\partial r}\right) = 0, \text{ or } &\left\{D_{11} - 2F_{22} + F_{33} + A_{552} - F_{572} + \frac{F_{552}}{4}\right\} r\frac{\partial^2 w}{\partial r^2} + \left\{D_{11}^* - 2F_{22}^* + F_{33}^* - A_{552} + F_{572} - \frac{F_{552}}{4}\right\}\frac{\partial w}{\partial r} \\ &+ (F_{11} - B_{11})r\frac{\partial u}{\partial r} + (F_{11}^* - B_{11}^*)u + \left\{F_{22} - F_{33} + \frac{F_{572}}{2} - \frac{F_{552}}{4}\right\} r\frac{\partial \phi}{\partial r} \\ &+ \left\{F_{22}^* - F_{33}^* - \frac{F_{572}}{2} + \frac{F_{552}}{4}\right\}\phi + r\left(M^T - N_f^T\right) = 0,\end{aligned} \tag{14c}$$

$$\delta\phi = 0, \text{ or } \left\{F_{22} - F_{33} + \frac{F_{572}}{2} - \frac{F_{552}}{4}\right\} r\frac{\partial^2 w}{\partial r^2} + \left\{F_{22}^* - F_{33}^* - \frac{F_{572}}{2} + \frac{F_{552}}{4}\right\}\frac{\partial w}{\partial r} - F_{11} r\frac{\partial u}{\partial r} - F_{11}^* u + \left\{F_{33} + \frac{F_{552}}{4}\right\} r\frac{\partial \phi}{\partial r} + \left\{F_{33}^* - \frac{F_{552}}{4}\right\}\phi + rN_f^T = 0. \tag{14d}$$

Coefficients and thermal loading terms in Eqs. (13) and (14) are found to be,

$$\{A_{11}, B_{11}, D_{11}, F_{11}, F_{22}, F_{33}\} = \int_{-h/2}^{h/2} \frac{E(z)}{1-\nu^2(z)}\left\{1, z, z^2, f, zf, f^2\right\} dz, \tag{15a}$$

$$\{A_{552}, F_{552}, F_{572}, F_{662}\} = \int_{-h/2}^{h/2} \frac{E(z) l^2(z)}{2(1+\nu(z))}\left\{1, f'^2, f', f''^2\right\} dz, \tag{15b}$$

$$F_{55} = \int_{-h/2}^{h/2} \frac{E(z)}{2(1+\nu(z))} f'^2 dz, \tag{15c}$$

$$\{A_{11}^*, B_{11}^*, D_{11}^*, F_{11}^*, F_{22}^*, F_{33}^*\} = \int_{-h/2}^{h/2} \frac{E(z)\nu(z)}{1-\nu^2(z)}\left\{1, z, z^2, f, zf, f^2\right\} dz, \tag{15d}$$

$$\left\{N^T, M^T, N_f^T\right\} = \int_{-h/2}^{h/2} \frac{E(z)\alpha(z)\theta(z)}{1-\nu(z)}\{1, z, f\} dz. \tag{15e}$$

k_s in Eqs. (13b) and (14b) is the shear correction factor, which assumes a value of unity in third-order shear deformation theory and $\pi^2/12$ in Mindlin plate theory. A shear correction factor is not included in the formulation based on Kirchhoff plate theory. Thermally induced force and moment given in Eq. (15e) appear in the boundary conditions (14a), (14c), and (14d). Note that governing equations conveyed by Eq. (13) are applicable for both annular and circular micro-plates. But, boundary condition specifications are dependent upon the plate type as will be delineated in Section 3.

2.2. Formulation for free vibrations

In this section, we consider free vibrations of a graded annular or circular micro-plate, that is subjected to an initial thermal stress field. Micro-plate geometry is given in Fig. 1. For the free vibrations problem, Hamilton's principle is expressed as

$$\delta\int_{t_1}^{t_2} (K - U) dt = 0, \tag{16}$$

where K is kinetic energy, and U is strain energy. U is a sum of two energy terms in the form [31,32]

$$U = U_S + U_T. \tag{17}$$

U_S is the strain energy corresponding to deformation field and U_T is due to initial thermal stresses. The expressions of K, U_S, and U_T are as follows:

$$K = \frac{1}{2}\int_{-h/2}^{h/2}\int_{R_i}^{R_o} \rho(z)(\dot{u}_r^2 + \dot{u}_z^2) 2\pi r dr dz, \tag{18a}$$

$$U_S = \frac{1}{2}\int_{-h/2}^{h/2}\int_{R_i}^{R_o} \left\{(\sigma_{rr} + \sigma_{rr}^T)\varepsilon_{rr} + (\sigma_{\theta\theta} + \sigma_{\theta\theta}^T)\varepsilon_{\theta\theta} + 2(\sigma_{rz}\varepsilon_{rz} + m_{r\theta}\chi_{r\theta} + m_{z\theta}\chi_{z\theta})\right\} 2\pi r dr dz, \tag{18b}$$

$$U_T = -\frac{1}{2}\int_{-h/2}^{h/2}\int_{R_i}^{R_o} \sigma_{rr}^T \left(\frac{dw}{dr}\right)^2 2\pi r dr dz. \tag{18c}$$

σ_{rr}^T and $\sigma_{\theta\theta}^T$ are defined by

$$\sigma_{rr}^T = \sigma_{\theta\theta}^T = \frac{E(z)\alpha(z)\theta(z)}{1-\nu(z)}. \tag{19}$$

Applying Hamilton's principle, we derive the governing partial differential equations as given below

$$(F_{11} - B_{11})\left\{\frac{\partial^3 w}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r}\right\} + A_{11}\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right\} - F_{11}\left\{\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} - \frac{\varphi}{r^2}\right\} = I_1\frac{\partial^2 u}{\partial t^2} + (I_4 - I_2)\frac{\partial^3 w}{\partial r \partial t^2} - I_4\frac{\partial^2 \varphi}{\partial t^2}, \tag{20a}$$

$$\left\{2F_{22} - D_{11} - F_{33} - A_{552} + F_{572} - \frac{F_{552}}{4}\right\} \times \left\{r\frac{\partial^4 w}{\partial r^4} + 2\frac{\partial^3 w}{\partial r^3} - \frac{1}{r}\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial w}{\partial r}\right\} + \left\{\frac{F_{662}}{4} + k_s F_{55} - N^T\right\}\left\{r\frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{\partial r}\right\} + (B_{11} - F_{11})\left\{r\frac{\partial^3 u}{\partial r^3} + 2\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{u}{r^2}\right\} + \left\{F_{33} - F_{22} - \frac{F_{572}}{2} + \frac{F_{552}}{4}\right\}\left\{r\frac{\partial^3 \phi}{\partial r^3} + 2\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\phi}{r^2}\right\} - \left\{k_s F_{55} + \frac{F_{662}}{4}\right\}\left\{r\frac{\partial \phi}{\partial r} + \phi\right\} = I_1 r\frac{\partial^2 w}{\partial t^2} + (2I_5 - I_3 - I_6)\left\{\frac{\partial^3 w}{\partial r \partial t^2} + r\frac{\partial^4 w}{\partial r^2 \partial t^2}\right\} + (I_2 - I_4)\left\{\frac{\partial^2 u}{\partial t^2} + r\frac{\partial^3 u}{\partial r \partial t^2}\right\} + (I_6 - I_5)\left\{\frac{\partial^2 \varphi}{\partial t^2} + r\frac{\partial^3 \varphi}{\partial r \partial t^2}\right\}, \tag{20b}$$

$$\left\{F_{33} + \frac{F_{552}}{4}\right\}\left\{\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} - \frac{\phi}{r^2}\right\} + \left\{F_{22} - F_{33} + \frac{F_{572}}{2} - \frac{F_{552}}{4}\right\}\left\{\frac{\partial^3 w}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r}\right\} + \left\{k_s F_{55} + \frac{F_{662}}{4}\right\}\left\{\frac{\partial w}{\partial r} - \varphi\right\} - F_{11}\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right\} = I_6\frac{\partial^2 \phi}{\partial t^2} + (I_5 - I_6)\frac{\partial^3 w}{\partial r \partial t^2} - I_4\frac{\partial^2 u}{\partial t^2}. \tag{20c}$$

And, boundary conditions at $r = R_i, R_o$ are obtained as

$$\delta u = 0, \text{ or } (F_{11} - B_{11}) r\frac{\partial^2 w}{\partial r^2} + (F_{11}^* - B_{11}^*)\frac{\partial w}{\partial r} + A_{11} r\frac{\partial u}{\partial r} + A_{11}^* u - F_{11} r\frac{\partial \phi}{\partial r} - F_{11}^* \phi = 0, \tag{21a}$$

$$\delta w = 0, \text{ or } \left\{2F_{22} - D_{11} - F_{33} - A_{552} + F_{572} - \frac{F_{552}}{4}\right\} \times \left\{\frac{\partial^3 w}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r}\right\} + \left\{\frac{F_{662}}{4} + k_s F_{55} - N^T\right\}\frac{\partial w}{\partial r} + (B_{11} - F_{11})\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right\} + \left\{F_{33} - F_{22} - \frac{F_{572}}{2} + \frac{F_{552}}{4}\right\}\left\{\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} - \frac{\phi}{r^2}\right\} - \left\{\frac{F_{662}}{4} + k_s F_{55}\right\}\phi = (2I_5 - I_3 - I_6)\frac{\partial^3 w}{\partial r \partial t^2} + (I_2 - I_4)\frac{\partial^2 u}{\partial t^2}, \tag{21b}$$

$$\delta\left(\frac{\partial w}{\partial r}\right)=0,\ \text{or}\ \left\{D_{11}-2F_{22}+F_{33}+A_{552}-F_{572}+\frac{F_{552}}{4}\right\}r\frac{\partial^2 w}{\partial r^2}+\left\{D^*_{11}-2F^*_{22}+F^*_{33}-A_{552}+F_{572}-\frac{F_{552}}{4}\right\}\frac{\partial w}{\partial r}+(F_{11}-B_{11})r\frac{\partial u}{\partial r}+(F^*_{11}-B^*_{11})u+\left\{F_{22}-F_{33}+\frac{F_{572}}{2}-\frac{F_{552}}{4}\right\}r\frac{\partial \phi}{\partial r}+\left\{F^*_{22}-F^*_{33}-\frac{F_{572}}{2}+\frac{F_{552}}{4}\right\}\phi=0, \tag{21c}$$

$$\delta\phi=0,\ \text{or}\ \left\{F_{22}-F_{33}+\frac{F_{572}}{2}-\frac{F_{552}}{4}\right\}r\frac{\partial^2 w}{\partial r^2}+\left\{F^*_{22}-F^*_{33}-\frac{F_{572}}{2}+\frac{F_{552}}{4}\right\}\frac{\partial w}{\partial r}-F_{11}r\frac{\partial u}{\partial r}-F^*_{11}u+\left\{F_{33}+\frac{F_{552}}{4}\right\}r\frac{\partial \phi}{\partial r}+\left\{F^*_{33}-\frac{F_{552}}{4}\right\}\phi=0. \tag{21d}$$

Inertia coefficients in Eqs. (20) and (21) are of the forms

$$\{I_1,I_2,I_3,I_4,I_5,I_6\}=\int_{-h/2}^{h/2}\rho(z)\left\{1,z,z^2,f,zf,f^2\right\}dz. \tag{22}$$

Notice that thermally induced constant force N^T enters free vibration formulation through Eqs. (20b) and (21b). Thermally induced moment however does not affect governing equations and boundary conditions because of the assumed form of strain energy due to initial thermal stresses. This form, which is given by Eq. (18c) and proposed by Raju and Rao [33,34], includes only the resultant thermal force. We also note that the terms $(\sigma_{rr}+\sigma^T_{rr})$ and $(\sigma_{\theta\theta}+\sigma^T_{\theta\theta})$ in Eq. (18b), where σ_{rr} and $\sigma_{\theta\theta}$ are given by Eq. (6a), are independent of temperature. Thus, use of Eqs. (18b) and (18c) in Hamilton's principle does not introduce terms involving thermal bending moment in the resulting governing equations and boundary conditions.

3. Numerical solution

Numerical solutions are developed for an annular micro-plate simply supported at $r=R_i,\ R_o$ as illustrated in Fig. 1 and a simply-supported circular micro-plate. Differential quadrature method is employed in the solution of the partial differential equations and associated boundary conditions. First step in the solution is definition of normalized quantities, which are as follows:

$$\xi=\frac{r-R_i}{R_o-R_i},\quad \gamma=\frac{R_i}{R_o-R_i},\quad \eta=\frac{R_o-R_i}{h},\quad \chi=\xi+\gamma, \tag{23a}$$

$$\tau=\frac{1}{R_o-R_i}\sqrt{\frac{A_{110}}{I_{10}}}t, \tag{23b}$$

$$\{\bar{u},\bar{w}\}=\frac{\{u,w\}}{h},\quad \varphi=\phi, \tag{23c}$$

$$\{\bar{A}_{11},\bar{B}_{11},\bar{D}_{11},\bar{F}_{11},\bar{F}_{22},\bar{F}_{33},\bar{F}_{55}\}=\left\{\frac{A_{11}}{A_{110}},\frac{B_{11}}{hA_{110}},\frac{D_{11}}{h^2A_{110}},\frac{F_{11}}{hA_{110}},\frac{F_{22}}{h^2A_{110}},\frac{F_{33}}{h^2A_{110}},\frac{F_{55}}{A_{110}}\right\}, \tag{23d}$$

$$\{\bar{A}_{552},\bar{F}_{552},\bar{F}_{572},\bar{F}_{662}\}=\left\{\frac{A_{552}}{h^2A_{110}},\frac{F_{552}}{h^2A_{110}},\frac{F_{572}}{h^2A_{110}},\frac{F_{662}}{A_{110}}\right\}, \tag{23e}$$

$$\{\bar{A}^*_{11},\bar{B}^*_{11},\bar{D}^*_{11},\bar{F}^*_{11},\bar{F}^*_{22},\bar{F}^*_{33}\}=\left\{\frac{A^*_{11}}{A_{110}},\frac{B^*_{11}}{hA_{110}},\frac{D^*_{11}}{h^2A_{110}},\frac{F^*_{11}}{hA_{110}},\frac{F^*_{22}}{h^2A_{110}},\frac{F^*_{33}}{h^2A_{110}}\right\}, \tag{23f}$$

$$\left\{\bar{N}^T,\bar{M}^T,\bar{N}^T_f\right\}=\left\{\frac{N^T}{A_{110}},\frac{M^T}{hA_{110}},\frac{N^T_f}{hA_{110}}\right\}, \tag{23g}$$

$$\{\bar{I}_1,\bar{I}_2,\bar{I}_3,\bar{I}_4,\bar{I}_5,\bar{I}_6\}=\left\{\frac{I_1}{I_{10}},\frac{I_2}{hI_{10}},\frac{I_3}{h^2I_{10}},\frac{I_4}{hI_{10}},\frac{I_5}{h^2I_{10}},\frac{I_6}{h^2I_{10}}\right\}. \tag{23h}$$

A_{110} and I_{10} in the above definitions are reference values of A_{11} and I_1 calculated by considering a homogeneous plate, whose properties are equal to those of the functionally graded plate at $z=-h/2$.

In differential quadrature method, an mth order partial derivative is approximated as

$$\left.\frac{\partial^m u(r,t)}{\partial r^m}\right|_{r=r_i}=\sum_{k=1}^{N}C^{(m)}_{ik}u(r_k,t),\quad i=1,\ldots,N, \tag{24}$$

where N is number of nodes, and $C^{(m)}_{ik}$ are weighting coefficients [35]. Nodal points are enforced to be shifted Chebyshev–Gauss–Lobatto points [36] through the relation

$$r_k=\frac{1}{2}\left\{1-\cos\left(\frac{\pi(k-1)}{N-1}\right)\right\},\quad k=1,\ldots,N. \tag{25}$$

3.1. Numerical solution for static bending

By utilizing Eq. (23) and implementing the differential quadrature method, governing partial differential equations for the static bending problem are reduced to series forms, which are given by

$$\frac{1}{\eta}(\bar{F}_{11}-\bar{B}_{11})\left\{\sum_{k=1}^{N}C^{(3)}_{ik}\bar{w}_{0k}+\frac{1}{\chi_i}\sum_{k=1}^{N}C^{(2)}_{ik}\bar{w}_{0k}-\frac{1}{\chi_i^2}\sum_{k=1}^{N}C^{(1)}_{ik}\bar{w}_{0k}\right\}+\bar{A}_{11}\left\{\sum_{k=1}^{N}C^{(2)}_{ik}\bar{u}_{0k}+\frac{1}{\chi_i}\sum_{k=1}^{N}C^{(1)}_{ik}\bar{u}_{0k}-\frac{\bar{u}_{0i}}{\chi_i^2}\right\}-\bar{F}_{11}\left\{\sum_{k=1}^{N}C^{(2)}_{ik}\varphi_{0k}+\frac{1}{\chi_i}\sum_{k=1}^{N}C^{(1)}_{ik}\varphi_{0k}-\frac{\varphi_{0i}}{\chi_i^2}\right\}=0, \tag{26a}$$

$$\frac{1}{\eta}\left\{2\bar{F}_{22}-\bar{D}_{11}-\bar{F}_{33}-\bar{A}_{552}+\bar{F}_{572}-\frac{\bar{F}_{552}}{4}\right\}\times\left\{\chi_i\sum_{k=1}^{N}C^{(4)}_{ik}\bar{w}_{0k}+2\sum_{k=1}^{N}C^{(3)}_{ik}\bar{w}_{0k}-\frac{1}{\chi_i}\sum_{k=1}^{N}C^{(2)}_{ik}\bar{w}_{0k}+\frac{1}{\chi_i^2}\sum_{k=1}^{N}C^{(1)}_{ik}\bar{w}_{0k}\right\}+\eta\left\{\frac{\bar{F}_{662}}{4}+k_s\bar{F}_{55}\right\}\left\{\chi_i\sum_{k=1}^{N}C^{(2)}_{ik}\bar{w}_{0k}+\sum_{k=1}^{N}C^{(1)}_{ik}\bar{w}_{0k}\right\}+(\bar{B}_{11}-\bar{F}_{11})\left\{\chi_i\sum_{k=1}^{N}C^{(3)}_{ik}\bar{u}_{0k}+2\sum_{k=1}^{N}C^{(2)}_{ik}\bar{u}_{0k}-\frac{1}{\chi_i}\sum_{k=1}^{N}C^{(1)}_{ik}\bar{u}_{0k}+\frac{\bar{u}_{0i}}{\chi_i^2}\right\}+\left\{\bar{F}_{33}-\bar{F}_{22}-\frac{\bar{F}_{572}}{2}+\frac{\bar{F}_{552}}{4}\right\}\times\left\{\chi_i\sum_{k=1}^{N}C^{(3)}_{ik}\varphi_{0k}+2\sum_{k=1}^{N}C^{(2)}_{ik}\varphi_{0k}-\frac{1}{\chi_i}\sum_{k=1}^{N}C^{(1)}_{ik}\varphi_{0k}+\frac{\varphi_{0i}}{\chi_i^2}\right\}-\eta^2\left\{k_s\bar{F}_{55}+\frac{\bar{F}_{662}}{4}\right\}\left\{\chi_i\sum_{k=1}^{N}C^{(1)}_{ik}\varphi_{0k}+\varphi_{0i}\right\}=0, \tag{26b}$$

$$\left\{\bar{F}_{33}+\frac{\bar{F}_{552}}{4}\right\}\left\{\sum_{k=1}^{N}C_{ik}^{(2)}\varphi_{0k}+\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(1)}\varphi_{0k}-\frac{\varphi_{0i}}{\chi_i^2}\right\}+\frac{1}{\eta}\left\{\bar{F}_{22}-\bar{F}_{33}+\frac{\bar{F}_{572}}{2}-\frac{\bar{F}_{552}}{4}\right\}\times\left\{\sum_{k=1}^{N}C_{ik}^{(3)}\bar{w}_{0k}+\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(2)}\bar{w}_{0k}-\frac{1}{\chi_i^2}\sum_{k=1}^{N}C_{ik}^{(1)}\bar{w}_{0k}\right\}+\eta\left\{k_s\bar{F}_{55}+\frac{\bar{F}_{662}}{4}\right\}\left\{\sum_{k=1}^{N}C_{ik}^{(1)}\bar{w}_{0k}-\eta\varphi_{0i}\right\}-\bar{F}_{11}\left\{\sum_{k=1}^{N}C_{ik}^{(2)}\bar{u}_{0k}+\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(1)}\bar{u}_{0k}-\frac{\bar{u}_{0i}}{\chi_i^2}\right\}=0. \quad (26c)$$

