Scattering and Bound State Solutions of Asymmetric Hulthén Potential

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Abstract

One-dimensional time-independent Schrödinger equation is solved for the asymmetric Hulthén potential. Reflection and transmission coefficients and bound state solutions are obtained in terms of the hypergeometric functions. It is observed that the unitary condition is satisfied in non-relativistic region.

Keywords: Scattering states, bound states, asymmetric Hulthén Potential, Schrödinger equation

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I. INTRODUCTION

The solutions including scattering and/or bound states of the wave equations have been great interest in quantum mechanical systems [1-22]. To achieve full information about the system under consideration one has to investigate the bound as well as scattering state problem. In Ref. [3], the authors have obtained the analytical scattering state solutions of the ℓ-wave Schrödinger equation for the Eckart potential. ℓ-wave continuum states of the Schrödinger equation for the modified Morse potential have been studied by Wei *et.al.* [7] where they have obtained the normalized analytical radial wave functions and derived a corresponding calculation of phase shifts. Chen *et.al.* have found the exact solutions of scattering states for the s-wave Schrödinger equation with the Manning-Rosen potential by using standard method [4]. In view of a spatially one-dimensional Woods-Saxon potential, the scattering solutions of the Klein-Gordon equation have been obtained in terms of hypergeometric functions by Rojas *et.al.* and they have derived the condition for the existence of transmission resonances [10]. In an arbitrary dimension, Chen *et.al.* have presented the properties of scattering state solutions of the Klein-Gordon equation for a Coulomb-like scalar plus vector potentials [16]. In Ref. [19], low-momentum scattering in the Dirac equation have been studied. Villalba and Greiner [17] have investigated the transmission resonances and supercritical states by solving the two-component Dirac equation for the cusp potential. In this manner, we intend to search the transmission and reflection coefficients and eigenvalues of the one-dimensional Schrödinger equation for the asymmetric Hulthén potential (ASHp).

The "usual" Hulthén potential [23] is one of the significant exponential potential which behaves like a Coulomb potential for small values of spatially coordinate. The Hulthén potential has many application areas in physics such as atomic physics [24, 25], nuclear and high energy physics [26], solid state physics [27] and chemical physics [28]. In addition, the Hulthén potential and its various forms are used in relativistic and non-relativistic regions [8, 9, 12, 13, 14]. In Ref. [8], the approximate analytical scattering state solutions of the Schrödinger equation with the generalized Hulthén potential for any ℓ-state have been obtained. Saad [9] has studied the bound states of a spin-0 particle in D-dimensions and found the normalization constant in terms of incomplete Beta function. The scattering solutions of the Klein-Gordon equation for the general Hulthén potential have been obtained and transmission resonances investigated in Ref. [13]. Guo *et.al.* [21] have found the transmission resonances for a Dirac particle in the presence of the Hulthén potential in one-dimension. On the other hand, solutions of the bound and scattering states of the wave equations for the asymmetric potentials have been recently examined [18, 22, 29]. In Ref. [18], the authors have investigated the low-momentum scattering of a Dirac particle in the presence of cusp potential. In $(1+1)$-dimensions, transmission resonances in the Duffin-Kemmer-Petiau (DKP) equation for an asymmetric cusp potential have also been obtained [29]. Recently, Sogut *et.al.* have examined the scattering and bound state solutions of the DKP equation in the presence of the ASHp [22]. In the present work, we study the scattering and bound state solutions of the one-dimensional Schrödinger equation for the ASHp [22]

$$V(x) = V_0 \left[\theta(-x)\frac{e^{ax}}{1-qe^{ax}} + \theta(x)\frac{e^{bx}}{1-\tilde{q}e^{bx}}\right], \tag{1}$$

where V_0 is the strength of the potential and a, b, q and $\tilde{q}$ are positive parameters. $\theta(x)$ is the Heaviside step function and for the parameters q and $\tilde{q}$ hold $q < 1$ and $\tilde{q} < 1$. Fig. (1) shows dependence of the ASHp barrier on these parameters.

