

COMPUTATIONAL NETWORKS AND SYSTEMS –  
HOMOGENIZATION OF SELF-ADJOINT DIFFERENTIAL  
OPERATORS IN VARIATIONAL FORM ON PERIODIC  
NETWORKS AND MICRO-ARCHITECTURED SYSTEMS

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**ABSTRACT.** Micro-architected systems and periodic network structures play an import role in multi-scale physics and material sciences. Mathematical modeling leads to challenging problems on the analytical and the numerical side. Previous studies focused on averaging techniques that can be used to reveal the corresponding macroscopic model describing the effective behavior. This study aims at a mathematical rigorous proof within the framework of homogenization theory. As a model example, the variational form of a self-adjoint operator on a large periodic network is considered. A notion of two-scale convergence for network functions based on a so-called two-scale transform is applied. It is shown that the sequence of solutions of the variational microscopic model on varying networked domains converges towards the solution of the macroscopic model. A similar result is achieved for the corresponding sequence of tangential gradients. The resulting homogenized variational model can be easily solved with standard PDE-solvers. In addition, the homogenized coefficients provide a characterization of the physical system on a global scale. In this way, a mathematically rigorous concept for the homogenization of self-adjoint operators on periodic manifolds is achieved. Numerical results illustrate the effectiveness of the presented approach.

**1. Introduction.** This research is motivated by our studies on flow and transport through extremely large *capillary systems* in the fields of *groundmotion prediction* in geo-engineering and *groundwater contamination monitoring* in environmental

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sciences ([35]). Huge capillary networks can be represented by periodic graphs with a very small length of periodicity when compared to the total extension of the domain under consideration. The physical process on the periodic networked is modeled by a system of second order differential equations that define a *self-adjoint operator*. The solution of this *diffusion-reaction system* describes for example the spatial distribution of a certain substance.

Network differential equations on very large networks with an inherent periodic microstructure are very challenging from both the analytical and the numerical perspective. Primarily, this is caused by

- the large number of branches,
- the huge amount of singularities (i.e., the nodes of the network),
- the highly oscillating coefficients of the system.

In addition, in case of very small diffusion coefficients, *boundary layers* arise at the end of *each* branch. That means, a very fine *discretization scheme* has to be applied in order to receive an adequate numerical solution. In case of capillary networks, such a discretization leads to a huge computational effort and an approximate solution can no longer be obtained in a reasonable amount of time.

In our study, we take advantage of the very small length of periodicity and apply principles of *homogenization theory*. In homogenization theory, parameter-dependent *microscopic models* are considered that describe the problem under consideration in all its details. In particular, the effects of the inherent microscopic structures, that are usually leading to highly oscillating coefficients, are integrated in such models. Methods from homogenization theory are applied to identify the so-called *homogenized model* in a mathematical rigorous way. The *macroscopic model* provides an approximate model that describes the process under consideration on a global scale. The so-called *homogenized coefficients* characterize the effective properties. In particular, they comprise the effects of the inherent microscopic structure. Various notions of convergence such as *G-convergence*, *H-convergence* or  $\Gamma$ -convergence have been developed for the analysis of a wide range of applications. These notions of convergence connect the macroscopic and the microscopic model in the sense that the sequence of solutions of the microscopic model converges to the solution of the macroscopic model. In addition, the limit function of the sequence of gradients of the solutions of the microscopic model can be determined (see Figure 1).

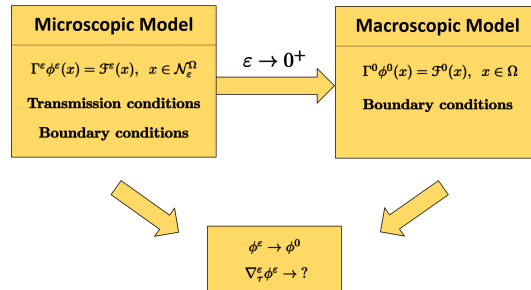


FIGURE 1. *Homogenization theory*: The limit process.

**Homogenization of network differential equations on periodic manifolds** Despite the vast amount of literature on various theoretical aspects and applications, there exists no generic concept for the homogenization of network differential equations. In particular, a number of issues arise in context of network differential equations that can be summarized as follows:

- no extension operators to the full domain are known,
- traditional notions of convergence are not applicable,
- huge number of singularities of the network,
- highly oscillating coefficients,
- development of boundary layers on *each* branch.

Important traditional approaches in homogenization theory rely on so-called *extension operators*. These operators transform a function on the domain of the microscopic model into a function on the global domain of the macroscopic model while constraints on the norm of the function and its gradient are preserved. Appropriate extension operators for network functions, that transform a function from the network domain to the global domain in the physical space, are not known by now.

In addition, generic homogenization methods and the corresponding *notions of convergence* are not directly applicable for microscopic models on periodic networks. The microscopic models are defined on domains depending on the parameter  $\varepsilon > 0$ . An appropriate *inclusion relation* for these domains and the corresponding function spaces is not fulfilled.

The structure of the periodic networks leads to further complications. For example, a huge number of *singularities* in the domain (i.e., the nodes of the graph) have to be integrated in the homogenization process. The *singular perturbations* act as connecting points of the network, where additional *transmission conditions* such as *continuity conditions* and *Kirchhoff laws* are imposed.

During the homogenization process, the length of periodicity and, thus, the length of each branch tends to zero. This immediately causes the problem of the *limit behaviour of the tangential derivatives* along the vanishing edges. Furthermore, the limit behaviour of the sequence of solutions of the microscopic model has to be addressed, because the physical domain (i.e., the global domain) will be increasingly filled with singularities. In other words, a notion of convergence for network functions is required; that allows us to decide whether or not the sequence of solutions of the microscopic model and their corresponding tangential gradients converge towards a limit function.

The singularities of the networked domain also cause major difficulties on the numerical side. As mentioned before, an extremely fine discretization scheme is required to provide appropriate numerical solutions on the networks with a vanishing length of periodicity (see [12], [37]). The transmission conditions at the nodes often destroy the symmetric (and diagonal) structure of the system's matrix of the numerical scheme. The periodicity of the network leads to highly oscillating (non-continuous) coefficients. Furthermore, for small values of the diffusion coefficient, boundary layers arise at the end of each single edge ([10]) and an adaptation of the discretization scheme is required.

For this reason, homogenization theory has to be applied to network models in order to identify approximate *macroscopic models* and their characteristic properties. Until now, only a few authors have addressed the problem of homogenization on periodic and singularly perturbed networks. Apparently, the most difficult aspect

seems to be the lack of an appropriate notion of two-scale convergence for network functions.

In [32], the problem of *identification* of the macroscopic model in case of a self-adjoint differential operator has been discussed. This approach has been further extended to *non self-adjoint operators* and *diffusion-advection-reaction models* in ([20, 21, 22, 23, 24]). In particular, *transport-dominated systems* are addressed where the diffusion part vanishes. Here, a *two-scale averaging technique* has been applied that is based on a *two-scale asymptotic expansion* of the solution of the microscopic model. Nevertheless, this method is a *purely formal approach* that can reveal the type and structure of the macroscopic model. However, no formal notion of convergence for network functions has been applied.

In particular, the mathematical sense of convergence of the solution of the microscopic models towards the solution of the macroscopic model is not clear. The same is true for the sequence of tangential gradients of the solutions of the microscopic models.

In this paper, we address the situation of [32] and aim at a mathematical rigorous proof of the homogenized model. We apply a notion of two-scale convergence for network functions and identify the homogenized model. In addition, the limit functions for sequence of solutions of the microscopic models as well as their tangential gradients are derived with respect to this notion of convergence.

The rest of the paper is organized as follows: In *Section 2*, relevant approaches for the averaging and homogenization of network differential equations are briefly reviewed. In *Section 3* various network structures are introduced and then, in *Section 4*, appropriate function spaces on scalable networks with a periodic micro-geometry are presented. In *Section 5*, the variational microscopic model is introduced. In the following *Section 6* the notions of two-scale convergence and two-scale transform for network functions are discussed. The homogenization result and the macroscopic model is derived in *Section 7*. Finally, in *Section 8*, an illustrative numerical example of a self-adjoint operator on a network is presented that demonstrates the effectiveness of this approach. We conclude with a discussion of potential developments and future works.

**2. Literature Review.** Differential equations on large periodic networks are applied in various disciplines ranging from mechanics and engineering sciences to material sciences and nanotechnology. They often arise in context of so-called *micro-architected systems*, where periodic *cellular structures* play an important role. In modern *multi-scale physics* and *material sciences*, specimens and devices with an *inherent micro-geometry* are of considerable importance. These applications are concerned for example with *photonic crystals* in nanotechnology and the development of new *lightweight materials* in engineering.

The major part of the current research in this field is devoted to differential equations on networks with a *positive thickness of the branches*. Such network structures are often called *thin domains* or *fattened graphs*. Generally, the corresponding publications apply partial differential equations on the full domain that depend on (several) small parameters such as the *length of periodicity*, the (positive) *thickness of the branches* and the *thickness of the physical device*. Several authors addressed applications of such models. For example, *elasticity problems for gridworks* and *thermal problems on lattice-type structures* and *reticulated structures* are discussed in [8]. In [36], elasticity problems for *thin planar box structures* are

addressed. Further applications deal with (*ultra-*) *lightweight materials* and *space antennas* (see [11]).

Multi-parameter systems can be analyzed by so-called *multiple-scale methods*. Such approaches are often based on *asymptotic expansions* that are used to reveal the asymptotic behaviour with respect to these parameters. Limit analysis leads to *simplified continuous models* supported by appropriate *convergence results*. These approaches are heavily used in *nanotechnology*, where so-called *photonic crystals* and their *optical characteristics* such as *spectral band gaps* are analyzed ([16, 25, 26, 27]).

Many approaches and applications in mechanics and material sciences fall into the framework of so-called *porous media* ([3, 9, 15]). The underlying physical system can be described as a *heterogeneous material* ([38]) consisting of a large number of periodically distributed inclusions. *Capillary systems* are an important class of models in the field of porous media. For example, they are applied for the modeling of *transport problems* in *soil mechanics* and *oil recovery* (see [1, 2]). Corresponding simulations of *microflows* and *nanoflows* are discussed in [17].

*Measure theoretic (fattening) approaches* are a part of a more recent methodology in the field of homogenization of thin domains. The basic idea of these approaches consists in an identification of the physical system's topology with an appropriate measure (see [4, 5, 40]). With respect to a representation of the physical domain by measures, *Sobolev spaces on measures* and the fields of potential and solenoidal (divergence free) vector-functions have been introduced. In particular, *periodic wire-networks* are analyzed in [7]. In [18], measure theoretic homogenization approaches have been used for the homogenization of *optimal control problems on periodic graphs*.

### Applications on periodic networks

The situation on networks that do not show a positive thickness of the branches is much more complicated. Only a few authors addressed this type of networks with particular regard to homogenization theory. A first model for a class of *resistive networks* in two-dimensional domains is presented by Vogelius ([39]). *Microelectromechanical systems* consisting of *recurrent cellular electronic circuits* are addressed by Lenczner et al. ([30, 31]). Lenczner also discussed applications in the fields of *smart materials* ([6]) and *field effect microscopy* ([29]). In [14, 33, 34], Göktepe and his colleagues applied a micro-mechanically based network model to analyze the viscoelasticity and elastic response of *rubbery polymers*.

**3. Periodic Networked Domains.** The mathematical models under consideration are defined on a finite periodic network with a small length of periodicity  $\varepsilon > 0$ . The physical domain of the material or device is represented by the polyhedral domain  $\Omega \subset \mathbb{R}^d$ ,  $d \in \mathbb{N}$ .

**Definition 3.1 (Infinite periodic networks).** Let  $\varepsilon > 0$  be a given *length of periodicity* and  $\Omega \subset \mathbb{R}^d$ ,  $d \in \mathbb{N}$ , is a *polyhedral domain* that describes the physical domain of the material or device under consideration. With  $\mathcal{N}_\varepsilon \subset \mathbb{R}^d$  we denote an *infinite  $\varepsilon$ -periodic network* in  $\mathbb{R}^d$ . The corresponding set of vertices is defined by  $\mathcal{V}_\varepsilon$ . For each bounded domain  $\mathcal{B} \subset \mathbb{R}^d$ , the condition  $|\mathcal{V}_\varepsilon \cap \mathcal{B}| < \infty$  has to be satisfied. In addition, we assume that  $\partial\Omega \cap \mathcal{V}_\varepsilon = \emptyset$ . This means that no vertex of the infinite periodic network  $\mathcal{N}_\varepsilon$  is located on the boundary of the domain  $\Omega$ .

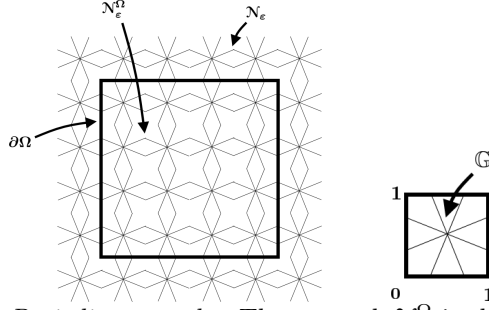


FIGURE 2. *Periodic networks*: The network  $\mathcal{N}_\varepsilon^\Omega$  is the segment of the infinite network  $\mathcal{N}_\varepsilon$  contained in the polyhedral domain  $\Omega$ , that is obtained by copying and scaling from the reference graph  $\mathbb{G}$  in the unit cell  $[0, 1]^d$ .

**Definition 3.2 (Finite Network).** The microscopic model is defined on the restriction  $\mathcal{N}_\varepsilon^\Omega := \mathcal{N}_\varepsilon \cap \Omega$  of the infinite network  $\mathcal{N}_\varepsilon$  to the domain  $\Omega$ . The associated (now finite) set of nodes of  $\mathcal{N}_\varepsilon^\Omega$  is defined by  $\mathcal{V}_\varepsilon^\Omega$ . With  $\partial^B(\mathcal{N}_\varepsilon^\Omega) := \mathcal{N}_\varepsilon \cap \partial\Omega$  we denote the *boundary nodes* (or *outer nodes*) of  $\mathcal{N}_\varepsilon^\Omega$  and  $\partial^R(\mathcal{N}_\varepsilon^\Omega) := \mathcal{V}_\varepsilon \cap \Omega$  are the *ramification nodes* (or *inner nodes*). Since both sets of nodes are distinct, we get  $\mathcal{V}_\varepsilon^\Omega = \partial^R(\mathcal{N}_\varepsilon^\Omega) \dot{\cup} \partial^B(\mathcal{N}_\varepsilon^\Omega)$ . The set of branches of the finite network  $\mathcal{N}_\varepsilon^\Omega$  is given by  $\mathfrak{J}_\varepsilon^\Omega$ . The edges  $\mathfrak{B}_j^\varepsilon \subset \mathbb{R}^d$  with index  $j \in \mathfrak{J}_\varepsilon^\Omega$  are parameterized in terms of their arc length  $\mathcal{L}_j^\varepsilon$  with regard to the interval  $\mathfrak{J}_j^\varepsilon := (0, \mathcal{L}_j^\varepsilon)$ . The tips of the branch  $\mathfrak{B}_j^\varepsilon$  are denoted by  $\mathfrak{E}^-(\mathfrak{B}_j^\varepsilon)$  and  $\mathfrak{E}^+(\mathfrak{B}_j^\varepsilon)$ .

**Assumption 3.3 (Length scales).**

(P) *The length of periodicity  $\varepsilon > 0$  of the network  $\mathcal{N}_\varepsilon^\Omega$  is considered as “very small” when compared to the diameter of the polyhedral domain  $\Omega \subset \mathbb{R}^d$  (i.e.,  $\text{diam}(\Omega) \gg \varepsilon > 0$ ).*

**Remark 1.** The periodic networks introduced before are so-called *singularly perturbed manifolds*. Here, the junctions of distinct branches - the nodes of the graph - are the singular perturbations of the networked domain.