In all three equations $i=1,\ldots,N$. For a simply supported annular micro-plate as depicted in Fig. 1, boundary conditions at $r=R_i,\ R_o$ read:

$$\bar{u}_{01}=\bar{u}_{0N}=0, \quad (27a)$$

$$\bar{w}_{01}=\bar{w}_{0N}=0, \quad (27b)$$

$$\left\{\bar{D}_{11}-\bar{F}_{22}+\bar{A}_{552}-\frac{\bar{F}_{572}}{2}\right\}\gamma\sum_{k=1}^{N}C_{1k}^{(2)}\bar{w}_{0k}+\left\{\bar{D}_{11}^{*}-\bar{F}_{22}^{*}-\bar{A}_{552}+\frac{\bar{F}_{572}}{2}\right\}\sum_{k=1}^{N}C_{1k}^{(1)}\bar{w}_{0k}-\eta\bar{B}_{11}\gamma\sum_{k=1}^{N}C_{1k}^{(1)}\bar{u}_{0k}+\eta\left\{\bar{F}_{22}+\frac{\bar{F}_{572}}{2}\right\}\gamma\sum_{k=1}^{N}C_{1k}^{(1)}\varphi_{0k}+\eta\left\{\bar{F}_{22}^{*}-\frac{\bar{F}_{572}}{2}\right\}\varphi_{01}+\eta^2\gamma\bar{M}^T=\left\{\bar{D}_{11}-\bar{F}_{22}+\bar{A}_{552}-\frac{\bar{F}_{572}}{2}\right\}(1+\gamma)\sum_{k=1}^{N}C_{Nk}^{(2)}\bar{w}_{0k}+\left\{\bar{D}_{11}^{*}-\bar{F}_{22}^{*}-\bar{A}_{552}+\frac{\bar{F}_{572}}{2}\right\}\sum_{k=1}^{N}C_{Nk}^{(1)}\bar{w}_{0k}-\eta\bar{B}_{11}(1+\gamma)\sum_{k=1}^{N}C_{Nk}^{(1)}\bar{u}_{0k}+\eta\left\{\bar{F}_{22}+\frac{\bar{F}_{572}}{2}\right\}(1+\gamma)\sum_{k=1}^{N}C_{Nk}^{(1)}\varphi_{0k}+\eta\left\{\bar{F}_{22}^{*}-\frac{\bar{F}_{572}}{2}\right\}\varphi_{0N}+\eta^2(1+\gamma)\bar{M}^T=0, \quad (27c)$$

$$\left\{\bar{F}_{22}-\bar{F}_{33}+\frac{\bar{F}_{572}}{2}-\frac{\bar{F}_{552}}{4}\right\}\gamma\sum_{k=1}^{N}C_{1k}^{(2)}\bar{w}_{0k}+\left\{\bar{F}_{22}^{*}-\bar{F}_{33}^{*}-\frac{\bar{F}_{572}}{2}+\frac{\bar{F}_{552}}{4}\right\}\sum_{k=1}^{N}C_{1k}^{(1)}\bar{w}_{0k}-\eta\bar{F}_{11}\gamma\sum_{k=1}^{N}C_{1k}^{(1)}\bar{u}_{0k}+\eta\left\{\bar{F}_{33}+\frac{\bar{F}_{552}}{4}\right\}\gamma\sum_{k=1}^{N}C_{1k}^{(1)}\varphi_{0k}+\eta\left\{\bar{F}_{33}^{*}-\frac{\bar{F}_{552}}{4}\right\}\varphi_{01}+\eta\gamma\bar{N}_f^T=\left\{\bar{F}_{22}-\bar{F}_{33}+\frac{\bar{F}_{572}}{2}-\frac{\bar{F}_{552}}{4}\right\}(1+\gamma)\sum_{k=1}^{N}C_{Nk}^{(2)}\bar{w}_{0k}+\left\{\bar{F}_{22}^{*}-\bar{F}_{33}^{*}-\frac{\bar{F}_{572}}{2}+\frac{\bar{F}_{552}}{4}\right\}\sum_{k=1}^{N}C_{Nk}^{(1)}\bar{w}_{0k}-\eta\bar{F}_{11}(1+\gamma)\sum_{k=1}^{N}C_{Nk}^{(1)}\bar{u}_{0k}+\eta\left\{\bar{F}_{33}+\frac{\bar{F}_{552}}{4}\right\}(1+\gamma)\sum_{k=1}^{N}C_{Nk}^{(1)}\varphi_{0k}+\eta\left\{\bar{F}_{33}^{*}-\frac{\bar{F}_{552}}{4}\right\}\varphi_{0N}+\eta(1+\gamma)\bar{N}_f^T=0. \quad (27d)$$

In the numerical solution of the circular micro-plate problem, plate profile is assumed to be annular with a very small inner radius R_i. For a circular micro-plate simply-supported around the periphery, the aforementioned equations are applicable at $r=R_o$. Boundary conditions at $r=R_i$ however need to be rearranged as

$$\bar{u}_{01}=0, \quad (28a)$$

$$\left\{2\bar{F}_{22}-\bar{D}_{11}-\bar{F}_{33}-\bar{A}_{552}+\bar{F}_{572}-\frac{\bar{F}_{552}}{4}\right\}\left\{\sum_{k=1}^{N}C_{1k}^{(3)}\bar{w}_{0k}+\frac{1}{\gamma_\varepsilon}\sum_{k=1}^{N}C_{1k}^{(2)}\bar{w}_{0k}\right\}+\eta(\bar{B}_{11}-\bar{F}_{11})\left\{\sum_{k=1}^{N}C_{1k}^{(2)}\bar{u}_{0k}+\frac{1}{\gamma_\varepsilon}\sum_{k=1}^{N}C_{1k}^{(1)}\bar{u}_{0k}\right\}+\eta\left\{\bar{F}_{33}-\bar{F}_{22}-\frac{\bar{F}_{572}}{2}+\frac{\bar{F}_{552}}{4}\right\}\left\{\sum_{k=1}^{N}C_{1k}^{(2)}\varphi_{0k}+\frac{1}{\gamma_\varepsilon}\sum_{k=1}^{N}C_{1k}^{(1)}\varphi_{0k}\right\}=0, \quad (28b)$$

$$\sum_{k=1}^{N}C_{1k}^{(1)}\bar{w}_{0k}=0, \quad (28c)$$

$$\varphi_1=0. \quad (28d)$$

γ_ε in Eq. (28b) is a sufficiently small number.

For both annular and circular micro-plate problems, governing equations and boundary conditions are consolidated into a matrix equation of the form

$$\mathbf{K_S X_0}+\mathbf{Q}=\mathbf{0}, \quad (29)$$

where $\mathbf{K_S}$ is stiffness matrix, $\mathbf{Q}$ is generalized distributed thermal load vector, and $\mathbf{X_0}$ is nodal displacement vector, which is expressed as follows:

$$\mathbf{X_0}=\left\{\{\bar{u}_{0i}\}^{\mathrm{T}},\{\bar{w}_{0i}\}^{\mathrm{T}},\{\varphi_{0i}\}^{\mathrm{T}}\right\},\quad i=1,2,\ldots,N. \quad (30)$$

All required field variables can be determined once the solution of Eq. (29) is obtained.

3.2. Numerical solution for free vibrations

By using the normalizations given by Eq. (23), series forms of the governing equations are derived as presented below:

$$\frac{1}{\eta}(\bar{F}_{11}-\bar{B}_{11})\left\{\sum_{k=1}^{N}C_{ik}^{(3)}\bar{w}_k^{*}+\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(2)}\bar{w}_k^{*}-\frac{1}{\chi_i^2}\sum_{k=1}^{N}C_{ik}^{(1)}\bar{w}_k^{*}\right\}+\bar{A}_{11}\left\{\sum_{k=1}^{N}C_{ik}^{(2)}\bar{u}_k^{*}+\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(1)}\bar{u}_k^{*}-\frac{\bar{u}_i^{*}}{\chi_i^2}\right\}-\bar{F}_{11}\left\{\sum_{k=1}^{N}C_{ik}^{(2)}\varphi_k^{*}+\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(1)}\varphi_k^{*}-\frac{\varphi_i^{*}}{\chi_i^2}\right\}=\bar{I}_1\ddot{\bar{u}}_i^{*}+\frac{1}{\eta}(\bar{I}_4-\bar{I}_2)\sum_{k=1}^{N}C_{ik}^{(1)}\ddot{\bar{w}}_k^{*}-\bar{I}_4\ddot{\varphi}_i^{*}, \quad (31a)$$

$$\frac{1}{\eta}\left\{2\bar{F}_{22}-\bar{D}_{11}-\bar{F}_{33}-\bar{A}_{552}+\bar{F}_{572}-\frac{\bar{F}_{552}}{4}\right\}\times\left\{\chi_i\sum_{k=1}^{N}C_{ik}^{(4)}\bar{w}_k^{*}+2\sum_{k=1}^{N}C_{ik}^{(3)}\bar{w}_k^{*}-\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(2)}\bar{w}_k^{*}+\frac{1}{\chi_i^2}\sum_{k=1}^{N}C_{ik}^{(1)}\bar{w}_k^{*}\right\}+\eta\left\{\frac{\bar{F}_{662}}{4}+k_s\bar{F}_{55}-\bar{N}^T\right\}\left\{\chi_i\sum_{k=1}^{N}C_{ik}^{(2)}\bar{w}_k^{*}+\sum_{k=1}^{N}C_{ik}^{(1)}\bar{w}_k^{*}\right\}+(\bar{B}_{11}-\bar{F}_{11})\left\{\chi_i\sum_{k=1}^{N}C_{ik}^{(3)}\bar{u}_k^{*}+2\sum_{k=1}^{N}C_{ik}^{(2)}\bar{u}_k^{*}-\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(1)}\bar{u}_k^{*}+\frac{\bar{u}_i^{*}}{\chi_i^2}\right\}+\left\{\bar{F}_{33}-\bar{F}_{22}-\frac{\bar{F}_{572}}{2}+\frac{\bar{F}_{552}}{4}\right\}\times\left\{\chi_i\sum_{k=1}^{N}C_{ik}^{(3)}\varphi_k^{*}+2\sum_{k=1}^{N}C_{ik}^{(2)}\varphi_k^{*}-\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(1)}\varphi_k^{*}+\frac{\varphi_i^{*}}{\chi_i^2}\right\}-\eta^2\left\{k_s\bar{F}_{55}+\frac{\bar{F}_{662}}{4}\right\}\left\{\chi_i\sum_{k=1}^{N}C_{ik}^{(1)}\varphi_k^{*}+\varphi_i^{*}\right\}=\eta\chi_i\bar{I}_1\ddot{\bar{w}}_i^{*}+\frac{1}{\eta}(2\bar{I}_5-\bar{I}_3-\bar{I}_6)\left\{\sum_{k=1}^{N}C_{ik}^{(1)}\ddot{\bar{w}}_k^{*}+\chi_i\sum_{k=1}^{N}C_{ik}^{(2)}\ddot{\bar{w}}_k^{*}\right\}+(\bar{I}_2-\bar{I}_4)\left\{\ddot{\bar{u}}_i^{*}+\chi_i\sum_{k=1}^{N}C_{ik}^{(1)}\ddot{\bar{u}}_k^{*}\right\}+(\bar{I}_6-\bar{I}_5)\left\{\ddot{\varphi}_i^{*}+\chi_i\sum_{k=1}^{N}C_{ik}^{(1)}\ddot{\varphi}_k^{*}\right\}, \quad (31b)$$

$$\left\{\bar{F}_{33}+\frac{\bar{F}_{552}}{4}\right\}\left\{\sum_{k=1}^{N}C_{ik}^{(2)}\varphi_k^*+\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(1)}\varphi_k^*-\frac{\varphi_i^*}{\chi_i^2}\right\}$$
$$+\frac{1}{\eta}\left\{\bar{F}_{22}-\bar{F}_{33}+\frac{\bar{F}_{572}}{2}-\frac{\bar{F}_{552}}{4}\right\}$$
$$\times\left\{\sum_{k=1}^{N}C_{ik}^{(3)}\bar{w}_k^*+\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(2)}\bar{w}_k^*-\frac{1}{\chi_i^2}\sum_{k=1}^{N}C_{ik}^{(1)}\bar{w}_k^*\right\}$$
$$+\eta\left\{k_s\bar{F}_{55}+\frac{\bar{F}_{662}}{4}\right\}\left\{\sum_{k=1}^{N}C_{ik}^{(1)}\bar{w}_k^*-\eta\varphi_i^*\right\}$$
$$-\bar{F}_{11}\left\{\sum_{k=1}^{N}C_{ik}^{(2)}\bar{u}_k^*+\frac{1}{\chi_i}\sum_{k=1}^{N}C_{ik}^{(1)}\bar{u}_k^*-\frac{\bar{u}_i^*}{\chi_i^2}\right\}$$
$$=\bar{I}_6\ddot{\varphi}_i^*+\frac{1}{\eta}(\bar{I}_5-\bar{I}_6)\sum_{k=1}^{N}C_{ik}^{(1)}\ddot{\bar{w}}_k^*-\bar{I}_4\ddot{\bar{u}}_i^*. \quad (31c)$$

$i=1,\dots,N$ in all three equations. For the simply supported annular micro-plate shown in Fig. 1, boundary conditions at $r=R_i,\ R_o$ are:

$$\bar{u}_1^*=\bar{u}_N^*=0, \quad (32a)$$

$$\bar{w}_1^*=\bar{w}_N^*=0, \quad (32b)$$

$$\left\{\bar{D}_{11}-\bar{F}_{22}+\bar{A}_{552}-\frac{\bar{F}_{572}}{2}\right\}\gamma\sum_{k=1}^{N}C_{1k}^{(2)}\bar{w}_k^*$$
$$+\left\{\bar{D}_{11}^*-\bar{F}_{22}^*-\bar{A}_{552}+\frac{\bar{F}_{572}}{2}\right\}\sum_{k=1}^{N}C_{1k}^{(1)}\bar{w}_k^*-\eta\bar{B}_{11}\gamma\sum_{k=1}^{N}C_{1k}^{(1)}\bar{u}_k^*$$
$$+\eta\left\{\bar{F}_{22}+\frac{\bar{F}_{572}}{2}\right\}\gamma\sum_{k=1}^{N}C_{1k}^{(1)}\varphi_k^*+\eta\left\{\bar{F}_{22}^*-\frac{\bar{F}_{572}}{2}\right\}\varphi_1^*$$
$$=\left\{\bar{D}_{11}-\bar{F}_{22}+\bar{A}_{552}-\frac{\bar{F}_{572}}{2}\right\}(1+\gamma)\sum_{k=1}^{N}C_{Nk}^{(2)}\bar{w}_k^*$$
$$+\left\{\bar{D}_{11}^*-\bar{F}_{22}^*-\bar{A}_{552}+\frac{\bar{F}_{572}}{2}\right\}\sum_{k=1}^{N}C_{Nk}^{(1)}\bar{w}_k^*$$
$$-\eta\bar{B}_{11}(1+\gamma)\sum_{k=1}^{N}C_{Nk}^{(1)}\bar{u}_k^*+\eta\left\{\bar{F}_{22}+\frac{\bar{F}_{572}}{2}\right\}(1+\gamma)\sum_{k=1}^{N}C_{Nk}^{(1)}\varphi_k^*$$
$$+\eta\left\{\bar{F}_{22}^*-\frac{\bar{F}_{572}}{2}\right\}\varphi_N^*=0, \quad (32c)$$

$$\left\{\bar{F}_{22}-\bar{F}_{33}+\frac{\bar{F}_{572}}{2}-\frac{\bar{F}_{552}}{4}\right\}\gamma\sum_{k=1}^{N}C_{1k}^{(2)}\bar{w}_k^*$$
$$+\left\{\bar{F}_{22}^*-\bar{F}_{33}^*-\frac{\bar{F}_{572}}{2}+\frac{\bar{F}_{552}}{4}\right\}\sum_{k=1}^{N}C_{1k}^{(1)}\bar{w}_k^*-\eta\bar{F}_{11}\gamma\sum_{k=1}^{N}C_{1k}^{(1)}\bar{u}_k^*$$
$$+\eta\left\{\bar{F}_{33}+\frac{\bar{F}_{552}}{4}\right\}\gamma\sum_{k=1}^{N}C_{1k}^{(1)}\varphi_k^*+\eta\left\{\bar{F}_{33}^*-\frac{\bar{F}_{552}}{4}\right\}\varphi_1^*$$
$$=\left\{\bar{F}_{22}-\bar{F}_{33}+\frac{\bar{F}_{572}}{2}-\frac{\bar{F}_{552}}{4}\right\}(1+\gamma)\sum_{k=1}^{N}C_{Nk}^{(2)}\bar{w}_k^*$$
$$+\left\{\bar{F}_{22}^*-\bar{F}_{33}^*-\frac{\bar{F}_{572}}{2}+\frac{\bar{F}_{552}}{4}\right\}\sum_{k=1}^{N}C_{Nk}^{(1)}\bar{w}_k^*$$
$$-\eta\bar{F}_{11}(1+\gamma)\sum_{k=1}^{N}C_{Nk}^{(1)}\bar{u}_k^*+\eta\left\{\bar{F}_{33}+\frac{\bar{F}_{552}}{4}\right\}(1+\gamma)\sum_{k=1}^{N}C_{Nk}^{(1)}\varphi_k^*$$
$$+\eta\left\{\bar{F}_{33}^*-\frac{\bar{F}_{552}}{4}\right\}\varphi_N^*=0. \quad (32d)$$

In the case of the circular micro-plate, these conditions are applicable at $r=R_o$. At $r=R_i$ on the other hand, following equations need to be implemented:

$$\bar{u}_1^*=0, \quad (33a)$$

$$\left\{2\bar{F}_{22}-\bar{D}_{11}-\bar{F}_{33}-\bar{A}_{552}+\bar{F}_{572}-\frac{\bar{F}_{552}}{4}\right\}\left\{\sum_{k=1}^{N}C_{1k}^{(3)}\bar{w}_k^*+\frac{1}{\gamma_\varepsilon}\sum_{k=1}^{N}C_{1k}^{(2)}\bar{w}_k^*\right\}$$
$$+\eta(\bar{B}_{11}-\bar{F}_{11})\left\{\sum_{k=1}^{N}C_{1k}^{(2)}\bar{u}_k^*+\frac{1}{\gamma_\varepsilon}\sum_{k=1}^{N}C_{1k}^{(1)}\bar{u}_k^*\right\}$$
$$+\eta\left\{\bar{F}_{33}-\bar{F}_{22}-\frac{\bar{F}_{572}}{2}+\frac{\bar{F}_{552}}{4}\right\}\left\{\sum_{k=1}^{N}C_{1k}^{(2)}\varphi_k^*+\frac{1}{\gamma_\varepsilon}\sum_{k=1}^{N}C_{1k}^{(1)}\varphi_k^*\right\}=0, \quad (33b)$$

$$\sum_{k=1}^{N}C_{1k}^{(1)}\bar{w}_k^*=0, \quad (33c)$$

$$\varphi_1^*=0. \quad (33d)$$

For both of the annular and circular micro-plate problems, governing equations and boundary conditions are combined and recast into the general form:

$$\mathbf{K_D X}^*+\mathbf{M}\ddot{\mathbf{X}}^*=\mathbf{0}, \quad (34)$$

in which $\mathbf{K_D}$ is the stiffness matrix including initial thermal stresses, and $\mathbf{M}$ is mass matrix. Dynamic displacement vector is expressed as follows:

$$\mathbf{X}^*=\left\{\{\bar{u}_i^*\}^{\mathrm{T}},\{\bar{w}_i^*\}^{\mathrm{T}},\{\phi_i^*\}^{\mathrm{T}}\right\}^{\mathrm{T}},\quad i=1,2,\dots,N. \quad (35)$$

By assuming

$$\mathbf{X}^*=\hat{\mathbf{X}}^*e^{i\omega\tau}, \quad (36)$$

Eq. (34) is reduced to the linear homogeneous system

$$(\mathbf{K_D}-\omega^2\mathbf{M})\hat{\mathbf{X}}^*=\mathbf{0}, \quad (37)$$

where $\hat{\mathbf{X}}^*$ is mode shape vector and ω stands for dimensionless vibration frequency. $\hat{\mathbf{X}}^*$ and ω are given by

$$\hat{\mathbf{X}}^*=\left\{\left\{\hat{\bar{u}}_i^*\right\}^{\mathrm{T}},\left\{\hat{\bar{w}}_i^*\right\}^{\mathrm{T}},\left\{\hat{\phi}_i^*\right\}^{\mathrm{T}}\right\}^{\mathrm{T}}, \quad (38a)$$

$$\omega=\sqrt{\frac{I_{10}}{A_{110}}}(R_o-R_i)\Omega. \quad (38b)$$

Ω here is the frequency of free vibrations. The deformation mode that a computed frequency represents is identified in the numerical solution by examining the mode shape vector.