The organization of the present work is as follows. In Section 2, we search the reflection and transmission coefficients in terms of hypergeometric functions for the ASHp barrier by using the form of the wave functions for $x \to \pm\infty$. In Section 3, we obtain a condition for extracting energy eigenvalue for the ASHp well. This condition is a transcendental equation which can be solved numerically. We give some numerical values of the energy eigenvalues for the bound states for chosen values of the potential parameters. We summarize our results in Section 4.

II. REFLECTION AND TRANSMISSION COEFFICIENTS

The one-dimensional time-independent Schrödinger equation for a particle with mass m moving in a potential $V(x)$ reads

$$\left\{\frac{d^2}{dx^2} + 2m\left(E - V(x)\right)\right\}\psi(x) = 0\,. \tag{2}$$

Now we look for the solution of the ASHp barrier for the region $x < 0$. Inserting Eq. (1) into Eq. (2) gives

$$\left\{\frac{d^2}{dx^2} + 2m\left[E - \frac{V_0}{e^{-ax} - q}\right]\right\}\psi_L(x) = 0\,, \tag{3}$$

Using a new variable $y = qe^{ax}$ in Eq. (3) one obtains the following equation

$$y(1-y)\psi''_L(y) + (1-y)\psi'_L(y) + \frac{1}{y(1-y)}\left\{\beta_1 - \beta_2 y + \beta_3 y^2\right\}\psi_L(y) = 0\,, \tag{4}$$

where

$$\beta_1 = \frac{2mE}{a^2}\,;\;\; \beta_2 = \frac{4mE}{a^2} + \frac{2mV_0}{qa^2}\,;\;\; \beta_3 = \frac{2mE}{a^2} + \frac{2mV_0}{qa^2}\,. \tag{5}$$

Taking the trial wave function

$$\psi_L(y) = y^{\mu}(1-y)^{\nu} f(y)\,, \tag{6}$$

and inserting it into Eq. (4) we have

$$y(1-y)f''(y) + \left[1 + 2\mu - (2\mu + 2\nu + 1)y\right] f'(y) - (\mu + \nu + \gamma)(\mu + \nu - \gamma) f(y) = 0\,, \tag{7}$$

which has the form of the hypergeometric-type equation [30]

$$s(1-s)\chi'' + [\zeta_3 - (\zeta_1 + \zeta_2 + 1)s]\chi' - \zeta_1\zeta_2\chi = 0\,. \tag{8}$$

whose solution is given as ${}_2F_1 = (\zeta_1, \zeta_2; \zeta_3; s)$. So, comparing Eq. (7) with Eq. (8) gives us the solution

$$\begin{aligned} f(y) &= A_1\, {}_2F_1(\mu + \nu - \gamma, \mu + \nu + \gamma; 1 + 2\mu; y) \\ &+ A_2 y^{-2\mu}\, {}_2F_1(-\mu + \nu - \gamma, -\mu + \nu + \gamma; 1 - 2\mu; y)\,, \end{aligned} \tag{9}$$

and the whole solution for the region $x < 0$

$$\begin{aligned} \psi_L(y) &= A_1 y^{\mu}(1-y)^{\nu}\, {}_2F_1(\mu + \nu - \gamma, \mu + \nu + \gamma; 1 + 2\mu; y) \\ &+ A_2 y^{-\mu}(1-y)^{\nu}\, {}_2F_1(-\mu + \nu - \gamma, -\mu + \nu + \gamma; 1 - 2\mu; y)\,, \end{aligned} \tag{10}$$

where

$$\mu = i\frac{k}{a}\,;\;\; k = \sqrt{2mE}\,;\;\; \nu = 1\,;\;\; \gamma = \frac{i}{a}\sqrt{2m\left(E + \frac{V_0}{q}\right)}\,. \tag{11}$$

We have to obtain the asymptotic form of the above wave function since we search the reflection and transmission coefficients. As $x \to -\infty$, $y \to 0$ and $(1-y)^{\nu} \to 1$, we obtain from Eq. (10)

$$\psi_L(x \to -\infty) \sim A_1 q^{\mu} e^{a\mu x} + A_2 q^{-\mu} e^{-a\mu x} \sim A_1 q^{ik/a} e^{ikx} + A_2 q^{-ik/a} e^{-ikx}\,, \tag{12}$$

where we have used ${}_2F_1 = (\zeta_1, \zeta_2; \zeta_3; 0) = 1$.

