Axi-Dilaton Gravity in $D \geq 4$ Dimensional Space-times with Torsion

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Abstract

We study models of axi-dilaton gravity in space-time geometries with torsion. We discuss conformal rescaling rules in both Riemannian and non-Riemannian formulations. We give static, spherically symmetric solutions and examine their singularity behaviour.

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1 Introduction

Gravitational interactions are formulated on a space-time manifold M equipped with a metric tensor field g and a metric compatible connection ∇ defined on the bundle of orthonormal frames. Most commonly, interactions coupled with gravity are studied in a geometry where the connection ∇ is constrained to be the unique torsion-free Levi-Civita connection. In this context, massive test particles are postulated to follow time-like geodesics associated with space-time metric and torsion-free connection. On the other hand a metric compatible connection with torsion provides new independent degrees of freedom. It has been shown that the scalar field interactions coupled with gravity can yield connections with non-zero torsion [1]. In that case, space-time history of particles may be determined by the autoparallels of a connection with torsion [2, 3, 4]. We know that the independent variation of any action with respect to connection determines space-time torsion. In particular, bosonic part of effective superstring interactions can produce a torsion that is proportional to the gradient of the dilaton (scalar) field. Hence, it would be of interest to formulate such type of interactions in frames where torsion exists.

It is an exciting conjecture that all superstring models belong to an eleven dimensional M-theory that accommodates their apparent dualities. M-theory as a classical theory can be considered in a low-energy limit where only the low-lying excitation modes contribute to an effective field theory. As such it would be the same as $D = 11$ dimensional supergravity theory. A subsequent Kaluza-Klein reduction to $D = 10$ dimensions would bring it to a string model whose gravitational sector consists of space-time metric tensor g, dilaton scalar ϕ and the axion potential $(p+1)$-form A that would minimally couple to p-branes. We call such an effective gravitational field theory an axi-dilaton gravity in D dimensions. Axi-dilaton gravity theory can be studied in the Einstein frame. However, by working out the theory in Brans-Dicke frame [5], one can see the difference between formulation of theory with a torsion-free connection and formulation with a connection with torsion. In the latter case, we vary the action treating the metric and the connection as independent variables. We have shown that the corresponding field equations in both cases with or without torsion are equivalent provided a shift in the Brans-Dicke coupling parameter ω is allowed. We further assume a direct coupling of the kth power of the dilaton scalar with the axionic kinetic term. The conformal scaling properties are examined in both geometries. In Section:3 we investigate a class of static, spherically symmetric solutions

which depend on the coupling parameters ω and k in dimensions $D \geq 4$. In particular, we point out a class of conformal black hole solutions obtained for the scale invariant parameter values.

2 Axi-dilaton Gravity In D Dimensions

We start with an action

$$I[g, \phi, A] = \int_M \mathcal{L} \tag{1}$$

where the Lagrangian density D-form $\mathcal{L}$ is given in Brans-Dicke frame in a geometry based on Riemannian formulation, by imposing as a constraint that the connection is Levi-Civita:

$$\mathcal{L} = \frac{\phi}{2} R^{ab} \wedge * (e_a \wedge e_b) - \frac{\omega}{2\phi} d\phi \wedge *d\phi - \frac{\phi^k}{2} H \wedge *H. \tag{2}$$

Here the basic gravitational variables are the co-frame 1-forms e^a in terms of which the space-time metric $g = \eta_{ab} e^a \otimes e^b$ where $\eta_{ab} = diago(-+++++...)$. Hodge $*$-map is defined so that the oriented volume form $*1 = e^0 \wedge e^1 \wedge ... \wedge e^{D-1}$. Levi-Civita connection 1-forms $^{(0)}\omega^a{}_b$ are obtained from the first Cartan structure equations

$$de^a +^{(0)} \omega^a{}_b \wedge e^b = 0 \tag{3}$$

where the metric compatibility requires $^{(0)}\omega_{ab} = -^{(0)}\omega_{ba}$ and corresponding curvature 2-forms are obtained from the second Cartan structure equations

$$^{(0)}R^{ab} = d\,^{(0)}\omega^{ab} +^{(0)} \omega^a{}_c \wedge^{(0)} \omega^{cb}. \tag{4}$$