4. Numerical results

The particular ceramic and metallic constituents considered are silicon carbide (SiC) and aluminum (Al), whose properties are given by

$$E_c=427\ \text{GPa},\quad \nu_c=0.17,\quad \rho_c=3100\ \text{kg/m}^3,$$
$$k_c=65\ \text{W/(m K)},\quad \alpha_c=4.3(10)^{-6}\ 1/\text{K}, \quad (39a)$$

$$E_m=70\ \text{GPa},\quad \nu_m=0.3,\quad \rho_m=2702\ \text{kg/m}^3,$$
$$k_m=204\ \text{W/(m K)},\quad \alpha_m=23.0(10)^{-6}\ 1/\text{K}. \quad (39b)$$

For a given material, the value of the length scale parameter can be determined experimentally. This requires development of a modified couple stress theory based solution considering a simple problem such as bending or torsion. Length scale parameter can then be measured by matching the theoretical results to those obtained from experiments. By adopting such a technique, the length scale parameter for epoxy is determined as 17.6 μm [22,37]. However, due to lack of experimental data for length scale parameters of SiC and Al, in the present study we use approximate values. Length scale parameter of the metallic phase is specified as

$l_m = 15$ μm, which is a reference value suggested in the literature [20,38]. l_c is taken as 22.5 μm in a number of parametric analyses. In the other cases, it is varied to be able to examine the influence of the length scale parameter variation.

In order to verify the solution methodologies developed, we first present comparisons to the results available in the literature. Fig. 2 shows results pertaining to static bending of a simply-supported annular homogeneous macro-scale plate under the influence of thermal loading. Plate material is silicon carbide, and Kirchhoff plate theory is employed in the computations. l_c and l_m are taken as zero to be able to generate results for the macro-scale annular plate. The figure illustrates that static deformation profile we compute is in perfect agreement with that found by Noda et al. [39]. The second set of comparisons comprises natural frequencies of a freely vibrating simply-supported aluminum circular macro-scale plate subjected to initial thermal stresses. First dimensionless natural frequencies of the homogeneous plate are given in Table 1 for four different loading conditions. Kirchhoff plate theory is used in these parametric analyses. Our findings are in excellent agreement with those given by Raju and Rao [34]. Hence, we conclude that proposed techniques lead to numerical results to within a high degree of accuracy.

Numerical results on static bending of FGM micro-plates under thermal loading are provided in Figs. 3–8. In Fig. 3, we present deflections of an annular FGM micro-plate computed utilizing Kirchhoff, Mindlin and third-order shear deformation theories. The theories are seen to lead to almost identical deformation profiles. Thus, in the computation of the results given in Figs. 4–8, we employed third-order shear deformation theory. Fig. 4 depicts deflections of an annular FGM micro-plate generated by considering three different thermal loading conditions. When the temperature of the upper surface is greater than or equal to that of the lower surface, the micro-plate becomes concave downward. In the case of a lower upper surface temperature, concavity is reversed.

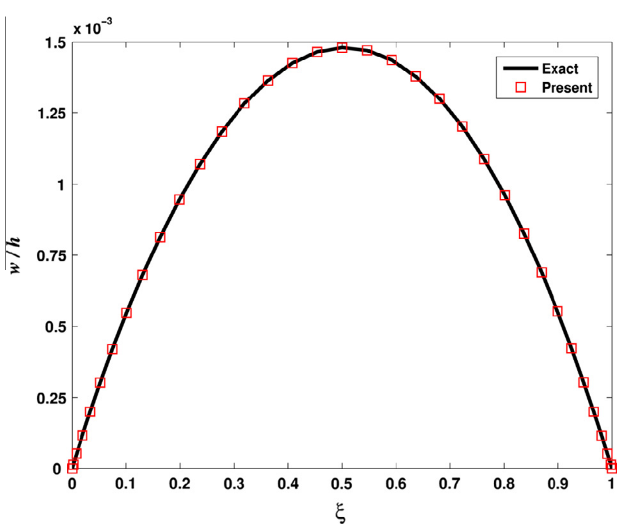

Fig. 2. Static bending deformation of a simply-supported annular silicon carbide plate. $R_o/h = 10$, $R_o/R_i = 4$, $\theta_L = 20\,^\circ$C, $\theta_U = 70\,^\circ$C.

Table 1
Comparisons of the first dimensionless natural frequencies computed for a simply-supported aluminum circular plate. $R_o/h = 10$, $\nu = 0.3$.

	$\theta_L = \theta_U$	0 °C	45.6 °C	91.3 °C	110.9 °C
ω	Raju and Rao [34]	0.1425	0.1113	0.0668	0.0326
	Present study	0.1422	0.1111	0.0666	0.0326

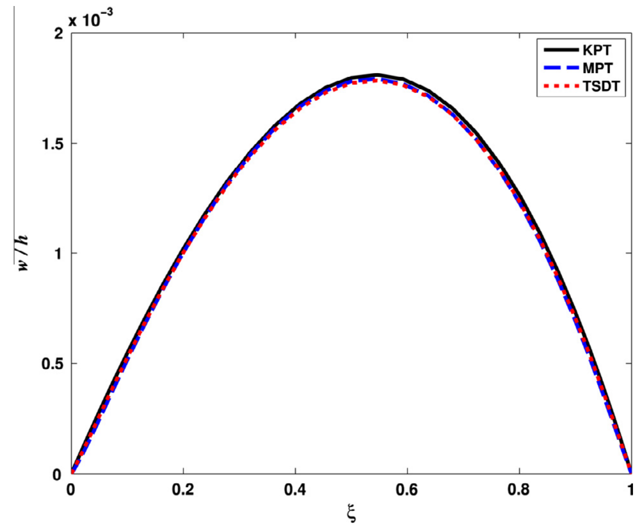

Fig. 3. Static deformation profiles of a simply-supported annular FGM micro-plate generated by considering three different plate theories. $R_o/h = 10$, $R_o/R_i = 4$, $h/l_m = 2$, $l_c/l_m = 3/2$, $l_m = 15$ μm, $\lambda = 2$, $\theta_L = 20\,^\circ$C, $\theta_U = 70\,^\circ$C.

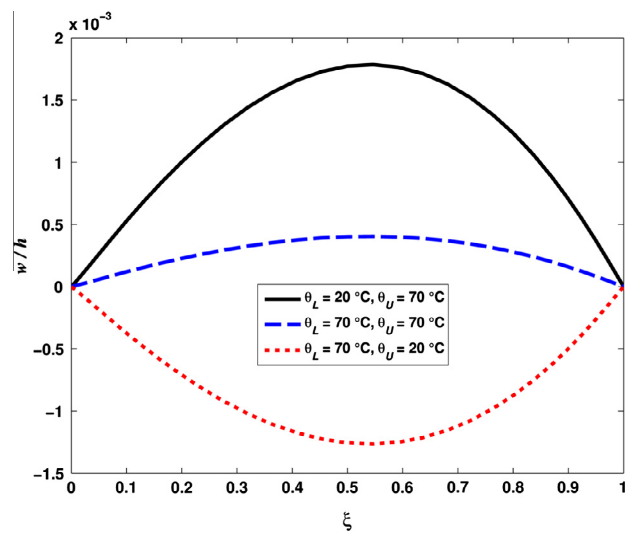

Fig. 4. Static deformation profiles of a simply-supported annular FGM micro-plate generated by considering different boundary temperature specifications. $R_o/h = 10$, $R_o/R_i = 4$, $h/l_m = 2$, $l_c/l_m = 3/2$, $l_m = 15$ μm, $\lambda = 2$.

The results provided in Figs. 5 and 6 demonstrate the impact of the length scale parameter ratio l_c/l_m upon static deformation profiles of functionally graded annular and circular micro-plates, respectively. Referring to Eq. (3b), it can be deduced that when $l_c = l_m$, length scale parameter l is constant within the plate. But, if $l_c \neq l_m$, l is a function of the spatial coordinate z; and the ratio l_c/l_m quantifies the degree of length scale parameter variation. From Fig. 5 it is seen that, for a thermally loaded annular micro-plate, static deflection decreases significantly as l_c/l_m increases. This implies that the micro-plate displays a stiffer behavior when the length scale parameter ratio is larger. However, as Fig. 6 indicates, a thermally loaded circular micro-plate is not sensitive to the variations in l_c/l_m. Figs. 7 and 8 show distributions of normalized circumferential and radial stresses in a simply supported annular FGM micro-plate. Both of the stress components are compressive under the given thermal loading condition. Largest magnitudes are computed near the upper surface. These magnitudes are greater for micro-plates displaying a stiffer behavior, i.e., for those

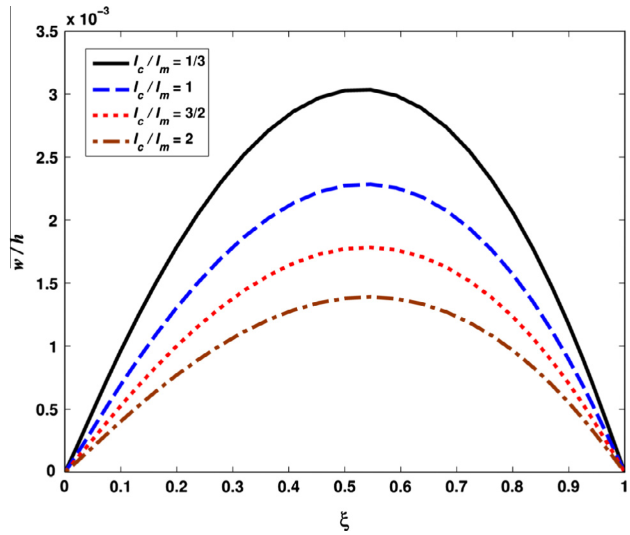

Fig. 5. Static deformation profiles of a simply-supported annular FGM micro-plate generated by considering different length scale parameter ratios. $R_o/h = 10$, $R_o/R_i = 4$, $h/l_m = 2$, $l_m = 15$ μm, $\lambda = 2$, $\theta_L = 20\,^\circ$C, $\theta_U = 70\,^\circ$C.

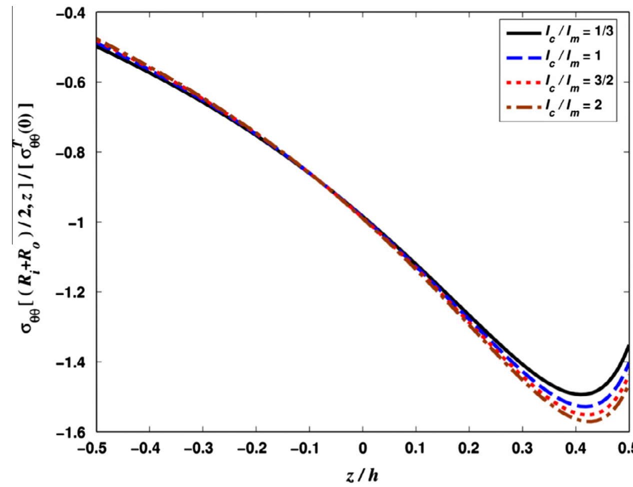

Fig. 7. Normalized circumferential stress distributions in a simply-supported annular FGM micro-plate generated by considering different length scale parameter ratios. $R_o/h = 10$, $R_o/R_i = 4$, $h/l_m = 2$, $l_m = 15$ μm, $\lambda = 2$, $\theta_L = 20\,^\circ$C, $\theta_U = 70\,^\circ$C.

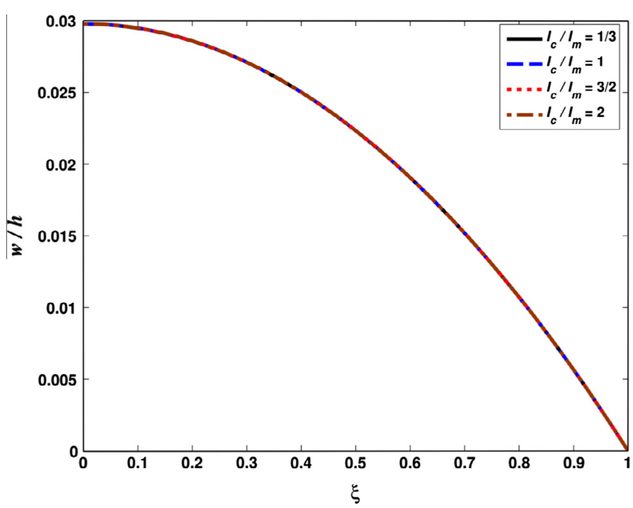

Fig. 6. Static deformation profiles of a simply-supported circular FGM micro-plate generated by considering different length scale parameter ratios. $R_o/h = 10$, $h/l_m = 2$, $l_m = 15$ μm, $\lambda = 2$, $\theta_L = 20\,^\circ$C, $\theta_U = 70\,^\circ$C.

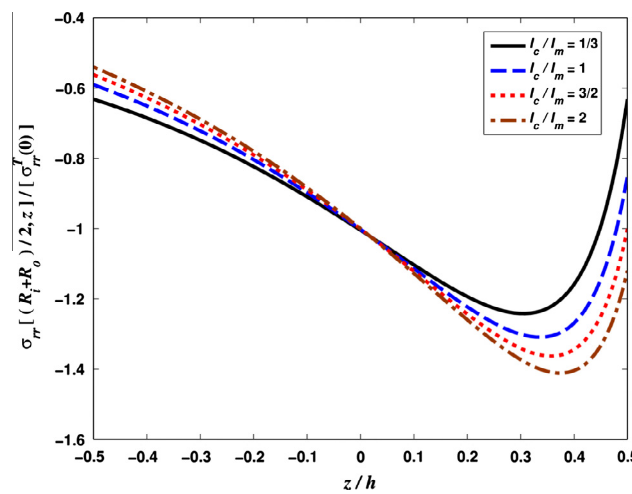

Fig. 8. Normalized radial stress distributions in a simply-supported annular FGM micro-plate generated by considering different length scale parameter ratios. $R_o/h = 10$, $R_o/R_i = 4$, $h/l_m = 2$, $l_m = 15$ μm, $\lambda = 2$, $\theta_L = 20\,^\circ$C, $\theta_U = 70\,^\circ$C.

possessing larger l_c/l_m values. The influence of the length scale parameter ratio on the radial stress component is in general more pronounced compared to its effect on circumferential stress.

Outcomes of parametric analyses regarding free vibrations under initial thermal stresses are presented in Tables 2 and 3; and Figs. 9–13. All of these results are produced by using third order shear deformation theory. Table 2 tabulates first three dimensionless natural frequencies of a simply supported annular FGM micro-plate. The frequencies correspond to the transverse deformation mode. The findings indicate that each natural frequency is an increasing function of the length scale parameter ratio l_c/l_m, and a decreasing function of the inhomogeneity parameter λ. The rise in l_c/l_m again causes a stiffer response demonstrated by the corresponding rise in the natural frequency. This behavior is consistent with our findings pertaining to static analysis. Similar trends are found to be valid for a simply supported circular FGM micro-plate as indicated by Table 3.

Table 2
First three dimensionless transverse deformation natural frequencies of a simply-supported annular FGM micro-plate. $R_o/h = 10$, $R_o/R_i = 4$, $h/l_m = 2$, $\theta_L = 20\,^\circ$C, $\theta_U = 70\,^\circ$C.

Mode	λ	l_c/l_m			
		1/3	1.0	3/2	2.0
First	0.5	0.7556	0.9867	1.2100	1.4548
	1.0	0.7163	0.8899	1.0586	1.2456
	2.0	0.6836	0.8036	0.9217	1.0547
	5.0	0.6459	0.7113	0.7773	0.8535
Second	0.5	2.6159	3.5185	4.3658	5.1598
	1.0	2.4443	3.1296	3.7703	4.3959
	2.0	2.3322	2.8127	3.2667	3.7331
	5.0	2.2556	2.5274	2.7920	3.0839
Third	0.5	5.3549	7.3128	9.0908	11.7195
	1.0	5.0445	6.5120	7.8643	10.2959
	2.0	4.7799	5.8484	6.8239	7.6958
	5.0	4.6024	5.2375	5.8336	6.4504

Table 3
First three dimensionless transverse deformation natural frequencies of a simply-supported circular FGM micro-plate. $R_o/h = 10$, $h/l_m = 2$, $\theta_L = 20\,°C$, $\theta_U = 70\,°C$.

Mode	λ	l_c/l_m			
		1/3	1.0	3/2	2.0
First	0.5	0.2234	0.2334	0.2386	0.2421
	1.0	0.2066	0.2147	0.2194	0.2228
	2.0	0.1915	0.1971	0.2009	0.2039
	5.0	0.1712	0.1740	0.1762	0.1783
Second	0.5	1.4875	1.9409	2.3861	2.8757
	1.0	1.3909	1.7349	2.0724	2.4467
	2.0	1.3281	1.5655	1.8010	2.0665
	5.0	1.2781	1.4059	1.5357	1.6862
Third	0.5	3.5587	4.7329	5.8428	7.1996
	1.0	3.3227	4.2186	5.0629	5.6589
	2.0	3.1687	3.7979	4.3968	4.9378
	5.0	3.0587	3.4125	3.7584	4.1340

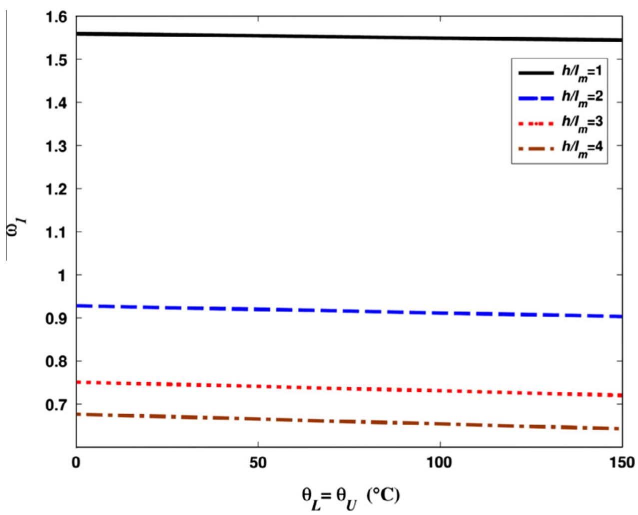

Fig. 9. First dimensionless natural frequency of a functionally graded annular micro-plate vs. temperature and h/l_m. $R_o/h = 10$, $l_m = 15$ μm, $\lambda = 2$, $l_c/l_m = 3/2$, $R_o/R_i = 4$.

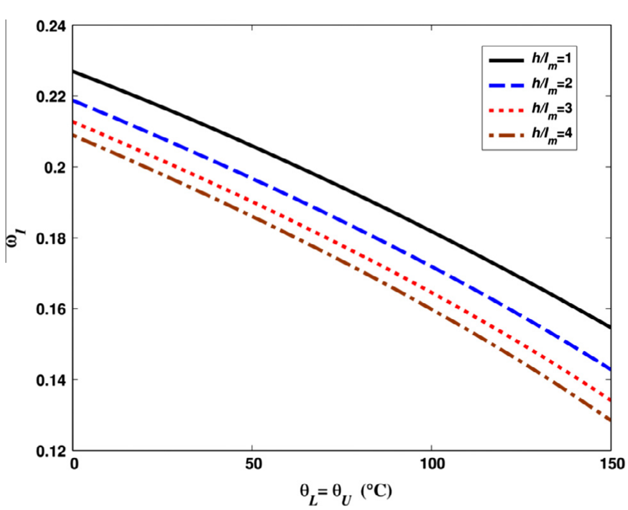

Fig. 10. First dimensionless natural frequency of a functionally graded circular micro-plate vs. temperature and h/l_m. $R_o/h = 10$, $l_m = 15$ μm, $\lambda = 2$, $l_c/l_m = 3/2$.