To obtain the solution of the ASHp barrier for the region $x > 0$ we insert Eq. (1) into Eq. (2) and get

$$\left\{ \frac{d^2}{dx^2} + 2m \left[E - \frac{V_0}{e^{bx} - \tilde{q}} \right] \right\} \psi_R(x) = 0\,. \tag{13}$$

Defining the new variable $z = \tilde{q} e^{-\,bx}$ gives us

$$z(1-z)\psi_R''(z) + (1-z)\psi_R'(z) + \frac{1}{z(1-z)} \left\{ \tilde{\beta}_1 - \tilde{\beta}_2 z + \tilde{\beta}_3 z^2 \right\} \psi_R(z) = 0\,, \tag{14}$$

where

$$\tilde{\beta}_1 = \frac{2mE}{b^2}\,;\;\; \tilde{\beta}_2 = \frac{4mE}{b^2} + \frac{2mV_0}{\tilde{q}b^2}\,;\;\; \tilde{\beta}_3 = \frac{2mE}{b^2} + \frac{2mV_0}{\tilde{q}b^2}\,. \tag{15}$$

By using a trial wave function $\psi_R(z) = z^{\mu_1}(1-z)^{\nu_1} h(z)$ in Eq. (14) we obtain the whole solution of the ASHp for the region $x > 0$

$$\begin{aligned} \psi_R(z) &= A_3 z^{\mu_1}(1-z)^{\nu_1}\, {}_2F_1(\mu_1+\nu_1-\gamma_1, \mu_1+\nu_1+\gamma_1; 1+2\mu_1; z) \\ &+ A_4 z^{-\mu_1}(1-z)^{\nu_1}\, {}_2F_1(-\mu_1+\nu_1-\gamma_1, -\mu_1+\nu_1+\gamma_1; 1-2\mu_1; z)\,, \end{aligned} \tag{16}$$

where

$$\mu_1 = i\frac{k}{b}\,;\;\; k = \sqrt{2mE}\,;\;\; \nu_1 = 1\,;\;\; \gamma_1 = \frac{i}{b}\sqrt{2m\left(E + \frac{V_0}{\tilde{q}}\right)}\,. \tag{17}$$

In order to define a plane wave travelling from left to right we have to set $A_3 = 0$ in Eq. (16), so

$$\psi_R(z) = A_4 z^{-\mu_1}(1-z)^{\nu_1}\, {}_2F_1(-\mu_1+\nu_1-\gamma_1, -\mu_1+\nu_1+\gamma_1; 1-2\mu_1; z)\,. \tag{18}$$

Now we give the form of the wave function at $x \to +\infty$ for region $x > 0$. As $x \to +\infty$, $z \to 0$ and $(1-z)^{\nu_1} \to 1$, we have from Eq. (18)

$$\psi_R(x \to +\infty) \sim A_4(\tilde{q})^{-\mu_1} e^{b\mu_1 x} \sim A_4(\tilde{q})^{-ik/b} e^{ikx}\,. \tag{19}$$

As a result we can summarize the wave function for the limit $x \to \pm\infty$ from Eq. (12) and Eq. (19) as

$$\psi(x) = \begin{cases} A_1 q^{ik/a} e^{ikx} + A_2 q^{-ik/a} e^{-ikx} & x \to -\infty, \\ A_4(\tilde{q})^{-ik/b} e^{ikx} & x \to +\infty. \end{cases} \tag{20}$$

The wave function in Eq. (10) can be written as $\psi_L = \psi_{inc} + \psi_{ref}$ in the limit $x \to -\infty$ where ψ_{inc} is the incident and ψ_{ref} is the reflected wave. Similarly, as $x \to +\infty$ the wave function in Eq. (18) is $\psi_R = \psi_{trans}$ where ψ_{trans} is the transmitted wave. These definitions give us the reflection and transmission coefficients as

$$\begin{aligned} R &= \left|\frac{\psi_{ref}}{\psi_{inc}}\right|^2 = \frac{|A_2|^2}{|A_1|^2}\,, \\ T &= \left|\frac{\psi_{trans}}{\psi_{inc}}\right|^2 = \frac{|A_4|^2}{|A_1|^2}\,. \end{aligned} \tag{21}$$

