“Self Dual” Solutions of Topologically Massive Gravity Coupled with the Maxwell-Chern-Simons Theory

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We give a general class of exact solutions to the (1+2)-dimensional topologically massive gravity model coupled with Maxwell-Chern-Simons theory where a “self duality” condition is imposed on the Maxwell field.

It is well known that general relativity in (1+2) dimensions has no degrees of freedom and the gravitational field is determined solely by the matter sources (see [1] and the references therein). However a dynamical model is provided by the topologically massive gravity (TMG) theory which is obtained by the addition of the gravitational Chern-Simons term to the usual Einstein-Hilbert piece in the action [2]. Recently, a general class of exact black hole solutions to TMG with a negative cosmological constant has been obtained [3] and it has also been shown that these solutions are supersymmetric and asymptotically approach the extremal BTZ black hole solution [4]. Here, we present a new class of solutions to TMG coupled with Maxwell-Chern-Simons theory where the Maxwell field has been constrained to obey the “self duality” condition.

We begin with the action $I[e,\omega,A] = \int_M L$ where the Lagrangian 3-form is given by

$$L = \frac{1}{\mu}(\omega^a{}_b \wedge d\omega^b{}_a + \frac{2}{3}\omega^a{}_b \wedge \omega^b{}_c \wedge \omega^c{}_a) + \frac{1}{2}\mathcal{R} * 1 - \lambda * 1 - \frac{1}{2}F \wedge *F - \frac{1}{2}m\, A \wedge F \quad . \tag{1}$$

Apart from the usual Einstein-Hilbert term and the negative cosmological constant $\lambda = -1/l^2 < 0$, the gravitational Chern-Simons term with the coupling constant μ, which has dimensions of mass, is written in terms of the connection 1-forms $\omega^a{}_b$; there is also the standard Maxwell Lagrangian given in terms of the Maxwell field $F \equiv dA$ along with the vector Chern-Simons term with the coupling constant m. The variation of I with respect to the orthonormal coframes e^a and the electromagnetic potential A yields

$$\frac{1}{\mu}C_a + G_a + \lambda * e_a = -\tau_a[A] \quad , \tag{2}$$

$$d * dA + m\, dA = 0 \quad , \tag{3}$$

respectively. Here

$$\tau_a[A] = -\frac{1}{2}(\iota_a dA \wedge *dA - dA \wedge \iota_a * dA)$$

is the electromagnetic field energy momentum 2-form along with the standard Einstein 2-forms $G_a \equiv G_{ab} * e^b = -\frac{1}{2}R^{bc} * e_{abc}$ and the Cotton 2-forms $C_a \equiv DY_a = dY_a + \omega_a{}^b \wedge Y_b$, where $Y_a \equiv (Ric)_a - \frac{1}{4}\mathcal{R}e_a$ is defined in terms of the Ricci 1-forms $(Ric)_b \equiv \iota_a R^a{}_b$ and the curvature scalar $\mathcal{R} \equiv \iota_a (Ric)^a$. The $R^a{}_b$, of course, are the curvature 2-forms $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ of the connection 1-forms $\omega^a{}_b$, which satisfy the Cartan structure equations $de^a + \omega^a{}_b \wedge e^b = 0$, so that there is no torsion present in the theory. The Hodge duality operation is specified with the oriented volume element $*1 = e^0 \wedge e^1 \wedge e^2$ and e_{abc} is a short hand notation for $e_a \wedge e_b \wedge e_c$.

We choose the local coordinates (t, ρ, ϕ) and make a general ansatz for the Maxwell field

$$F = dA = E(\rho)\, e^0 \wedge e^1 + B(\rho)\, e^1 \wedge e^2 \quad , \tag{4}$$

and for the orthonormal coframe 1-forms as

$$e^0 = f(\rho)dt \quad , \quad e^1 = d\rho \quad , \quad e^2 = h(\rho)(d\phi + a(\rho)dt) \quad , \tag{5}$$

so that the metric takes the form

$$ds^2 = -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 \quad , \tag{6}$$

which is most suitable for a study of rotationaly symmetric solutions.

This choice gives

$$- Z + \frac{1}{l^2} + \frac{1}{\mu}\left(X' + \gamma X + \frac{1}{2}\beta(Y - W)\right) = \frac{1}{2}(E^2 + B^2) \ , \tag{7}$$

$$-X + \frac{1}{\mu}\left(\frac{1}{2}(Z - W - Y)' + \alpha(Z - Y) + \frac{3}{2}\beta X\right) = -EB \ , \tag{8}$$

$$Y - \frac{1}{l^2} + \frac{1}{\mu}\left((\gamma - \alpha)X + \frac{1}{2}\beta(W - Z)\right) = -\frac{1}{2}(E^2 - B^2) \ , \tag{9}$$

$$W - \frac{1}{l^2} + \frac{1}{\mu}\left(X' + \alpha X + \frac{1}{2}\beta(Y + Z - 2W)\right) = \frac{1}{2}(E^2 + B^2) \ , \tag{10}$$

for the gravitational field equations (2), whereas the modified Maxwell equations (3) become:

$$B' + \alpha B - (\beta + m)E = 0 \ , \tag{11}$$

$$E' + \gamma E - mB = 0 \ , \tag{12}$$

where we denote derivatives with respect to ρ by a prime.