In Figs. 9 and 10, we provide first dimensionless natural frequency ω_1 as a function of boundary temperature difference for simply-supported annular and circular micro-plates, respectively.

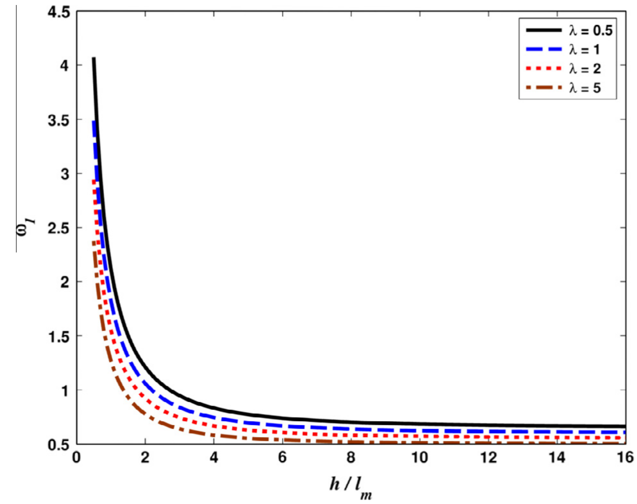

Fig. 11. First dimensionless natural frequency of a functionally graded annular micro-plate as functions of λ and h/l_m. $R_o/h = 10$, $l_m = 15$μm, $l_c/l_m = 3/2$, $R_o/R_i = 4$, $\theta_L = 20\,°C$, $\theta_U = 70\,°C$.

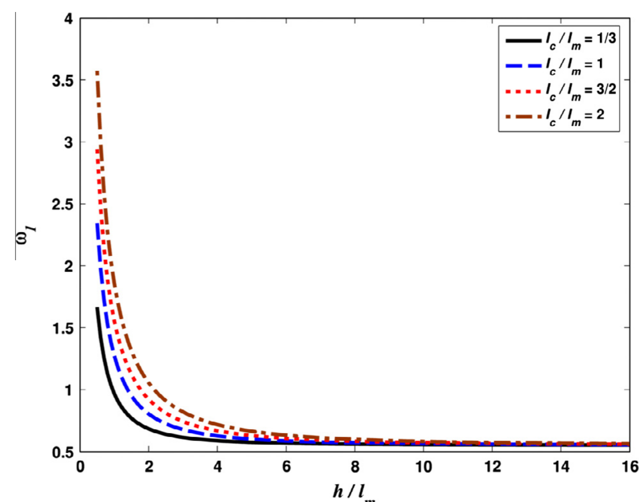

Fig. 12. First dimensionless natural frequency of a functionally graded annular micro-plate as functions of l_c/l_m and h/l_m. $R_o/h = 10$, $l_m = 15$ μm, $\lambda = 2$, $R_o/R_i = 4$, $\theta_L = 20\,°C$, $\theta_U = 70\,°C$.

In each case, ω_1 curve is plotted for four different values of the ratio h/l_m. Note that temperature differences of lower and upper plate surfaces are assumed to be equal, i.e. $\theta_L = \theta_U$. Results given in Fig. 9 reveal that first natural frequency of a simply-supported annular micro-plate is not that sensitive to the variation in temperature difference. However, ω_1 of a circular micro-plate drops significantly as temperature difference increases, as can be seen from Fig. 10. The ratio h/l_m is representative of the degree of size effect. Size dependence is more prevalent when this ratio is relatively small. For both annular and circular micro-plates, first dimensionless frequency increases notably as h/l_m decreases from 4 to 1. This finding is in complete agreement with the physics of the problem since as the component size approaches the realm of micro-scale, size effect causes it to behave in a stiffer manner and leads to significantly smaller deflections and larger natural frequencies.

The effect of h/l_m over a wider domain for a simply-supported annular micro-plate is depicted in Fig. 11. The variation is plotted for four different values of the inhomogeneity parameter λ. ω_1

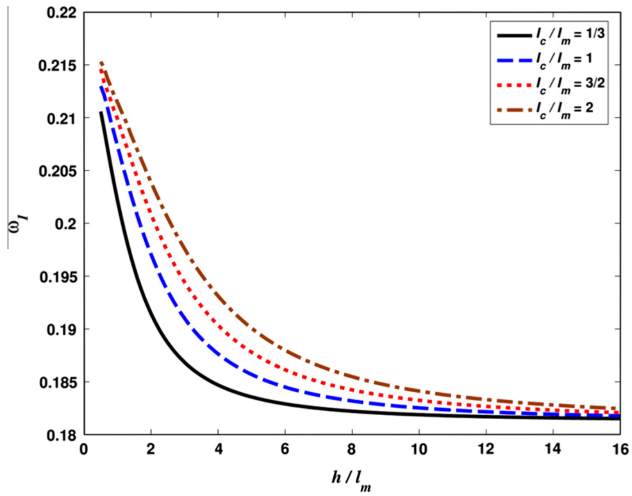

Fig. 13. First dimensionless natural frequency of a functionally graded circular micro-plate as functions of l_c/l_m and h/l_m. $R_o/h = 10$, $l_m = 15$ μm, $\lambda = 2$, $\theta_L = 20\,^\circ$C, $\theta_U = 70\,^\circ$C.

rises sharply as the ratio h/l_m tends to zero, which is indicative of the size effect. Inhomogeneity parameter is also influential and an increase in its value causes a drop in the dimensionless first natural frequency. From Eq. (2), it follows that, annular and circular micro-plates are both ceramic-rich when $\lambda < 1$, and metal-rich when $\lambda > 1$. Thus, ceramic-rich micro-plates display a stiffer vibration behavior compared to metal-rich micro-plates.

Finally, in Figs. 12 and 13, ω_1 is given as functions of h/l_m and the length scale parameter ratio l_c/l_m. The results provided in Fig. 12 for an annular plate and those presented in Fig. 13 for a circular one are in general similar. In both cases, as l_c/l_m is increased from 1/3 to 2, dimensionless first natural frequency becomes larger. As is the case for static loading, the increase in the length scale parameter ratio causes the micro-plates to display a stiffer behavior.

5. Concluding remarks

We presented new techniques that facilitate solution of static bending and free vibrations problems involving thermally loaded functionally graded annular and circular micro-plates. Governing partial differential equations and corresponding boundary conditions are derived by applying Hamilton's principle in conjunction with modified couple stress theory. Posed problems are then solved numerically utilizing the differential quadrature method, which converts partial derivatives into finite sums in terms of weighting coefficients and functional values. Comparisons to the results provided by Noda et al. [39] and Raju and Rao [34] do verify the proposed thermal analysis procedures. Extensive parametric analyses are conducted to be able to assess the influences of factors such as applied thermal loading, length scale parameter ratio, problem geometry, and material inhomogeneity.

Our findings illustrate that both type and magnitude of thermal loading have a substantive effect on the mechanical response of graded annular and circular micro-plates. A graded micro-plate under static loading bends concave downwards when the temperature of the upper surface is greater than or equal to that of the lower surface. On the other hand, for freely vibrating annular and circular micro-plates that are under the influence of initial thermal stresses, increase in the body temperature leads to drops in the first dimensionless natural frequency.

Each of the material and geometric parameters l_c/l_m, h/l_m, and λ possesses a strong bearing on the behavior of annular and circular micro-plates. The ratio h/l_m quantifies the extent of size effect since when it is around or smaller than unity, size effect is expected to be noticeable. Our results demonstrate a sharp rise in natural frequency with a drop in h/l_m which is indicative of stiffening at the micro-scale. Such a strengthening in mechanical response is also observed for larger l_c/l_m values and for ceramic-rich plates, for which λ assumes values less than unity.

Annular and circular micro-plates are fabricated for a wide variety of micro-electro-mechanical-systems including acoustic energy harvesters, micro-scale resonators, optical and pressure sensors, and stiction valves. The methods presented could prove useful in accounting for temperature related effects and material inhomogeneity in design studies involving such components. Proposed framework is open to further extension especially through incorporation of large deformations by means of a suitable nonlinear plate theory.

Acknowledgements

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(Konu ile ilgili bir dergiye gönderilecektir)

Free vibration analysis of functionally graded rectangular nano-plates considering spatial variation of the nonlocal parameter

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ABSTRACT

This study presents a new nonlocal elasticity based analysis method for free vibrations of functionally graded rectangular nano-plates. The method allows taking into account spatial variation of the nonlocal parameter. Governing partial differential equations and associated boundary conditions are derived by employing the variational approach and applying Hamilton's principle. All required material properties are assumed to be functions of thickness coordinate in the derivations. Displacement field is expressed in a unified way to be able to produce numerical results pertaining to three different plate theories, namely Kirchhoff, Mindlin, and third-order shear deformation theories. The equations are solved numerically by means of the generalized differential quadrature method. Proposed procedures are verified through comparisons made to the results available in the literature. Further numerical results are generated by considering functionally graded simply-supported and cantilever nano-plates undergoing free vibrations. These findings demonstrate influences of factors such as dimensionless plate length, plate theory, nonlocal parameter ratio, and power-law index upon natural vibration frequencies.

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Keywords: A. Nano-structures; A. Plates; B. Vibration; C. Analytical modelling; Functionally graded materials.

1. Introduction

Functionally graded materials (FGMs) are a special class of composites, which possess smooth spatial variations in the volume fractions of the constituent phases. They find applications in a wide variety of technological fields including thermal barrier coatings, solid oxide fuel cells, high performance cutting tools, and biomedical materials. Deployment of functionally graded components in small-scale systems has recently become feasible with advances in fabrication technologies such as magnetron sputtering [1], chemical vapor deposition and plasma enhanced chemical vapor deposition [2], and modified soft lithography [3]. These developments are accompanied by theoretical and computational studies directed towards understanding mechanical behavior of small-scale FGM composite structures. Classical continuum theories fail to account for the size effect observable at small-scales, whereas molecular dynamics based simulations need to confront an immense computational effort. As a result, higher order continuum theories have been commonly utilized to examine behavior of small-scale FGM beams, plates, and shells. Among such theories, we can mention nonlocal elasticity [4,5], strain gradient elasticity [6–9], and couple stress theories [10–12].

Nonlocal elasticity theory has been widely used to investigate small-scale effects on free vibrations, bending and buckling of small-sized functionally graded beams and plates. Work on functionally graded nonlocal beams encompass free vibrations [13,14], buckling [15,16], nonlinear vibrations [17], surface effects [18], forced vibrations [19], and thermomechanical vibrations [20]. Finite element analysis based solutions are presented by Eltaher et al. [21–23] and Reddy et al. [24]. Structural theories utilized in these articles to describe nonlocal FGM beam behavior include Euler-Bernoulli, Timoshenko, and third-order beam theories.

Studies pertaining to nonlocal FGM plates consider free vibrations [25–28], buckling [29,30], effect of distributed nanoparticles [31] and Winkler-Pasternak elastic foundation [32]. A three dimensional nonlocal elasticity solution for small-scale FGM plates is proposed by Salehipour et al. [33]. Novel computational approaches are discussed by Nguyen et al. [34] and Ansari et al. [35]. Commonly used structural theories in analysis of nonlocal FGM plates are Kirchhoff, Mindlin, and third-order shear deformation theories.

In all work mentioned in the above paragraphs, the nonlocal parameter of the nonlocal elasticity theory is assumed to be constant. However, the nonlocal parameter is essentially a material property [36] and thus varies as a function of spatial coordinates in a functionally graded composite structure. The primary objective in this study, is to reveal the influence of the *spatial variation of the nonlocal parameter* upon free vibration behavior of small-scale rectangular functionally graded plates. For this purpose, a set of governing partial differential equations and boundary conditions are derived by employing the nonlocal elasticity theory and variational principles. All material properties, including the nonlocal parameter, are assumed to be functions of the thickness coordinate in the derivations. Displacement field is expressed in a unified way to be able to produce numerical results for Kirchhoff, Mindlin, and third-order plate theories. The equations are solved numerically by means of the generalized differential quadrature method. Developed procedures are verified through comparisons made to the findings available in the literature. Simply-supported and cantilever nano-plates are considered in parametric analyses. Results presented for these configurations illustrate influences of material and geometric parameters upon natural vibration frequencies.

2. Formulation

The geometry of the functionally graded rectangular nano-plate is depicted in Fig. 1. The plate is of thickness h and assumed to possess property variations in z-direction. In nonlocal elasticity theory, stress at a point is expressed as a function of the strain field in the material domain as follows:

$$\sigma_{ij} = \iiint_V \alpha(|x' - x|, \tau) t_{ij}(x') dV(x'), \tag{1}$$

where α is the nonlocal modulus or kernel function, $|x' - x|$ represents the distance, τ is a material property that depends on internal and external characteristic lengths (such as the lattice spacing and wavelength); and σ_{ij} and t_{ij} stand for nonlocal and local stress tensors, respectively. Eringen [37] proposed an equivalent differential form of the nonlocal constitutive equation in the form:

$$(1 - \mu \nabla^2)\sigma_{ij} = t_{ij} . \tag{2}$$

μ in this equation is the nonlocal parameter defined by

$$\mu = (e_0 l)^2, \tag{3}$$

where l is internal characteristic length and e_0 is a material property found through experimental characterization. Because of its dependence on l and e_0, μ is a material property and should be expressed as a function of the z-coordinate as well.

The relation between local stress tensor t_{ij} and strain tensor ε_{ij} is expressed by

$$\begin{Bmatrix} t_{xx} \\ t_{yy} \\ t_{xy} \\ t_{xz} \\ t_{yz} \end{Bmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{66} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{pmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \end{Bmatrix}, \tag{4}$$

where,

$$Q_{11} = Q_{22} = \frac{E(z)}{1-\nu(z)^2}, \tag{5a}$$

$$Q_{12} = Q_{21} = \frac{E(z)\upsilon(z)}{1-\nu(z)^2}, \tag{5b}$$

$$Q_{66} = \frac{E(z)}{2(1+\nu(z))}. \tag{5c}$$

E and ν are respectively modulus of elasticity and Poisson's ratio. All material properties including the nonlocal parameter are functions of the thickness coordinate and their spatial variations are described by

$$E(z) = E_c V_c(z) + E_m V_m(z), \tag{6a}$$

$$\nu(z) = \nu_c V_c(z) + \nu_m V_m(z), \tag{6b}$$

$$\rho(z) = \rho_c V_c(z) + \rho_m V_m(z), \tag{6c}$$

$$\mu(z) = \mu_c V_c(z) + \mu_m V_m(z). \tag{6d}$$

The subscripts c and m stand for ceramic and metallic phases; and V_c and V_m are volume fractions. ρ in Eq. (6c) is mass density. Spatial variations of the volume fractions are represented as follows:

$$V_c(z) = (\frac{1}{2} + \frac{z}{h})^n, \tag{7a}$$

$$V_m(z) = 1 - V_c(z). \tag{7b}$$

The power-law index n defines property distribution profiles. When n is less than 1 the nano-plate is ceramic-rich, whereas if n is greater than unity plate has a metal-rich FGM profile.

Displacement field of the nano-plate is expressed in a unified way as given below:

$$u(x,y,z,t) = u_0(x,y,t) - zw_{,x} + f(z)(\phi_x + w_{,x}), \tag{8a}$$

$$v(x,y,z,t) = v_0(x,y,t) - zw_{,y} + f(z)(\phi_y + w_{,y}), \tag{8b}$$

$$w(x,y,z,t) = w_0(x,y,t), \tag{8c}$$

where,

$$f(z) = \begin{cases} 0, & \text{for Kirchhof plate theory,} \\ z, & \text{for Mindlin plate theory,} \\ z\left(1 - \dfrac{4z^2}{3h^2}\right), & \text{for Third-order shear deformation plate theory.} \end{cases} \tag{8d}$$

In this representation *u*, *v*, and *w* are displacement components in *x*, *y*, and *z* directions, respectively; u_0, v_0, and w_0 are displacements of a point on the midplane $z = 0$; ϕ_x and ϕ_y are the rotations of a transverse normal about y and x axes, respectively; and a comma stands for differentiation. Strain field corresponding to these displacements is then found in the form:

$$\varepsilon_{xx} = \frac{1}{2}(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}) = u_{0,x} - zw_{,xx} + f(\phi_{x,x} + w_{,xx}), \tag{9a}$$

$$\varepsilon_{yy} = \frac{1}{2}(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}) = v_{0,y} - zw_{,yy} + f(\phi_{y,y} + w_{,yy}), \tag{9b}$$

$$\varepsilon_{zz} = \frac{1}{2}(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial z}) = 0, \tag{9c}$$

$$\varepsilon_{xy} = \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) = \frac{1}{2}\left\{(u_{0,y} + v_{0,x}) - 2zw_{,xy} + f(\phi_{x,y} + 2w_{,xy} + \phi_{y,x})\right\}, \tag{9d}$$

$$\varepsilon_{yz} = \frac{1}{2}(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) = \frac{1}{2} f'(\phi_y + w_{,y}), \tag{9e}$$

$$\varepsilon_{xz} = \frac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) = \frac{1}{2} f'(\phi_x + w_{,x}). \tag{9f}$$

For an FGM composite nano-plate undergoing free vibrations, Hamilton's principle requires that

$$\delta \int_{t_1}^{t_2} (K - U) dt = 0, \tag{10}$$

where *U* is strain energy and *K* is kinetic energy. Variations of the energy terms are written as:

$$\delta U = \iiint_V (\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + 2\sigma_{xy}\delta\varepsilon_{xy} + 2\sigma_{xz}\delta\varepsilon_{xz} + +2\sigma_{yz}\delta\varepsilon_{yz})dV, \tag{11a}$$

$$\delta K = \iiint_V \rho(z)(\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v} + \dot{w}\delta\dot{w})dV. \tag{11b}$$

Using Eqs. (9)-(11) and variational principles, governing partial differential equations are derived as follows:

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \left\{ I_0 \frac{\partial^2 u_0}{\partial t^2} - I_1 \frac{\partial^3 w}{\partial t^2 \partial x} + I_3 (\frac{\partial^2 \phi_x}{\partial t^2} + \frac{\partial^3 w}{\partial t^2 \partial x}) \right\} - \nabla^2 \left\{ L_0 \frac{\partial^2 u_0}{\partial t^2} - L_1 \frac{\partial^3 w}{\partial t^2 \partial x} + L_3 (\frac{\partial^2 \phi_x}{\partial t^2} + \frac{\partial^3 w}{\partial t^2 \partial x}) \right\}, \tag{12a}$$

$$\frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} = \left\{ I_0 \frac{\partial^2 v_0}{\partial t^2} - I_1 \frac{\partial^3 w}{\partial t^2 \partial y} + I_3 (\frac{\partial^2 \phi_y}{\partial t^2} + \frac{\partial^3 w}{\partial t^2 \partial y}) \right\} - \nabla^2 \left\{ L_0 \frac{\partial^2 v_0}{\partial t^2} - L_1 \frac{\partial^3 w}{\partial t^2 \partial y} + L_3 (\frac{\partial^2 \phi_y}{\partial t^2} + \frac{\partial^3 w}{\partial t^2 \partial y}) \right\}, \tag{12b}$$

$$\begin{aligned}
&\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial^2 M_{yy}}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} - \frac{\partial^2 P_{xx}}{\partial x^2} - \frac{\partial^2 P_{yy}}{\partial y^2} - 2\frac{\partial^2 P_{xy}}{\partial x \partial y} + \frac{\partial R_{yz}}{\partial y} + \frac{\partial R_{xz}}{\partial x} = \\
&+\left\{ I_0 \ddot{w} + (I_1 - I_3)(\frac{\partial \ddot{u}_0}{\partial x} + \frac{\partial \ddot{v}_0}{\partial y}) + (-I_2 + 2I_4 - I_5)(\frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\partial^2 \ddot{w}}{\partial y^2}) + (I_4 - I_5)(\frac{\partial \ddot{\phi}_x}{\partial x} + \frac{\partial \ddot{\phi}_y}{\partial y}) \right\} \\
&-\nabla^2 \left\{ L_0 \ddot{w} + (L_1 - L_3)(\frac{\partial \ddot{u}_0}{\partial x} + \frac{\partial \ddot{v}_0}{\partial y}) + (-L_2 + 2L_4 - L_5)(\frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\partial^2 \ddot{w}}{\partial y^2}) + (L_4 - L_5)(\frac{\partial \ddot{\phi}_x}{\partial x} + \frac{\partial \ddot{\phi}_y}{\partial y}) \right\},
\end{aligned} \tag{12c}$$

$$\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} - R_{xz} = \left\{ I_3 \frac{\partial^2 u_0}{\partial t^2} - I_4 \frac{\partial^3 w}{\partial t^2 \partial x} + I_5 (\frac{\partial^2 \phi_x}{\partial t^2} + \frac{\partial^3 w}{\partial t^2 \partial x}) \right\} - \nabla^2 \left\{ L_3 \frac{\partial^2 u_0}{\partial t^2} - L_4 \frac{\partial^3 w}{\partial t^2 \partial x} + L_5 (\frac{\partial^2 \phi_x}{\partial t^2} + \frac{\partial^3 w}{\partial t^2 \partial x}) \right\}, \tag{12d}$$

$$\frac{\partial P_{yy}}{\partial y} + \frac{\partial P_{xy}}{\partial x} - R_{yz} = \left\{ I_3 \frac{\partial^2 v_0}{\partial t^2} - I_4 \frac{\partial^3 w}{\partial t^2 \partial y} + I_5 (\frac{\partial^2 \phi_y}{\partial t^2} + \frac{\partial^3 w}{\partial t^2 \partial y}) \right\} - \nabla^2 \left\{ L_3 \frac{\partial^2 v_0}{\partial t^2} - L_4 \frac{\partial^3 w}{\partial t^2 \partial y} + L_5 (\frac{\partial^2 \phi_y}{\partial t^2} + \frac{\partial^3 w}{\partial t^2 \partial y}) \right\}. \tag{12e}$$