**Definition 3.4 (Unit cell and reference graph).** With  $\square = [0, 1]^d$  we denote the *unit cell* in  $\mathbb{R}^d$ . The graph  $\mathbb{G} := \mathcal{N}_1 \cap \square$  is called the *reference graph* in  $\square$ . The *set of nodes* of the reference graph are given by  $\mathcal{V}(\mathbb{G})$ . The set of *boundary nodes* (or *outer nodes*) is defined by  $\partial^B(\mathbb{G}) := \mathcal{N}_1 \cap \partial\square$  and the set of *ramification nodes* (or *inner nodes*) is denoted by  $\partial^R(\mathbb{G}) := \mathcal{V}_1 \cap \text{int}\square$  (i.e.,  $\mathcal{V}(\mathbb{G}) = \partial^R(\mathbb{G}) \dot{\cup} \partial^B(\mathbb{G})$ ). In addition, the corresponding *set of edges* of the reference graph is denoted  $\mathfrak{J}^\mathbb{G}$ . Each branch  $j \in \mathfrak{J}^\mathbb{G}$  is parameterized in terms of its arc length  $\mathcal{L}_j^\mathbb{G}$  with respect to the interval  $\mathfrak{J}_j^\mathbb{G} := (0, \mathcal{L}_j^\mathbb{G})$ .

Both networks  $\mathcal{N}_\varepsilon$  and  $\mathcal{N}_\varepsilon^\Omega$  are composed of *recurrent elements* that are obtained from the reference graph  $\mathbb{G}$  by copying and scaling with factor  $\varepsilon$ . Finally, we introduce an assumption regarding the topology of the reference graph.

**Assumption 3.5 (Geometry).**

- (G1) Each edge of the reference graph  $\mathbb{G}$  crosses the unit cell  $\square$  completely and connects two opposite boundary nodes  $\tau^-, \tau^+ \in \partial^B(\mathbb{G})$ .
- (G2) A (directed) edge of the reference graph  $\mathbb{G}$  can intersect the boundary of the unit cell  $\square$  only with its vertices.

**4. Function Spaces on Scalable Networks with a Periodic Microgeometry.** The homogenization process of the microscopic model of the physical process on the  $\varepsilon$ -periodic network  $\mathcal{N}_\varepsilon^\Omega$  requires appropriate function spaces. The periodic coefficients of the variational microscopic model are defined on the infinite periodic network  $\mathcal{N}_1$ . The solutions of the variational network models are elements of some Hilbert spaces on networks that eventually guarantee the existence of solutions. The two-scale limit analysis of the sequence of solutions of the microscopic models and the corresponding sequence of tangential gradients leads to limit functions on the domain  $\Omega \times \mathbb{G}$ . For this reason, functions spaces are considered which are associated with domains combining the microscopic level and the macroscopic scale.

**Notation.** Let  $\phi^\varepsilon : \mathcal{N}_\varepsilon^\Omega \rightarrow \mathbb{R}$  be a network function. Then,  $\nabla_\tau^\varepsilon \phi^\varepsilon : \mathcal{N}_\varepsilon^\Omega \rightarrow \mathbb{R}$  denotes the *tangential derivative* of  $\phi^\varepsilon$  on  $\mathcal{N}_\varepsilon^\Omega$ . Similarly, the *tangential derivative* of a function  $\phi : \mathbb{G} \rightarrow \mathbb{R}$  on the reference graph is given by  $\nabla_\tau^\mathbb{G} \phi$ . By  $u * v$  we denote the inner product of  $u, v \in \mathbb{R}^d$ .

**Function spaces on the network  $\mathcal{N}_\varepsilon^\Omega$**

The function spaces associated with the variational microscopic model on  $\mathcal{N}_\varepsilon^\Omega$  are derived from  $L^p$ -spaces.

**Definition 4.1.** Let  $1 \leq p < \infty$ . The function space  $L^p(\mathcal{N}_\varepsilon^\Omega)$ , equipped with the norm

$$\|\phi^\varepsilon\|_{L^p(\mathcal{N}_\varepsilon^\Omega)}^p := \int_{\mathcal{N}_\varepsilon^\Omega} |\phi^\varepsilon(x)|^p dx,$$

is introduced on the edges of the  $\varepsilon$ -periodic network  $\mathcal{N}_\varepsilon^\Omega$ .

The two-scale limit analysis of the variational microscopic model depends on the  $L^2$ -inner product of network functions on  $\mathcal{N}_\varepsilon^\Omega$ .

**Definition 4.2.** Let  $\phi, \psi \in L^2(\mathcal{N}_\varepsilon^\Omega)$ . The  $L^2$ -inner product of network functions on  $\mathcal{N}_\varepsilon^\Omega$  is defined by

$$\langle \phi, \psi \rangle_{\mathcal{N}_\varepsilon^\Omega} := \int_{\mathcal{N}_\varepsilon^\Omega} \phi(x) \cdot \psi(x) dx.$$

On the restricted  $\varepsilon$ -periodic network  $\mathcal{N}_\varepsilon^\Omega$  two types of Sobolev-spaces are introduced.

**Definition 4.3.** On the  $\varepsilon$ -periodic network  $\mathcal{N}_\varepsilon^\Omega$  the Hilbert spaces

$$\widetilde{\mathcal{H}}^1(\mathcal{N}_\varepsilon^\Omega) := H^1(\mathcal{N}_\varepsilon^\Omega)$$

and

$$\mathcal{H}^1(\mathcal{N}_\varepsilon^\Omega) := \left\{ \phi^\varepsilon \in \widetilde{\mathcal{H}}^1(\mathcal{N}_\varepsilon^\Omega) \mid \phi^\varepsilon \text{ is continuous at } x \in \mathcal{V}_\varepsilon^\Omega \right\}$$

are introduced. Each function  $\phi \in \mathcal{H}^1(\mathcal{N}_\varepsilon^\Omega)$  is continuous at the inner nodes. Microscopic models with *Dirichlet boundary conditions* at the outer nodes depend on the following Hilbert space:

$$\mathcal{H}_0^1(\mathcal{N}_\varepsilon^\Omega) := \left\{ \phi^\varepsilon \in \mathcal{H}^1(\mathcal{N}_\varepsilon^\Omega) \mid \phi^\varepsilon(x) = 0 \text{ for } x \in \partial^B(\mathcal{N}_\varepsilon^\Omega) \right\}.$$

These function spaces can be equipped with the equivalent norms

$$\|\phi^\varepsilon\|_{\mathcal{H}_1^1(\mathcal{N}_\varepsilon^\Omega)} := \left\{ \int_{\mathcal{N}_\varepsilon^\Omega} [\nabla_\tau^\varepsilon \phi^\varepsilon(x)]^2 + [\phi^\varepsilon(x)]^2 dx \right\}^{\frac{1}{2}}$$

and

$$\|\phi^\varepsilon\|_{\mathcal{H}_{1,*}^1(\mathcal{N}_\varepsilon^\Omega)} := \left\{ \int_{\mathcal{N}_\varepsilon^\Omega} [\nabla_\tau^\varepsilon \phi^\varepsilon(x)]^2 dx \right\}^{\frac{1}{2}}.$$

In order to guarantee the existence of a solution of the microscopic variational model  $\mathcal{N}_\varepsilon^\Omega$ , the next function space is introduced.

**Definition 4.4.** The solution of the variational microscopic model with Dirichlet boundary conditions will be an element of the function space

$$\mathcal{H}_{0,*}^1 := (\mathcal{H}_0^1(\mathcal{N}_\varepsilon^\Omega), \|\cdot\|_{\mathcal{H}_{1,*}^1(\mathcal{N}_\varepsilon^\Omega)}).$$

**Remark 2.** In some situations, a scaled norm on  $\mathcal{H}^1(\mathcal{N}_\varepsilon^\Omega)$  has to be applied:

$$\|\phi^\varepsilon\|_{\mathcal{H}_{\varepsilon,*}^1(\mathcal{N}_\varepsilon^\Omega)} := \varepsilon^{\frac{d-1}{2}} \cdot \|\phi^\varepsilon\|_{\mathcal{H}_{1,*}^1(\mathcal{N}_\varepsilon^\Omega)}.$$

### Periodic functions

The microscopic variational model on the  $\varepsilon$ -periodic network  $\mathcal{N}_\varepsilon^\Omega$  depends on highly oscillating (periodic) coefficients. They are defined on the infinite network  $\mathcal{N}_1$ . Since the  $\varepsilon$ -periodic network  $\mathcal{N}_\varepsilon$  is obtained by scaling from  $\mathcal{N}_1$ , the corresponding coefficients on  $\mathcal{N}_\varepsilon$  are determined by  $\phi^\varepsilon(x) := u(\varepsilon^{-1}x)$  for each  $x \in \mathcal{N}_\varepsilon$ .

**Definition 4.5.** The spaces of periodic functions

$$\begin{aligned} L_{per}^2 &:= \{\phi \in L^2(\mathcal{N}_1) \mid \phi \text{ is } \mathbb{G}\text{-periodic}\}, \\ \mathcal{H}_{per}^1 &:= \{\phi \in \mathcal{H}^1(\mathcal{N}_1) \mid \phi \text{ is } \mathbb{G}\text{-periodic}\}, \end{aligned}$$

are defined on the infinite network  $\mathcal{N}_1$ .

### Functions on $\Omega \times \mathbb{G}$

The two-scale limit analysis of sequences of functions on the scalable network  $\mathcal{N}_\varepsilon^\Omega$  leads to limit functions on the domain  $\Omega \times \mathbb{G}$ . Again, appropriate  $L^p$ -spaces are required.

**Definition 4.6.** Let  $1 \leq p < \infty$ . The function space  $L^p(\Omega \times \mathbb{G})$  is equipped with the norm

$$\|\phi\|_{L^p(\Omega \times \mathbb{G})}^p := \int_{\Omega} \int_{\mathbb{G}} |\phi(z, y)|^p dy dz.$$

The two-scale limit analysis of the sequence of solutions of the variational microscopic models on the  $\varepsilon$ -periodic networks reveals that this sequence converges to a function in the following space.

**Definition 4.7.** Let  $\Omega \subset \mathbb{R}^d$  be the superior domain and  $\mathbb{G}$  denotes the reference graph. We set

$$\begin{aligned} \mathcal{H}_{0,\tau\mathbb{G}}^1(\Omega, \mathbb{G}) &:= \{\phi \in L^2(\Omega \times \mathbb{G}) \mid \nabla_z \phi(z, y) * \tau^{\mathbb{G}}(y) \in L^2(\Omega) \text{ for } y \in \mathbb{G}, \\ &\quad \nabla_\tau^{\mathbb{G}} \phi(z, y) = 0 \text{ for } (z, y) \in \Omega \times \mathbb{G}\}. \end{aligned}$$



**5. The Variational Form of the Microscopic Model.** In this study, the physical system under consideration is represented by a *diffusion-reaction process* on an extremely fine periodic network like a *capillary system*. Such models are motivated by our studies on *groundwater contamination monitoring* and *groundmotion prediction* ([35]). The *microscopic model* on the periodic network  $\mathcal{N}_\varepsilon^\Omega$  is given in the variational form of a *diffusion-reaction process*. We assume that *continuity conditions* at the ramification nodes and *Dirichlet boundary conditions* are imposed.

**Definition 5.1 (Variational microscopic model).**

Let the assumptions (P), (B), (G1), (G2) and the condition (C) defined below be fulfilled. The *variational microscopic model* is given by

$$\left. \begin{array}{l} \text{Find } \phi^\varepsilon \in \mathcal{H}_{0,*}^1(\mathcal{N}_\varepsilon^\Omega), \text{ such that} \\ \Gamma^\varepsilon(\phi^\varepsilon, \psi^\varepsilon) = \langle f^\varepsilon, \psi^\varepsilon \rangle_{\mathcal{N}_\varepsilon^\Omega} \quad (\forall \psi^\varepsilon \in \mathcal{H}_{0,*}^1(\mathcal{N}_\varepsilon^\Omega)), \end{array} \right\} (\text{VMP}_\varepsilon)$$

where the self-adjoint operator is given by the *bilinear form*  $\Gamma^\varepsilon : \mathcal{H}_0^1(\mathcal{N}_\varepsilon^\Omega) \times \mathcal{H}_0^1(\mathcal{N}_\varepsilon^\Omega) \rightarrow \mathbb{R}$  with

$$\Gamma^\varepsilon(\phi^\varepsilon, \psi^\varepsilon) := \int_{\mathcal{N}_\varepsilon^\Omega} a^\varepsilon(x) \nabla_\tau^\varepsilon \phi^\varepsilon(x) \nabla_\tau^\varepsilon \psi^\varepsilon(x) + d^\varepsilon(x) \phi^\varepsilon(x) \psi^\varepsilon(x) dx.$$

**Assumption 5.2 (Coefficients of the variational microscopic model (VMP<sub>ε</sub>)).**

(C) *The coefficients  $a$ ,  $d$  and the function  $\mathcal{F}$  are  $\mathbb{G}$ -periodic and they fulfill the following conditions:*

$$\begin{aligned} a &\in \mathcal{H}_{per}^1(\mathcal{N}_1), \quad a_{max} \geq a \geq a_0 > 0, \\ d &\in \mathcal{H}_{per}^1(\mathcal{N}_1), \quad d_{max} \geq d \geq 0, \\ f &\in L_{per}^2(\mathcal{N}_1). \end{aligned}$$

**Remark 3.** In the rest of the paper, we assume that the assumptions (P), (B), (G1), (G2), (C) are always satisfied.

In the sequel, some properties of the bilinear form  $\Gamma$  are summarized. Firstly, we refer to an inequality of *Poincaré-type*, that is fulfilled for network functions in  $\mathcal{H}_0^1(\mathcal{N}_\varepsilon^\Omega)$ .

**Theorem 5.3.** *Let  $\phi^\varepsilon \in \mathcal{H}_0^1(\mathcal{N}_\varepsilon^\Omega)$ . Then, the inequality*

$$\int_{\mathcal{N}_\varepsilon^\Omega} [\phi^\varepsilon(x)]^2 dx \leq C \cdot \int_{\mathcal{N}_\varepsilon^\Omega} [\nabla_\tau^\varepsilon \phi^\varepsilon(x)]^2 dx$$

*is fulfilled, where  $C \in \mathcal{O}(1)$ .*<sup>1</sup>

Next, we recall the *Lax-Milgram-Theorem*.

**Theorem 5.4 (Lax-Milgram-Theorem<sup>2</sup>).** *Let  $\Gamma : H \times H \rightarrow \mathbb{R}$  be a bounded and coercive (with constant  $\gamma_0$ ) bilinear form on the Hilbert space  $H$  and let  $F \in H'$ . Then there exists a unique  $\phi \in H$ , such that*

$$\Gamma(\phi, \psi) = F(\psi) \quad (\forall \psi \in H).$$

<sup>1</sup>See Mazja/Nasarow/Plamenevski 1991, Theorem 19.2.2, p. 261.

<sup>2</sup>See [13], Theorem 5.8, p. 78.

In addition, the following inequality is fulfilled:

$$\|\phi\|_H \leq \frac{1}{\gamma_0} \cdot \|F\|_{H'}.$$

Firstly, we prove that the bilinear form  $\Gamma^\varepsilon$  is *coercive*.

**Theorem 5.5.** *The bilinear form  $\Gamma^\varepsilon$  is  $\mathcal{H}_{0,*}^1(\mathcal{N}_\varepsilon^\Omega)$ -coercive with constant  $\delta a_0$ .*

*Proof.* Let  $\phi^\varepsilon \in \mathcal{H}_0^1(\mathcal{N}_\varepsilon^\Omega)$ . Then, we get

$$\Gamma^\varepsilon(\phi^\varepsilon, \phi^\varepsilon) = \langle a^\varepsilon, [\nabla_\tau^\varepsilon \phi^\varepsilon]^2 \rangle_{\mathcal{N}_\varepsilon^\Omega} + \langle d^\varepsilon, [\phi^\varepsilon]^2 \rangle_{\mathcal{N}_\varepsilon^\Omega},$$

and with assumption (C) we obtain  $\Gamma^\varepsilon(\phi^\varepsilon, \phi^\varepsilon) \geq a_0 \cdot \|\phi^\varepsilon\|_{\mathcal{H}_{1,*}^1(\mathcal{N}_\varepsilon^\Omega)}$ .  $\square$

Now, we prove that the bilinear form  $\Gamma^\varepsilon$  is *bounded*.