Here we introduced the functions:

$$W \equiv \alpha' + \alpha^2 - \frac{3}{4}\beta^2 \quad , \quad X \equiv \frac{1}{2}\beta' + \gamma\beta \quad , \quad Y \equiv \alpha\gamma + \frac{1}{4}\beta^2 \quad , \quad Z \equiv \gamma' + \gamma^2 + \frac{1}{4}\beta^2 \quad , \tag{13}$$

which actually describe the curvature 2-forms $R^a_{\ b}$, and

$$\alpha \equiv \frac{f'}{f} \quad , \quad \beta \equiv \frac{a'h}{f} \quad , \quad \gamma \equiv \frac{h'}{h} \quad , \tag{14}$$

that come from the connection 1-forms $\omega^a_{\ b}$ (see [3] for details).

Assuming the "self duality" of the electromagnetic field as

$$E = kB \quad \text{with} \ \ k^2 = 1 \tag{15}$$

and substituting this into (11) and (12), we find $\gamma + k\beta = \alpha$, and using this in (9) gives $(\gamma + \frac{k}{2}\beta)^2 = \frac{1}{l^2}$. Hence one finds that α and γ are determined by β alone as

$$\alpha = \frac{k}{2}\beta + \frac{1}{l} \quad , \quad \gamma = -\frac{k}{2}\beta + \frac{1}{l} \ . \tag{16}$$

These ubiquitous conditions have first appeared in the study of the general self dual solutions of the Einstein-Maxwell-Chern-Simons theory in (1+2) dimensions [5]. They are in fact the necessary and sufficient conditions that any solution to TMG of the form (5), (6) has to satisfy in order to be supersymmmetric as well [3].

The above conditions (16) simplify W, X, Y and Z which now take the form

$$W = u + \frac{1}{l^2} \quad , \quad X = ku \quad , \quad Y = \frac{1}{l^2} \quad , \quad Z = \frac{1}{l^2} - u \tag{17}$$

where

$$u \equiv -\frac{1}{2}\,\beta^2 + \frac{k}{2}\,\beta' + k\,\frac{\beta}{l} \ . \tag{18}$$

Finally, we find that equations (7)...(12) are satisfied simultaneously provided

$$u' - k\,\beta\,u + (\frac{1}{l} + \mu k)u = y \ , \tag{19}$$

$$y' - k\,\beta\,y + 2(\frac{1}{l} - mk)y = 0 \ , \tag{20}$$

where we defined $y \equiv k\,\mu\,B^2$.

By setting $y = k\frac{u}{z}$ in (20), we find a linear first order differential equation for z as

$$z' + (\mu k + 2mk - 1/l)z = k \ . \tag{21}$$

Integrating this, one finds

$$z = \frac{k}{\mu k + 2mk - 1/l} \left(1 + y_0 e^{(1/l - \mu k - 2mk)\rho}\right) \tag{22}$$

for some integration constant y_0. Substituting this back into (19), one finds that

$$u' - k\,\beta\,u + \left(\frac{1}{l} + \mu\,k - \frac{\mu k + 2mk - 1/l}{1 + y_0 e^{(1/l - \mu k - 2mk)\rho}}\right) u = 0 \tag{23}$$

and taking $u = k\frac{\beta}{v}$, this simplifies to a linear first order differential equation for v as

$$v' + \left(\frac{1}{l} - \mu\,k + \frac{\mu k + 2mk - 1/l}{1 + y_0 e^{(1/l - \mu k - 2mk)\rho}}\right) v = 2 \quad . \tag{24}$$

This can be integrated easily and one finds

$$v = \frac{1}{e^{(1/l - k\mu)\rho} + \frac{1}{y_0} e^{2km\rho}} \left(v_0 + \frac{2}{1/l - \mu k} e^{(1/l - \mu k)\rho} + \frac{1}{kmy_0} e^{2km\rho}\right) \tag{25}$$

with a new integration constant v_0.