The boundary conditions are obtained as:

$$u_0 = 0, \quad \text{or} \qquad N_{xx} n_x + N_{xy} n_y = 0, \tag{13a}$$

$$v_0 = 0, \quad \text{or} \qquad N_{xy} n_x + N_{yy} n_y = 0, \tag{13b}$$

$$\phi_x = 0, \quad \text{or} \qquad P_{xx} n_x + P_{xy} n_y = 0, \tag{13c}$$

$$\phi_y = 0, \quad \text{or} \qquad P_{xy} n_x + P_{yy} n_y = 0, \tag{13d}$$

$w = 0$, or

$$\begin{aligned}
&\left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - \frac{\partial P_{xx}}{\partial x} - \frac{\partial P_{xy}}{\partial y} + R_{xz}\right) n_x + \left(\frac{\partial M_{yy}}{\partial y} + \frac{\partial M_{xy}}{\partial x} - \frac{\partial P_{yy}}{\partial y} - \frac{\partial P_{xy}}{\partial x} + R_{yz}\right) n_y = \\
&\left\{ \left\{ (I_1 - I_3)\ddot{u}_0 + (-I_2 + 2I_4 - I_5)\frac{\partial \ddot{w}}{\partial x} + (I_4 - I_5)\ddot{\phi}_x \right\} - \nabla^2 \left\{ (L_1 - L_3)\ddot{u}_0 + (-L_2 + 2L_4 - L_5)\frac{\partial \ddot{w}}{\partial x} + (L_4 - L_5)\ddot{\phi}_x \right\} \right\} n_x \\
&+ \left\{ \left\{ (I_1 - I_3)\ddot{v}_0 + (-I_2 + 2I_4 - I_5)\frac{\partial \ddot{w}}{\partial y} + (I_4 - I_5)\ddot{\phi}_y \right\} - \nabla^2 \left\{ (L_1 - L_3)\ddot{v}_0 + (-L_2 + 2L_4 - L_5)\frac{\partial \ddot{w}}{\partial y} + (L_4 - L_5)\ddot{\phi}_y \right\} \right\} n_y,
\end{aligned} \tag{13e}$$

$$\frac{\partial w}{\partial x} = 0, \quad \text{or} \qquad (M_{xx} - P_{xx}) n_x + (M_{xy} - P_{xy}) n_y = 0, \tag{13f}$$

$$\frac{\partial w}{\partial y} = 0, \quad \text{or} \qquad (M_{xy} - P_{xy}) n_x + (M_{yy} - P_{yy}) n_y = 0, \tag{13g}$$

where n_x and n_y are the components of the unit outward normal vector.

Stress resultants and coefficient terms in the governing equations and boundary conditions are defined by

$$\begin{Bmatrix} N_{\alpha\beta} \\ M_{\alpha\beta} \\ P_{\alpha\beta} \end{Bmatrix} = \int_{-h/2}^{h/2} (1-\mu(z)\nabla^2)\sigma_{\alpha\beta} \begin{Bmatrix} 1 \\ z \\ f \end{Bmatrix} dz, \qquad \alpha = x, y, \qquad \beta = x, y, \tag{14a}$$

$$\begin{Bmatrix} N_{\alpha z} \\ R_{\alpha z} \end{Bmatrix} = \int_{-h/2}^{h/2} (1-\mu(z)\nabla^2)\sigma_{\alpha z} \begin{Bmatrix} 1 \\ f' \end{Bmatrix} dz, \qquad \alpha = x, y, \tag{14b}$$

$$\begin{Bmatrix} I_0 \\ I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{Bmatrix} = \int_{-h/2}^{h/2} \rho(z) \begin{Bmatrix} 1 \\ z \\ z^2 \\ f \\ zf \\ f^2 \end{Bmatrix} dz, \tag{14c}$$

$$\begin{Bmatrix} L_0 \\ L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{Bmatrix} = \int_{-h/2}^{h/2} \mu(z)\rho(z) \begin{Bmatrix} 1 \\ z \\ z^2 \\ f \\ zf \\ f^2 \end{Bmatrix} dz. \tag{14d}$$

3. Numerical solution

Generalized differential quadrature method (GDQM) [38] is used to solve the equation system comprising governing partial differential equations and boundary conditions. According to GDQM, n^{th}- derivative of a function f is expressed as follows:

$$\frac{\partial^n f(x,t)}{\partial x^n}\Big|_{x=x_i} = \sum_{j=1}^{N} c_{ij}^{(n)} f(x_j, t), \qquad i = 1, 2, ..., N, \tag{15}$$

where $c_{ij}^{(n)}$ are weighting coefficients [39] for the n^{th}-order derivative and N is the number of nodes. In parametric analyses, we consider two different types of nano-plate configurations: A nano-plate simply-supported over all edges; and a cantilever nano-plate fixed at $x = 0$. For both simply-supported and cantilever nano-plates, nodal points are identified as Chebyshev-Gauss-Lobatto points, which are given by

$$x_i = \frac{1}{2}\left\{1-\cos\left(\frac{\pi\left(i-1\right)}{N-1}\right)\right\}, \qquad i = 1, 2, ..., N. \tag{16}$$

Applying the representation in Eq. (15) to the differential operators, series forms of the governing equations are derived as:

$$\begin{aligned}
&A_0\sum_{k=1}^{N_x} c_{ik}{}^{(2)} u_{0_{k,j}} + C_0\sum_{k=1}^{N_y} c_{jk}{}^{(2)} u_{0_{i,k}} + (B_0+C_0)\sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} v_{0_{k,m}} + (A_3-A_1)\sum_{k=1}^{N_x} c_{ik}{}^{(3)} w_{k,j} \\
&+(B_3-B_1-2C_1+2C_3)\sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} w_{k,m} + A_3\sum_{k=1}^{N_x} c_{ik}{}^{(2)} \phi_{x_{k,j}} + C_3\sum_{k=1}^{N_y} c_{jk}{}^{(2)} \phi_{x_{i,k}} \\
&+(B_3+C_3)\sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{y_{k,m}} = I_0\ddot{u}_0 + (I_3-I_1)\sum_{k=1}^{N_x} c_{ik}{}^{(1)} \ddot{w}_{k,j} + I_3\ddot{\phi}_x - L_0\sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{u}_{0_{k,j}} \\
&-(L_3-L_1)\sum_{k=1}^{N_x} c_{ik}{}^{(3)} \ddot{w}_{k,j} - L_3\sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{\phi}_{x_{k,j}} - L_0\sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{u}_{0_{i,k}} - (L_3-L_1)\sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \ddot{w}_{k,m} - L_3\sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{\phi}_{x_{i,k}},
\end{aligned} \tag{17a}$$

$$\begin{aligned}
&C_0\sum_{k=1}^{N_x} c_{ik}{}^{(2)} v_{0_{k,j}} + A_0\sum_{k=1}^{N_y} c_{jk}{}^{(2)} v_{0_{i,k}} + (B_0+C_0)\sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} u_{0_{k,m}} + (A_3-A_1)\sum_{k=1}^{N_y} c_{jk}{}^{(3)} w_{i,k} \\
&+(B_3-B_1-2C_1+2C_3)\sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} w_{k,m} + C_3\sum_{k=1}^{N_x} c_{ik}{}^{(2)} \phi_{y_{k,j}} + A_3\sum_{k=1}^{N_y} c_{jk}{}^{(2)} \phi_{y_{i,k}} \\
&+(B_3+C_3)\sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{x_{k,m}} = I_0\ddot{v}_0 + (I_3-I_1)\sum_{k=1}^{N_y} c_{jk}{}^{(1)} \ddot{w}_{i,k} + I_3\ddot{\phi}_y - L_0\sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{v}_{0_{k,j}} \\
&-(L_3-L_1)\sum_{k=1}^{N_y} c_{jk}{}^{(3)} \ddot{w}_{i,k} - L_3\sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{\phi}_{y_{k,j}} - L_0\sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{v}_{0_{i,k}} - (L_3-L_1)\sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{w}_{k,m} - L_3\sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{\phi}_{y_{i,k}},
\end{aligned} \tag{17b}$$

$$
\begin{aligned}
&(A_1-A_3)\Big(\sum_{k=1}^{N_y} c_{jk}{}^{(3)} v_{0_{i,k}} + \sum_{k=1}^{N_x} c_{ik}{}^{(3)} u_{0_{k,j}}\Big) + (2A_4-A_2-A_5)\Big(\sum_{k=1}^{N_x} c_{ik}{}^{(4)} w_{k,j} + \sum_{k=1}^{N_y} c_{jk}{}^{(4)} w_{i,k}\Big) + (A_4-A_5)\Big(\sum_{k=1}^{N_x} c_{ik}{}^{(3)} \phi_{x_{k,j}} \\
&+\sum_{k=1}^{N_y} c_{jk}{}^{(3)} \phi_{y_{i,k}}\Big) + (B_1-B_3+2C_1-2C_3)\Big(\sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} v_{0_{k,m}} + \sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} u_{0_{k,m}}\Big) \\
&+2(2B_4-B_2-B_5+4C_4-2C_2-2C_5)\sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} w_{k,m} + (B_4-B_5+2C_4-2C_5)\Big(\sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \phi_{y_{k,m}} \\
&+\sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{x_{k,m}}\Big) + C_6\Big(\sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{x_{k,j}} + \sum_{k=1}^{N_y} c_{jk}{}^{(1)} \phi_{y_{i,k}}\Big) + C_6\Big(\sum_{k=1}^{N_x} c_{ik}{}^{(2)} w_{k,j} + \sum_{k=1}^{N_y} c_{jk}{}^{(2)} w_{i,k}\Big) = \\
&I_0\ddot{w} + (I_1-I_3)\Big(\sum_{k=1}^{N_x} c_{ik}{}^{(1)} \ddot{u}_{0_{k,j}} + \sum_{k=1}^{N_y} c_{jk}{}^{(1)} \ddot{v}_{0_{i,k}}\Big) + (-I_2+2I_4-I_5)\Big(\sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{w}_{k,j} + \sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{w}_{i,k}\Big) \\
&+(I_4-I_5)\Big(\sum_{k=1}^{N_x} c_{ik}{}^{(1)} \ddot{\phi}_{x_{k,j}} + \sum_{k=1}^{N_y} c_{jk}{}^{(1)} \ddot{\phi}_{y_{i,k}}\Big) - L_0 \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{w}_{k,j} - (L_1-L_3)\Big(\sum_{k=1}^{N_x} c_{ik}{}^{(3)} \ddot{u}_{0_{k,j}} + \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{v}_{0_{k,m}}\Big) \\
&-(-L_2+2L_4-L_5)\Big(\sum_{k=1}^{N_x} c_{ik}{}^{(4)} \ddot{w}_{k,j} + \sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{w}_{k,m}\Big) - (L_4-L_5)\Big(\sum_{k=1}^{N_x} c_{ik}{}^{(3)} \ddot{\phi}_{x_{k,j}} + \sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{\phi}_{y_{k,m}}\Big) \\
&-L_0 \sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{w}_{i,k} - (L_1-L_3)\Big(\sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \ddot{u}_{0_{k,m}} + \sum_{k=1}^{N_y} c_{jk}{}^{(3)} \ddot{v}_{0_{i,k}}\Big) \\
&-(-L_2+2L_4-L_5)\Big(\sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{w}_{k,m} + \sum_{k=1}^{N_y} c_{jk}{}^{(4)} \ddot{w}_{i,k}\Big) - (L_4-L_5)\Big(\sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \ddot{\phi}_{x_{k,m}} + \sum_{k=1}^{N_y} c_{jk}{}^{(3)} \ddot{\phi}_{y_{i,k}}\Big),
\end{aligned}
\tag{17c}
$$

$$
\begin{aligned}
&A_3\sum_{k=1}^{N_x} c_{ik}{}^{(2)} u_{0_{k,j}} + C_3\sum_{k=1}^{N_y} c_{jk}{}^{(2)} u_{0_{i,k}} + (B_3+C_3)\sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} v_{0_{k,m}} + (A_5-A_4)_3\sum_{k=1}^{N_x} c_{ik}{}^{(3)} w_{k,j} \\
&+(B_5-B_4-2C_4+2C_5)\sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} w_{k,m} + A_5\sum_{k=1}^{N_x} c_{ik}{}^{(2)} \phi_{x_{k,j}} + C_5\sum_{k=1}^{N_y} c_{jk}{}^{(2)} \phi_{x_{i,k}} \\
&+(B_5+C_5)\sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{y_{k,m}} - C_6\phi_x - C_6\sum_{k=1}^{N_x} c_{ik}{}^{(1)} w_{k,j} = I_3\ddot{u}_0 + (I_5-I_4)\sum_{k=1}^{N_x} c_{ik}{}^{(1)} \ddot{w}_{k,j} + I_5\ddot{\phi}_x - L_3\sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{u}_{0_{k,j}} \\
&-(L_5-L_4)\sum_{k=1}^{N_x} c_{ik}{}^{(3)} \ddot{w}_{k,j} - L_5\sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{\phi}_{x_{k,j}} - L_3\sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{u}_{0_{i,k}} - (L_5-L_4)\sum_{m=1}^{N_y} c_{jm}{}^{(2)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \ddot{w}_{k,m} - L_5\sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{\phi}_{x_{i,k}},
\end{aligned}
\tag{17d}
$$

$$
\begin{aligned}
&C_3\sum_{k=1}^{N_x} c_{ik}{}^{(2)} v_{0_{k,j}} + A_3\sum_{k=1}^{N_y} c_{jk}{}^{(2)} v_{0_{i,k}} + (B_3+C_3)\sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} u_{0_{k,m}} + (A_5-A_4)\sum_{k=1}^{N_y} c_{jk}{}^{(3)} w_{i,k} \\
&+(B_5-B_4-2C_4+2C_5)\sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} w_{k,m} + C_5\sum_{k=1}^{N_x} c_{ik}{}^{(2)} \phi_{y_{k,j}} + A_5\sum_{k=1}^{N_y} c_{jk}{}^{(2)} \phi_{y_{i,k}} \\
&+(B_5+C_5)\sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{x_{k,m}} - C_6\phi_y - C_6\sum_{k=1}^{N_y} c_{jk}{}^{(1)} w_{i,k} = I_3\ddot{v}_0 + (I_5-I_4)\sum_{k=1}^{N_y} c_{jk}{}^{(1)} \ddot{w}_{i,k} + I_5\ddot{\phi}_y - L_3\sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{v}_{0_{k,j}} \\
&-(L_5-L_4)\sum_{k=1}^{N_y} c_{jk}{}^{(3)} \ddot{w}_{i,k} - L_5\sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{\phi}_{y_{k,j}} - L_3\sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{v}_{0_{i,k}} - (L_5-L_4)\sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(2)} \ddot{w}_{k,m} - L_5\sum_{k=1}^{N_y} c_{jk}{}^{(2)} \ddot{\phi}_{y_{i,k}}.
\end{aligned}
\tag{17e}
$$

N_x and N_y above are number of nodal points in *x*- and *y*-directions, respectively.

For a simply-supported nano-plate, boundary conditions at $y=0$ and $y=b$ read:

$$u_0 = v_0 = w = \phi_x = 0, \tag{18a}$$

$$A_3 \sum_{k=1}^{N_y} c_{jk}{}^{(1)} v_{0_{i,k}} + (A_5 - A_4) \sum_{k=1}^{N_y} c_{jk}{}^{(2)} w_{i,k} + A_5 \sum_{k=1}^{N_y} c_{jk}{}^{(1)} \phi_{y_{i,k}} = 0, \tag{18b}$$

$$A_1 \sum_{k=1}^{N_y} c_{jk}{}^{(1)} v_{0_{i,k}} + (A_4 - A_2) \sum_{k=1}^{N_y} c_{jk}{}^{(2)} w_{i,k} + A_4 \sum_{k=1}^{N_y} c_{jk}{}^{(1)} \phi_{y_{i,k}} = 0, \tag{18c}$$

and at $x=0,\ x=a,$ we have

$$u_0 = v_0 = w = \phi_y = 0, \tag{19a}$$

$$A_3 \sum_{k=1}^{N_x} c_{ik}{}^{(1)} u_{0_{k,j}} + (A_5 - A_4) \sum_{k=1}^{N_x} c_{ik}{}^{(2)} w_{k,j} + A_5 \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{x_{k,j}} = 0, \tag{19b}$$

$$A_1 \sum_{k=1}^{N_x} c_{ik}{}^{(1)} u_{0_{k,j}} + (A_4 - A_2) \sum_{k=1}^{N_x} c_{ik}{}^{(2)} w_{k,j} + A_4 \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{x_{k,j}} = 0. \tag{19c}$$

For the cantilever nano-plate fixed at $x=0,$ boundary conditions at the cantilever edge are:

$$u_0 = v_0 = w = \phi_x = \phi_y = \frac{\partial w}{\partial x} = 0. \tag{20}$$