In order to give the explicit expressions for the coefficients used in the above equations we need to use the continuity conditions on the wave function given as $\psi_R(x=0) = \psi_L(x=0)$ and $\psi'_R(x=0) = \psi'_L(x=0)$ where prime denotes derivative with respect to x . The matching of the wave functions at $x = 0$ gives

$$A_1C_1F_1 + A_2C_2F_2 = A_4C_3F_3\,, \tag{22}$$

and the matching of derivatives of the wave functions reads

$$\begin{aligned} aqA_1C_1(D_1F_1 + D_4F_4) &+ aqA_2C_2(D_2F_2 + D_5F_5) \\ &= b\tilde{q}A_4C_3(D_3F_3 - D_6F_6)\,. \end{aligned} \tag{23}$$

where we have used the property of the hypergeometric functions as $\frac{d}{ds}\,{}_2F_1(\zeta_1, \zeta_2; \zeta_3; s) = \frac{\zeta_1\zeta_2}{\zeta_3}\,{}_2F_1(\zeta_1 + 1, \zeta_2 + 1; \zeta_1 + 1; s)$.

Combining last two equations we obtain the followings for the coefficients written in Eq. (21)

$$\frac{A_2}{A_1} = \frac{C_1\left[b\tilde{q}F_1(D_3F_3 - D_6F_6) - aqF_3(D_1F_1 + D_4F_4)\right]}{C_2\left[aqF_3(D_2F_2 + D_5F_5) - b\tilde{q}F_2(D_3F_3 - D_6F_6)\right]}\,, \tag{24}$$

$$\frac{A_4}{A_1} = \frac{aqC_1\left[F_1(D_2F_2 + D_5F_5) - F_2(D_1F_1 + D_4F_4)\right]}{C_3\left[aqF_3(D_2F_2 + D_5F_5) - b\tilde{q}F_2(D_3F_3 - D_6F_6)\right]}\,. \tag{25}$$

where the following abbreviations in the above equations have been used

$$
\begin{aligned}
C_1 &= q^{\mu}(1-q)^{\nu}\,;\ C_2 = q^{-\mu}(1-q)^{\nu}\,;\ C_3 = (\tilde{q})^{-\mu_1}(1-\tilde{q})^{\nu_1}\,, \qquad (26)\\
D_1 &= \frac{\mu}{q} - \frac{\nu}{1-q}\,;\ D_2 = -\frac{\mu}{q} - \frac{\nu}{1-q}\,;\ D_3 = \frac{\mu_1}{\tilde{q}} + \frac{\nu_1}{1-\tilde{q}}\,,\\
D_4 &= \frac{(\mu+\nu-\gamma)(\mu+\nu+\gamma)}{1+2\mu}\,; D_5 = \frac{(-\mu+\nu-\gamma)(-\mu+\nu+\gamma)}{1-2\mu}\,;\\
D_6 &= \frac{(-\mu_1+\nu_1-\gamma_1)(-\mu_1+\nu_1+\gamma_1)}{1-2\mu_1}\,, \qquad (27)\\
F_1(\mu,\nu,\gamma,q) &= {}_2F_1(\mu+\nu-\gamma, \mu+\nu+\gamma; 1+2\mu; q)\,,\\
F_2(\mu,\nu,\gamma,q) &= {}_2F_1(-\mu+\nu-\gamma, -\mu+\nu+\gamma; 1-2\mu; q)\,,\\
F_3(\mu_1,\nu_1,\gamma_1,\tilde{q}) &= {}_2F_1(-\mu_1+\nu_1-\gamma_1, -\mu_1+\nu_1+\gamma_1; 1-2\mu_1; \tilde{q})\,,\\
F_4(\mu,\nu,\gamma,q) &= {}_2F_1(\mu+\nu-\gamma+1, \mu+\nu+\gamma+1; 2+2\mu; q)\,,\\
F_5(\mu,\nu,\gamma,q) &= {}_2F_1(-\mu+\nu-\gamma+1, -\mu+\nu+\gamma+1; 2-2\mu; q)\,,\\
F_6(\mu_1,\nu_1,\gamma_1,\tilde{q}) &= {}_2F_1(-\mu_1+\nu_1-\gamma_1+1, -\mu_1+\nu_1+\gamma_1+1; 2-2\mu_1; \tilde{q})\,. \qquad (28)
\end{aligned}
$$