Going back to the definition of u, (18), gives a differential equation for β and by setting $\omega \equiv \frac{1}{\beta}$, it becomes:

$$\omega' + \left(\frac{2e^{(1/l - \mu k)\rho} + \frac{2}{y_0} e^{2km\rho}}{v_0 + \frac{2}{1/l - \mu k} e^{(1/l - \mu k)\rho} + \frac{1}{kmy_0} e^{2km\rho}} - \frac{2}{l}\right) \omega + k = 0 \quad . \tag{26}$$

This is of the form

$$\omega' + \left(\frac{\varphi'}{\varphi} - \frac{2}{l}\right) \omega + k = 0 \quad , \tag{27}$$

with

$$\varphi = v_0 + \frac{2}{1/l - \mu k} e^{(1/l - \mu k)\rho} + \frac{1}{kmy_0} e^{2km\rho} \quad , \tag{28}$$

and its solution is given by (see [5])

$$\omega = \frac{1}{\beta} = \frac{k\Omega}{\varphi e^{-2\rho/l}} \quad \text{where} \quad \Omega \equiv c_0 - \int^{\rho} d\tilde{\rho}\; \varphi(\tilde{\rho})\; e^{-2\tilde{\rho}/l} \tag{29}$$

for some integration constant c_0. In this case, Ω is:

$$\Omega = c_0 + \frac{v_0 l}{2} e^{-2\rho/l} + \frac{2}{1/l^2 - \mu^2} e^{-(1/l + \mu k)\rho} - \frac{1}{2kmy_0(km - 1/l)} e^{2(km - 1/l)\rho} \quad . \tag{30}$$

Finally, integrating for the metric functions using (14) and (16), we find

$$f = f_0\; e^{\rho/l}\; \Omega^{-1/2} \quad , \tag{31}$$

$$h = h_0\; e^{\rho/l}\; \Omega^{1/2} \quad , \tag{32}$$

$$a = -a_0 + k\, \frac{f_0}{h_0}\, \Omega^{-1} \quad , \tag{33}$$

for some integration constants f_0, h_0, a_0 whereas the magnetic field becomes

$$B^2 = \frac{\mu k + 2mk - 1/l}{k\mu y_0}\; e^{2(km - 1/l)\rho}\; \Omega^{-1} \; . \tag{34}$$

As a first check of this solution, we look at the $k\mu \to \infty$ limit. Then the contributions from the gravitational Chern-Simons term vanish and we must arrive at the "self dual" solutions of the Einstein-Maxwell-Chern-Simons theory which was studied earlier in [5]. Taking $k\mu \to \infty$ in (30), one gets

$$\Omega \quad \to \quad c_0 + \frac{v_0 l}{2} e^{-2\rho/l} - \frac{1}{2kmy_0(km-1/l)} e^{2(km-1/l)\rho} \quad . \tag{35}$$

To compare this result with [5], take equation (23) of [5] and substitute in (21) to find

$$\Omega = c_0 + \frac{l}{2} e^{-2\rho/l} + \frac{u_0}{2(km-1/l)} e^{2(km-1/l)\rho} \quad . \tag{36}$$

Hence our solution has the correct limit provided $v_0 = 1$ and $u_0 = -\frac{1}{kmy_0}$.

As for the physical properties of our solution, it is obvious that depending on the values of the integration constants c_0, v_0 and y_0, and also on the values of l, μ and m, the metric functions may have singularities. One needs to carefully study the causal structure of our solution to understand its nature and the geometry that it describes. However, just as explained in [3], this is again rendered impossible since one cannot invert the functional relation $r = h(\rho)$ which is crucial to put the line element given by (5), (6) and (31)..(33) into the well studied form of the BTZ (and hence the AdS) metric.

Nevertheless one can still work out the quasilocal mass and the angular momentum corresponding to this solution. The analysis is very similar to the ones given in [5] and [3], and we refer the reader to these articles for the missing details below. The quasilocal angular momentum is

$$j(r) = kh_0^2 \varphi(r) \quad , \tag{37}$$

whereas the quasilocal mass is given by

$$m(r) = a_0 j(r) = ka_0 h_0^2 \varphi(r) \tag{38}$$

in an AdS background. Here φ is as given in (28) and it is understood that $r = h(\rho)$ has to be inverted so that φ is a function of r.

The total angular momentum J and the total mass M are defined by the limits $J \equiv j(r)|_{r\to\infty}$ and $M \equiv m(r)|_{r\to\infty}$, respectively. We assume that $r \to \infty$ limits can be found by taking the $\rho \to \infty$ limits in our solution, i.e. that $\varphi(r)|_{r\to\infty} = \varphi(\rho)|_{\rho\to\infty}$. Just as was done in [5] and [3], start by examining $a(r)$. Depending on the values of l, μ and m, a either goes to $-a_0$ or $-a_0 + \frac{kf_0}{h_0 c_0}$ as $r \to \infty$. For a to vanish asymptotically as $r \to \infty$, a_0 has to be chosen either as 0 or as $\frac{kf_0}{h_0 c_0}$. When $1/l > k\mu$ or $km > 0$, $a_0 = 0$ and hence $M = 0$ whereas $J \to \infty$. For $1/l < k\mu$ and $km < 0$, $J = kh_0^2 v_0$. Then $a_0 = \frac{kf_0}{h_0 c_0}$, $M = a_0 J$ and both M and J are finite. As $k\mu \to \infty$, this solution approaches the extremal BTZ solution. We again refer the reader to [5] and [3] for the details.