**Theorem 5.6.** *The bilinear form  $\Gamma^\varepsilon$  is Lipschitz-continuous on  $\mathcal{H}_{0,*}^1(\mathcal{N}_\varepsilon^\Omega) \times \mathcal{H}_{0,*}^1(\mathcal{N}_\varepsilon^\Omega)$ .*

*Proof.* Let  $\phi^\varepsilon, \psi^\varepsilon \in \mathcal{H}_0^1(\mathcal{N}_\varepsilon^\Omega)$  and  $M := \max\{a_{max}, d_{max}\}$ . Applying the Cauchy-Schwarz inequality leads to

$$\begin{aligned} |a^\varepsilon(\phi^\varepsilon, \psi^\varepsilon)| &\leq a_{max} \cdot \|\phi^\varepsilon\|_{\mathcal{H}_{1,*}^1(\mathcal{N}_\varepsilon^\Omega)} \cdot \|\psi^\varepsilon\|_{\mathcal{H}_1^1(\mathcal{N}_\varepsilon^\Omega)} + d_{max} \cdot \|\phi^\varepsilon\|_{\mathcal{H}_1^1(\mathcal{N}_\varepsilon^\Omega)} \cdot \|\psi^\varepsilon\|_{\mathcal{H}_1^1(\mathcal{N}_\varepsilon^\Omega)} \\ &\leq \{a_{max} + d_{max}\} \left\{ \|\phi^\varepsilon\|_{\mathcal{H}_{1,*}^1(\mathcal{N}_\varepsilon^\Omega)} \cdot \|\psi^\varepsilon\|_{\mathcal{H}_{1,*}^1(\mathcal{N}_\varepsilon^\Omega)} \right\} \\ &\leq 2M \cdot \|\phi^\varepsilon\|_{\mathcal{H}_{1,*}^1(\mathcal{N}_\varepsilon^\Omega)} \cdot \|\psi^\varepsilon\|_{\mathcal{H}_{1,*}^1(\mathcal{N}_\varepsilon^\Omega)}. \end{aligned}$$

Because of

$$|\Gamma^\varepsilon(\phi^\varepsilon, \psi^\varepsilon)| \leq 2M \cdot \|\phi^\varepsilon\|_{\mathcal{H}_1^1(\mathcal{N}_\varepsilon^\Omega)} \cdot \|\psi^\varepsilon\|_{\mathcal{H}_1^1(\mathcal{N}_\varepsilon^\Omega)},$$

the bilinear form  $\Gamma^\varepsilon$  is bounded on  $\mathcal{H}_0^1(\mathcal{N}_\varepsilon^\Omega)$ . The norms  $\|\cdot\|_{\mathcal{H}_1^1(\mathcal{N}_\varepsilon^\Omega)}$  and  $\|\cdot\|_{\mathcal{H}_{1,*}^1(\mathcal{N}_\varepsilon^\Omega)}$  are equivalent. That means, there exists a value  $C > 0$  such that

$$|\Gamma^\varepsilon(\phi^\varepsilon, \psi^\varepsilon)| \leq 2MC \cdot \|\phi^\varepsilon\|_{\mathcal{H}_{1,*}^1(\mathcal{N}_\varepsilon^\Omega)} \cdot \|\psi^\varepsilon\|_{\mathcal{H}_{1,*}^1(\mathcal{N}_\varepsilon^\Omega)}.$$

$\square$

Because of Theorem 5.5 and Theorem 5.6 the bilinear form  $\Gamma^\varepsilon$  is continuous and coercive. The *Lax-Milgram-theorem* leads to the next result.

**Theorem 5.7. (Existence and uniqueness)**

*The variational problem  $(VMP)_\varepsilon$  has a unique solution in  $\mathcal{H}_{0,*}^1(\mathcal{N}_\varepsilon^\Omega)$ .*

We note that *a priori estimations* of the solution of the variational formulation of the microscopic model can be derived by some standard calculations.

**Theorem 5.8. (A priori estimations)**

*The solution  $\phi^\varepsilon \in \mathcal{H}_{0,*}^1(\mathcal{N}_\varepsilon^\Omega)$  of the variational problem  $(VMP)_\varepsilon$  satisfies*

$$\|\phi^\varepsilon\|_{\mathcal{H}_{1,*}^1(\mathcal{N}_\varepsilon^\Omega)} \leq \frac{\sqrt{C}}{\gamma_0} \|f^\varepsilon\|_{L^2(\mathcal{N}_\varepsilon^\Omega)},$$

where  $C \in \mathcal{O}(1)$ .

The sequence  $\{\phi^\varepsilon\}_{\varepsilon \in E}$  of solutions of the variational microscopic models is also *uniformly bounded*.

**Theorem 5.9. (Uniformly boundedness)**

Let  $\phi^\varepsilon \in \mathcal{H}_{0,*}^1(\mathcal{N}_\varepsilon^\Omega)$  be the solution of the variational problem (VMP $_\varepsilon$ ). Then, there exists a parameter  $K \in \mathcal{O}(1)$  such that the following inequality is satisfied:

$$\varepsilon^{\frac{d-1}{2}} \cdot \|\phi^\varepsilon\|_{\mathcal{H}_{1,*}^1(\mathcal{N}_\varepsilon^\Omega)} \leq \frac{K}{a_0} \cdot \|f\|_{L^2(\mathbb{G})}.$$

*Proof.* Let  $\phi^\varepsilon \in \mathcal{H}_{0,*}^1(\mathcal{N}_\varepsilon^\Omega)$  be the solution of (VMP $_\varepsilon$ ). With Theorem 5.8 and the periodicity of  $f$  we get

$$\begin{aligned} \|\phi^\varepsilon\|_{\mathcal{H}_{1,*}^1(\mathcal{N}_\varepsilon^\Omega)} &\leq \frac{\sqrt{C}}{\gamma_0} \cdot \|f^\varepsilon\|_{L^2(\mathcal{N}_\varepsilon^\Omega)} = \frac{\sqrt{C}}{\gamma_0} \cdot \varepsilon^{\frac{1}{2}} \cdot \|f\|_{L^2(\mathcal{N}^{\varepsilon^{-1}\Omega})} \\ &= \frac{\sqrt{C}}{\gamma_0} \cdot \varepsilon^{\frac{1}{2}} \cdot C(\varepsilon)^{\frac{1}{2}} \|f\|_{L^2(\mathbb{G})} = \frac{\sqrt{C} \cdot \tilde{C}}{\gamma_0} \cdot \varepsilon^{\frac{1}{2}} \cdot \varepsilon^{-\frac{d}{2}} \cdot \|f\|_{L^2(\mathbb{G})} \\ &= \frac{\sqrt{C} \cdot \tilde{C}}{\gamma_0} \cdot \varepsilon^{\frac{1-d}{2}} \cdot \|f\|_{L^2(\mathbb{G})}, \end{aligned}$$

where  $C$  is the constant from Theorem 5.3 (i.e.,  $C \in \mathcal{O}(1)$ ) and  $\mathcal{N}^{\varepsilon^{-1}\Omega} := \varepsilon^{-1} \cdot \mathcal{N}_\varepsilon^\Omega$ . In addition, the parameter  $C(\varepsilon) = \tilde{C} \cdot \varepsilon^{-d} \in \mathcal{O}(\varepsilon^{-d})$  describes the number of cells that contain the set  $\mathcal{N}_\varepsilon^\Omega$ . Multiplication with  $\varepsilon^{\frac{d-1}{2}}$  leads to

$$\varepsilon^{\frac{d-1}{2}} \cdot \|\phi^\varepsilon\|_{\mathcal{H}_{1,*}^1(\mathcal{N}_\varepsilon^\Omega)} \leq \frac{K_1}{\gamma_0} \cdot \|f\|_{L^2(\mathbb{G})},$$

where  $K := \sqrt{C} \cdot \tilde{C}$ . □

**6. Two-Scale Transform and Two-Scale Convergence.** The limit analysis  $\varepsilon \rightarrow 0$  for the microscopic variational models on the varying networks  $\mathcal{N}_\varepsilon^\Omega$  and the corresponding sequence of solutions is based on the notion of *two-scale convergence* for network functions. Here, we discuss this two-scale convergence and the associated *two-scale transform*. The two-scale transform is applied to establish a connection to functions on the fixed domain  $\Omega \times \mathbb{G}$  which does not depend on the parameter  $\varepsilon \in E$ . In this way, traditional notions of convergence for functions on the domain  $\Omega \times \mathbb{G}$  can be made applicable for network functions.

**6.1. Coverings of domains and networks.** The notion of two-scale convergence and the associated two-scale transform depend on specific coverings with cells of the domain  $\Omega$  segments of the networks  $\mathcal{N}_\varepsilon$  and  $\mathcal{N}_\varepsilon^\Omega$ .

**Covering of  $\mathbb{R}^d$  with  $\varepsilon$ -cells**

For a given length of periodicity  $\varepsilon > 0$  we set  $\square_\varepsilon := [0, \varepsilon]^d$ . The *corner point*  $c_i^\varepsilon := \varepsilon i \in \mathbb{R}^d$  is defined for each *multi-index*  $i = (i_1, \dots, i_d) \in \mathbb{Z}^d$ . Then, the  **$\varepsilon$ -cell in  $\mathbb{R}^d$  with corner point  $c_i^\varepsilon$**  is introduced by  $C_i^\varepsilon := \varepsilon(i + \square) = c_i^\varepsilon + \square_\varepsilon$ . The union of the cells  $C_i^\varepsilon$  provides a disjoint covering of  $\mathbb{R}^d$ :

$$\mathbb{R}^d = \bigcup_{i \in \mathbb{Z}^d} C_i^\varepsilon.$$

**Covering of  $\mathcal{N}_\varepsilon$** 

We introduce the sets  $\mathbb{G}_\varepsilon := \varepsilon \mathbb{G}$  and  $\mathbb{G}_i^\varepsilon := \varepsilon(i + \mathbb{G}) = c_i^\varepsilon + \mathbb{G}_\varepsilon$ . Then,  $\mathbb{G}_i^\varepsilon$  represents

the segment of the network  $\mathcal{N}_\varepsilon$  contained in the  $\varepsilon$ -cell  $\mathcal{C}_i^\varepsilon$ . Thus, the infinite network  $\mathcal{N}_\varepsilon$  is as a disjoint union of all these sets:

$$\mathcal{N}_\varepsilon = \bigcup_{i \in \mathbb{Z}^d} \mathbb{G}_i^\varepsilon.$$

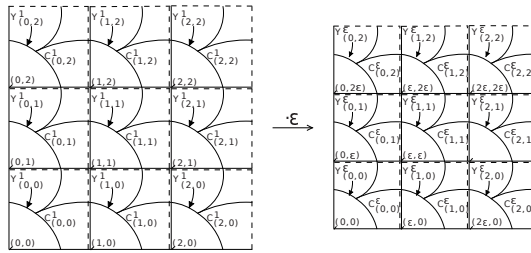


FIGURE 3. *Covering with cells*: The  $\varepsilon$ -cells and the corresponding graphs.

### Covering of $\Omega$ with $\varepsilon$ -cells

For each  $\varepsilon > 0$ , the *index set of all  $\varepsilon$ -cells in  $\Omega$*  is defined as the set of multi-indices  $\mathcal{I}_\varepsilon^\Omega := \{i \in \mathbb{R}^d \mid \mathcal{C}_i^\varepsilon \cap \Omega \neq \emptyset\}$ . The  *$\varepsilon$ -cell in  $\Omega$*  is given by  $\mathcal{C}_i^{\Omega, \varepsilon} := \mathcal{C}_i^\varepsilon \cap \Omega$  for all  $i \in \mathcal{I}_\varepsilon^\Omega$ . The **feasible lengths of periodicity** are denoted by

$$E := \left\{ \varepsilon \in (0, 1] \mid \Omega = \bigcup_{i \in \mathcal{I}_\varepsilon^\Omega} \mathcal{C}_i^{\Omega, \varepsilon} \right\}.$$

For each *feasible length of periodicity*  $\varepsilon \in E$ , the domain  $\Omega$  is covered by pairwise disjoint  $\varepsilon$ -cells. We refer to Figure 4 for an example.

### Covering of $\mathcal{N}_\varepsilon^\Omega$

For each  $\varepsilon \in E$  the sets  $\mathbb{G}_i^{\Omega, \varepsilon} := \mathbb{G}_i^\varepsilon \cap \Omega$  can be introduced for each  $i \in \mathcal{I}_\varepsilon^\Omega$ . Then we obtain

$$\mathcal{N}_\varepsilon^\Omega = \bigcup_{i \in \mathcal{I}_\varepsilon^\Omega} \mathbb{G}_i^{\Omega, \varepsilon}. \quad (1)$$

*Feasible networks* are illustrated in Figure 4.

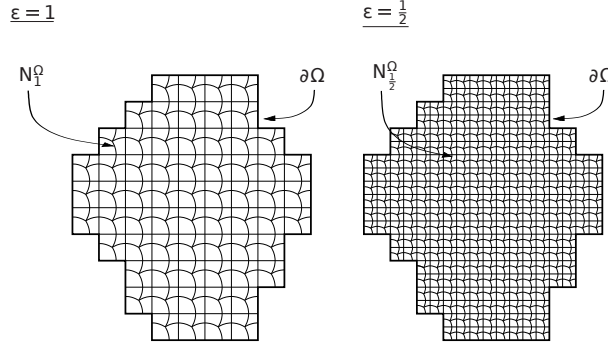


FIGURE 4. *Feasible networks*: The network  $\mathcal{N}_1^\Omega$  (left figure) and the network  $\mathcal{N}_{\frac{1}{2}}^\Omega$  are feasible networks where the corresponding  $\varepsilon$ -cells cover the domain  $\Omega$ .

**6.2. The two-scale transform.** After these preparations, the *two-scale transform*  $\widehat{\phi}^\varepsilon \in L^2(\Omega \times \mathbb{G})$  of a network function  $\phi^\varepsilon \in L^2(\mathcal{N}_\varepsilon^\Omega)$  can be introduced. This notion is based on the so-called *inverse two-scale transform* - a *surjective* mapping from the product  $\Omega \times \mathbb{G}$  of the microscopic and the macroscopic scale to the set  $\mathcal{N}_\varepsilon^\Omega$ . It is important to note that the domain of the *two-scale transform*  $\widehat{\phi}^\varepsilon$  does not depend on the parameter  $\varepsilon$ . In other words,  $\widehat{\phi}^\varepsilon \in L^2(\Omega \times \mathbb{G})$  is defined on a fixed domain that does not vary with  $\varepsilon$ . For this reason, the limit behaviour of the sequence  $\{\widehat{\phi}^\varepsilon\}_{\varepsilon \in E}$  can be analyzed with respect to traditional notions of convergence such as the *weak* and *strong two scale-convergence* (see Section 6.3).

### Two-scale transform

Let  $\varepsilon \in E$  be a feasible length of periodicity, let  $z \in \Omega$  be an element of the macroscopic space and  $y \in \mathbb{G}$  be an element of the microscopic space. The domain  $\Omega$  is covered by pairwise disjoint  $\varepsilon$ -cells by Equation (1). Thus, for each  $z \in \Omega$  there exists a unique cell index  $i \in \mathcal{I}_\varepsilon^\Omega$  such that  $z \in \mathcal{C}_i^{\Omega, \varepsilon}$  which can be used to define the point

$$x(z, y) := \varepsilon(i + y) = c_i^\varepsilon + \varepsilon y \in \mathcal{N}_\varepsilon^\Omega.$$

Now, assume that  $\widehat{y} \in \mathbb{G}$  is fixed. Then,  $x(z, \widehat{y})$  takes the same value for all  $z \in \mathcal{C}_i^{\Omega, \varepsilon}$ . This means,  $x(z, \widehat{y})$  only depends on the  $\varepsilon$ -cell  $\mathcal{C}_i^{\Omega, \varepsilon}$  with  $z \in \mathcal{C}_i^{\Omega, \varepsilon}$ , but not on the location of  $z$  within the  $\varepsilon$ -cell  $\mathcal{C}_i^{\Omega, \varepsilon}$ . It follows:

$$z_1, z_2 \in \mathcal{C}_i^{\Omega, \varepsilon} \text{ for } i \in \mathcal{I}_\varepsilon^\Omega, \widehat{y} \in \mathbb{G} \Rightarrow x(z_1, \widehat{y}) = x(z_2, \widehat{y}).$$

The parameter  $z \in \mathcal{C}_i^{\Omega, \varepsilon}$  can be replaced by each other element of the  $\varepsilon$ -cell  $\mathcal{C}_i^{\Omega, \varepsilon}$  in  $x(\cdot, \widehat{y})$  and the value of  $x$  does not change. In particular, the corner point  $c_i^\varepsilon = \varepsilon i$  of the  $\varepsilon$ -cell  $\mathcal{C}_i^{\Omega, \varepsilon}$  can be inserted.

