Lens optics as an optical computer for group contractions

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Abstract

It is shown that the one-lens system in para-axial optics can serve as an optical computer for contraction of Wigner's little groups and an analogue computer which transforms analytically computations on a spherical surface to those on a hyperbolic surface. It is shown possible to construct a set of Lorentz transformations which leads to a two-by-two matrix whose expression is the same as those in the para-axial lens optics. It is shown that the lens focal condition corresponds to the contraction of the $O(3)$-like little group for a massive particle to the $E(2)$-like little group for a massless particle, and also to the contraction of the $O(2,1)$-like little group for a space-like particle to the same $E(2)$-like little group. The lens-focusing transformations presented in this paper allow us to continue analytically the spherical $O(3)$ world to the hyperbolic $O(2,1)$ world, and vice versa.

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I. INTRODUCTION

The six-parameter Lorentz group was initially introduced to physics as a group of Lorentz transformations applicable to the four-dimensional Minkowskian space. However, the Lorentz group can serve as the basic mathematical language for many branches of physics. It serves as the backbone for the theory of coherent and squeezed states of light [1,2]. Recently we are realizing that the Lorentz group can serve as the standard language for classical ray optics, including polarization optics [3], interferometers [4], layer optics [5,6], lens optics [7], and cavity optics [8].

Since the Lorentz group provides the underlying scientific language to classical optics, it is not unreasonable to examine whether we can construct optical devices which will perform computations in the Lorentz group. This group has many subgroups, and we are particularly interested in Wigner's little groups which dictate the internal space-time symmetries of relativistic particles [9]. While these groups play the fundamental role in particle physics, they had a stormy history in connection with their role in explaining the space-time symmetry of massless particles [10].

Wigner's little group is defined to be the maximal subgroup of the Lorentz group whose transformations leave the four-momentum of a given particle invariant. The little groups for massive, massless, and space-like momentum are like $O(3)$, $E(2)$, and $O(2,1)$ respectively [11]. The $O(3)$ group is the three-dimensional rotation group and can provide computations on the numbers distributed on a spherical surface. The $O(2,1)$ group provides transformations in the Minkowskian space of two space-like and one time-like dimensions. Thus, this group deals with the numbers on a hyperbolic surface. The $E(2)$ group stands for Euclidean transformations on a flat surface. It consists of two translational degrees of freedom as well as the rotation around the origin.

The transitions from $O(3)$ to $E(2)$, and from $O(2,1)$ to $E(2)$ are called the group contractions in the literature [12,13], and they are known to be singular transformations which forbid analytic continuation from $O(3)$ to $E(2)$. After the $O(3)$ or $O(2,1)$ is contracted to $E(2)$, it is not possible to recover either of the two groups from $E(2)$. In addition, the little groups are not exactly the $O(3)$, $E(2)$, and $O(2,1)$ groups whose geometry is quite transparent to us. They are only "like" [11]. The question is then whether the contraction of $O(3)$ to $E(2)$ necessarily mean the contraction of the $O(3)$-like little group to $E(2)$-like little group. This conceptual question also has been discussed extensively in the literature [10,13,14]. In this paper, we only use the results which can be represented in the two-by-two matrix representation of the Lorentz group.

The one-lens system consists of one lens matrix and two translation matrices [15]. The combined matrix can be written in terms of the two-by-two matrices corresponding to four-by-four Lorentz-transformation matrices which constitute the transformations of the little groups [7]. However, unlike the case of the little groups, the parameters of the two-by-two matrices are analytic, especially in the neighborhood of the focal condition in which the upper-right element vanishes. On the other hand, from the little group point of view, this is precisely where the group contraction occurs, and this transformation is singular as was mentioned above. Then how can we establish the correspondence between singular and non-singular representations?

Indeed, if we can represent those three little groups using one convex lens, the result would be quite interesting, especially in view of the fact that computations in a hyperbolic world can be performed in a spherical world, and vice versa. In order to achieve this goal, we have to develop a mathematical device which establishes a bridge between lens optics and the little groups, starting from Wigner's original idea of finding the maximal subgroups of the Lorentz group which will leave the four-momentum of a given particle invariant.

With this point in mind, we present a different set of Lorentz transformations achieving the same purpose [18]. We then show that the representation of the little groups in this set coincides with the matrix representation of the one-lens system. It is then seen that the focal condition corresponds to the transition from one little group to another. The transition is analytic. In this way, we achieve an analytic transformation of computations on a hyperbolic surface to a spherical surface.

In this paper, we are employing many sophisticated mathematical items such as the Lie algebra, compact groups, non-compact groups, solvable groups, as well as group contractions. However, we are very fortunate to be able to avoid these words and get directly into the computational world using only familiar two-by-two matrices without complex numbers. Our mathematics starts from the well-known two-by-two matrix formulation of the one-lens system.