ϕ is the dilaton 0-form and H is a $(p+2)$-form field that is derived from the axion potential $(p+1)$-form A so that $H = dA$. ω and k are real coupling parameters. Co-frame e^a variations of this action lead to the Einstein field equations

$$\frac{1}{2}\phi\,^{(0)}R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) = -\frac{\omega}{2\phi}\tau_c[\phi] - \frac{\phi^k}{2}\tau_c[H] -^{(0)} D(\iota_c(*d\phi)) \tag{5}$$

where dilaton and axion stress-energy $(D-1)$-forms are given, respectively, by

$$\tau_c[\phi] = (\iota_c d\phi \wedge *d\phi + d\phi \wedge \iota_c(*d\phi)) \tag{6}$$

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and
$$\tau_c[H] = (\iota_c H \wedge *H - (-1)^p H \wedge \iota_c(*H)). \tag{7}$$

ϕ variation of (2) yields

$$\frac{1}{2}\,^{(0)}R^{ab} \wedge *(e_a \wedge e_b) = -\omega d\left(\frac{*d\phi}{\phi}\right) - \frac{\omega}{2\phi^2}d\phi \wedge *d\phi + k\frac{\phi^{k-1}}{2}H \wedge *H \tag{8}$$

We trace (5) by considering its exterior multiplication by e^c and multiply (8) by $(D-2)\phi$. The resulting two equations are then subtracted side by side to obtain the dilaton field equation

$$\left(\omega + \frac{D-1}{D-2}\right)d*d\phi = \frac{\phi^k}{2}\alpha H \wedge *H, \tag{9}$$

where $\alpha = \frac{2p-(D-4)}{D-2} + k$. Finally, independent axion potential A variations lead to

$$d(\phi^k * H) = 0 \tag{10}$$

with $dH = 0$.

Next we consider the following action in which connection 1-forms are varied independently of the metric of space-time:

$$\mathcal{L} = \frac{\phi}{2}R^{ab} \wedge * (e_a \wedge e_b) - \frac{c}{2\phi}d\phi \wedge *d\phi - \frac{\phi^k}{2}H \wedge *H. \tag{11}$$

Co-frame variations of this action give the Einstein field equations

$$\frac{1}{2}\phi R^{ab} \wedge (e_a \wedge e_b \wedge e_c) = -\frac{c}{2\phi}\tau_c[\phi] - \frac{1}{2}\phi^k\tau_c[H]. \tag{12}$$

with $\tau_c[\phi]$ and $\tau_c[H]$ are as given by (6) and (7), respectively. Scalar field variations of the action give

$$\frac{1}{2}R^{ab} \wedge (e_a \wedge e_b) = -cd\left(\frac{*d\phi}{\phi}\right) - \frac{c}{2\phi^2}d\phi \wedge *d\phi + k\frac{\phi^{k-1}}{2}H \wedge *H. \tag{13}$$

When we trace (12) and compare it with (13) multiplied by $(D-2)\phi$, we obtain the dilaton field equation

$$cd*d\phi = \frac{\alpha}{2}\phi^k H \wedge *H. \tag{14}$$

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Independent connection variations of (11) lead to

$$D\left(\frac{\phi}{2} * (e^a \wedge e^b)\right) = 0 \qquad (15)$$

from which we can readily solve for the torsion 2-forms:

$$T^a = e^a \wedge \frac{d\phi}{(D-2)\phi}. \qquad (16)$$

We can decompose the connection 1-forms as

$$\omega^a{}_b = {}^0\omega^a{}_b + K^a{}_b \qquad (17)$$

where contorsion one-forms $K^a{}_b$ satisfy

$$K^a{}_b \wedge e^b = T^a. \qquad (18)$$

Substitution of (16) into (18) gives

$$K^a{}_b = \frac{1}{(D-2)\phi}(e^a\iota_b d\phi - e_b\iota^a d\phi). \qquad (19)$$