Inserting Eqs. (26), (27) and (28) into Eq. (21) gives the explicit expressions of the transmission and reflection coefficients. Figs. (2)-(4) show different variations of these coefficients according to the energy and also potential strength for various potential parameter values. It is seen in Fig. (2) that the unitarity condition, $R+T=1$, is certainly satisfied. Fig. (3) and left panel of Fig. (4) show that the dependence of the transmission coefficient on the potential parameters is very similar which means that it goes to zero for relatively lower values of energy while goes to unity for higher values of energy. The right panel of Fig. (4) displays the dependence of the transmission coefficient on the strength of the potential. According to this plot, the transmission probability of the particle from the barrier is exactly one if the height of the potential is zero as expected. This probability goes to zero with increasing value of the strength of the potential.

III. ENERGY EIGENVALUES

In this section, we deal with the bound state solutions of the ASHp well which means that $V_0 \to -V_0$. The equation (3) for $x<0$ turns into

$$\left\{\frac{d^2}{dx^2} + 2m\left[E + \frac{V_0}{e^{-\,ax} - q}\right]\right\}\psi(x) = 0\,, \qquad (29)$$

Using the transformation $y = qe^{ax}$ and taking the trial wavefunction $\psi(y) = y^{\mu_2}(1-y)^{\nu_2}g(y)$, the solution of Eq. (29) becomes

$$g(y) = A_5\,{}_2F_1(\mu_2+\nu_2-\gamma_2, \mu_2+\nu_2+\gamma_2; 1+2\mu_2; y)$$
$$+A_6 y^{-2\mu_2}\,{}_2F_1(-\mu_2+\nu_2-\gamma_2, -\mu_2+\nu_2+\gamma_2; 1-2\mu_2; y)\,, \tag{30}$$

with the parameters

$$\mu_2 = \frac{i}{a}\sqrt{2mE} = \mu\,;\;\; \nu_2 = \nu = 1\,;\;\; \gamma_2 = \frac{i}{a}\sqrt{2m\left(E - \frac{V_0}{q}\right)}\,. \tag{31}$$

and the complete solution of Eq. (29) is given

$$\psi_L(y) = A_5 y^{\mu_2}(1-y)^{\nu_2}\,{}_2F_1(\mu_2+\nu_2-\gamma_2, \mu_2+\nu_2+\gamma_2; 1+2\mu_2; y)$$
$$+A_6 y^{-\mu_2}(1-y)^{\nu_2}\,{}_2F_1(-\mu_2+\nu_2-\gamma_2, -\mu_2+\nu_2+\gamma_2; 1-2\mu_2; y)\,. \tag{32}$$

Next, we search the solutions of the following form of Eq. (13) for $x > 0$

$$\left\{\frac{d^2}{dx^2} + 2m\left[E + \frac{V_0}{e^{bx} - \tilde{q}}\right]\right\}\psi(x) = 0\,, \tag{33}$$

By using the variable $z = \tilde{q}e^{-bx}$ and putting $\psi(z) = z^{\mu_3}(1-z)^{\nu_3}\omega(z)$ in Eq. (33) we obtain

$$\omega(z) = A_7\,{}_2F_1(\mu_3+\nu_3-\gamma_3, \mu_3+\nu_3+\gamma_3; 1+2\mu_3; z)$$
$$+A_8 z^{-2\mu_3}\,{}_2F_1(-\mu_3+\nu_3-\gamma_3, -\mu_3+\nu_3+\gamma_3; 1-2\mu_3; z)\,, \tag{34}$$

with the parameters

$$\mu_3 = \frac{i}{b}\sqrt{2mE} = \mu_1\,;\;\; \nu_3 = \nu_1 = 1\,;\;\; \gamma_3 = \frac{i}{b}\sqrt{2m\left(E - \frac{V_0}{\tilde{q}}\right)}\,. \tag{35}$$