Because of

$$x(c_i^\varepsilon, y) = x(z, y) \quad (\forall z \in \mathcal{C}_i^{\Omega, \varepsilon}) (\forall y \in \mathbb{G}),$$

the corner point  $c_i^\varepsilon = \varepsilon i$  is a representative of the  $\varepsilon$ -cell  $\mathcal{C}_i^{\Omega, \varepsilon}$ . Furthermore, we get

$$\overline{\mathbb{G}_i^{\Omega, \varepsilon}} = \left\{ x(c_i^\varepsilon, y) \mid y \in \mathbb{G} \right\} = \left\{ x(z, y) \mid z \in \mathcal{C}_i^{\Omega, \varepsilon}, y \in \mathbb{G} \right\}$$

for each  $i \in \mathcal{I}_\varepsilon^\Omega$ , and because of Equation (1) we achieve

$$\mathcal{N}_\varepsilon^\Omega = \bigcup_{i \in \mathcal{I}_\varepsilon^\Omega} \left\{ x(c_i^\varepsilon, y) \mid y \in \mathbb{G} \right\} = \bigcup_{i \in \mathcal{I}_\varepsilon^\Omega} \left\{ x(z, y) \mid z \in \mathcal{C}_i^{\Omega, \varepsilon}, y \in \mathbb{G} \right\}.$$

The function

$$\begin{aligned} \Omega \times \mathbb{G} &\rightarrow \mathcal{N}_\varepsilon^\Omega, \\ (z, y) &\mapsto x(z, y), \end{aligned}$$

is *surjective*, but *not injective*. For each  $s \in \{1, \dots, d\}$  we introduce the vector

$$\mathbf{1}_s := (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^d.$$

↑  
position  $s$

For each  $i, j \in \mathcal{I}_\varepsilon^\Omega$  with  $\mathcal{C}_j^{\Omega, \varepsilon} = \mathcal{C}_i^{\Omega, \varepsilon} + \mathbf{1}_s$  and  $v^-, v^+ \in \partial \mathcal{V}_\mathbb{G}$  with  $v^+ = v^- + \mathbf{1}_s$  we have  $x(c_i^\varepsilon, v^+) = x(c_j^\varepsilon, v^-)$ . This means, the function  $x(\cdot, \cdot)$  is not injective (see Figure 5).

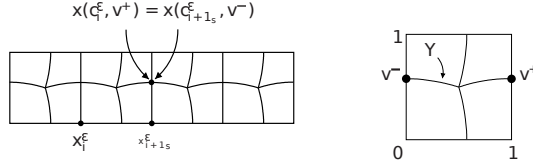


FIGURE 5. *Two-scale transform*: The function  $x : \Omega \times \mathbb{G} \rightarrow \mathcal{N}_\varepsilon^\Omega$  is surjective, but not injective.

### Two-scale transform and inverse two-scale transform

Summarizing the previous observations, it follows that the set  $\mathcal{C}_i^{\Omega, \varepsilon} \times \{y\}$  can be assigned to each point  $x \in \mathcal{N}_\varepsilon^\Omega$  on the network, where

$$x = x(c_i^\varepsilon, y) = x(z, y) \quad \text{for all } z \in \mathcal{C}_i^{\Omega, \varepsilon}.$$

This can be used to define the **two-scale transform**  $\mathcal{T}_{TS}$  on  $\mathcal{N}_\varepsilon^\Omega$  with

$$\mathcal{T}_{TS}(x) := \mathcal{C}_i^{\Omega, \varepsilon} \times \{y\}.$$

In addition, each element  $(z, y) \in \mathcal{C}_i^{\Omega, \varepsilon} \times \{y\}$  leads to the point  $x \in \overline{\mathbb{G}_i^{\Omega, \varepsilon}} (\subset \mathcal{N}_\varepsilon^\Omega)$ . The function

$$\begin{aligned} (\mathcal{T}_{TS}^i)^{-1} : \mathcal{C}_i^{\Omega, \varepsilon} \times \mathbb{G} &\rightarrow \overline{\mathbb{G}_i^\varepsilon}, \\ (z, y) &\mapsto (\mathcal{T}_{TS}^i)^{-1}(z, y) := x(z, y) \end{aligned}$$

can be introduced for each  $i \in \mathcal{I}_\varepsilon^\Omega$ .

By extending this function to the  $\varepsilon$ -periodic network  $\mathcal{N}_\varepsilon^\Omega$  and the combination  $\Omega \times \mathbb{G}$  of the microscopic scale and the macroscopic scale we achieve

$$\begin{aligned} (\mathcal{T}_{TS})^{-1} : \Omega \times \mathbb{G} &\rightarrow \mathcal{N}_\varepsilon^\Omega, \\ (z, y) &\mapsto (\mathcal{T}_{TS})^{-1}(z, y) := (\mathcal{T}_{TS}^i)^{-1}(z, y), \end{aligned}$$

for all  $z \in \mathcal{C}_i^{\Omega, \varepsilon}$ . The surjective function  $(\mathcal{T}_{TS})^{-1}$  is called the **inverse two-scale transform**.

After these technical preparations, the *two-scale transform* of a function on a network is introduced. Basically, the two-scale transform assigns a measurable function on  $\Omega \times \mathbb{G}$  to a measurable function on  $\mathcal{N}_\varepsilon^\Omega$ .

**Definition 6.1.** Let  $\phi^\varepsilon \in L^1(\mathcal{N}_\varepsilon^\Omega)$ . The function

$$\begin{aligned} \widehat{\phi}^\varepsilon : \Omega \times \mathbb{G} &\rightarrow \mathbb{R}, \\ (z, y) &\mapsto \widehat{\phi}^\varepsilon(z, y) := \phi^\varepsilon(x), \end{aligned}$$

with  $x = (\mathcal{T}_{TS})^{-1}(z, y)$  is called the **two-scale transform** of  $\phi^\varepsilon$ .

**Remark 4.** The notion of a *two-scale transform* for network functions on a one-dimensional manifold has been introduced by *Lenzner et al.* in [28, 31]. In this paper, this transform has been technically adapted to the situation of our applications in terms of network structure and topology. Nevertheless, the two-scale convergence of network functions is still a difficult task, because no *extension operators* are known that directly connect network functions on  $\mathcal{N}_\varepsilon^\Omega$  and functions on the domain  $\Omega$ . The two-scale transform applied here is based on the so-called *dilatation operator* ”~“ introduced by *Arbogast/Douglas/Hornung* in [1]. The authors are concerned with flow and transport through thin networks of channels in the soil of a petroleum reservoir. In these applications, the dilatation-operator transforms a function  $\phi^\varepsilon \in L^2(\Omega_\varepsilon)$  with domain  $\Omega_\varepsilon \subset \Omega$  into a function  $\widehat{\phi}^\varepsilon \in L^2(\Omega \times \mathbb{G})$ .

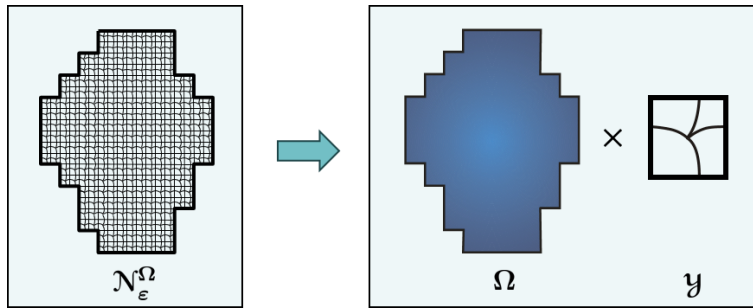


FIGURE 6. *Two-scale transform*: Mapping from  $\mathcal{N}_\varepsilon^\Omega$  to the product  $\Omega \times \mathbb{G}$ .

**Properties of the two-scale transform**

In the sequel, we are summarizing some results about functions on the network  $\mathcal{N}_\varepsilon^\Omega$  and the corresponding two-scale transform ([28, 31]). Let  $\varepsilon \in E$  be a feasible length of periodicity. Then, the macroscopic domain  $\Omega$  is covered by pairwise disjoint  $\varepsilon$ -cells  $\mathcal{C}_i^{\Omega, \varepsilon}$ :

$$\Omega = \bigcup_{i \in \mathcal{I}_\varepsilon^\Omega} \mathcal{C}_i^{\Omega, \varepsilon}. \tag{2}$$

With Equation (2) the following lemma is obtained.

**Lemma 6.2.** *Let  $\phi^\varepsilon \in L^1(\mathcal{N}_\varepsilon^\Omega)$  and let  $i \in \mathcal{I}_\varepsilon^\Omega$ . Then,*

$$\widehat{\phi}^\varepsilon(z, y) = \widehat{\phi}^\varepsilon(c_i^\varepsilon, y)$$

for each  $z \in \mathcal{C}_i^{\Omega, \varepsilon}$  and all  $y \in \mathbb{G}$ .

Now, we state a first connection between the  $L^p(\mathcal{N}_\varepsilon^\Omega)$ -norm of a function on a network and the  $L^p(\Omega \times \mathbb{G})$ -norm of the corresponding two-scale transform.

**Lemma 6.3.** *Let  $\phi^\varepsilon \in L^p(\mathcal{N}_\varepsilon^\Omega)$  with  $1 \leq p < \infty$ . Then,  $\widehat{\phi}^\varepsilon \in L^p(\Omega \times \mathbb{G})$  and*

$$\|\phi^\varepsilon\|_{L^p(\mathcal{N}_\varepsilon^\Omega)}^p = \varepsilon^{1-d} \cdot \|\widehat{\phi}^\varepsilon\|_{L^p(\Omega \times \mathbb{G})}^p.$$

*Proof.* Let  $\phi^\varepsilon \in L^p(\mathcal{N}_\varepsilon^\Omega)$ . With Equation (2) and Lemma 6.2 it follows

$$\begin{aligned} \varepsilon^{1-d} \cdot \|\widehat{\phi}^\varepsilon\|_{L^p(\Omega \times \mathbb{G})}^p &= \varepsilon^{1-d} \cdot \int_{\Omega} \int_{\mathbb{G}} |\widehat{\phi}^\varepsilon(z, y)|^p dy dz \\ &= \varepsilon^{1-d} \cdot \sum_{i \in \mathcal{I}_\varepsilon^\Omega} \int_{\mathcal{C}_i^{\Omega, \varepsilon}} \int_{\mathbb{G}} |\widehat{\phi}^\varepsilon(c_i^\varepsilon, y)|^p dy dz \\ &= \varepsilon^{1-d} \cdot \sum_{i \in \mathcal{I}_\varepsilon^\Omega} \underbrace{\int_{\mathcal{C}_i^{\Omega, \varepsilon}}}_{\varepsilon^d} dz \int_{\mathbb{G}} |\phi^\varepsilon(c_i^\varepsilon + \varepsilon y)|^p dy \\ &= \varepsilon^{1-d} \cdot \frac{\varepsilon^d}{\varepsilon} \cdot \sum_{i \in \mathcal{I}_\varepsilon^\Omega} \int_{\mathcal{X}=\varepsilon\mathbb{G}} |\phi^\varepsilon(c_i^\varepsilon + x)|^p dx \\ &= \sum_{i \in \mathcal{I}_\varepsilon^\Omega} \int_{\mathcal{X}=\varepsilon\mathbb{G}} |\phi^\varepsilon(c_i^\varepsilon + x)|^p dx = \int_{\mathcal{N}_\varepsilon^\Omega} |\phi^\varepsilon(x)|^p dx = \|\phi^\varepsilon\|_{L^p(\mathcal{N}_\varepsilon^\Omega)}^p. \end{aligned}$$

□

For  $p = 2$  we achieve an important case of the previous lemma.

**Lemma 6.4.** *Let  $\phi^\varepsilon \in L^2(\mathcal{N}_\varepsilon^\Omega)$ , then  $\widehat{\phi}^\varepsilon \in L^2(\Omega \times \mathbb{G})$  and*

$$\|\phi^\varepsilon\|_{L^2(\mathcal{N}_\varepsilon^\Omega)}^2 = \varepsilon^{1-d} \cdot \|\widehat{\phi}^\varepsilon\|_{L^2(\Omega \times \mathbb{G})}^2.$$

A similar result is obtained for the  $L^2$ -inner products.

**Lemma 6.5.** *Let  $\phi^\varepsilon, \psi^\varepsilon \in L^2(\mathcal{N}_\varepsilon^\Omega)$ . Then,*

$$\langle \phi^\varepsilon, \psi^\varepsilon \rangle_{L^2(\mathcal{N}_\varepsilon^\Omega)} = \varepsilon^{1-d} \cdot \langle \widehat{\phi}^\varepsilon, \widehat{\psi}^\varepsilon \rangle_{L^2(\Omega \times \mathbb{G})}.$$

*Proof.* Let  $\phi^\varepsilon, \psi^\varepsilon \in L^2(\mathcal{N}_\varepsilon^\Omega)$ . With Equation (2) and Lemma 6.2 we get:

$$\begin{aligned} \varepsilon^{1-d} \cdot \langle \widehat{\phi}^\varepsilon, \widehat{\psi}^\varepsilon \rangle_{L^2(\Omega \times \mathbb{G})} &= \varepsilon^{1-d} \cdot \int_{\Omega} \int_{\mathbb{G}} \widehat{\phi}^\varepsilon(z, y) \cdot \widehat{\psi}^\varepsilon(z, y) dy dz \\ &= \varepsilon^{1-d} \cdot \sum_{i \in \mathcal{I}_\varepsilon^\Omega} \int_{\mathcal{C}_i^{\Omega, \varepsilon}} \int_{\mathbb{G}} \widehat{\phi}^\varepsilon(c_i^\varepsilon, y) \cdot \widehat{\psi}^\varepsilon(c_i^\varepsilon, y) dy dz \end{aligned}$$



$$\begin{aligned}
&= \varepsilon^{1-d} \cdot \sum_{i \in \mathcal{I}_\varepsilon^\Omega} \underbrace{\int_{C_i^{\Omega, \varepsilon}} dz}_{\varepsilon^d} \int_{\mathbb{G}} \phi^\varepsilon(c_i^\varepsilon + \varepsilon y) \cdot \psi^\varepsilon(c_i^\varepsilon + \varepsilon y) dy \\
&= \varepsilon^{1-d} \cdot \frac{\varepsilon^d}{\varepsilon} \cdot \sum_{i \in \mathcal{I}_\varepsilon^\Omega} \int_{\mathcal{X} = \varepsilon \mathbb{G}} \phi^\varepsilon(c_i^\varepsilon + x) \cdot \psi^\varepsilon(c_i^\varepsilon + x) dx \\
&= \sum_{i \in \mathcal{I}_\varepsilon^\Omega} \int_{\mathcal{X} = \varepsilon \mathbb{G}} \phi^\varepsilon(c_i^\varepsilon + x) \cdot \psi^\varepsilon(c_i^\varepsilon + x) dx \\
&= \int_{\mathcal{N}_\varepsilon^\Omega} \phi^\varepsilon(x) \cdot \psi^\varepsilon(x) dx = \langle \phi^\varepsilon, \psi^\varepsilon \rangle_{L^2(\mathcal{N}_\varepsilon^\Omega)}.
\end{aligned}$$

□

For the limit analysis of the variational microscopic model, the two-scale transform of the tangential derivatives of functions on the network  $\mathcal{N}_\varepsilon^\Omega$  and functions on the reference graph have to be considered.