Curvature 2-forms R^{ab} can be similarly decomposed as

$$R^{ab} = {}^{(0)}R^{ab} + {}^{(0)}DK^{ab} + K^a{}_c \wedge K^{cb} \qquad (20)$$

where
$$^{(0)}DK^{ab} = dK^{ab} + {}^{(0)}\omega^b{}_c \wedge K^{ac} + {}^{(0)}\omega^a{}_c \wedge K^{cb}. \qquad (21)$$

Then we calculate

$$R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) = {}^{(0)}R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) + \frac{2}{\phi}\,{}^{(0)}D(\iota_c(*d\phi))$$
$$-\frac{2(D-1)}{(D-2)\phi^2}d\phi \wedge \iota_c(*d\phi) - \frac{D-1}{(D-2)\phi^2}\iota_c(d\phi \wedge *d\phi) \qquad (22)$$

and

$$R^{ab} \wedge *(e_a \wedge e_b) = {}^{(0)}R^{ab} \wedge *(e_a \wedge e_b) - \frac{2(D-1)}{(D-2)}d\left(*\frac{d\phi}{\phi}\right) - \frac{D-1}{(D-2)\phi^2}d\phi \wedge *d\phi. \qquad (23)$$

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If we insert (23) into (11), action density reduces to

$$\mathcal{L} = \frac{1}{2}\phi\ ^{(0)}R^{ab} \wedge *(e_a \wedge e_b) - \left(c - \frac{D-1}{D-2}\right)\frac{1}{2\phi}d\phi \wedge *d\phi - \frac{\phi^k}{2}H \wedge *H \quad (24)$$

upto a closed form. Substituting (22) into the Einstein field equations (12), we obtain

$$\frac{1}{2}\phi\ ^{(0)}R^{ab} \wedge *(e_a \wedge e_b \wedge e_c) = -\left(c - \frac{D-1}{D-2}\right)\frac{1}{2\phi}\tau_c[\phi]$$
$$-\frac{\phi^k}{2}\tau_c[H] -^0 D\left(\iota_c(*d\phi)\right). \quad (25)$$

Similarly, substituting (23) into the dilaton field equation (13), we obtain

$$\frac{1}{2}\ ^{(0)}R^{ab} \wedge *(e_a \wedge e_b) = \frac{\left(c - \frac{D-1}{D-2}\right)}{2\phi^2}d\phi \wedge *d\phi - \left(c - \frac{D-1}{D-2}\right)\frac{1}{\phi}d(*d\phi)$$
$$+k\frac{\phi^{k-1}}{2}H \wedge *H. \quad (26)$$

We have thus shown that if the coupling constants are identified as

$$\omega = c - \frac{D-1}{(D-2)}, \quad (27)$$

the field equations (25) and (26) are equivalent to the field equations (5) and (8).

Let us now consider conformal rescalings of the metric induced by the co-frame rescalings

$$e^a \rightarrow e^{\sigma(x)}e^a. \quad (28)$$

These imply the transformation

$$^{(0)}\omega_{ab} \rightarrow^{(0)} \omega_{ab} - e_b\iota_a d\sigma + \iota_b d\sigma e_a \quad (29)$$

of the Levi-Civita connection 1-forms. If we also postulate the following rescaling of the Brans-Dicke scalar field

$$\phi \rightarrow e^{-(D-2)\sigma}\phi, \quad (30)$$

then a straightforward calculation shows that the action (2) is scale invariant for $\omega = -\frac{D-1}{D-2}$ and $k = -\frac{2p+4-D}{D-2}$, or for $c = 0$ and $\alpha = 0$. In terms of the