Finally, we obtain the complete bound state solution of the Schrödinger equation for $x > 0$

$$\psi_R(z) = A_7 z^{\mu_3}(1-z)^{\nu_3}\,{}_2F_1(\mu_3+\nu_3-\gamma_3, \mu_3+\nu_3+\gamma_3; 1+2\mu_3; z)$$
$$+A_8 z^{-\mu_3}(1-z)^{\nu_3}\,{}_2F_1(-\mu_3+\nu_3-\gamma_3, -\mu_3+\nu_3+\gamma_3; 1-2\mu_3; z)\,. \tag{36}$$

In order to represent the wavefunctions in Eq. (32) and Eq. (36) of the bound state solutions they satisfy the boundary condition being zero at infinity which gives $A_6 = A_8 = 0$ and we obtain

$$\psi_L(y) \sim A_5 y^{\mu_2}(1-y)^{\nu_2}\,{}_2F_1(\mu_2+\nu_2-\gamma_2, \mu_2+\nu_2+\gamma_2; 1+2\mu_2; y)\,, \tag{37}$$

$$\psi_R(z) \sim A_7 z^{\mu_3}(1-z)^{\nu_3}\,{}_2F_1(\mu_3+\nu_3-\gamma_3, \mu_3+\nu_3+\gamma_3; 1+2\mu_3; z)\,. \tag{38}$$

Matching last two expressions in $x = 0$ requiring continuity of the wavefunction and of its first derivative gives

$$A_5 F_1(\mu_2, \nu_2, \gamma_2, q) - A_7 F_2(\mu_3, \nu_3, \gamma_3, \tilde{q}) = 0\,, \tag{39a}$$

$$A_5 \left\{ \left(\frac{\mu_2}{q} - \frac{\nu_2}{1-q} \right) F_1(\mu_2, \nu_2, \gamma_2, q) + F_3(\mu_2, \nu_2, \gamma_2, q) \right\}$$
$$-A_7 \left\{ \left(\frac{\mu_3}{\tilde{q}} - \frac{\nu_3}{1-\tilde{q}} \right) F_2(\mu_3, \nu_3, \gamma_3, \tilde{q}) + F_4(\mu_3, \nu_3, \gamma_3, \tilde{q}) \right\} = 0\,. \tag{39b}$$

where

$$\begin{aligned}
F_1(\mu_2, \nu_2, \gamma_2, q) &= q^{\mu_2}(1-q)^{\nu_2}\, {}_2F_1(\mu_2 + \nu_2 - \gamma_2, \mu_2 + \nu_2 + \gamma_2; 1 + 2\mu_2; q)\,, \\
F_2(\mu_3, \nu_3, \gamma_3, \tilde{q}) &= (\tilde{q})^{\mu_3}(1-\tilde{q})^{\nu_3}\, {}_2F_1(\mu_3 + \nu_3 - \gamma_3, \mu_3 + \nu_3 + \gamma_3; 1 + 2\mu_3; \tilde{q})\,, \\
F_3(\mu_2, \nu_2, \gamma_2, q) &= q^{\mu_2}(1-q)^{\nu_2}\, \frac{(\mu_2 + \nu_2 - \gamma_2)((\mu_2 + \nu_2 + \gamma_2)}{1 + 2\mu_2} \\
&\times\; {}_2F_1(\mu_2 + \nu_2 - \gamma_2 + 1, \mu_2 + \nu_2 + \gamma_2 + 1; 2 + 2\mu_2; q)\,, \\
F_4(\mu_3, \nu_3, \gamma_3, \tilde{q}) &= (\tilde{q})^{\mu_3}(1-\tilde{q})^{\nu_3}\, \frac{(\mu_3 + \nu_3 - \gamma_3)((\mu_3 + \nu_3 + \gamma_3)}{1 + 2\mu_3} \\
&\times\; {}_2F_1(\mu_3 + \nu_3 - \gamma_3 + 1, \mu_3 + \nu_3 + \gamma_3 + 1; 2 + 2\mu_3; \tilde{q})\,.
\end{aligned} \tag{40}$$

Equation (39) has a nontrivial solution only if its determinant is zero. Using this equation, one can determine the energy eigenvalues of the ASHp well numerically. Here, we give our numerical results for the energy eigenvalues as a list for some values of the parameters, for example, $m = 1, a = 0.5, b = 0.75, V_0 = 5, q = 0.1$ and $\tilde{q} = 0.5$ taking into account that $-|V_0| < E < 0$: $E_1 = -2.453010, E_2 = -2.251290, E_3 = -0.924802, E_4 = -0.491271, E_5 = -0.001356$ (in atomic unit).