The conditions at $x=a$ are derived as:

$$A_0\sum_{k=1}^{N_x}c_{ik}{}^{(1)}u_{0_{k,j}}+(A_3-A_1)\sum_{k=1}^{N_x}c_{ik}{}^{(2)}w_{k,j}+A_3\sum_{k=1}^{N_x}c_{ik}{}^{(1)}\phi_{x_{k,j}}$$
$$+B_0\sum_{k=1}^{N_y}c_{jk}{}^{(1)}v_{0_{i,k}}+(B_3-B_1)\sum_{k=1}^{N_y}c_{jk}{}^{(2)}w_{i,k}+B_3\sum_{k=1}^{N_y}c_{jk}{}^{(1)}\phi_{y_{i,k}}=0, \tag{21a}$$

$$C_0(\sum_{k=1}^{N_y}c_{jk}{}^{(1)}u_{0_{i,k}}+\sum_{k=1}^{N_x}c_{ik}{}^{(1)}v_{0_{k,j}})+2(C_3-C_1)\sum_{m=1}^{N_y}c_{jm}{}^{(1)}\sum_{k=1}^{N_x}c_{ik}{}^{(1)}w_{k,m}$$
$$+C_3(\sum_{k=1}^{N_y}c_{jk}{}^{(1)}\phi_{x_{i,k}}+\sum_{k=1}^{N_x}c_{ik}{}^{(1)}\phi_{y_{k,j}})=0, \tag{21b}$$

$$A_3\sum_{k=1}^{N_x}c_{ik}{}^{(1)}u_{0_{k,j}}+(A_5-A_4)\sum_{k=1}^{N_x}c_{ik}{}^{(2)}w_{k,j}+A_5\sum_{k=1}^{N_x}c_{ik}{}^{(1)}\phi_{x_{k,j}}$$
$$+B_3\sum_{k=1}^{N_y}c_{jk}{}^{(1)}v_{0_{i,k}}+(B_5-B_4)\sum_{k=1}^{N_y}c_{jk}{}^{(2)}w_{i,k}+B_5\sum_{k=1}^{N_y}c_{jk}{}^{(1)}\phi_{y_{i,k}}=0, \tag{21c}$$

$$C_3(\sum_{k=1}^{N_y}c_{jk}{}^{(1)}u_{0_{i,k}}+\sum_{k=1}^{N_x}c_{ik}{}^{(1)}v_{0_{k,j}})+2(C_5-C_4)\sum_{m=1}^{N_y}c_{jm}{}^{(1)}\sum_{k=1}^{N_x}c_{ik}{}^{(1)}w_{k,m}$$
$$+C_5(\sum_{k=1}^{N_y}c_{jk}{}^{(1)}\phi_{x_{i,k}}+\sum_{k=1}^{N_x}c_{ik}{}^{(1)}\phi_{y_{k,j}})=0, \tag{21d}$$

$$(A_1-A_3)\sum_{k=1}^{N_x}c_{ik}{}^{(2)}u_{0_{k,j}}+(-A_2+2A_4-A_5)\sum_{k=1}^{N_x}c_{ik}{}^{(3)}w_{k,j}+(A_4-A_5)\sum_{k=1}^{N_x}c_{ik}{}^{(2)}\phi_{x_{k,j}}$$
$$+(B_1-B_3)\sum_{m=1}^{N_y}c_{jm}{}^{(1)}\sum_{k=1}^{N_x}c_{ik}{}^{(1)}v_{0_{k,m}}+(-B_2+2B_4-B_5)\sum_{m=1}^{N_y}c_{jm}{}^{(2)}\sum_{k=1}^{N_x}c_{ik}{}^{(1)}w_{k,m}+(B_4-B_5)\sum_{m=1}^{N_y}c_{jm}{}^{(1)}\sum_{k=1}^{N_x}c_{ik}{}^{(1)}\phi_{y_{k,m}}$$
$$+(C_1-C_3)(\sum_{k=1}^{N_y}c_{jk}{}^{(2)}u_{0_{i,k}}+\sum_{m=1}^{N_y}c_{jm}{}^{(1)}\sum_{k=1}^{N_x}c_{ik}{}^{(1)}v_{0_{k,m}})+2(-C_2+2C_4-C_5)\sum_{m=1}^{N_y}c_{jm}{}^{(2)}\sum_{k=1}^{N_x}c_{ik}{}^{(1)}w_{k,m}$$
$$+(C_4-C_5)(\sum_{k=1}^{N_y}c_{jk}{}^{(2)}\phi_{x_{i,k}}+\sum_{m=1}^{N_y}c_{jm}{}^{(1)}\sum_{k=1}^{N_x}c_{ik}{}^{(1)}\phi_{y_{k,m}})+C_6(\phi_x+\sum_{k=1}^{N_x}c_{ik}{}^{(1)}w_{k,j})=$$
$$(I_1-I_3)\ddot{u}_0+(-I_2+2I_4-I_5)\sum_{k=1}^{N_x}c_{ik}{}^{(1)}\ddot{w}_{k,j}+(I_4-I_5)\ddot{\phi}_x$$
$$-(L_1-L_3)\sum_{k=1}^{N_x}c_{ik}{}^{(2)}\ddot{u}_{0_{k,j}}-(-L_2+2L_4-L_5)\sum_{k=1}^{N_x}c_{ik}{}^{(3)}\ddot{w}_{k,j}-(L_4-L_5)\sum_{k=1}^{N_x}c_{ik}{}^{(2)}\ddot{\phi}_{x_{k,j}}$$
$$-(L_1-L_3)\sum_{k=1}^{N_y}c_{jk}{}^{(2)}\ddot{u}_{0_{i,k}}-(-L_2+2L_4-L_5)\sum_{m=1}^{N_y}c_{jm}{}^{(2)}\sum_{k=1}^{N_x}c_{ik}{}^{(1)}\ddot{w}_{k,m}-(L_4-L_5)\sum_{k=1}^{N_y}c_{jk}{}^{(2)}\ddot{\phi}_{x_{i,k}},$$
(21e)

$$(A_1 - A_3)\sum_{k=1}^{N_x} c_{ik}{}^{(1)} u_{0_{k,j}} + (-A_2 + 2A_4 - A_5)\sum_{k=1}^{N_x} c_{ik}{}^{(2)} w_{k,j} + (A_4 - A_5)\sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{x_{k,j}}$$
$$+(B_1 - B_3)\sum_{k=1}^{N_y} c_{jk}{}^{(1)} v_{0_{i,k}} + (-B_2 + 2B_4 - B_5)\sum_{k=1}^{N_y} c_{jk}{}^{(2)} w_{i,k} + (B_4 - B_5)\sum_{k=1}^{N_y} c_{jk}{}^{(1)} \phi_{y_{i,k}} = 0. \tag{21f}$$

And, the conditions at $y = 0, y = b$ are of the forms:

$$C_0(\sum_{k=1}^{N_y} c_{jk}{}^{(1)} u_{0_{i,k}} + \sum_{k=1}^{N_x} c_{ik}{}^{(1)} v_{0_{k,j}}) + 2(C_3 - C_1)\sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} w_{k,m}$$
$$+C_3(\sum_{k=1}^{N_y} c_{jk}{}^{(1)} \phi_{x_{i,k}} + \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{y_{k,j}}) = 0, \tag{22a}$$

$$B_0\sum_{k=1}^{N_x} c_{ik}{}^{(1)} u_{0_{k,j}} + (B_3 - B_1)\sum_{k=1}^{N_x} c_{ik}{}^{(2)} w_{k,j} + B_3\sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{x_{k,j}}$$
$$+A_0\sum_{k=1}^{N_y} c_{jk}{}^{(1)} v_{0_{i,k}} + (A_3 - A_1)\sum_{k=1}^{N_y} c_{jk}{}^{(2)} w_{i,k} + A_3\sum_{k=1}^{N_y} c_{jk}{}^{(1)} \phi_{y_{i,k}} = 0, \tag{22b}$$

$$C_3(\sum_{k=1}^{N_y} c_{jk}{}^{(1)} u_{0_{i,k}} + \sum_{k=1}^{N_x} c_{ik}{}^{(1)} v_{0_{k,j}}) + 2(C_5 - C_4)\sum_{m=1}^{N_y} c_{jm}{}^{(1)} \sum_{k=1}^{N_x} c_{ik}{}^{(1)} w_{k,m}$$
$$+C_5(\sum_{k=1}^{N_y} c_{jk}{}^{(1)} \phi_{x_{i,k}} + \sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{y_{k,j}}) = 0, \tag{22c}$$

$$B_3\sum_{k=1}^{N_x} c_{ik}{}^{(1)} u_{0_{k,j}} + (B_5 - B_4)\sum_{k=1}^{N_x} c_{ik}{}^{(2)} w_{k,j} + B_5\sum_{k=1}^{N_x} c_{ik}{}^{(1)} \phi_{x_{k,j}}$$
$$+A_3\sum_{k=1}^{N_y} c_{jk}{}^{(1)} v_{0_{i,k}} + (A_5 - A_4)\sum_{k=1}^{N_y} c_{jk}{}^{(2)} w_{i,k} + A_5\sum_{k=1}^{N_y} c_{jk}{}^{(1)} \phi_{y_{i,k}} = 0, \tag{22d}$$

$$(B_1-B_3)\sum_{m=1}^{N_y}c_{jm}^{(1)}\sum_{k=1}^{N_x}c_{ik}^{(1)}u_{0_{k,m}}+(-B_2+2B_4-B_5)\sum_{m=1}^{N_y}c_{jm}^{(1)}\sum_{k=1}^{N_x}c_{ik}^{(2)}w_{k,m}+(B_4-B_5)\sum_{m=1}^{N_y}c_{jm}^{(1)}\sum_{k=1}^{N_x}c_{ik}^{(1)}\phi_{x_{k,m}}$$

$$+(A_1-A_3)\sum_{k=1}^{N_y}c_{jk}^{(2)}v_{0_{i,k}}+(-A_2+2A_4-A_5)\sum_{k=1}^{N_y}c_{jk}^{(3)}w_{i,k}+(A_4-A_5)\sum_{k=1}^{N_y}c_{jk}^{(2)}\phi_{y_{i,k}}$$

$$+(C_1-C_3)(\sum_{m=1}^{N_y}c_{jm}^{(1)}\sum_{k=1}^{N_x}c_{ik}^{(1)}u_{0_{k,m}}+\sum_{k=1}^{N_x}c_{ik}^{(2)}v_{0_{k,j}})+2(-C_2+2C_4-C_5)\sum_{m=1}^{N_y}c_{jm}^{(1)}\sum_{k=1}^{N_x}c_{ik}^{(2)}w_{k,m}$$

$$+(C_4-C_5)(\sum_{m=1}^{N_y}c_{jm}^{(1)}\sum_{k=1}^{N_x}c_{ik}^{(1)}\phi_{x_{k,m}}+\sum_{k=1}^{N_x}c_{ik}^{(2)}\phi_{y_{k,j}})+C_6(\phi_y+\sum_{k=1}^{N_y}c_{jk}^{(1)}w_{i,k})=$$

$$(I_1-I_3)\ddot{v}_0+(-I_2+2I_4-I_5)\sum_{k=1}^{N_y}c_{jk}^{(1)}\ddot{w}_{i,k}+(I_4-I_5)\ddot{\phi}_y$$

$$-(L_1-L_3)\sum_{k=1}^{N_x}c_{ik}^{(2)}\ddot{v}_{0_{k,j}}-(-L_2+2L_4-L_5)\sum_{m=1}^{N_y}c_{jm}^{(1)}\sum_{k=1}^{N_x}c_{ik}^{(2)}\ddot{w}_{k,m}-(L_4-L_5)\sum_{k=1}^{N_x}c_{ik}^{(2)}\ddot{\phi}_{y_{k,j}}$$

$$-(L_1-L_3)\sum_{k=1}^{N_y}c_{jk}^{(2)}\ddot{v}_{0_{i,k}}-(-L_2+2L_4-L_5)\sum_{k=1}^{N_y}c_{jk}^{(3)}\ddot{w}_{i,k}-(L_4-L_5)\sum_{k=1}^{N_y}c_{jk}^{(2)}\ddot{\phi}_{y_{i,k}},$$

(22e)

$$(B_1-B_3)\sum_{k=1}^{N_x}c_{ik}^{(1)}u_{0_{k,j}}+(-B_2+2B_4-B_5)\sum_{k=1}^{N_x}c_{ik}^{(2)}w_{k,j}+(B_4-B_5)\sum_{k=1}^{N_x}c_{ik}^{(1)}\phi_{x_{k,j}}$$

$$+(A_1-A_3)\sum_{k=1}^{N_y}c_{jk}^{(1)}v_{0_{i,k}}+(-A_2+2A_4-A_5)\sum_{k=1}^{N_y}c_{jk}^{(2)}w_{i,k}+(A_4-A_5)\sum_{k=1}^{N_y}c_{jk}^{(1)}\phi_{y_{i,k}}=0.$$

(22f)

In addition to these boundary conditions, $w_{,xy}=0$ should be imposed at the points where free edges intersect. The coefficients in the governing equations and boundary conditions conveyed by Eqs. (17)-(22) are given by:

$$\begin{Bmatrix}A_0\\A_1\\A_2\\A_3\\A_4\\A_5\end{Bmatrix}=\int_{-h/2}^{h/2}Q_{11}\begin{Bmatrix}1\\z\\z^2\\f(z)\\zf(z)\\(f(z))^2\end{Bmatrix}dz,\quad \begin{Bmatrix}B_0\\B_1\\B_2\\B_3\\B_4\\B_5\end{Bmatrix}=\int_{-h/2}^{h/2}Q_{12}\begin{Bmatrix}1\\z\\z^2\\f(z)\\zf(z)\\(f(z))^2\end{Bmatrix}dz,\quad \begin{Bmatrix}C_0\\C_1\\C_2\\C_3\\C_4\\C_5\\C_6\end{Bmatrix}=\int_{-h/2}^{h/2}Q_{66}\begin{Bmatrix}1\\z\\z^2\\f(z)\\zf(z)\\(f(z))^2\\(f'(z))^2\end{Bmatrix}dz. \quad (23)$$

For both simply-supported and cantilever nano-plates, governing equations and boundary conditions are consolidated into the following matrix form:

$$(\mathbf{K}-\Omega^2\mathbf{M})\ \mathbf{d}^* = \mathbf{0}, \tag{24}$$

where Ω is natural frequency, $\mathbf{K}$ is the stiffness matrix, $\mathbf{M}$ is the mass matrix and $\mathbf{d}^*$ is mode shape vector expressed as

$$\mathbf{d}^* = \left\{ \left\{u_i^*\right\}^T, \left\{v_i^*\right\}^T, \left\{w_i^*\right\}^T, \left\{\phi_{x_i}^*\right\}^T, \left\{\phi_{y_i}^*\right\}^T \right\}^T, \quad i = 1, 2, ..., N_x \times N_y. \tag{25}$$

4. Numerical results

In parametric analyses, we examine free vibrations of ceramic-metal functionally graded composite nano-plates, whose constituents are silicon nitride (Si_3N_4) and stainless steel. Properties for this material pair are given by

$$E_c = 348.43 \text{ GPa}, \quad \nu_c = 0.3, \quad \rho_c = 2370 \text{ kg/m}^3, \tag{26a}$$

$$E_m = 201.04 \text{ GPa}, \quad \nu_m = 0.3, \quad \rho_m = 8166 \text{ kg/m}^3. \tag{26b}$$

Nonlocal parameter of the metallic phase is taken as $\mu_m = 2 \text{ nm}^2$, which is a reference value adopted in various studies in the literature [26,28]. The degree of variation in the nonlocal parameter is quantified by the ratio μ_c/μ_m. When the nonlocal parameter is assumed to be constant μ_c/μ_m is equal to unity, whereas when μ varies across the thickness $\mu_c/\mu_m \neq 1$.

We set μ_c/μ_m as 2 in a number of parametric analyses. In remaining cases it is varied to be able to assess the influence of the nonlocal parameter variation.

To be able to verify theoretical and computational developments, in Table 1 we provide comparisons to the results given in the article by Zare et al. [25]. This article presents solutions regarding free vibrations of functionally graded rectangular nano-plates developed under the assumption of constant nonlocal parameter. Material properties used in [25] are same as those given by Eq. (26). Table 1 presents comparisons of first three dimensionless natural frequencies of a simply-supported functionally graded nano-plate. Dimensionless natural frequency is defined as

$$\omega = \Omega h \sqrt{\frac{2\left(1+\nu_c\right)\rho_c}{E_c}}. \tag{27}$$

Both our results and those provided in [25] are generated by using Kirchhoff plate theory. Natural frequencies computed in the present study are in very good agreement with those given by Zare et al. [25], which is indicative of the high degree of accuracy attained by the application of the proposed procedures.

In Figs. 2-10 and Tables 2 and 3, we present results of our parametric analyses for functionally graded rectangular nano-plates possessing a spatially variable nonlocal parameter. Fig. 2 depicts dimensionless first natural frequencies of simply-supported and cantilever nano-plates as a function of the dimensionless plate length $a/\sqrt{\mu_m}$. Dimensionless natural frequency is defined by Eq. (27). The frequencies are calculated in accordance with Kirchhoff plate theory (KPT), Mindlin plate theory (MPT), and third-order shear deformation theory (TSDT). For both simply-supported and cantilever nano-plates, ω_1 increases with a corresponding increase in $a/\sqrt{\mu_m}$; and levels off around a constant attained at larger values of dimensionless plate length. The constants are equal to the vibration frequencies of the classical macro-scale plate, for which numerical results are found by setting $\mu_c = \mu_m = 0$. Thus, as expected, size effect turns out to be important especially for relatively smaller values of the ratio $a/\sqrt{\mu_m}$. In the case of the simply-supported nano-plate, frequencies found for different plate theories are close to each other; whereas for the cantilever nano-plate, differences are slightly larger. Curves generated by KPT and MPT envelope that computed by utilizing TSDT. Numerical results given in Fig. 3 regarding second natural frequency point out to similar trends, except for more pronounced differences among results obtained for the simply-supported nano-plate. The first two mode shapes of simply-supported and cantilever nano-plates for $a/\sqrt{\mu_m} = 10$ are provided in Fig. 4. Third-order shear deformation theory is used in the generation of the mode shapes and the remaining sets of results presented in this section.

Figs. 5 and 6 depict the influence of the nonlocal parameter ratio μ_c/μ_m on respectively ω_1 and ω_2. For both simply-supported and cantilever nano-plates, nonlocal parameter ratio has a significant impact on the dimensionless natural frequency. This finding implies that variation of the nonlocal parameter needs to be taken into account to be able to produce more realistic numerical results. Both of the dimensionless natural frequencies ω_1 and ω_2 get smaller as the ratio μ_c/μ_m is increased from 0.5 to 4.0. Notice that natural frequencies merge at a single value as the dimensionless plate length $a/\sqrt{\mu_m}$ gets larger. This single value is again the frequency obtained for a classical plate by taking $\mu_c = \mu_m = 0$. Further results regarding the influence of μ_c/μ_m on dimensionless frequencies ω_1 and ω_2 are provided in Tables 2 and 3. Dependence on μ_c/μ_m is examined by considering different values of dimensionless plate length $a/\sqrt{\mu_m}$ and aspect ratio a/b. In all cases, dimensionless frequencies are decreasing functions of nonlocal parameter ratio μ_c/μ_m. Increase in the aspect ratio a/b however causes an increase in the dimensionless natural frequencies.

In Figs. 7 and 8, we present ω_1 and ω_2 as functions of the power-law index n and dimensionless plate length $a/\sqrt{\mu_m}$. The index n controls the variation of the ceramic volume fraction as indicated by Eq. (7a). The nano-plate considered is ceramic-rich if $n<1$, and metal-rich if $n>1$. Both of the dimensionless natural frequencies ω_1 and ω_2 decrease as the exponent n is increased from 0.5 to 8. Thus, ceramic-rich functionally graded nano-plates possess larger natural frequencies. This is the expected result since the ceramic phase of the FGM composite nano-plate has larger elastic modulus and lower density compared to the metallic phase. The constants attained for larger $a/\sqrt{\mu_m}$ are equal to the frequencies predicted by the classical plate theory.

Further parametric analyses regarding the effects of the power-law index n and the nonlocal parameter ratio μ_c/μ_m on the first four dimensionless natural frequencies of a simply-supported FGM nano-plate are presented in Figs. 9 and 10. The figures clearly show how the natural frequencies increase as the power-law index n gets smaller. Ceramic-rich functionally graded composite nano-plates are again shown to possess significantly larger natural frequencies than the metal-rich nano-plates. Nonlocal parameter ratio μ_c/μ_m also imparts a notable influence upon all four natural frequencies. Sensitivity of the frequencies to the change in the nonlocal parameter ratio points to the fact that nonlocal parameter variation should be accounted for in dynamic analysis, which is the basic premise of the present study.

5. Concluding remarks

In this article, we outline a nonlocal elasticity based method for free vibration analysis of functionally graded rectangular composite nano-plates. The method developed is capable of accounting for the spatial variation of the nonlocal parameter. Governing equations and boundary conditions are derived by following the variational approach and applying Hamilton's principle. All material properties, including the nonlocal parameter, are assumed to be functions of the thickness coordinate in the derivations. Displacement field is expressed in a unified way in the formulation to be able to generate results for three plate theories, which are Kirchhoff, Mindlin, and third order shear deformation theories. The equations derived are solved numerically by means of generalized differential quadrature method. Proposed procedure is verified through comparisons made to the results available in the literature. Further numerical results are provided to be able to demonstrate the effects of dimensionless nano-plate length, nonlocal parameter ratio, and power-law index upon the natural vibration frequencies of simply-supported and cantilever nano-plates.