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geometry described by the action (11), the above rescaling rules imply the transformation

$$K_{ab} \to K_{ab} + \iota_a d\sigma e_b - \iota_b d\sigma e_a \tag{31}$$

so that the connection with torsion does not scale:

$$\omega_{ab} \to \omega_{ab}. \tag{32}$$

Hence

$$R_{ab} \to R_{ab} \tag{33}$$

and

$$T^a \to e^\sigma (T^a + d\sigma \wedge e^a). \tag{34}$$

We can reformulate our axi-dilaton gravity in the so-called Einstein frame by adopting the co-frames

$$\tilde{e}^a = \left(\frac{\phi}{\phi_0} \right)^{\frac{1}{(n-1)}} e^a \tag{35}$$

where ϕ_0 is a constant. The new co-frames $\tilde{e}^a$ become orthonormal with respect to space-time metric

$$\tilde{g} = \left(\frac{\phi}{\phi_0} \right)^{\frac{2}{(n-1)}} g. \tag{36}$$

In terms of this metric the associated Hodge dual is denoted by $\tilde{*}$. For an arbitrary frame independent p-form Ω,

$$* \Omega = \left(\frac{\phi}{\phi_0} \right)^{\frac{2p-(n+1)}{(n-1)}} \tilde{*} \Omega \tag{37}$$

In the reformulation of action (2) in terms of $\tilde{g}$, new connection fields $\tilde{\omega}^{ab}$ can be written in terms of $^{(0)}\omega^{ab}$ as

$$\tilde{\omega}^{ab} = \Gamma^{ab} + {}^{(0)}\omega^{ab} \tag{38}$$

where,

$$\Gamma^{ab} = \frac{1}{(D-2)\phi} (e^a \iota^b d\phi - e^b \iota^a d\phi). \tag{39}$$

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The corresponding curvature 2-forms become

$$\tilde{R}^{ab} =^{(0)} R^{ab} +^{(0)} D\Gamma^{ab} + \Gamma^{ac} \wedge \Gamma_c{}^b. \tag{40}$$

In terms of $\tilde{g}$, (2) becomes

$$\mathcal{L} = \frac{1}{2}\phi_0 \tilde{R}^{ab} \wedge \tilde{*}(\tilde{e}_a \wedge \tilde{e}_b) - \frac{c}{2}\phi_0 \frac{1}{\phi^2}d\phi \wedge \tilde{*}d\phi - \frac{\phi^\alpha}{2}\phi_0^{(k-\alpha)} H \wedge \tilde{*}H, \tag{41}$$

upto a closed form. Introducing a massless scalar field $\Phi = \ln|\frac{\phi}{\phi_0}|$, (41) reads

$$\mathcal{L} = \frac{1}{2}\phi_0 \tilde{R}^{ab} \wedge \tilde{*}(\tilde{e}_a \wedge \tilde{e}_b) - \frac{c}{2}\phi_0 d\Phi \wedge \tilde{*}d\Phi - \frac{1}{2}(\phi_0)^k \exp(\alpha\Phi)H \wedge \tilde{*}H. \tag{42}$$

Einstein field equations obtained by co-frame variations of (42) are

$$\frac{1}{2}\phi_0 \tilde{R}^{ab} \wedge \tilde{*}(\tilde{e}_a \wedge \tilde{e}_b \wedge \tilde{e}_c) = -\frac{c}{2}\phi_0 \tilde{\tau}_c[\Phi] - \frac{1}{2}(\phi_0)^k e^{\alpha\Phi}\tilde{\tau}_c[H], \tag{43}$$

where

$$\tilde{\tau}_c[\Phi] = \{\tilde{\iota}_c d\Phi \wedge \tilde{*}d\Phi + d\Phi \wedge \tilde{\iota}_c(\tilde{*}d\Phi)\} \tag{44}$$

and

$$\tilde{\tau}_c[H] = \{\tilde{\iota}_c H \wedge \tilde{*}H - (-1)^p H \wedge \tilde{\iota}_c(\tilde{*}H)\}. \tag{45}$$