**Lemma 6.6.** *Let  $\phi^\varepsilon \in \mathcal{H}^1(\mathcal{N}_\varepsilon^\Omega)$  and let  $x \in \mathcal{N}_\varepsilon^\Omega$ . For each  $(z, y) \in \Omega \times \mathbb{G}$  with  $(\mathcal{T}_{TS})^{-1}(z, y) = x$  we obtain*

$$\nabla_\tau^\varepsilon \phi^\varepsilon(x) = \widehat{\nabla_\tau^\varepsilon \phi^\varepsilon}(z, y) = \frac{1}{\varepsilon} \cdot \nabla_\tau^{\mathbb{G}} \widehat{\phi^\varepsilon}(z, y).$$

*Proof.* Let  $\phi^\varepsilon \in \mathcal{H}^1(\mathcal{N}_\varepsilon^\Omega)$  and let  $x \in \mathcal{N}_\varepsilon^\Omega$ . For  $(z, y) \in \Omega \times Y$  with  $(\mathcal{T}_{TS})^{-1}(z, y) = x$ , there exists a unique  $i \in \mathcal{I}_\varepsilon^\Omega$  with  $z \in C_i^{\Omega, \varepsilon}$  as can be seen by Equation (2). With Lemma 6.2 it follows:

$$\begin{aligned}
\frac{1}{\varepsilon} \cdot \nabla_\tau^{\mathbb{G}} \widehat{\phi^\varepsilon}(z, y) &= \frac{1}{\varepsilon} \cdot \nabla_\tau^{\mathbb{G}} \widehat{\phi^\varepsilon}(c_i^\varepsilon, y) = \frac{1}{\varepsilon} \cdot \nabla_\tau^{\mathbb{G}} \phi^\varepsilon(c_i^\varepsilon + \varepsilon y) \\
&= \frac{1}{\varepsilon} \cdot \varepsilon \cdot \nabla_\tau^\varepsilon \phi^\varepsilon(x) = \nabla_\tau^\varepsilon \phi^\varepsilon(x) = \widehat{\nabla_\tau^\varepsilon \phi^\varepsilon}(z, y).
\end{aligned}$$

□

There is also a connection between the  $L^2(\mathcal{N}_\varepsilon^\Omega)$ -norm of the tangential derivative  $\nabla_\tau^\varepsilon \phi^\varepsilon$  and the  $L^2(\Omega \times \mathbb{G})$ -norm of  $\nabla_\tau^{\mathbb{G}} \widehat{\phi^\varepsilon}$ .

**Lemma 6.7.** *Let  $\phi^\varepsilon \in \mathcal{H}^1(\mathcal{N}_\varepsilon^\Omega)$ . Then the following equation is satisfied:*

$$\varepsilon^{\frac{d-1}{2}} \cdot \|\nabla_\tau^\varepsilon \phi^\varepsilon\|_{L^2(\mathcal{N}_\varepsilon^\Omega)} = \frac{1}{\varepsilon} \cdot \|\nabla_\tau^{\mathbb{G}} \widehat{\phi^\varepsilon}\|_{L^2(\Omega \times \mathbb{G})}.$$

*Proof.* Let  $\phi^\varepsilon \in \mathcal{H}^1(\mathcal{N}_\varepsilon^\Omega)$ . With Equation (2), Lemma 6.2 and Lemma 6.6 we obtain:

$$\begin{aligned}
\|\nabla_\tau^{\mathbb{G}} \widehat{\phi^\varepsilon}\|_{L^2(\Omega \times \mathbb{G})} &= \left\{ \int_{\Omega} \int_{\mathbb{G}} \left[ \nabla_\tau^{\mathbb{G}} \widehat{\phi^\varepsilon}(z, y) \right]^2 dy dz \right\}^{\frac{1}{2}} \\
&= \left\{ \sum_{i \in \mathcal{I}_\varepsilon^\Omega} \int_{C_i^{\Omega, \varepsilon}} \int_Y \left[ \nabla_\tau^{\mathbb{G}} \widehat{\phi^\varepsilon}(c_i^\varepsilon, y) \right]^2 dy dz \right\}^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
&= \left\{ \sum_{i \in \mathcal{I}_\varepsilon^\Omega} \int_{\underbrace{C_i^{\Omega, \varepsilon}}_{=\varepsilon^d}} dz \int_{\mathbb{G}} \left[ \nabla_\tau^{\mathbb{G}} \widehat{\phi^\varepsilon}(c_i^\varepsilon, y) \right]^2 dy \right\}^{\frac{1}{2}} \\
&= \varepsilon^{\frac{d}{2}} \cdot \left\{ \sum_{i \in \mathcal{I}_\varepsilon^\Omega} \int_{\mathbb{G}} \left[ \nabla_\tau^{\mathbb{G}} \widehat{\phi^\varepsilon}(c_i^\varepsilon, y) \right]^2 dy \right\}^{\frac{1}{2}} \\
&= \varepsilon^{\frac{d}{2}} \cdot \frac{\varepsilon}{\varepsilon^{\frac{1}{2}}} \cdot \left\{ \int_{\mathcal{N}_\varepsilon^\Omega} \left[ \nabla_\tau^\varepsilon \phi^\varepsilon(x) \right]^2 dx \right\}^{\frac{1}{2}} = \varepsilon^{\frac{d+1}{2}} \cdot \left\| \nabla_\tau^\varepsilon \phi^\varepsilon \right\|_{L^2(\mathcal{N}_\varepsilon^\Omega)}.
\end{aligned}$$

Division by  $\varepsilon$  leads to

$$\frac{1}{\varepsilon} \cdot \left\| \nabla_\tau^{\mathbb{G}} \widehat{\phi^\varepsilon} \right\|_{L^2(\Omega \times \mathbb{G})} = \varepsilon^{\frac{d-1}{2}} \cdot \left\| \nabla_\tau^\varepsilon \phi^\varepsilon \right\|_{L^2(\mathcal{N}_\varepsilon^\Omega)}.$$

□

Finally, the two-scale transform is *additive*, *multiplicative* and *homogeneous*.

**Lemma 6.8.** *Let  $\phi^\varepsilon, \psi^\varepsilon \in L^2(\mathcal{N}_\varepsilon^\Omega)$ . For each  $(z, y) \in \Omega \times \mathbb{G}$  we get*

1.  $\widehat{\phi^\varepsilon + \psi^\varepsilon}(z, y) = \widehat{\phi^\varepsilon}(z, y) + \widehat{\psi^\varepsilon}(z, y),$  *(additive)*
2.  $\widehat{\phi^\varepsilon \cdot \psi^\varepsilon}(z, y) = \widehat{\phi^\varepsilon}(z, y) \cdot \widehat{\psi^\varepsilon}(z, y),$  *(multiplicative)*
3.  $\widehat{\phi^\varepsilon}(z, y) = \widehat{\phi^\varepsilon}(z, y).$  *(homogeneous)*

**Remark 5.** Because of the previous lemma, the function

$$\widehat{\cdot} : L^2(\mathcal{N}_\varepsilon^\Omega) \rightarrow L^2(\Omega \times \mathbb{G}),$$

that assigns a two-scale transform to each function in  $L^2(\mathcal{N}_\varepsilon^\Omega)$ , is linear.

**6.3. The two-scale convergence.** The notion of *two-scale convergence* of a sequence of network functions in the sense of Lenczner et al. (see [28, 31]) is based on the *two-scale transform* introduced above. In this way, the parameter-dependent domain  $\mathcal{N}_\varepsilon^\Omega$  is connected with the fixed domain  $\Omega \times \mathbb{G}$ , where traditional notions of convergence can be applied.

**Definition 6.9.** Let  $\{\phi^\varepsilon \in L^2(\mathcal{N}_\varepsilon^\Omega)\}_{\varepsilon \in E}$  be a sequence of network functions and let  $\phi_0 \in L^2(\Omega \times \mathbb{G})$ .

1. The sequence  $\{\phi^\varepsilon \in L^2(\mathcal{N}_\varepsilon^\Omega)\}_{\varepsilon \in E}$  is **weakly two-scale convergent** to  $u_0$ , in symbols  $\phi^\varepsilon \xrightarrow{2} \phi_0$ , if  $\{\widehat{\phi^\varepsilon} \in L^2(\Omega \times \mathbb{G})\}_{\varepsilon \in E}$  is weakly convergent in  $L^2(\Omega \times \mathbb{G})$  to  $\phi_0$ .
2. The sequence  $\{\phi^\varepsilon \in L^2(\mathcal{N}_\varepsilon^\Omega)\}_{\varepsilon \in E}$  is **strongly two-scale convergent** to  $\phi_0$  in symbols  $\phi^\varepsilon \xrightarrow{2} \phi_0$ , if  $\{\widehat{\phi^\varepsilon} \in L^2(\Omega \times \mathbb{G})\}_{\varepsilon \in E}$  strongly converges in  $L^2(\Omega \times \mathbb{G})$  to  $\phi_0$ .

As usual, the strong two-scale convergence in  $L^2(\Omega \times \mathbb{G})$  immediately implies the weak two-scale convergence.

**Corollary 1.** *If the sequence  $\{\phi^\varepsilon \in L^2(\mathcal{N}_\varepsilon^\Omega)\}_{\varepsilon \in E}$  is strongly two-scale convergent to  $\phi_0 \in L^2(\Omega \times \mathbb{G})$ , then it is also weakly two-scale convergent to  $\phi_0$ .*

The limit analysis of the solutions of the microscopic models in Section 7 is based on the following theorem on the two-scale convergence of network functions<sup>3</sup>.

**Theorem 6.10 (Two-scale convergence).**

Let the assumptions (P), (B), (G1), (G2) be fulfilled. Let  $\{\phi^\varepsilon \in \mathcal{H}_0^1(\mathcal{N}_\varepsilon^\Omega)\}_{\varepsilon \in E}$  be sequence of network functions such that the sequence  $\{\|\phi^\varepsilon\|_{\mathcal{H}_0^1(\mathcal{N}_\varepsilon^\Omega)} \in \mathbb{R}\}_{\varepsilon \in E}$  is bounded, then there exists a subsequence  $\{\phi^{\varepsilon_k} \in \mathcal{H}_0^1(\mathcal{N}_{\varepsilon_k}^\Omega)\}_{\varepsilon_k \in E_k}$ , such that:

- (i)  $\{\phi^{\varepsilon_k} \in L^2(\mathcal{N}_{\varepsilon_k}^\Omega)\}_{\varepsilon_k \in E_k}$  is weakly two-scale convergent to  $\phi_0 \in \mathcal{H}_{0,\tau^\mathbb{G}}^1(\Omega, \mathbb{G})$ .
- (ii)  $\{\nabla_\tau^{\varepsilon_k} \phi^{\varepsilon_k} \in L^2(\mathcal{N}_{\varepsilon_k}^\Omega)\}_{\varepsilon_k \in E_k}$  is weakly two-scale convergent to

$$\nabla_z \phi_0 * \tau^\mathbb{G} + \nabla_\tau^\mathbb{G} \phi_1,$$

where  $\phi_1 \in L^2(\Omega; \mathcal{H}_\mathbb{G}^1)$ .

**7. The Macroscopic Model.** In this section, the limit behavior of the sequence of solutions of the microscopic variational models (VMP<sub>ε</sub>) associated with the self-adjoint operators on the network  $\mathcal{N}_\varepsilon^\Omega$  is addressed. The *two-scale limit analysis* leads to the *macroscopic variational problem* on  $\Omega \times \mathbb{G}$ . This *two-scale homogenized problem* provides the characterization of the original model on a global scale.

Let  $\varepsilon \in E$  be a feasible length of periodicity. The variational microscopic model (VMP)<sub>ε</sub> has a unique solution in  $\mathcal{H}_{0,*}^1$ . Because of Theorem 5.9, the corresponding sequence of solutions is uniformly bounded with respect to the  $\|\cdot\|_{\mathcal{H}_{\varepsilon,*}^1(\mathcal{N}_\varepsilon^\Omega)}$ -norm. Theorem 6.10 gives us the following result.

**Theorem 7.1. (Two-scale convergence)**

Let  $\{\phi^\varepsilon \in \mathcal{H}^1(\mathcal{N}_\varepsilon^\Omega)\}_{\varepsilon \in E}$  be the sequence of solutions of the variational microscopic model (VMP)<sub>ε</sub>. Then, there exists a subsequence  $\{\phi^{\varepsilon_k} \in \mathcal{H}^1(\mathcal{N}_{\varepsilon_k}^\Omega)\}_{\varepsilon_k \in E_k}$  such that

- (i)  $\{\phi^{\varepsilon_k} \in L^2(\mathcal{N}_{\varepsilon_k}^\Omega)\}_{\varepsilon_k \in E_k}$  is weakly two-scale convergent to  $\phi_0 \in \mathcal{H}_{\tau^\mathbb{G}}^1(\Omega, \mathbb{G})$ .
- (ii)  $\{\nabla_\tau^{\varepsilon_k} \phi^{\varepsilon_k} \in L^2(\mathcal{N}_{\varepsilon_k}^\Omega)\}_{\varepsilon_k \in E_k}$  is weakly two-scale convergent to

$$\nabla_z \phi_0 * \tau^\mathbb{G}(y) + \nabla_\tau^\mathbb{G} \phi_1,$$

where  $\phi_1 \in L^2(\Omega; \mathcal{H}_\mathbb{G}^1)$ .

**Two-scale limit analysis**

The *two-scale limit analysis* of the microscopic variational models for a vanishing length of periodicity applies *test functions* of the form

$$\psi^\varepsilon = \psi^{0\varepsilon} + \varepsilon \cdot \psi^{1\varepsilon}.$$

In order to obtain the desired homogenization results, we have to prove that their strong two-scale limit exists (see, e.g., [31]).

**The test function  $\psi^{0\varepsilon}$**

Firstly, we introduce the test function  $\psi^{0\varepsilon} : \mathcal{N}_\varepsilon^\Omega \rightarrow \mathbb{R}$ . Let  $\xi^0 \in H_{0,\tau^\mathbb{G}}^1(\Omega, \mathbb{G})$ . Then, the function  $\xi^0(z, y)$  is independent from  $y$  for each  $z \in \Omega$ . In addition, we obtain  $\nabla_z \xi^0(z, y) * \tau^\mathbb{G}(y) \in L^2(\Omega)$  for each  $y \in \mathbb{G}$ . At each node of the network  $\mathcal{N}_\varepsilon^\Omega$ , the test function  $\psi^{0\varepsilon}$  is equal to  $\xi^0$ :

$$\psi^{0\varepsilon}(x) := \xi^0(x) \quad \text{for each } x \in \mathcal{V}_\varepsilon^\Omega.$$

<sup>3</sup>The theorem on the two-scale convergence for network functions has been stated in [31, 28] with respect to models electrical networks. We also refer to [20] for the corresponding theorem with respect to the variational problems discussed in this paper.

For each edge  $\mathfrak{B}_j^\varepsilon$  with  $j \in \mathfrak{J}_\varepsilon^\Omega$  and tips  $x^- \in \mathcal{E}^-(\mathfrak{B}_j^\varepsilon)$ ,  $x^+ \in \mathcal{E}^+(\mathfrak{B}_j^\varepsilon)$  at the ends  $\mathcal{E}^-, \mathcal{E}^+$  of branch  $\mathfrak{B}_j^\varepsilon$  we set

$$\psi^{0\varepsilon}(x) := \xi^0(x^-) + H^\varepsilon(x) \cdot \delta^\varepsilon(x) \quad \text{for each } x \in \mathfrak{B}_j^\varepsilon,$$

where

$$H^\varepsilon(x) := |x - x^-| > 0 \quad \text{for each } x \in \mathfrak{B}_j^\varepsilon$$

and

$$\delta^\varepsilon(x) := \frac{\xi^0(x^+) - \xi^0(x^-)}{|x^+ - x^-|} \quad \text{for each } x \in \mathfrak{B}_j^\varepsilon.$$

At each boundary point we define

$$\psi^{0\varepsilon}(x) := \xi^0(x) = 0 \quad \text{for each } x \in \partial^{\mathfrak{B}} \mathcal{N}_\varepsilon^\Omega.$$

The function  $\psi^{0\varepsilon}$  is continuous at each node of the network  $\mathcal{N}_\varepsilon^\Omega$ . This follows from

$$\lim_{\substack{x \rightarrow x^- \\ x \in \mathfrak{B}_j^\varepsilon}} \psi^{0\varepsilon}(x) = \lim_{\substack{x \rightarrow x^- \\ x \in \mathfrak{B}_j^\varepsilon}} \{ \xi^0(x^-) + |x - x^-| \cdot \delta^\varepsilon(x) \} = \xi^0(x^-)$$

and

$$\begin{aligned} \lim_{\substack{x \rightarrow x^+ \\ x \in \mathfrak{B}_j^\varepsilon}} \psi^{0\varepsilon}(x) &= \lim_{\substack{x \rightarrow x^+ \\ x \in \mathfrak{B}_j^\varepsilon}} \left\{ \xi^0(x^-) + |x - x^-| \cdot \frac{\xi^0(x^+) - \xi^0(x^-)}{|x^+ - x^-|} \right\} \\ &= \xi^0(x^-) + \xi^0(x^+) - \xi^0(x^-) = \xi^0(x^+) \end{aligned}$$

for  $j \in \mathfrak{J}_\varepsilon^\Omega$ . Now we have proved that  $\psi^{0\varepsilon} \in \mathcal{H}_0^1(\mathcal{N}_\varepsilon^\Omega)$ . Let  $(z, y) \in \Omega \times Y$  with  $z \in \mathcal{C}_i^{\Omega, \varepsilon}$  for the index  $i \in \mathfrak{J}_\varepsilon^\Omega$  and  $\mathcal{T}^{-1}(z, y) = x \in \mathfrak{B}_j^\varepsilon$  for  $j \in \mathfrak{J}_\varepsilon^\Omega$ , then

$$\widehat{\psi}^{0\varepsilon}(z, y) = \widehat{\psi}^{0\varepsilon}(x_i^\varepsilon, y^-) + \widehat{H}^\varepsilon(z, y) \cdot \widehat{\delta}^\varepsilon(z, y) = \xi^0(x_i^\varepsilon + \varepsilon y^-) + \mathcal{O}(\varepsilon)$$

and

$$\widehat{\nabla}_\tau^\varepsilon \widehat{\psi}^{0\varepsilon}(z, y) = \frac{1}{\varepsilon} \cdot \nabla_\tau^\mathbb{G} \widehat{\psi}^{0\varepsilon}(z, y) = \nabla_z \xi^0(x_i^\varepsilon + \varepsilon y^-) * \tau^\mathbb{G}(y) + \mathcal{O}(\varepsilon),$$

where  $y^- \in \mathcal{E}^-(\mathfrak{B}_t^\mathbb{G})$  for  $t \in \mathfrak{J}^\mathbb{G}$  with  $\mathfrak{B}_j^\varepsilon = \varepsilon \cdot \mathfrak{B}_t^\mathbb{G}$ . For a vanishing length of periodicity it follows

$$\lim_{\varepsilon \rightarrow 0} \widehat{\psi}^{0\varepsilon}(z, y) = \xi^0(z)$$

and

$$\lim_{\varepsilon \rightarrow 0} \widehat{\nabla}_\tau^\varepsilon \widehat{\psi}^{0\varepsilon}(z, y) = \nabla_z \xi^0(z) * \tau^\mathbb{G}(y).$$

Finally, we receive the following lemma.