Vibration frequencies calculated for three different plate theories show that Kirchhoff plate theory predicts higher natural frequencies and stiffer nano-plate behavior. The results for third-order shear deformation theory lie in between those calculated for Kirchhoff and Mindlin plate theories. In all cases, as the dimensionless plate length $a/\sqrt{\mu_m}$ is increased, vibration frequencies first increase and then tend to approach constant values. For a given configuration, the constant attained at large $a/\sqrt{\mu_m}$ is equal to the frequency computed for a plate whose nonlocal parameter μ is zero. In other words, results corresponding to macro-scale plates are recovered for larger $a/\sqrt{\mu_m}$, which is indicative of size effect's dominance for smaller plate length. The influence of the nonlocal parameter ratio μ_c/μ_m is shown to be significant. An increase in μ_c/μ_m causes drops in the dimensionless natural frequencies. This finding implies that assuming a constant nonlocal parameter might lead to results of reduced accuracy. Thus, reliable results regarding free vibrations of functionally graded rectangular nano-plates can be produced by taking into account nonlocal parameter variation. The methods presented in this article could prove useful in this respect and in analysis, design, and optimization of graded composite nano-plate structures.

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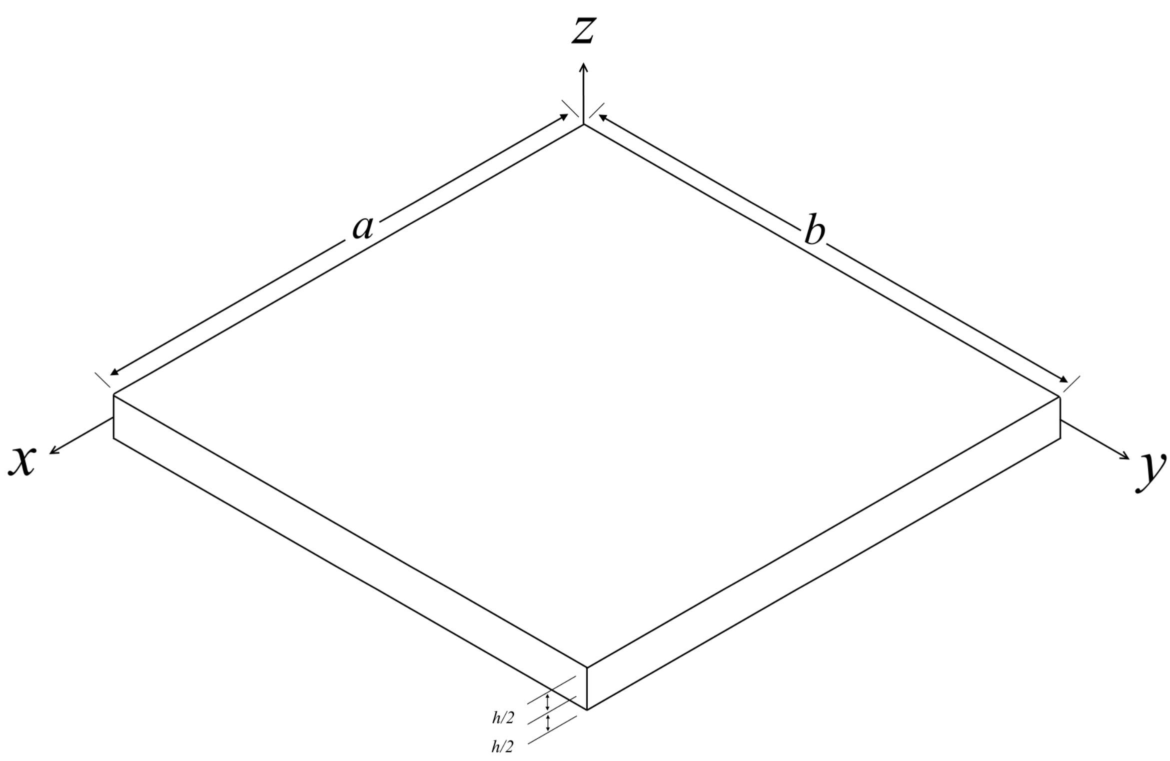

Fig.1. The geometry of the functionally graded rectangular nanoplate.

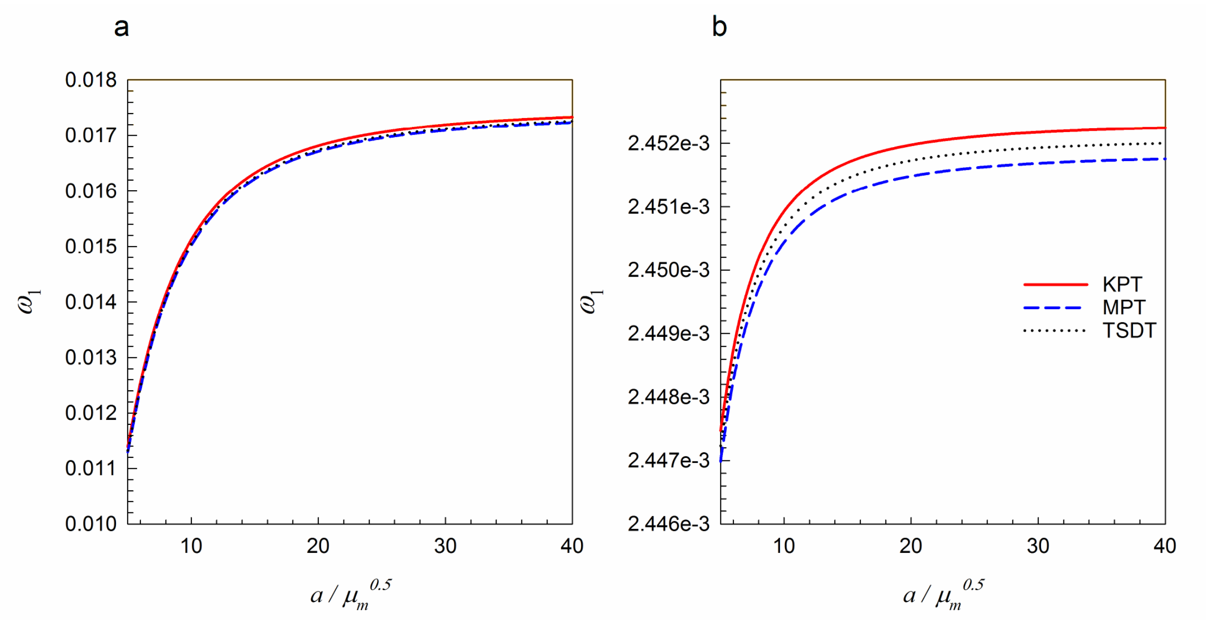

Fig. 2. Dimensionless first natural frequency ω_1 as a function of $a/\sqrt{\mu_m}$ for three different plate theories: (a) Simply-supported nano-plate; (b) cantilever nano-plate. $a/b = 4/3$, $a/h = 20$, $\mu_m = 2\ \text{nm}^2$, $\mu_c/\mu_m = 2$, $n = 2$.

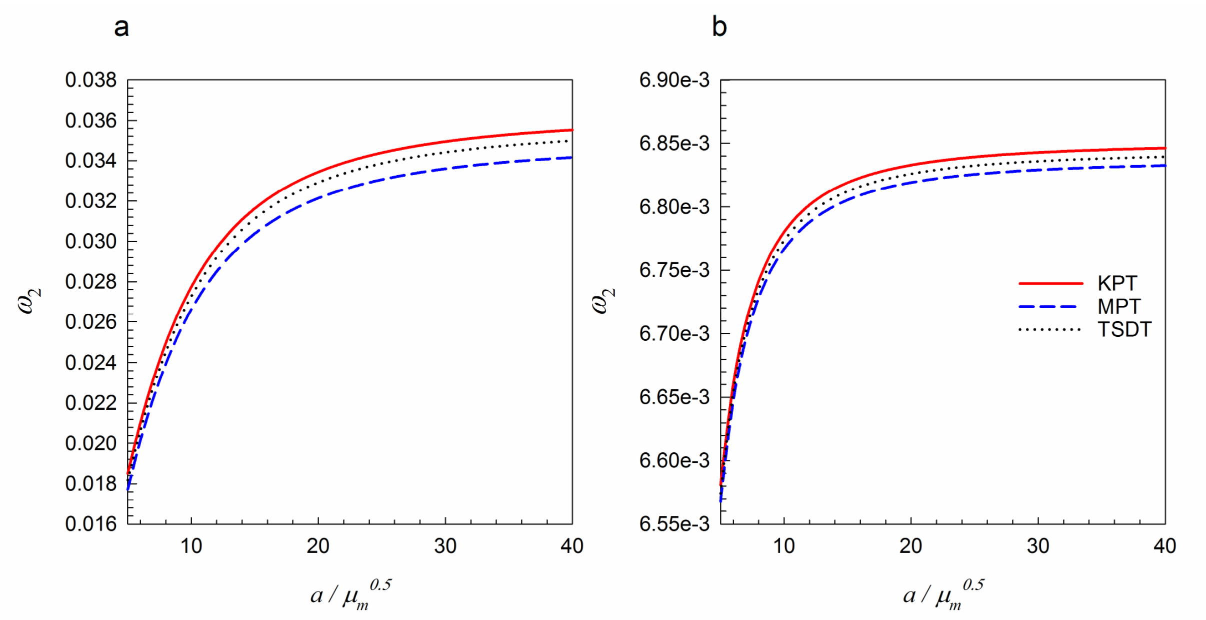

Fig. 3. Dimensionless second natural frequency ω_2 as a function of $a/\sqrt{\mu_m}$ for three different plate theories: (a) Simply-supported nano-plate; (b) cantilever nano-plate. $a/b = 4/3$, $a/h = 20$, $\mu_m = 2\ \text{nm}^2$, $\mu_c/\mu_m = 2$, $n = 2$.

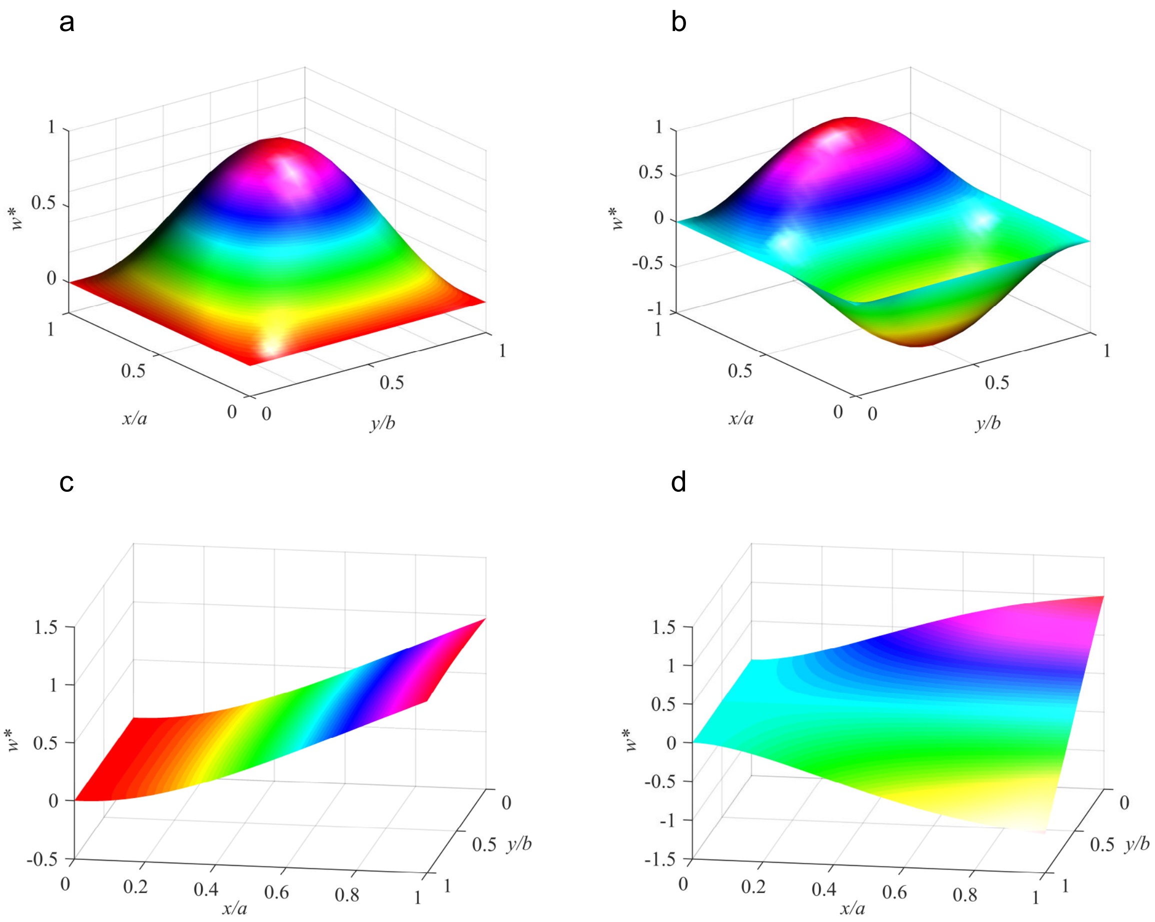

Fig. 4. First two mode shapes of simply-supported and cantilever nano-plates: (a) First mode shape of simply supported nano-plate, $\omega_1 = 0.015$, (b) second mode-shape of simply-supported nano-plate, $\omega_2 = 0.027$, (c) first mode shape of cantilever nano-plate, $\omega_1 = 2.451\times10^{-3}$, (d) second mode shape of cantilever nano-plate, $\omega_2 = 6.774\times10^{-3}$. $a/\sqrt{\mu_m} = 10$, $a/b = 4/3$, $a/h = 20$, $\mu_m = 2\ \text{nm}^2$, $\mu_c/\mu_m = 2$, $n = 2$.

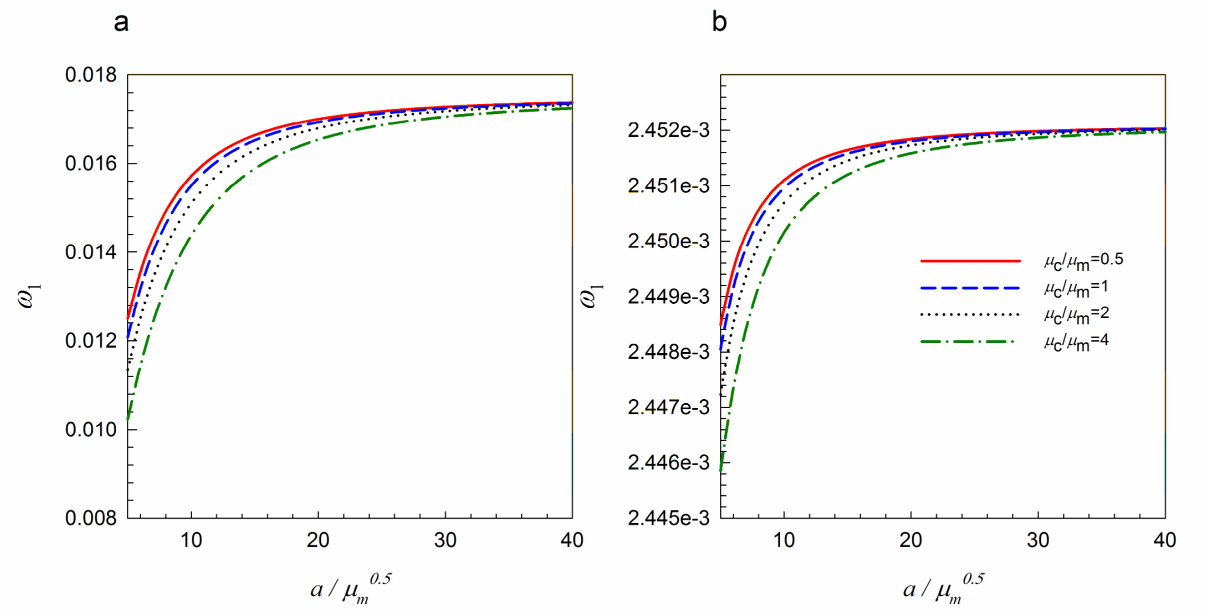

Fig. 5. Dimensionless first natural frequency ω_1 versus $a/\sqrt{\mu_m}$ according to different nonlocal conditions: (a) Simply-supported nano-plate; (b) cantilever nano-plate. $a/b = 4/3$, $a/h = 20$, $n = 2$, $\mu_m = 2$ nm^2.

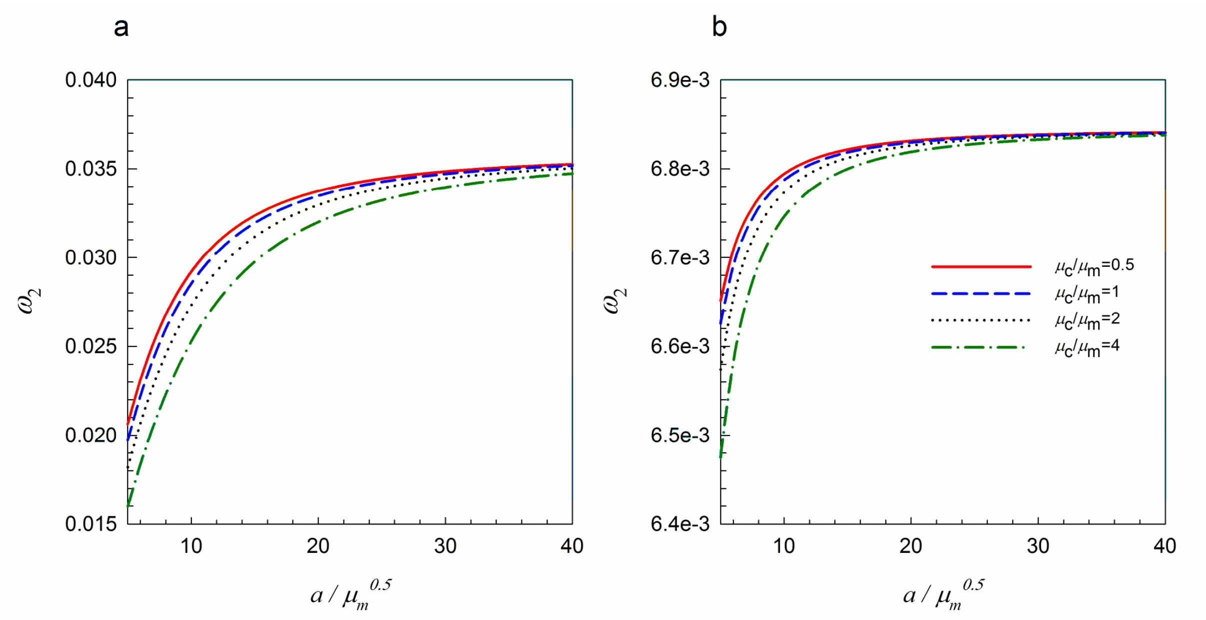

Fig. 6. Dimensionless second natural frequency ω_2 versus $a/\sqrt{\mu_m}$ according to different nonlocal conditions: (a) Simply-supported nano-plate; (b) cantilever nano-plate. $a/b = 4/3$, $a/h = 20$, $n = 2$, $\mu_m = 2$ nm^2.

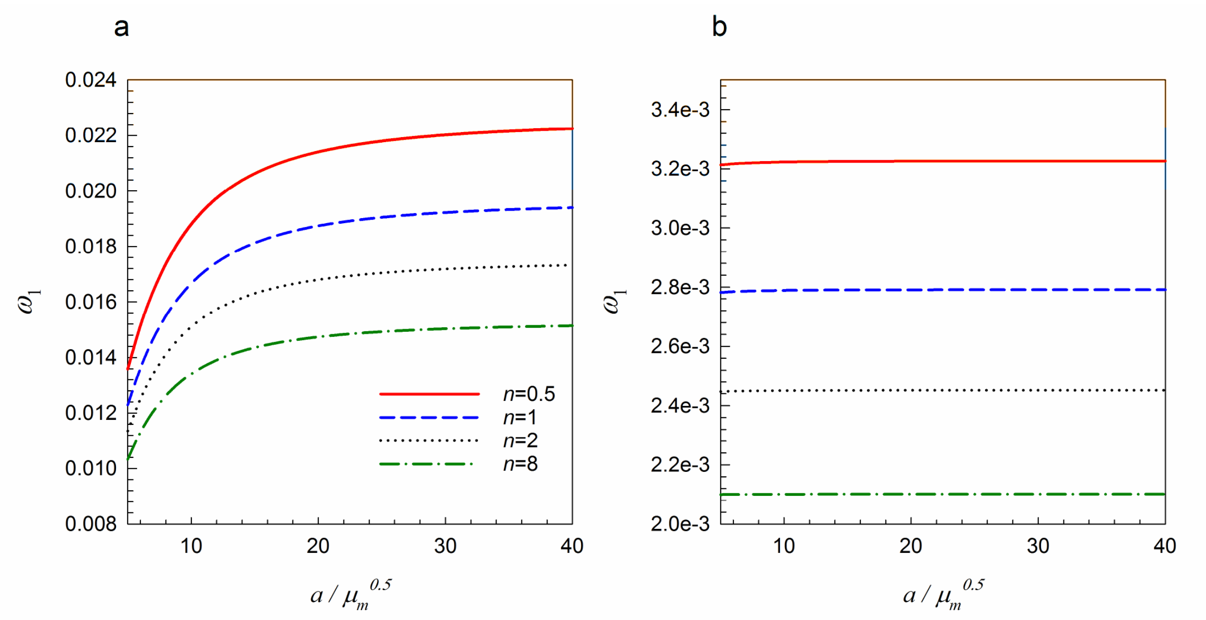

Fig. 7. Dimensionless first natural frequency ω_1 versus $a/\sqrt{\mu_m}$ for various values of the power-law index n: (a) Simply-supported nano-plate; (b) cantilever nano-plate. $a/b = 4/3$, $a/h = 20$, $\mu_m = 2\ \text{nm}^2$, $\mu_c/\mu_m = 2\ \text{nm}^2$.