On the other hand variations with respect to connection 1-forms $\tilde{\omega}^{ab}$ yield

$$D(\tilde{\omega})(\tilde{*}(\tilde{e}_a \wedge \tilde{e}_b)) = 0, \tag{46}$$

from which we obtain $\tilde{T}^a = 0$. Finally, we give the scalar field equation

$$c\phi_0 d(\tilde{*}d\Phi) = \frac{1}{2}(\phi_0)^k \alpha e^{\alpha\Phi} H \wedge \tilde{*}H. \tag{47}$$

and the axion field equation

$$d\left(e^{\alpha\Phi}\tilde{*}H\right) = 0. \tag{48}$$

Interestingly, by another conformal rescaling of the co-frames in Einstein frame, we can obtain the so-called string frame action. Applying the transformation

$$\hat{e}^a = \exp(\frac{2\Phi}{D-2})\tilde{e}^a \tag{49}$$

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where $\hat{e}^a$ are assumed to satisfy the torsion-free structure equations

$$d\hat{e}^a + \hat{\omega}^a{}_b \wedge \hat{e}^b = 0 \,, \tag{50}$$

the action density (42) becomes

$$\mathcal{L} = e^{-2\Phi} \left\{ \frac{1}{2}\phi_0 \hat{R}^{ab} \wedge \hat{*}(\hat{e}_a \wedge \hat{e}_b) - \frac{1}{2}\phi_0 \hat{k} d\Phi \wedge \hat{*} d\Phi \right\} - \frac{1}{2}(\phi_0)^k \exp(\alpha_0 \Phi) H \wedge \hat{*} H, \tag{51}$$

upto a closed form where coupling parameters are redefined as

$$\alpha_0 = (2p + 4 - D)\frac{3}{D - 2} + k \tag{52}$$

and

$$\hat{k} = c - \frac{4(D - 1)}{(D - 2)}. \tag{53}$$

Action density (51) is called the string frame action in D dimensions. We would like to remark that it is possible to start directly from (51) and make independent co-frame $\hat{e}^a$ and connection $\hat{\omega}^{ab}$ variations. Independent connection variations yield

$$D(\hat{\omega})(e^{-2\Phi}\hat{*}(\hat{e}_a \wedge \hat{e}_b)) = 0 \tag{54}$$

from which we can obtain torsion 2-forms $\hat{T}^a = \frac{2}{D-2}d\Phi \wedge \hat{e}^a$ [6]. Co-frame variations on the other hand yield

$$\begin{aligned}
\frac{1}{2}\phi_0 e^{-2\Phi}\hat{R}^{ab} \wedge \hat{*}(\hat{e}_a \wedge \hat{e}_b \wedge \hat{e}_c) &= -\frac{1}{2}\phi_0 \hat{k} e^{-2\Phi}\hat{\tau}_c[\Phi] \\
&\quad -\frac{1}{2}(\phi_0)^k e^{\alpha_0 \Phi}\hat{\tau}_c[H],
\end{aligned} \tag{55}$$

where

$$\hat{\tau}_c[\Phi] = \{\hat{\imath}_c d\Phi \wedge \hat{*} d\Phi + d\Phi \wedge \hat{\imath}_c(\hat{*} d\Phi)\} \tag{56}$$

and

$$\hat{\tau}_c[H] = \{\hat{\imath}_c H \wedge \hat{*} H - (-1)^p H \wedge \hat{\imath}_c(\hat{*} H)\}. \tag{57}$$