**Lemma 7.2.** *The sequence  $\{\psi^{0\varepsilon}\}_{\varepsilon \in E}$  is strongly two-scale convergent to  $\xi^0$ , and the sequence  $\{\nabla_\tau^\varepsilon \psi^{0\varepsilon}\}_{\varepsilon \in E}$  is strongly two-scale convergent to  $\nabla_z \xi^0 * \tau^Y$ .*

**The test function  $\psi^{1\varepsilon}$**

Let  $\psi^1 \in \mathcal{H}_\mathbb{G}^1$  and let  $\rho \in \mathcal{D}(\Omega)$ . The function  $\rho^\varepsilon : \mathcal{N}_\varepsilon^\Omega \rightarrow \mathbb{R}$  be defined by  $\rho^\varepsilon := \rho|_{\mathcal{N}_\varepsilon^\Omega}$  and the function  $\psi^\varepsilon : \mathcal{N}_\varepsilon^\Omega \rightarrow \mathbb{R}$  is introduced by

$$\psi^\varepsilon(x) = \widehat{\psi}^\varepsilon(z, y) := \psi^1(y) \quad \text{for } x = \mathcal{T}_{TS}^{-1}(z, y).$$

The test function  $\psi^{1\varepsilon} : \mathcal{N}_\varepsilon^\Omega \rightarrow \mathbb{R}$  is the product

$$\psi^{1\varepsilon}(x) := \rho^\varepsilon(x) \cdot \psi^\varepsilon(x).$$

Because of the definition of  $\psi^{1\varepsilon}$ , we obtain  $\psi^{1\varepsilon}|_{B_j^\varepsilon} \in \mathcal{H}^1(\mathfrak{B}_j^\varepsilon)$  for each  $j \in \mathfrak{J}_\varepsilon^\Omega$ . In addition,  $\psi^{1\varepsilon}$  is continuous at the nodes of the network  $\mathcal{N}_\varepsilon^\Omega$  and  $\psi^{1\varepsilon}(x) = 0$  is satisfied for each  $x \in \partial^{\mathfrak{B}} \mathcal{V}_\varepsilon^\Omega$ , because of  $\rho \in \mathcal{D}(\Omega)$ . Thus,  $\psi^{1\varepsilon} \in \mathcal{H}_0^1(\mathcal{N}_\varepsilon^\Omega)$ .

With  $\xi^1 : \Omega \times Y \rightarrow \mathbb{R}$ , defined by

$$\xi^1(z, y) := \rho(z) \cdot \psi^1(y),$$

we obtain the next lemma.

**Lemma 7.3.** *The sequence  $\{\psi^{1\varepsilon}\}_{\varepsilon \in E}$  is strongly two-scale convergent to  $\psi^1$ , and the sequence  $\{\varepsilon \nabla_\tau^\varepsilon \psi^{1\varepsilon}\}_{\varepsilon \in E}$  is strongly two-scale convergent to  $\nabla_\tau^{\mathbb{G}} \psi^1$ .*

*Proof.* We consider the sequence  $\{\psi^{1\varepsilon}\}_{\varepsilon \in E}$ . Because of

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left\| \widehat{\psi^{1\varepsilon}} - \xi^1 \right\|_{L^2(\Omega \times \mathbb{G})} &= \lim_{\varepsilon \rightarrow 0} \left\{ \int_{\Omega} \int_{\mathbb{G}} \left[ \widehat{\psi^{1\varepsilon}}(z, y) - \psi^1(z, y) \right]^2 dy dz \right\}^{\frac{1}{2}} \\ &= \lim_{\varepsilon \rightarrow 0} \left\{ \sum_{i \in \mathfrak{J}_\varepsilon^\Omega} \int_{\mathcal{C}_i^{\Omega, \varepsilon}} \left[ \widehat{\rho^\varepsilon \cdot \psi^\varepsilon}(z, y) - \psi^1(z, y) \right]^2 dz \right\}^{\frac{1}{2}} \\ &= \lim_{\varepsilon \rightarrow 0} \left\{ \sum_{i \in \mathfrak{J}_\varepsilon^\Omega} \int_{\mathcal{C}_i^{\Omega, \varepsilon}} \left[ \widehat{\rho^\varepsilon \cdot \psi^\varepsilon}(z, y) - \rho(z) \cdot \psi^1(y) \right]^2 dz \right\}^{\frac{1}{2}} \\ &= \left\{ \int_{\mathbb{G}} (\psi^1(y))^2 \cdot \lim_{\varepsilon \rightarrow 0} \left\{ \sum_{i \in \mathfrak{J}_\varepsilon^\Omega} \int_{\mathcal{C}_i^{\Omega, \varepsilon}} [\widehat{\rho^\varepsilon}(z, y) - \rho(z)]^2 dz \right\} dy \right\}^{\frac{1}{2}} \\ &= \left\{ \int_{\mathbb{G}} (\psi^1(y))^2 \cdot \lim_{\varepsilon \rightarrow 0} \left\{ \sum_{i \in \mathfrak{J}_\varepsilon^\Omega} \int_{\mathcal{C}_i^{\Omega, \varepsilon}} [\rho^\varepsilon(x_i^\varepsilon + \varepsilon y) - \rho(z)]^2 dz \right\} dy \right\}^{\frac{1}{2}} \\ &= \left\{ \int_{\mathbb{G}} (\psi^1(y))^2 \cdot 0 dy \right\}^{\frac{1}{2}} = 0, \end{aligned}$$

we can see that the sequence is strongly two-scale convergent to  $\psi^1$ . Next, we consider the sequence  $\{\varepsilon \nabla_\tau^\varepsilon \psi^{1\varepsilon}\}_{\varepsilon \in E}$ . With

$$\nabla_\tau^\varepsilon \psi^{1\varepsilon}(x) = \nabla_\tau^\varepsilon \rho^\varepsilon(x) \cdot \psi^\varepsilon(x) + \rho^\varepsilon(x) \cdot \nabla_\tau^\varepsilon \psi^\varepsilon(x),$$

Lemma 6.8 and Lemma 6.6, we obtain

$$\begin{aligned} \widehat{\nabla_\tau^\varepsilon \psi^{1\varepsilon}}(z, y) &= \widehat{\nabla_\tau^\varepsilon \rho^\varepsilon}(z, y) \cdot \widehat{\psi^\varepsilon}(z, y) + \widehat{\rho^\varepsilon}(z, y) \cdot \widehat{\nabla_\tau^\varepsilon \psi^\varepsilon}(z, y) \\ &= \widehat{\nabla_\tau^\varepsilon \rho^\varepsilon}(z, y) \cdot \widehat{\psi^\varepsilon}(z, y) + \widehat{\rho^\varepsilon}(z, y) \cdot \frac{1}{\varepsilon} \cdot \nabla_\tau^{\mathbb{G}} \widehat{\psi^\varepsilon}(z, y) \\ &= \widehat{\nabla_\tau^\varepsilon \rho^\varepsilon}(z, y) \cdot \psi^1(y) + \widehat{\rho^\varepsilon}(z, y) \cdot \frac{1}{\varepsilon} \cdot \nabla_\tau^{\mathbb{G}} \psi^1(y), \end{aligned}$$

for  $x = (\mathcal{T}_{TS})^{-1}(z, y)$ . Then, we get

$$\varepsilon \cdot \widehat{\nabla_\tau^\varepsilon \psi^{1\varepsilon}}(z, y) = \varepsilon \cdot \widehat{\nabla_\tau^\varepsilon \rho^\varepsilon}(z, y) \cdot \psi^1(y) + \widehat{\rho^\varepsilon}(z, y) \cdot \nabla_\tau^{\mathbb{G}} \psi^1(y).$$

Finally, we see

$$\lim_{\varepsilon \rightarrow 0} \left\| \varepsilon \widehat{\nabla_\tau^\varepsilon \psi^{1\varepsilon}} - \nabla_\tau^{\mathbb{G}} \psi^1 \right\|_{L^2(\Omega \times \mathbb{G})}$$

$$\begin{aligned}
&= \lim_{\varepsilon \rightarrow 0} \left\{ \int_{\Omega} \int_{\mathbb{G}} \left[ \widehat{\varepsilon \nabla_{\tau}^{\varepsilon} \psi^{1\varepsilon}}(z, y) - \nabla_{\tau}^{\mathbb{G}} \psi^1(z, y) \right]^2 dy dz \right\}^{\frac{1}{2}} \\
&= \lim_{\varepsilon \rightarrow 0} \left\{ \sum_{i \in \mathcal{J}_{\varepsilon}^{\Omega}} \int_{C_i^{\Omega, \varepsilon}} \left[ \int_{\mathbb{G}} \left[ \widehat{\varepsilon \nabla_{\tau}^{\varepsilon} \rho^{\varepsilon}}(z, y) \cdot \psi^1(y) + \widehat{\rho^{\varepsilon}}(z, y) \cdot \nabla_{\tau}^{\mathbb{G}} \psi^1(y) \right. \right. \right. \\
&\quad \left. \left. \left. - \rho(z) \cdot \nabla_{\tau}^{\mathbb{G}} \psi^1(y) \right]^2 dy \right] dz \right\}^{\frac{1}{2}} \\
&= \left\{ \int_{\mathbb{G}} \lim_{\varepsilon \rightarrow 0} \left\{ \sum_{i \in \mathcal{J}_{\varepsilon}^{\Omega}} \int_{ZC_i^{\Omega, \varepsilon}} \left[ \varepsilon \nabla_z \rho(x_i^{\varepsilon} + \varepsilon y) * \tau(y) + (\rho^{\varepsilon}(x_i^{\varepsilon} + \varepsilon y) - \rho(z)) \right. \right. \right. \\
&\quad \left. \left. \left. \times \nabla_{\tau}^{\mathbb{G}} \psi^1(y) \right]^2 dz \right\} dy \right\}^{\frac{1}{2}} = 0.
\end{aligned}$$

This means, the sequence  $\{\varepsilon \nabla_{\tau}^{\varepsilon} \psi^{1\varepsilon}\}_{\varepsilon \in E}$  is strongly two-scale convergent to  $\nabla_{\tau}^{\mathbb{G}} \psi^1$ .  $\square$

### The macroscopic homogenized model

The *two-scale homogenized problem* is obtained by a limit analysis for vanishing values of  $\varepsilon$ . The coefficients and the right hand side of the variational microscopic problem have satisfy some additional conditions.

#### Assumption 7.4 (Coefficients).

The coefficients and the right hand side of the variational microscopic model fulfill the following assumptions:

$$(K) \quad a^{\varepsilon} \xrightarrow{2} a^0, \quad d^{\varepsilon} \xrightarrow{2} d^0, \quad \mathcal{F}^{\varepsilon} \xrightarrow{2} \mathcal{F}^0.$$

**Remark 6.** Each strongly two-scale convergent sequence is also weakly two-scale convergent by Theorem 1. For this reason, it is sufficient to assume that  $\mathcal{F}^{\varepsilon} \xrightarrow{2} \mathcal{F}^0$ .

The next theorem states the *main homogenization result*.

#### Theorem 7.5. (Homogenization result)

Let the assumptions (C), (G1), (G2), (P), (B), and (K) be fulfilled. Let  $\{\phi^{\varepsilon} \in \mathcal{H}_0^1(\mathcal{N}_{\varepsilon}^{\Omega})\}_{\varepsilon \in E}$  be the sequence of solutions of the variational problems (VPM $_{\varepsilon}$ ). Then, there exists a subsequence  $\{\phi^{\varepsilon_k} \in \mathcal{H}_0^1(\mathcal{N}_{\varepsilon_k}^{\Omega})\}_{\varepsilon_k \in E_k}$  such that

$$\begin{aligned}
\phi^{\varepsilon_k} &\xrightarrow{2} \phi_0, \\
\nabla_{\tau}^{\varepsilon_k} \phi^{\varepsilon_k} &\xrightarrow{2} \nabla_z \phi_0 * \tau^{\mathbb{G}} + \nabla_{\tau}^{\mathbb{G}} \phi_1.
\end{aligned}$$

Here,  $(\phi_0, \phi_1)$  is the unique solution of the **variational two-scale homogenized problem**:

Find  $(\phi_0, \phi_1) \in \mathcal{H}_{0, \tau^G}^1(\Omega, \mathbb{G}) \times L^2(\Omega; \mathcal{H}_{\mathbb{G}}^1)$ , such that

$$\begin{aligned} & \int_{\Omega} \int_{\mathbb{G}} a^0(z, y) \phi_0(z, y) \cdot (\nabla_z \phi_0(z) * \tau^{\mathbb{G}}(y) + \nabla_{\tau}^{\mathbb{G}} \phi_1(z, y)) \\ & \quad \cdot (\nabla_z \psi^0(z) * \tau^{\mathbb{G}}(y) + \nabla_{\tau}^{\mathbb{G}} \psi^1(z, y)) \\ & \quad + d^0(z, y) \phi_0(z, y) \cdot \phi_0(z) \cdot \psi^0(z) dy dz \\ & = \int_{\Omega} \int_{\mathbb{G}} \mathcal{F}^0(z, y) \cdot \psi^0(z) dy dz \\ & \text{for all } (\psi^0, \psi^1) \in \mathcal{H}_{0, \tau^G}^1(\Omega, \mathbb{G}) \times L^2(\Omega; \mathcal{H}_{\mathbb{G}}^1). \quad (\text{VHP}_0) \end{aligned}$$