IV. RESULTS AND CONCLUSIONS

We solve the one-dimensional Schrödinger equation for the asymmetric Hulthén potential. We find the transmission and reflection coefficients for the ASHp barrier in terms of hypergeometric functions and give some plots showing the dependence of these coefficients on the potential parameters $a, b, q, \tilde{q}, V_0$ and on the energy E. We observe that the unitarity condition is exactly satisfied in all cases. We also compute the energy eigenvalues for the bound states extracting an eigenvalue equation which can be solved numerically. We calculate five different energy eigenvalues by taking into account that $-|V_0| < E < 0$.

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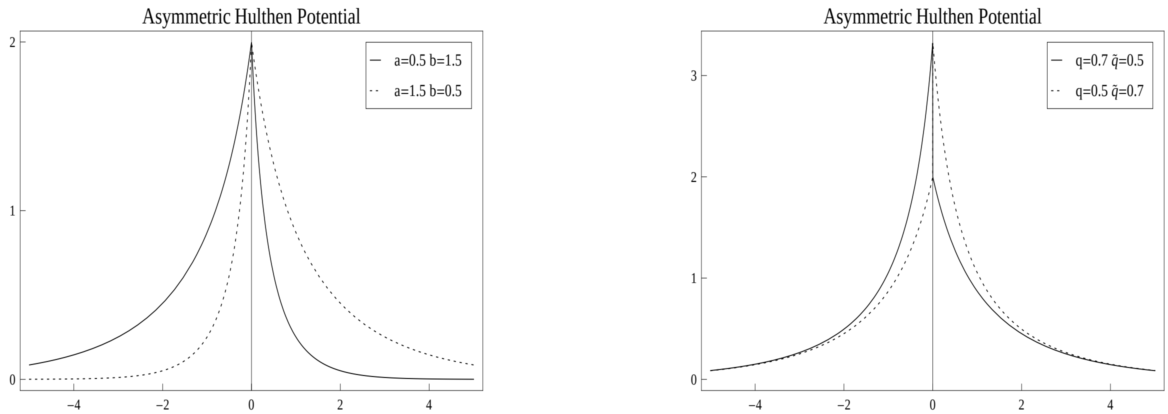

FIG. 1: plots of ASHp for different values of the potential parameters for $V_0 = 1, q = \tilde{q} = 0.5$ (left panel) and $V_0 = 1, a = b = 0.5$ (right panel).

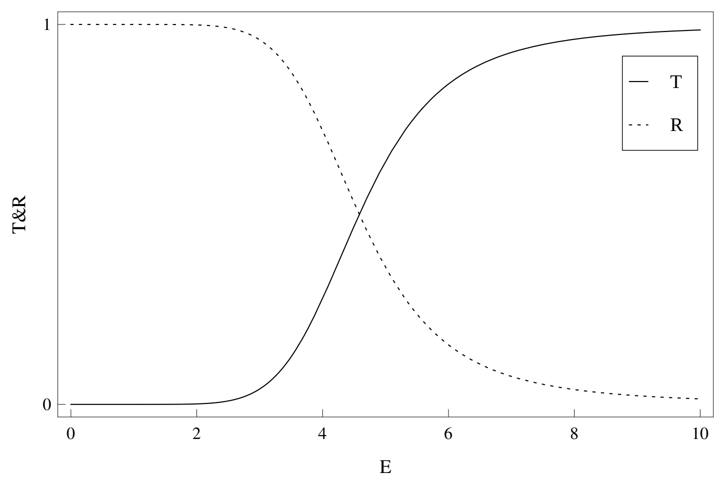

FIG. 2: transmission (T) and reflection (R) coefficients varying with E for $a = 0.4, b = 0.5, q = 0.6, \tilde{q} = 0.7, m = 1, V_0 = 2$.