The scalar field Φ variation of (51) gives

$$\begin{aligned}
\phi_0 e^{-2\Phi}\hat{R}^{ab} \wedge \hat{*}(\hat{e}_a \wedge \hat{e}_b) &= \phi_0 \hat{k} e^{-2\Phi} d\Phi \wedge \hat{*} d\Phi \\
&\quad + \hat{k}\phi_0 d\left(e^{-2\Phi}\hat{*} d\Phi\right) - \frac{1}{2}(\phi_0)^k \alpha_0 e^{\alpha_0 \Phi} H \wedge \hat{*} H.
\end{aligned} \tag{58}$$

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We consider exterior multiplication of (55) by $\hat{e}^c$ and then multiply the equation by $\frac{2}{2-D}$. If we subtract the resulting equation from (58) and use (52), we obtain the scalar field equation

$$\phi_0 \hat{k} d \left(e^{-2\Phi} \hat{*} d\Phi \right) = \frac{1}{2}(\phi_0)^k \alpha e^{\alpha_0 \Phi} H \wedge \hat{*} H. \tag{59}$$

Finally, the gauge field A variation yields

$$d \left(e^{\alpha_0 \Phi} \hat{*} H \right) = 0. \tag{60}$$

The field equations without torsion in the string frame can be determined exactly in the same way we explained above.

3 Static, Spherically Symmetric Solutions

In this section we investigate a class of static, spherically symmetric solutions of the axi-dilaton field equations with $p = D - 4$, in the Brans-Dicke frame. Such solutions were studied previously in the Einstein and string frames [7, 8, 9, 10] in Riemannian geometries. We emphasize again that classical solutions of the coupled field equations given in the Brans-Dicke, Einstein and string frames, whether we consider a space-time geometry with or without torsion, are all conformally equivalent to each other. However, the scale invariant case can be most conveniently studied in the Brans-Dicke frame [11]. In terms of spherical polar coordinates $(t, r, \theta_i, i = 1, 2, 3, \cdots, D - 2)$, we take the metric

$$g = -f^2(r)dt \otimes dt + h^2(r)dr \otimes dr + R^2(r)d\Omega_{D-2}, \tag{61}$$

axion field $(D - 2)$-form

$$H = g(r)e^1 \wedge e^2 \wedge e^3 \wedge \cdots \wedge e^{D-2}, \tag{62}$$

and the dilaton scalar

$$\phi = \phi(r). \tag{63}$$

Case: $c \neq 0$, $k \neq -\frac{D-4}{D-2}$.

The solutions are given by the metric functions [11]

$$R(r) = r \left(1 - \left(\frac{C_1}{r} \right)^{n-2} \right)^{\alpha_3}$$
$$f(r) = \left(1 - \left(\frac{C_2}{r} \right)^{n-2} \right)^{\alpha_4} \left(1 - \left(\frac{C_1}{r} \right)^{n-2} \right)^{\alpha_5} \tag{64}$$
$$h(r) = \left(1 - \left(\frac{C_2}{r} \right)^{n-2} \right)^{\alpha_2} \left(1 - \left(\frac{C_1}{r} \right)^{n-2} \right)^{\alpha_1}$$

together with

$$\phi(r) = \left(1 - \left(\frac{C_1}{r}\right)^{n-2}\right)^{\frac{2\gamma}{\alpha}} \tag{65}$$

and

$$g(r) = \frac{Q}{R^{D-2}} \tag{66}$$

where the exponents are related by

$$\alpha_1 = \gamma\left(\frac{1}{(D-3)} - \frac{2}{(D-2)\alpha}\right) - \frac{1}{2} \quad , \quad \alpha_2 = -\frac{1}{2} \quad ,$$

$$\alpha_3 = \left(\frac{1}{(D-3)} - \frac{2}{(D-2)\alpha}\right)\gamma \quad ,$$

$$\alpha_4 = \frac{1}{2} \quad , \quad \alpha_5 = \frac{1}{2} - \left(1 + \frac{2}{(D-2)\alpha}\right)\gamma.$$