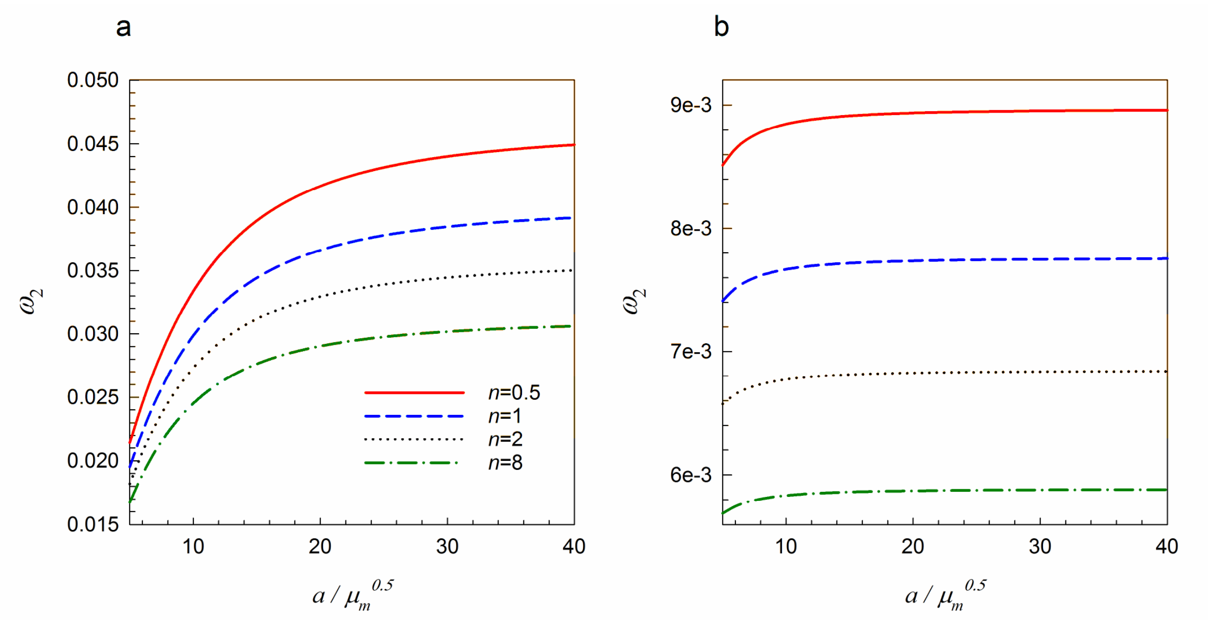

Fig. 8. Dimensionless second natural frequency ω_2 versus $a/\sqrt{\mu_m}$ for various values of the power-law index *n*: (a) Simply-supported nano-plate; (b) cantilever nano-plate. $a/b = 4/3$, $a/h = 20$, $\mu_m = 2\ \text{nm}^2$, $\mu_c/\mu_m = 2\ \text{nm}^2$.

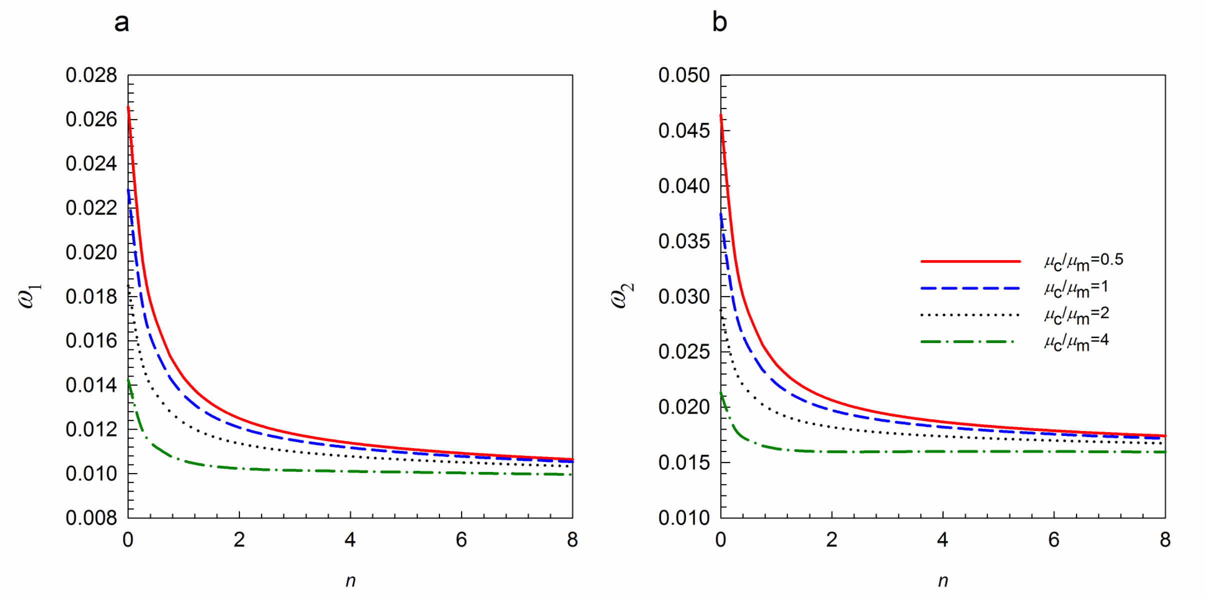

Fig. 9. Dimensionless natural frequencies of a simply-supported nano-plate as functions of n and μ_c/μ_m: (a) First natural frequency ω_1 ; (b) Second natural frequency ω_2. $a/\sqrt{\mu_m}=5$, $a/b=4/3$, $a/h=20$, $\mu_m=2\ \text{nm}^2$.

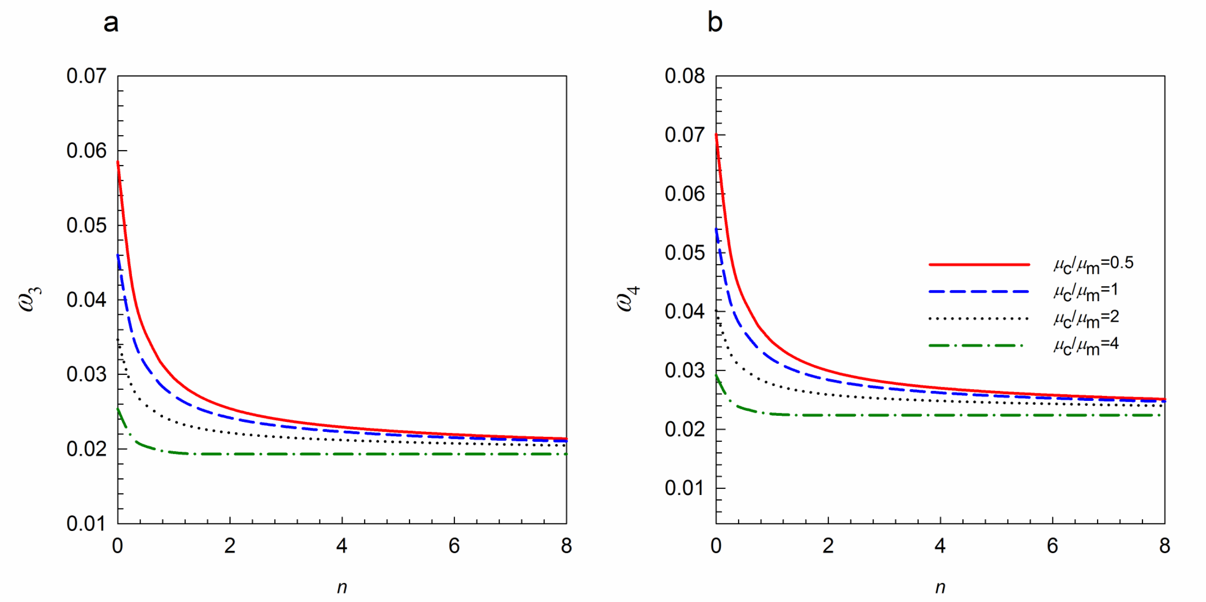

Fig. 10. Dimensionless natural frequencies of a simply-supported nano-plate as functions of n and μ_c/μ_m: (a) Third natural frequency ω_3; (b) Fourth natural frequency ω_4. $a/\sqrt{\mu_m} = 5$, $a/b = 4/3$, $a/h = 20$, $\mu_m = 2\ \text{nm}^2$.

Table 1

Comparisons of dimensionless first three natural frequencies calculated for a simply-supported functionally graded nano-plate possesing a constant nonlocal parameter μ. $n = 5$, $a = 10$ nm, $a/h = 20$.

a/b	$\mu_c = \mu_m = \mu$ (in nm^2)		ω_1	ω_2	ω_3
1	0	Present study	0.0114	0.0285	0.0285
		Zare et al. [25]	0.0114	0.0281	0.0281
	1	Present study	0.0104	0.0233	0.0233
		Zare et al. [25]	0.0104	0.0230	0.0230
	4	Present study	0.0085	0.0165	0.0165
		Zare et al. [25]	0.0085	0.0165	0.0165
2	0	Present study	0.0285	0.0454	0.0732
		Zare et al. [25]	0.0281	0.0443	0.0704
	1	Present study	0.0233	0.0340	0.0484
		Zare et al. [25]	0.0230	0.0330	0.0466
	4	Present study	0.0165	0.0223	0.0296
		Zare et al. [25]	0.0165	0.0218	0.0286

Table 2

Dimensionless first natural frequencies of simply-supported and cantilever nano-plates. $a/h = 20$, $n = 2$, $\mu_m = 2$ nm^2.

		Simply-supported nano-plate		Cantilever nano-plate	
		$\frac{a}{\sqrt{\mu_m}} = 5$	$\frac{a}{\sqrt{\mu_m}} = 10$	$\frac{a}{\sqrt{\mu_m}} = 5$	$\frac{a}{\sqrt{\mu_m}} = 10$
$\frac{a}{b} = \frac{1}{2}$	$\mu_c/\mu_m = 0.5$	6.6127e-3	7.5184e-3	1.9457e-3	1.9575e-3
	$\mu_c/\mu_m = 1$	6.4742e-3	7.4665e-3	1.9454e-3	1.9574e-3
	$\mu_c/\mu_m = 2$	6.2215e-3	7.3657e-3	1.9450e-3	1.9573e-3
	$\mu_c/\mu_m = 4$	5.7938e-3	7.1757e-3	1.9444e-3	1.9570e-3
$\frac{a}{b} = 1$	$\mu_c/\mu_m = 0.5$	9.7128e-3	1.1667e-2	2.2529e-3	2.2547e-3
	$\mu_c/\mu_m = 1$	9.4402e-3	1.1545e-2	2.2526e-3	2.2546e-3
	$\mu_c/\mu_m = 2$	8.9575e-3	1.1314e-2	2.2522e-3	2.2545e-3
	$\mu_c/\mu_m = 4$	8.1795e-3	1.0889e-2	2.2514e-3	2.2541e-3
$\frac{a}{b} = 2$	$\mu_c/\mu_m = 0.5$	1.8897e-2	2.6108e-2	2.4577e-3	2.4597e-3
	$\mu_c/\mu_m = 1$	1.8094e-2	2.5561e-2	2.4574e-3	2.4596e-3
	$\mu_c/\mu_m = 2$	1.6754e-2	2.4562e-2	2.4569e-3	2.4594e-3
	$\mu_c/\mu_m = 4$	1.4778e-2	2.2871e-2	2.4561e-3	2.4590e-3

Table 3

Dimensionless second natural frequencies of simply-supported and cantilever nano-plates. $a/h = 20$, $n = 2$, $\mu_m = 2$ nm^2.

		Simply-supported nano-plate		Cantilever nano-plate	
		$\frac{a}{\sqrt{\mu_m}} = 5$	$\frac{a}{\sqrt{\mu_m}} = 10$	$\frac{a}{\sqrt{\mu_m}} = 5$	$\frac{a}{\sqrt{\mu_m}} = 10$
$\frac{a}{b} = \frac{1}{2}$	$\mu_c/\mu_m = 0.5$	9.6723e-3	1.1618e-2	3.2856e-3	3.3026e-3
	$\mu_c/\mu_m = 1$	9.4009e-3	1.1497e-2	3.2825e-3	3.3018e-3
	$\mu_c/\mu_m = 2$	8.9204e-3	1.1267e-2	3.2764e-3	3.3001e-3
	$\mu_c/\mu_m = 4$	8.1457e-3	1.0844e-2	3.2649e-3	3.2968e-3
$\frac{a}{b} = 1$	$\mu_c/\mu_m = 0.5$	1.8760e-2	2.5917e-2	5.3673e-3	5.4606e-3
	$\mu_c/\mu_m = 1$	1.7963e-2	2.5374e-2	5.3502e-3	5.4561e-3
	$\mu_c/\mu_m = 2$	1.6634e-2	2.4383e-2	5.3165e-3	5.4471e-3
	$\mu_c/\mu_m = 4$	1.4673e-2	2.2706e-2	5.2510e-3	5.4291e-3
$\frac{a}{b} = 2$	$\mu_c/\mu_m = 0.5$	2.5295e-2	3.7799e-2	9.1438e-3	9.3824e-3
	$\mu_c/\mu_m = 1$	2.4063e-2	3.6737e-2	9.1006e-3	9.3708e-3
	$\mu_c/\mu_m = 2$	2.2058e-2	3.4858e-2	9.0159e-3	9.3476e-3
	$\mu_c/\mu_m = 4$	1.9203e-2	3.1828e-2	8.8526e-3	9.3018e-3

TÜBİTAK
PROJE ÖZET BİLGİ FORMU

Proje Yürütücüsü:	Prof. Dr. SERKAN DAĞ
Proje No:	213M606
Proje Başlığı:	Mikro-Plakların Modelleme Ve Analizi İçin Yeni Yöntemler
Proje Türü:	1001 - Araştırma
Proje Süresi:	24
Araştırmacılar:	ENDER CİĞEROĞLU
Danışmanlar:	
Projenin Yürütüldüğü Kuruluş ve Adresi:	ORTA DOĞU TEKNİK Ü. MÜHENDİSLİK F. MAKİNE MÜHENDİSLİĞİ B.
Projenin Başlangıç ve Bitiş Tarihleri:	15/04/2014 - 15/04/2016
Onaylanan Bütçe:	131977.0
Harcanan Bütçe:	83789.5

Öz:	Bu araştırma projesinin temel amacı mekanik veya termal yükleme altındaki mikro-plakların analizi için yeni yöntemler ortaya koymaktır. Malzemelerin makro-ölçekte mekanik analizini yapmakta kullanılan teoriler mikro-ölçekte geçerli değildir. Bunun nedeni uzunluk ölçeği küçüldükçe etkisi artış gösteren boyut etkisidir. Mikro-ölçekli yapıların analizi için gerinim gradayanı elastisite teorisi ve modifiye edilmiş kuvvet çifti gerilmesi teorisi gibi yüksek dereceden sürekli ortam teorileri geliştirilmiştir. Teknik literatürde, mikro-plakların yüksek dereceden sürekli ortam teorileri ile modellenmesi üzerine çeşitli çalışmalar bulunmaktadır. Bu araştırmalarda hem fonksiyonel derecelendirilmiş malzemelerden (FDM) yapılmış mikro-plaklar hem de homojen mikro-plaklar analiz edilmiştir. Ancak, yapısal mekanik ile ilgili bazı önemli problemler bu makalelerde incelenmemiştir. İlgili çalışmalar sadece mekanik yükleme altındaki mikro-plaklar için yapılmış; ve çevresel ve elektrik etkiler gibi nedenlerle oluşabilecek termal yüklemeler ele alınmamıştır. Ayrıca, fonksiyonel derecelendirilmiş mikro-plaklar üzerine yürütülen çalışmalarda, hacim oranlarındaki değişimler nedeniyle uzaysal koordinatların fonksiyonları olması gereken uzunluk ölçeği parametreleri sabit olarak kabul edilmiştir. Bu araştırma projesinde, termal etkiler ve FDM?lerin uzunluk ölçeği parametrelerindeki değişimler göz önüne alınarak yeni analiz yöntemleri geliştirilmiştir. Yeni yöntemler geliştirilirken, öncelikle termal yükleme altındaki uzunluk ölçeği parametreleri değişken fonksiyonel derecelendirilmiş mikro-plaklar için bağlaşık kısmi diferansiyel denklemler ve sınır koşulları türetilmiştir. Bu formülasyonda yüksek dereceden sürekli ortam teorisi olarak gerinim gradyanı elastisite teorisi kullanılmıştır. Kirchhoff, Mindlin, ve üçüncü dereceden plak teorileri olarak belirlenen üç farklı plak teorisi için sonuç üretebilmek amacıyla, genel bir formülasyon yaklaşımı ortaya konulmuştur. Matematiksel olarak, modifiye edilmiş kuvvet çifti gerilmesi teorisi, gerinim gradyanı elastisite teorisinin özel bir halidir; dolayısıyla basitleştirme yoluyla modifiye edilmiş kuvvet çifti gerilmesi teorisi için geçerli sonuçlar da bulunabilmektedir. Benzer şekilde, homojen mikro-plaklar için geçerli olan sonuçlar, FDM mikro-plaklar için türetilen formülasyon kullanılarak bulunabilmektedir. Sonuç itibariyle, geliştirilen formülasyon olabilecek en genel formda yapılandırılmış ve eğilme, burkulma, ve serbest titreşim gibi yapısal problemlerin çözümünde kullanılmıştır. Bağlaşık denklemleri sayısal olarak çözebilmek için, diferansiyel kare yapma metodunu baz alan sayısal algoritmalar hazırlanmıştır. Bu algoritmalar MATLAB adlı matematik yazılımına entegre edilmiştir. Formülasyonun ve sayısal çözüm tekniklerinin geçerliliklerini gösterebilmek amacıyla özel durumlar için geçerli olan ve literatürde bulunan sayısal sonuçlarla karşılaştırmalar yapılmıştır. Yürütülen detaylı sayısal analizler aracılığıyla, sıcaklık farkı, uzunluk ölçeği parametrelerindeki uzaysal değişimler, heterojenlik sabitleri, ve geometrik parametrelerin, mikro-plakların statik deformasyonları, burkulma yükleri, ve serbest titreşim doğal frekansları üzerlerindeki etkileri belirlenmiştir. Proje önerisinde tanımlanan bu çalışmalara ek olarak modifiye edilmiş kuvvet çifti teorisi kullanılarak halka şeklinde ve dairesel FDM mikro-plaklar için ve lokal olmayan elastisite teorisi aracılığı ile dikdörtgen FDM nano-plaklar için formülasyon ve sayısal çözüm çalışmaları yapılmıştır. Bu çalışmalarla statik eğilme ve serbest titreşim davranışları ile ilgili ek sonuçlar üretilmiştir.
Anahtar Kelimeler:	Mikro-plaklar, fonksiyonel derecelendirilmiş malzemeler, eğilme, serbest titreşimler, burkulma.
Fikri Ürün Bildirim Formu Sunuldu Mu?:	Hayır
Projeden Yapılan Yayınlar:	1- Consideration of spatial variation of the length scale parameter in static and dynamic analyses of functionally graded annular and circular micro-plates (Makale - Diğer Hakemli Makale),