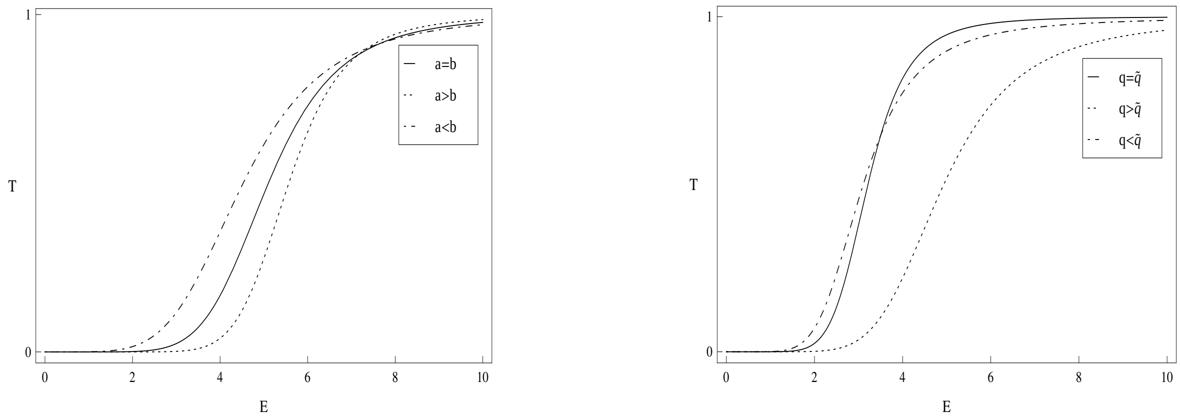

FIG. 3: variation of the transmission coefficient with different potential parameters for $m = 1, V_0 = 2$ (left panel:$a = b = 0.5; a = 0.8, b = 0.3; a = 0.3, b = 0.8; q = \tilde{q} = 0.7$ right panel:$q = \tilde{q} = 0.5; q = 0.6, \tilde{q} = 0.4; q = 0.4, \tilde{q} = 0.6; a = b = 0.5$).

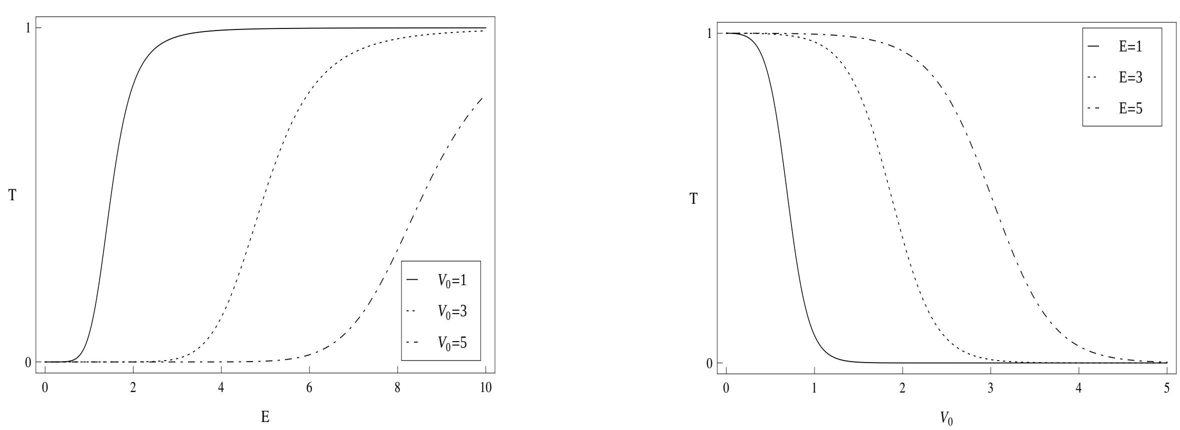

FIG. 4: variation of the transmission coefficient with energy E and potential parameter V_0 for $a = b = q = \tilde{q} = 0.5, m = 1$.