Here, we introduced parameters

$$\gamma = \frac{(D-2)\alpha^2}{4c(D-3) + (D-2)\alpha^2} \tag{67}$$

and

$$c = \omega + \frac{D-1}{D-2}. \tag{68}$$

The integration constants C_1 and C_2 should satisfy

$$Q^2 = \frac{4c(C_1C_2)^{D-3}(D-3)^2}{\alpha^2}. \tag{69}$$

These solutions are asymptotically flat as $r \to \infty$. Therefore the following physical constants can be identified:
We define the mass

$$2M \equiv \lim_{r\to\infty} r^{D-3}(1 - f^2) = (C_2)^{D-3} + \left(1 - \frac{4\gamma}{(D-2)\alpha} - 2\gamma\right)(C_1)^{D-3} \tag{70}$$

The scalar charge is

$$\Sigma \equiv \lim_{r\to\infty} \frac{\phi'}{\phi}r^{D-2} = 2(D-3)(C_1)^{D-3}\frac{\gamma}{\alpha}. \tag{71}$$

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Magnetic charge can be found from

$$Q \equiv \lim_{r \to \infty} g r^{D-2} = Q. \tag{72}$$

Eliminating the integration constants $(C_1)^{D-3}$ and $(C_2)^{D-3}$ above, we obtain one relationship among our three physical parameters:

$$Q^2 = \frac{2(D-3)\Sigma}{\alpha} c \left\{ \left(2\gamma + \frac{4\gamma}{(D-2)\alpha} - 1 \right) \frac{\Sigma\alpha}{2(D-3)\gamma} + 2M \right\} \tag{73}$$

The BPS bound is determined from this relationship since Σ is a real parameter. This implies the following inequality satisfied by the mass and charge:

$$M \geq \sqrt{\frac{4c(D-3) - 4\alpha - (D-2)\alpha^2}{4c(D-2)(D-3)^2}} |Q| \tag{74}$$

provided

$$\alpha^2(D-2) + 4\alpha \leq 4c(D-3). \tag{75}$$

Assume that $C_2 > C_1$. Then, for $Q \neq 0$, the metric functions admit an outer horizon at $r_+ = C_2$ and an inner horizon at $r_- = C_1$. The corresponding curvature scalar (of the Levi-Civita connection)

$$
^{(0)}\mathcal{R} = \frac{1}{r^{2(D-2)}} \left\{ \left(\frac{D-4}{D-2} - \frac{(D-1)\alpha}{c(D-2)} \right) Q^2 \left(1 - \left(\frac{C_1}{r} \right)^{D-3} \right)^{\{ \frac{2(k-1)\gamma}{\alpha} - 2(D-2)\alpha_3 \}} \right.
$$
$$
\left. - \omega \left(\frac{2\gamma}{\alpha} (C_1)^{D-3}(D-3) \right)^2 \left(1 - \left(\frac{C_1}{r} \right)^{D-3} \right)^{-2-2\alpha_1} \left(1 - \left(\frac{C_2}{r} \right)^{D-3} \right) \right\} \tag{76}
$$

is finite at $r_+ = C_2$. The calculation of quadratic curvature invariant on the other hand yields

$$* (R_{ab} \wedge *R^{ab}) \sim \left(1 - \left(\frac{C_1}{r} \right)^{D-3} \right)^{-4-4\alpha_1} r^{-4(D-2)}, \tag{77}$$

which shows that $r = 0$ is an essential singularity. So the solutions above describe black holes.

It is also interesting to see that if geometry of space-time is equipped with a connection with torsion, then the corresponding curvature scalar

$$
\mathcal{R} = \frac{1}{r^{2(D-2)}} \left\{ \left(\frac{D-4}{D-3} \right) Q^2 \left(1 - \left(\frac{C_1}{r} \right)^{D-3} \right)^{\{ \frac{2(k-1)\gamma}{\alpha} - 2(D-2)\alpha_3 \}} \right.
$$
$$
\left. - c \left(\frac{2\gamma}{\alpha} (C_1)^{D-3}(D-3) \right)^2 \left(1 - \left(\frac{C_1}{r} \right)^{D-3} \right)^{-2-2\alpha_1} \left(1 - \left(\frac{C_2}{r} \right)^{D-3} \right) \right\} \tag{78}
$$

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is again finite at $r_+ = C_2$ while $r = 0$ is an essential singularity. Hence, the nature of the horizon and the essential singularity are not affected by torsion.