We next examine the special cases i) $m = 0$ and $1/l \neq 0$, ii) $m \neq 0$ and $1/l = 0$, and iii) $m = 1/l = 0$. In all these cases, one has to go back to the original equations since simply taking the corresponding $m \to 0$ and (or) $1/l \to 0$ limits in the above expressions do not give the correct answers.

i) $m = 0$ and $1/l \neq 0$:

In this case, the equation for v becomes

$$v' + \left(\frac{y_0(1/l - \mu k) e^{(1/l-\mu k)\rho}}{1 + y_0 e^{(1/l-\mu k)\rho}} \right) v = 2 \tag{39}$$

which yields

$$v = \frac{1}{1 + y_0 e^{(1/l-\mu k)\rho}} \left(v_0 + \frac{2}{y_0}\rho + \frac{2}{1/l - \mu k} e^{(1/l-\mu k)\rho} \right) \tag{40}$$

with an integration constant v_0.

Again using the definition of u, (18), one gets a new differential equation for β and by setting $\omega \equiv \frac{1}{\beta}$, one again finds a differential equation of the form (27). Only this time

$$\varphi = v_0 + \frac{2}{y_0}\rho + \frac{2}{1/l - \mu k} e^{(1/l-\mu k)\rho} \quad , \tag{41}$$

and

$$\Omega = c_0 + \frac{v_0 l}{2} e^{-2\rho/l} + \frac{2}{1/l^2 - \mu^2} e^{-(1/l+\mu k)\rho} + \frac{l}{y_0}\, \rho \, e^{-2\rho/l} \quad . \tag{42}$$

The metric functions are again of the form (31)..(33) only that Ω is now given by (42). Now the magnetic field becomes

$$B^2 = \frac{\mu k - 1/l}{k\mu y_0} \, e^{-2\rho/l} \, \Omega^{-1} \; . \tag{43}$$

ii) $m \neq 0$ and $1/l = 0$:
For this case, following similar steps as was done in i), one gets

$$\varphi = v_0 - \frac{2}{k\mu} e^{-k\mu\rho} + \frac{1}{kmy_0} e^{2km\rho} \quad , \tag{44}$$

and

$$\Omega = c_0 - v_0 \rho - \frac{2}{\mu^2} e^{-k\mu\rho} - \frac{1}{2y_0 m^2} e^{2km\rho} \quad . \tag{45}$$

Using these, the metric functions are then found to be

$$f = f_0 \, \Omega^{-1/2} \quad , \tag{46}$$

$$h = h_0 \, \Omega^{1/2} \quad , \tag{47}$$

$$a = -a_0 + k \, \frac{f_0}{h_0} \, \Omega^{-1} \quad , \tag{48}$$

for some new integration constants f_0, h_0, a_0 whereas the magnetic field becomes

$$B^2 = \frac{\mu + 2m}{\mu y_0} \, e^{2km\rho} \, \Omega^{-1} \; . \tag{49}$$

iii) $m = 1/l = 0$:
In this case, the equation for v is

$$v' - \left(\frac{k\mu y_0 e^{-k\mu\rho}}{1 + y_0 e^{-k\mu\rho}} \right) v = 2 \tag{50}$$

which is easily integrated as

$$v = \frac{1}{1 + y_0 e^{-k\mu\rho}} \left(v_0 + \frac{2}{y_0}\rho - \frac{2}{k\mu} e^{-k\mu\rho} \right) \quad . \tag{51}$$

Following similar steps as in i) and ii), one finds

$$\varphi = v_0 + \frac{2}{y_0}\rho - \frac{2}{k\mu} e^{-k\mu\rho} \quad , \tag{52}$$

and

$$\Omega = c_0 - v_0 \rho - \frac{1}{y_0}\rho^2 - \frac{2}{\mu^2} e^{-k\mu\rho} \quad . \tag{53}$$

Now the metric functions are of the form (46)..(48) where it is understood that Ω is given by (53). Finally the magnetic field is simply

$$B^2 = \frac{1}{y_0} \Omega^{-1} \quad . \tag{54}$$

In this work, we have obtained a general class of “self dual” solutions to TMG model coupled with Maxwell-Chern-Simons theory which covers all the particular cases studied previously. Even though we couldn’t give a detailed analysis of the causal structure of our solutions, we were able to analyze the corresponding total angular momentum and the total mass. We also found the special solutions corresponding to taking the cosmological constant and (or) the Chern-Simons coupling constant to zero.

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