*Proof.* Let  $\{\phi^\varepsilon \in \mathcal{H}_0^1(\mathcal{N}_\varepsilon^\Omega)\}_{\varepsilon \in E}$  be the sequence of solutions of the variational problem (VPM $_\varepsilon$ ). Because of Theorem 5.9, the sequence

$$\{\|\phi^\varepsilon\|_{\mathcal{H}_{\varepsilon, *}}^1(\mathcal{N}_\varepsilon^\Omega) \in \mathbb{R}\}_{\varepsilon \in E}$$

is bounded. With Theorem 6.10 there exists a subsequence

$$\{\phi^{\varepsilon_k} \in \mathcal{H}_0^1(\mathcal{N}_{\varepsilon_k}^\Omega)\}_{\varepsilon_k \in E_k}$$

such that

$$\begin{aligned} \phi^{\varepsilon_k} & \xrightarrow{2} \phi_0, \\ \nabla_{\tau}^{\varepsilon_k} \phi^{\varepsilon_k} & \xrightarrow{2} \nabla_z \phi_0 * \tau^{\mathbb{G}} + \nabla_{\tau}^{\mathbb{G}} \phi_1, \end{aligned}$$

where  $\phi_0 \in \mathcal{H}_{0, \tau^G}^1(\Omega, \mathbb{G})$  and  $\phi_1 \in L^2(\Omega; \mathcal{H}_{\mathbb{G}}^1)$ . In order to show that  $(\phi_0, \phi_1)$  is the (unique) solution of the two-scaled homogenized problem we consider the test function

$$\psi^\varepsilon = \psi^{0\varepsilon} + \varepsilon \psi^{1\varepsilon} \in \mathcal{H}_0^1(\mathcal{N}_\varepsilon^\Omega)$$

in the variational problem (VMP $_\varepsilon$ ). It follows

$$\langle a^\varepsilon \nabla_{\tau}^{\varepsilon} \phi^\varepsilon, [\nabla_{\tau}^{\varepsilon} \psi^{0\varepsilon} + \varepsilon \nabla_{\tau}^{\varepsilon} \psi^{1\varepsilon}] \rangle_{\mathcal{N}_\varepsilon^\Omega} + \langle d^\varepsilon \phi^\varepsilon, [\psi^{0\varepsilon} + \varepsilon \psi^{1\varepsilon}] \rangle_{\mathcal{N}_\varepsilon^\Omega} = \langle \mathcal{F}^\varepsilon, [\psi^{0\varepsilon} + \varepsilon \psi^{1\varepsilon}] \rangle_{\mathcal{N}_\varepsilon^\Omega}.$$

Separating the strongly and weakly two-scale convergent parts, we get

$$\begin{aligned} & \langle \nabla_{\tau}^{\varepsilon} \phi^\varepsilon a^\varepsilon, \nabla_{\tau}^{\varepsilon} \psi^{0\varepsilon} \rangle_{\mathcal{N}_\varepsilon^\Omega} + \langle \nabla_{\tau}^{\varepsilon} \phi^\varepsilon a^\varepsilon, \varepsilon \nabla_{\tau}^{\varepsilon} \psi^{1\varepsilon} \rangle_{\mathcal{N}_\varepsilon^\Omega} + \langle \phi^\varepsilon d^\varepsilon, \psi^{0\varepsilon} \rangle_{\mathcal{N}_\varepsilon^\Omega} + \langle \phi^\varepsilon d^\varepsilon, \varepsilon \psi^{1\varepsilon} \rangle_{\mathcal{N}_\varepsilon^\Omega} \\ & = \langle \mathcal{F}^\varepsilon, \psi^{0\varepsilon} \rangle_{\mathcal{N}_\varepsilon^\Omega} + \langle \mathcal{F}^\varepsilon, \varepsilon \psi^{1\varepsilon} \rangle_{\mathcal{N}_\varepsilon^\Omega}. \end{aligned}$$

With Lemma 6.5 and Lemma 6.8 we obtain

$$\begin{aligned} & \varepsilon^{1-d} \cdot \left\{ \langle \widehat{\nabla_{\tau}^{\varepsilon} \phi^\varepsilon a^\varepsilon}, \widehat{\nabla_{\tau}^{\varepsilon} \psi^{0\varepsilon}} \rangle_{L^2(\Omega \times \mathbb{G})} + \langle \widehat{\nabla_{\tau}^{\varepsilon} \phi^\varepsilon a^\varepsilon}, \varepsilon \widehat{\nabla_{\tau}^{\varepsilon} \psi^{1\varepsilon}} \rangle_{L^2(\Omega \times \mathbb{G})} \right. \\ & \quad \left. + \langle \widehat{\phi^\varepsilon d^\varepsilon}, \varepsilon \widehat{\psi^{1\varepsilon}} \rangle_{L^2(\Omega \times \mathbb{G})} \right\} \\ & = \varepsilon^{1-d} \cdot \left\{ \langle \widehat{\mathcal{F}^\varepsilon}, \widehat{\psi^{0\varepsilon}} \rangle_{L^2(\Omega \times \mathbb{G})} + \langle \widehat{\mathcal{F}^\varepsilon}, \varepsilon \widehat{\psi^{1\varepsilon}} \rangle_{L^2(\Omega \times \mathbb{G})} \right\}. \end{aligned}$$

Multiplying both sides with  $\varepsilon^{d-1}$  and passing to the limit  $\varepsilon \rightarrow 0$  we obtain

$$\begin{aligned} & \int_{\Omega} \int_{\mathbb{G}} (\nabla_z \phi_0(z) * \tau^{\mathbb{G}}(y) + \nabla_{\tau}^{\mathbb{G}} \phi_1(z, y)) \cdot a^0(z, y) \cdot \nabla_z \psi^0(z) * \tau^{\mathbb{G}}(y) dy dz \\ & + \int_{\Omega} \int_{\mathbb{G}} (\nabla_z \phi_0(z) * \tau^{\mathbb{G}}(y) + \nabla_{\tau}^{\mathbb{G}} \phi_1(z, y)) \cdot a^0(z, y) \cdot \nabla_{\tau}^{\mathbb{G}} \psi^1(z, y) dy dz \end{aligned}$$

$$+ \int_{\Omega} \int_{\mathbb{G}} \phi_0(z) \cdot d^0(z, y) \cdot \psi^0(z) \, dy \, dz = \int_{\Omega} \int_{\mathbb{G}} \mathcal{F}^0(z, y) \cdot \psi^0(z) \, dy \, dz.$$

With

$$\begin{aligned} & \int_{\Omega} \int_{\mathbb{G}} a^0(z, y) \cdot (\nabla_z \phi_0(z) * \tau^{\mathbb{G}}(y) + \nabla_{\tau}^{\mathbb{G}} \phi_1(z, y)) \cdot (\nabla_z \psi^0(z) * \tau^{\mathbb{G}}(y) + \nabla_{\tau}^{\mathbb{G}} \psi^1(z, y)) \\ & + d^0(z, y) \cdot \phi_0(z) \cdot \psi^0(z) \, dy \, dz = \int_{\Omega} \int_{\mathbb{G}} \mathcal{F}^0(z, y) \cdot \psi^0(z) \, dy \, dz, \end{aligned}$$

we receive the variational equation of the two-scaled homogenized problem.

Next, we show that the homogenized equation is a variational problem in

$$\mathcal{H} := \mathcal{H}_{0, \tau^{\mathbb{G}}}^1(\Omega, \mathbb{G}) \times L^2(\Omega; \mathcal{H}_{\mathbb{G}}^1).$$

In addition, we prove that the Lax-Milgram-theorem is fulfilled such that there exists a unique solution  $(\phi_0, \phi_1) \in \mathcal{H}$ . On  $\mathcal{H}$  we define the norm

$$\|\Phi\|_{\mathcal{H}}^2 := \|\phi_0\|_{\mathcal{H}_{0, \tau^{\mathbb{G}}}^1(\Omega, \mathbb{G})}^2 + \|\phi_1\|_{L^2(\Omega; \mathcal{H}_{\mathbb{G}}^1)}^2$$

for each  $\Phi = (\phi_0, \phi_1) \in \mathcal{H}$ , where

$$\|\phi_0\|_{\mathcal{H}_{0, \tau^{\mathbb{G}}}^1(\Omega, \mathbb{G})} := \|\nabla_z \phi_0 * \tau^{\mathbb{G}}\|_{L^2(\Omega \times \mathbb{G})} = \left\{ \int_{\Omega} \int_{\mathbb{G}} [\nabla_z \phi_0(z) * \tau^{\mathbb{G}}(y)]^2 \, dy \, dz \right\}^{\frac{1}{2}}$$

and

$$\|\phi_1\|_{L^2(\Omega; \mathcal{H}_{\mathbb{G}}^1)} := \|\nabla_{\tau}^{\mathbb{G}} \phi_1\|_{L^2(\Omega \times \mathbb{G})} = \left\{ \int_{\Omega} \int_{\mathbb{G}} [\nabla_{\tau}^{\mathbb{G}} \phi_1(z, y)]^2 \, dy \, dz \right\}^{\frac{1}{2}}.$$

Let  $A : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$  be the continuous bilinear function

$$\begin{aligned} A(\Phi, \Psi) &:= \int_{\Omega} \int_{\mathbb{G}} a^0(z, y) \cdot (\nabla_z \phi_0(z) * \tau^{\mathbb{G}}(y) + \nabla_{\tau}^{\mathbb{G}} \phi_1(z, y)) \\ &\quad \cdot (\nabla_z \psi^0(z) * \tau^{\mathbb{G}}(y) + \nabla_{\tau}^{\mathbb{G}} \psi^1(z, y)) \\ &\quad + d^0(z, y) \cdot \phi_0(z) \cdot \psi^0(z) \, dy \, dz, \end{aligned}$$

where  $\Phi = (\phi_0, \phi_1) \in \mathcal{H}$  and  $\Psi = (\psi^0, \psi^1) \in \mathcal{H}$ .

For each  $\Phi = (\phi_0, \phi_1) \in \mathcal{H}$  we obtain

$$\begin{aligned} A(\Phi, \Phi) &= \int_{\Omega} \int_{\mathbb{G}} a^0(z, y) \cdot (\nabla_z \phi_0(z) * \tau^{\mathbb{G}}(y) + \nabla_{\tau}^{\mathbb{G}} \phi_1(z, y))^2 \\ &\quad + d^0(z, y) \cdot (\phi_0(z))^2 \, dy \, dz \\ &\geq \gamma_0 \cdot \int_{\Omega} \int_{\mathbb{G}} (\nabla_z \phi_0(z) * \tau^{\mathbb{G}}(y) + \nabla_{\tau}^{\mathbb{G}} \phi_1(z, y))^2 \, dy \, dz \\ &= a_0 \cdot \|\nabla_z \phi_0 * \tau^{\mathbb{G}} + \nabla_{\tau}^{\mathbb{G}} \phi_1\|_{L^2(\Omega \times \mathbb{G})}^2. \end{aligned}$$

In addition, we get

$$\|\nabla_z \phi_0 * \tau^{\mathbb{G}} + \nabla_{\tau}^{\mathbb{G}} \phi_1\|_{L^2(\Omega \times \mathbb{G})}^2$$



$$\begin{aligned}
 &= \int_{\Omega} \int_{\mathbb{G}} (\nabla_z \phi_0(z) * \tau^{\mathbb{G}}(y) + \nabla_{\tau}^{\mathbb{G}} \phi_1(z, y))^2 dy dz = \int_{\Omega} \int_{\mathbb{G}} [\nabla_z \phi_0(z) * \tau^{\mathbb{G}}(y)]^2 dy dz \\
 &+ \int_{\Omega} \int_{\mathbb{G}} [\nabla_{\tau}^{\mathbb{G}} \phi_1(z, y)]^2 dy dz + 2 \int_{\Omega} \int_{\mathbb{G}} \nabla_z \phi_0(z) * \tau^{\mathbb{G}}(y) \cdot \nabla_{\tau}^{\mathbb{G}} \phi_1(z, y) dy dz \\
 &= \|\phi_0\|_{\mathcal{H}_{0, \tau^{\mathbb{G}}}^1(\Omega, \mathbb{G})}^2 + \|\phi_1\|_{L^2(\Omega; \mathcal{H}_{\mathbb{G}}^1)}^2 + 2 \int_{\Omega} \int_{\mathbb{G}} \nabla_z \phi_0(z) * \tau^{\mathbb{G}}(y) \cdot \nabla_{\tau}^{\mathbb{G}} \phi_1(z, y) dy dz \\
 &= \|U\|_{\mathcal{H}}^2 + 2 \int_{\Omega} \int_{\mathbb{G}} \nabla_z \phi_0(z) * \tau^{\mathbb{G}}(y) \cdot \nabla_{\tau}^{\mathbb{G}} \phi_1(z, y) dy dz
 \end{aligned}$$

and the equation

$$\int_{\Omega} \int_{\mathbb{G}} \nabla_z \phi_0(z) * \tau^{\mathbb{G}}(y) \cdot \nabla_{\tau}^{\mathbb{G}} \phi_1(z, y) dy dz = 0,$$

which is satisfied because of the  $\mathbb{G}$ -periodicity of  $u_1$ , we obtain

$$A(\Phi, \Phi) \geq a_0 \|\Phi\|_{\mathcal{H}}^2.$$

Thus,  $A$  is  $\mathcal{H}$ -coercive. Since  $F : \mathcal{H} \rightarrow \mathbb{R}$  with

$$F(v) := \int_{\Omega} \int_{\mathbb{G}} \mathcal{F}^0(z, y) \cdot \psi^1(z, y) dy dz,$$

for any  $\Psi = (\psi^0, \psi^1) \in \mathcal{H}$ , is a continuous, linear functional on  $\mathcal{H}$ , the Lax-Milgram-theorem guarantees the existence and uniqueness of the solution

$$(\phi_0, \phi_1) \in \mathcal{H}_{0, \tau^{\mathbb{G}}}^1(\Omega, \mathbb{G}) \times L^2(\Omega; \mathcal{H}_{\mathbb{G}}^1)$$

of the two-scale homogenized problem.  $\square$

**8. Numerical Results.** In this section, we discuss an illustrative numerical example of the two-scale limit process. The microscopic problem is the variational form of a model with second-order differential equations on edge of the periodic network  $\mathcal{N}_{\varepsilon}^{\Omega}$  in the macroscopic domain  $\Omega = (0, 10)^2$ . The system of differential equations is associated with a self-adjoint (symmetric) operator. The reference graph  $\mathbb{G}$  of the two-dimensional network  $\mathcal{N}_{\varepsilon}^{\Omega}$  is shown in Figure 7.

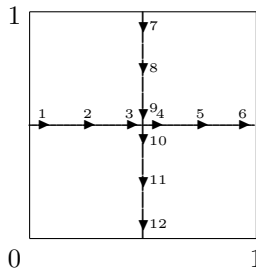


FIGURE 7. *Illustrative example:* The reference graph  $\mathbb{G}$ .

We introduce the index sets of branches

$$\mathfrak{J}_0^{\mathbb{G}} := \{1, 4, 7, 10\}, \quad \mathfrak{J}_1^{\mathbb{G}} := \{2, 5, 8, 11\}, \quad \mathfrak{J}_2^{\mathbb{G}} := \{3, 6, 9, 12\},$$

and the intervals

$$\mathfrak{J}_j^{\mathbb{G}} := \left(0, \frac{1}{8}\right) \quad \text{for } j \in \mathfrak{J}_0^{\mathbb{G}} \cup \mathfrak{J}_2^{\mathbb{G}}, \quad \mathfrak{J}_j^{\mathbb{G}} := \left(0, \frac{1}{4}\right) \quad \text{for } j \in \mathfrak{J}_1^{\mathbb{G}}.$$

The *coefficients* of the microscopic model are given by

$$a_j(\lambda_j) := \begin{cases} 0.2 & , j \in \mathfrak{J}_0^{\mathbb{G}}, \\ 1.0 & , j \in \mathfrak{J}_1^{\mathbb{G}}, \\ 0.2 & , j \in \mathfrak{J}_2^{\mathbb{G}}, \end{cases}$$

and

$$d_j(\lambda_j) := 1 \quad \text{for each } j \in \mathfrak{J}_0^{\mathbb{G}} \cup \mathfrak{J}_1^{\mathbb{G}} \cup \mathfrak{J}_2^{\mathbb{G}},$$

where  $\lambda_j \in \mathfrak{J}_j^{\mathbb{G}}$ . The *source term* is defined as

$$f_j(\lambda_j) := \begin{cases} 1 + \cos(4\pi \cdot \lambda_j) & , j \in \mathfrak{J}_0^{\mathbb{G}}, \\ 1 + \cos\left(4\pi \cdot \left(\lambda_j + \frac{1}{8}\right)\right) & , j \in \mathfrak{J}_1^{\mathbb{G}}, \\ 1 + \cos\left(4\pi \cdot \left(\lambda_j + \frac{3}{8}\right)\right) & , j \in \mathfrak{J}_2^{\mathbb{G}}. \end{cases}$$

Figure 8 shows the diffusion coefficient  $a$  and the source term  $f$ .

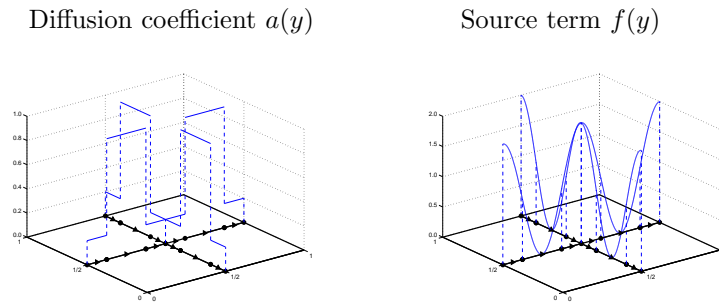


FIGURE 8. *Illustrative example*: Diffusion coefficient and source term.