Case: $c = 0$, $k = -\frac{D-4}{D-2}$.

A class of asymptotically flat solutions to conformally scale invariant theory has the following form:

$$R(r) = r\left(1 - \left(\frac{E_1}{r}\right)^{D-3}\right)^{-\frac{\beta}{(D-2)}}$$

$$f(r) = \left(1 - \left(\frac{E_1}{r}\right)^{D-3}\right)^{\frac{1}{2} - \frac{\beta}{(D-2)}}\left(1 - \left(\frac{E_2}{r}\right)^{D-3}\right)^{1/2}$$

$$h(r) = \left(1 - \left(\frac{E_1}{r}\right)^{D-3}\right)^{-\frac{1}{2} - \frac{\beta}{(D-2)}}\left(1 - \left(\frac{E_2}{r}\right)^{D-3}\right)^{-\frac{1}{2}} \tag{79}$$

$$\phi(r) = \left(1 - \left(\frac{E_1}{r}\right)^{D-3}\right)^{\beta}$$

$$g(r) = \frac{Q}{R^{D-2}}$$

where E_1 and E_2 are constants that satisfy,

$$(E_2 E_1)^{D-3} = \frac{Q^2}{(D-2)(D-3)}. \tag{80}$$

β is a free parameter. The special case of parameter values $Q = 0$ and $E_2 = 0$ brings (79) to Einstein-conformal scalar field solution of Bekenstein [12]. Bekenstein proposed a black hole interpretation of his solutions based on the study of conformal world lines [13]. The scalar particles are postulated to follow geodesic world-lines in Brans-Dicke theory. On the other hand, if space-time geometry is equipped with a connection with torsion, history of particles would be an autoparallel of a connection with torsion [3]. It has been shown that the conformal world-lines are nothing but the autoparallel curves in the non-Riemannian reformulation of the Brans-Dicke theory [2]. In this case, the scalar curvature of the connection with torsion is calculated as

$$\mathcal{R}_c = \frac{D-4}{D-2}Q^2\left(1 - \left(\frac{E_1}{r}\right)^{D-3}\right)^{\frac{2\beta}{(D-2)}}\frac{1}{r^{2(D-2)}}. \tag{81}$$

It is seen that $r = E_2$ is a regular event horizon, while $r = 0$ is an essential singularity. Therefore conformal solutions describe a black hole. Mass of the

black hole can be defined in terms of integration constants:

$$2M_c = (E_2)^{D-3} + \left(1 - \frac{2\beta}{(D-2)}\right)(E_1)^{D-3}.$$

(82)

β turns out to be proportional to the scalar charge.

4 Conclusion

In this paper we have studied axi-dilaton gravity theories in $D \geq 4$ dimensional space-times. We have shown by making use of the conformal rescaling properties of the space-time geometry, the equivalence of the variational field equations obtained in the Brans-Dicke, Einstein and string frames, with or without torsion.

We have investigated a class of asymptotically flat, static, spherically symmetric solutions in the Brans-Dicke frame. The black hole configurations found in the case of non-scale invariant axi-dilaton gravity generalize the well-known $D = 4$ Janis-Newman-Winicour solutions of the Einstein-Maxwell-massless scalar field equations [14]. The fact that we are working in the Brans-Dicke frame is essential to our discussion of the solutions of the scale invariant axi-dilaton gravity in D-dimensions. The solutions found in this case generalize the conformal black hole solutions of Bekenstein of $D = 4$ Einstein-conformal scalar field theory.

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