The  $\varepsilon$ -periodic network  $\mathcal{N}_\varepsilon^\Omega$  consists of  $k \geq 10$  copies of the reference graph  $\mathbb{G}$  for each coordinate direction. For this reason, the length of periodicity  $\varepsilon$  satisfies the equation  $\varepsilon = \frac{10}{k}$ . The network  $\mathcal{N}_\varepsilon^\Omega$  consists of  $12 \cdot k^2$  branches.

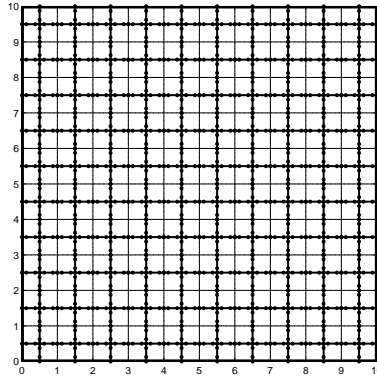


FIGURE 9. *Illustrative example:* The periodic network  $\mathcal{N}_1^\Omega$ .

The corresponding homogenized problem is given by

$$\left. \begin{aligned} -0.3 \cdot \frac{\partial^2}{\partial z_1^2} \phi_0(z) - 0.3 \cdot \frac{\partial^2}{\partial z_2^2} \phi_0(z) + 2 \cdot \phi_0(z) &= 2, \quad z \in \Omega, \\ \phi_0(z) &= 0, \quad z \in \partial\Omega. \end{aligned} \right\}$$

Figure 10 shows the behavior of the sequence of solutions of the microscopic variational model for a vanishing length of periodicity  $\varepsilon$ . The number of cells and the parameter  $\varepsilon$  are indicated above each subfigure in the form  $\#cells/\varepsilon$ .

Figure 11 illustrates the quality of the approximate homogenized model for the parameter  $\varepsilon = 0.2$  (i.e.,  $\hat{=} 2500$  cells). The numerical computations have been performed on a Pentium III-processor with 500 MHz and 4 GB random access memory. The computation time for the solution of the variational network model reached approximately 10.000 seconds. The homogenized model can be solved by standard PDE-solver in less than a second.

**9. Conclusion and Outlook.** The present study is a part of our investigations on mathematical problems in spatial data analysis related to large scale flow and transport models for capillary systems in the soil. Such problems are important for applications such as *groundwater contamination monitoring* in environmental sciences and *groundmotion prediction* ([35]). In these applications, the physical domain under considerations is large when compared to the very small length of periodicity of the capillary network. A numerical solution of the corresponding microscopic models in an acceptable amount of time is no longer possible. In many applications related to flow problems of capillary systems, the microscopic model describes the concentration values of a certain substance. The spatial distribution of this substance can be modeled by a diffusion-reaction system on the periodic network. Previous studies applied averaging approaches in order to derive an easy-to-solve approximating model on the scale of the physical domain. Two-scale averaging techniques are considered as purely formal approaches that provide a first picture of the corresponding macroscopic model. Sometimes it is also possible to provide some measure for the quality of the approximation. Nevertheless, this kind of approach does not refer to a mathematical analysis of the corresponding self-adjoint

(symmetric) operator and it does not rely on a particular notion of convergence. In order to overcome this drawback, an extension principle and a corresponding notion of convergence introduced by Lenczner et al. is applied. The microscopic model is defined in variational form of a diffusion-reaction process defined on the branches of the network. For a vanishing length of periodicity the homogenized model can

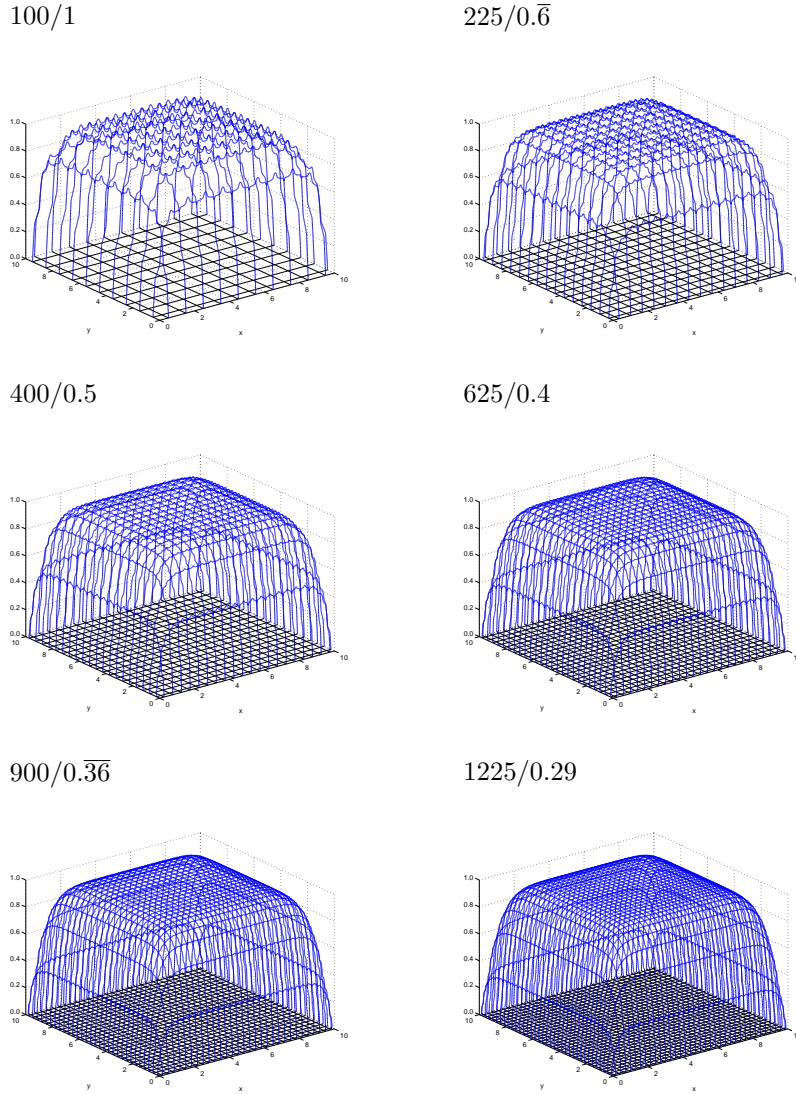


FIGURE 10. *Illustrative example:* Solutions of the microscopic model for different lengths of periodicity  $\varepsilon$ .

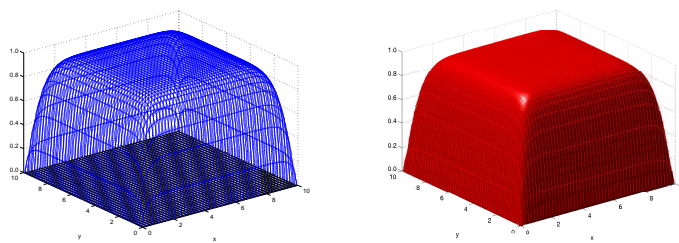


FIGURE 11. *Illustrative example:* The approximate homogenized model and the solution for  $\varepsilon = 0.2$ .

be homogenized model can be identified. Several directions of research can be pursued in future work. In a further study, we will address the problem of the homogenization of *non self-adjoint operators* on singularly perturbed networks. Currently, we are working on methods for a *topology optimization* for systems on large scale periodic networks. These approaches can be applied for example to develop new materials that - when their topology is optimized - possess optimal global properties. The homogenization of *stochastic process* on periodic networks have been completely ignored until now. We intend to apply such systems in problems related to *signal theory*. In this regard, homogenization approaches for *optimal control problems on networks* offer a promising avenue for further investigations. In previous studies conducted by Kogut and Leugering [19], the concept of *S-homogenization* has been used for a limit analysis of *optimal control problems on periodic graphs*. In further studies, we intend to show how the notion of two-scale convergence for network functions can be integrated into the framework of *S-homogenization*.

#### REFERENCES

- [1] T. Arbogast, J. Douglas Jr. and U. Hornung, Derivation of the double porosity model of single phase flow via homogenization theory, *SIAM Journal on Mathematical Analysis*, **21** (1990), 823–836.
- [2] J.-L. Auriault and J. Lewandoska, Diffusion/adsorption/advection macrotransport in soils, *European Journal of Mechanics, A/Solids*, **15**, 1996, 681–704.
- [3] J. Bear, *Dynamics of Fluids in Porous Media*, Dover Publications, New York, 1988.
- [4] G. Bouchitté and I. Fragalà, Homogenization of elastic thin structures: a measure-fattening approach, *Journal Of Convex Analysis*, **9** (2002), 1–24.
- [5] G. Bouchitté and I. Fragalà, Homogenization of thin structures by two-scale method with respect to measures, *SIAM Journal on Mathematical Analysis*, **32** (2002), 1198–1226.
- [6] E. Canon and M. Lenczner, Modelling of thin elastic plates with small piezoelectric inclusions and distributed electronic circuits. Models for inclusions that are small with respect to the thickness of the plate, *Journal of Elasticity*, **55** (1999), 111–141.
- [7] G. A. Chechkin, V. V. Jikov, D. Lukkassen and A. L. Piatnitski, On homogenization of networks and junctions, *Asymptotic Analysis*, **30** (2002), 61–80.
- [8] D. Cioranescu and J. Saint Jean Paulin, *Homogenization of Reticulated Structures*, Springer, New York, 1999.
- [9] F. Civan, *Porous Media Transport Phenomena*, John Wiley & Sons, Hoboken, New Jersey, 2011.
- [10] A. Fortin, J. M. Urquiza and R. Bois, A mesh adaptation method for 1D-boundary layer problems, *International Journal of Numerical Analysis and Modeling, Series B*, **3** (2012), 408–428.
- [11] L. Gibson and M. Ashby, *Cellular Solids. Structure and Properties*, Pergamon Press, New York, 1999.

- [12] I. G. Graham, T. Y. Hou, O. Lakkis and Robert Scheichl, *Numerical Analysis of Multiscale Problems*, Lecture Notes in Computational Science and Engineering, 83, Springer, Berlin, Heidelberg, 2012. Interdisciplinary Applied Mathematics, Springer, New York, 2003.
- [13] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer, Berlin, Heidelberg, 2001.
- [14] S. Göktepe and C. Miehe, A micro-macro approach to rubber-like materials. Part III: The micro-sphere model of anisotropic Mullins-type damage, *Journal of the Mechanics and Physics of Solids*, **53** (2005), 2259–2283.
- [15] U. Hornung, *Homogenization and Porous Media*, Interdisciplinary Applied Mathematics, Springer, Stuttgart, 1997.
- [16] J. D. Joannopoulos, R. D. Meade and J. N. Winn, *Photonic Crystals*, Princeton University Press, Princeton, NJ, 2008.
- [17] G. E. Karniadakis, A. Beskok and N. Aluru, *Microflows and Nanoflows. Fundamentals and Simulation*, Interdisciplinary Applied Mathematics, Springer, New York, 2005.
- [18] P. Kogut and G. Leugering, Homogenization of constrained optimal control problems for one-dimensional elliptic equations on periodic graphs, *ESAIM: Control, Optimisation and Calculus of Variations*, **15** (2009), 471–498.
- [19] P. Kogut and G. Leugering, Asymptotic Analysis of Optimal Control Problems on Periodic Singular Graphs, in *Optimal Control Problems for Partial Differential Equations on Reticulated Domains* (eds. P. Kogut and G. Leugering), Birkhäuser Boston, (2011), 409–440.
- [20] E. Kropat, *Über die Homogenisierung von Netzwerk-Differentialgleichungen*, Wissenschaftlicher Verlag Berlin, Berlin, 2007.
- [21] E. Kropat and S. Meyer-Nieberg, Homogenization of singularly perturbed diffusion-advection-reaction equations on periodic networks, in *Proceedings of the 15th IFAC Workshop on Control Applications of Optimization (CAO 2012)*, September 13-16, 2012, Rimini, Italy, (2012), 83–88.
- [22] E. Kropat, S. Meyer-Nieberg and G.-W. Weber, Two-scale asymptotic analysis of singularly perturbed elliptic differential equations on large periodic networks, *Dynamics of Continuous, Discrete and Impulsive Systems - Series B: Applications & Algorithms*, **22** (2015), 293–324.
- [23] E. Kropat, S. Meyer-Nieberg and G.-W. Weber, Singularly perturbed diffusion-advection-reaction processes on extremely large three-dimensional curvilinear networks with a periodic microstructure - efficient solution strategies based on homogenization theory, *Numerical Algebra, Control and Optimization*, **9** (2016), 183–219.
- [24] E. Kropat and S. Meyer-Nieberg and G.-W. Weber, A topology optimization approach for micro-architected systems on singularly perturbed periodic manifolds? Two-scale asymptotic analysis and the influence of the network topology, *Dynamics of Continuous, Discrete and Impulsive Systems - Series B: Applications & Algorithms*, **23** (2016), 155–193.
- [25] P. Kuchment, Quantum graphs I: Some basic structures, *Waves Random Media*, **14** (2004), 107–128.
- [26] P. Kuchment, Quantum graphs: An Introduction and a Brief Survey, In *Analysis on Graphs and Its Applications - Proceedings of Symposia in Pure Mathematics 77* (eds. P. Exner, J.P. Keating, P. Kuchment, T. Sunada and A. Teplyaev), American Mathematical Society, (2008), 291–312.
- [27] P. Kuchment and L. Kunyansky, Differential operators on graphs and photonic crystals, *Advances in Computational Mathematics*, **16** (2008), 263–290.
- [28] M. Lenczner, Homogénéisation d'un circuit électrique, *Comptes Rendus de l'Académie des Sciences - Series IIB - Mechanics-Physics-Chemistry-Astronomy*, **324** (1997), 537–542.
- [29] M. Lenczner, Multiscale model for atomic force microscope array mechanical behaviour, *Applied Physics Letters*, **90** (2007), 091908.
- [30] M. Lenczner and D. Mercier, Homogenization of periodic electrical networks including voltage to current amplifiers, *Multiscale Modeling and Simulation*, **2** (2004), 359–397.
- [31] M. Lenczner and G. Senouci-Bereksi, Homogenization of electrical networks including voltage-to-voltage amplifiers, *Mathematical Models and Methods in Applied Sciences*, **9** (1999), 899–932.
- [32] V. G. Mazja, S. A. Nasarow and B. A. Plamenewski, Asymptotische Theorie elliptischer Randwertaufgaben in singular gestörten Gebieten II, Wiley, Berlin, 1991.
- [33] C. Miehe and S. Göktepe, A micro-macro approach to rubber-like materials. Part II: The micro-sphere model of finite rubber viscoelasticity, *Journal of the Mechanics and Physics of Solids*, **53** (2005), 2231–2258.

- [34] C. Miehe, S. Göktepe and F. Lulei, A micro-macro approach to rubber-like materials - Part I: the non-affine micro-sphere model of rubber elasticity, *Journal of the Mechanics and Physics of Solids*, **52** (2004), 2617–2660.
- [35] E. Nuhn and E. Kropat and W. Reinhardt and S. Pckl, Preparation of Complex Landslide Simulation Results with Clustering Approaches for Decision Support and Early Warning, in *Proceedings of the 45th Annual Hawaii International Conference on System Sciences (HICSS-45)*, January 4-7, 2012, Grand Wailea, Maui, Hawaii, Ralph H. Sprague, Jr. (eds.), IEEE Computer Society, (2012), 1089–1096.
- [36] S. E. Pastukhova, Homogenization in Elasticity Problems on Combined Structures, *Journal of Mathematical Sciences*, **132** (2006), 313–330.
- [37] C. Pechstein, *Finite and Boundary Element Tearing and Interconnecting Solvers for Multiscale Problems*, Lecture Notes in Computational Science and Engineering, 90, Springer, Berlin, Heidelberg, 2013.
- [38] M. Sahimi, *Heterogeneous Materials. Nonlinear and Breakdown Properties and Atomistic Modeling*,
- [39] M. Vogelius, A homogenization result for planar, polygonal networks, *RAIRO Modélisation Mathématique et Analyse Numérique*, **25** (1991), 483–514.
- [40] V. V. Zhikov, On an extension of the method of two-scale convergence and its applications, *Sbornik: Mathematics*, **191** (2000), 973–1014.

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