In Sec. II, we start with one lens matrix and two translation matrices, and derive a core matrix to be studied in detail. In Sec. III, we introduce Wigner's little groups and their traditional two-by-two representations, and point out that they are not suitable for describing the core matrix in the one-lens system because the transition from one little group to another is a singular transformation. In Sec. IV, for the little groups, we introduce a different set of Lorentz transformations which can serve as a bridge between the symmetries of relativistic particles and the one-lens system. It is noted that the transition from one little group to another can be achieved analytically. It is noted that the group contraction is not always a singular transformation.

Then in Sec. V, we formulate the one-lens system in terms of the little groups and the analytic group contraction. In Sec VI, it is pointed out that the cavity optics is a special case of the one-lens system, but that this simplified system contains all the essential features of group contractions. In Sec. VII, it is shown how the abstract idea of group contractions leads to a tool of concrete numerical calculations which can be carried out by optical devices.

II. FORMULATION OF THE PROBLEM

The simplest lens system is of course the one-lens system with the lens matrix

$$\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}, \tag{1}$$

where f is the focal length. We assume that the focal length is positive throughout the paper. The translation matrix takes the form

$$\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}. \tag{2}$$

If the object and image are d_1 and d_2 from the lens respectively, the optical system is described by

$$\begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

The multiplication of these matrices leads to

$$\begin{pmatrix} 1 - d_2/f & d_1 + d_2 - d_1 d_2/f \\ -1/f & 1 - d_1/f \end{pmatrix}. \tag{4}$$

The image becomes focused when the upper right element of this matrix vanishes with

$$\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{f}. \tag{5}$$

The problem with this expression is that the off-diagonal elements are not dimensionless, but it can be decomposed into

$$\begin{pmatrix} (d_1 d_2)^{1/4} & 0 \\ 0 & (d_1 d_2)^{-1/4} \end{pmatrix} \begin{pmatrix} 1 - x_2 & 2\cosh\rho - x \\ -x & 1 - x_1 \end{pmatrix} \begin{pmatrix} (d_1 d_2)^{-1/4} & 0 \\ 0 & (d_1 d_2)^{1/4} \end{pmatrix}, \tag{6}$$

with

$$x_1 = \frac{d_1}{f}, \qquad x_2 = \frac{d_2}{f}, \qquad x = \frac{\sqrt{d_1 d_2}}{f},$$

$$\cosh\rho = \frac{1}{2}\left(\sqrt{d_1/d_2} + \sqrt{d_2/d_1}\right). \tag{7}$$

The matrix in the middle, the core matrix, can now be written as

$$\begin{pmatrix} 1 - x_2 & 2\cosh\rho - x \\ -x & 1 - x_1 \end{pmatrix}. \tag{8}$$

In the camera configuration, both the image and object distances are larger than the focal length, and both $(1 - x_1)$ and $(1 - x_2)$ are negative. Thus we start with the negative of the above matrix

$$\begin{pmatrix} x_2 - 1 & x - 2\cosh\rho \\ x & x_1 - 1 \end{pmatrix}. \tag{9}$$

We can further renormalize this matrix to make the two diagonal elements equal. For this purpose, we can write it as

$$\begin{pmatrix} b & 0 \\ 0 & 1/b \end{pmatrix} \begin{pmatrix} z - 1 & x - 2\cosh\rho \\ x & z - 1 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 1/b \end{pmatrix}, \tag{10}$$

with

$$b = \left(\frac{x_2 - 1}{x_1 - 1}\right)^{1/4}.$$

Then the core matrix becomes

$$\begin{pmatrix} z - 1 & x - 2\cosh\rho \\ x & z - 1 \end{pmatrix}, \tag{11}$$

with

$$z = 1 + \sqrt{(x_1 - 1)(x_2 - 1)}. \tag{12}$$

In terms of the ρ and x variables, z can be written as

$$z = 1 + \sqrt{x^2 - 2x \cosh \rho + 1}. \tag{13}$$

We shall use the core matrix of Eq.(11) as the starting point in this paper. If x is smaller than $2(\cosh \rho)$, the core matrix can be written as

$$\begin{pmatrix} \cos(\phi/2) & -e^{-\eta/2} \sin(\phi/2) \\ e^{\eta/2} \sin(\phi/2) & \cos(\phi/2) \end{pmatrix}, \tag{14}$$

where the range of the angle variable ϕ is between 0 and π, and η is positive.

This form of the core matrix serves a very useful purpose in laser optics which consists of chains of the one-lens system [15,16,17]. Its connection with the Lorentz group and Wigner rotations has been studied recently by the present authors [8]. Indeed, this is the starting point of this paper, where we intend to establish a connection between the one-lens system and a set of Lorentz transformations.

If $x = 2 \cosh \rho$, the above expression becomes

$$\begin{pmatrix} 1 & 0 \\ 2 \cosh \rho & 1 \end{pmatrix}, \tag{15}$$

and the focal condition of Eq.(5) is satisfied. If x is is greater than $2(\cosh \rho)$, all the elements in the core matrix of Eq.(11) become positive. Thus, it is appropriate to write it as

$$\begin{pmatrix} \cosh(\chi/2) & e^{-\eta/2} \sinh(\chi/2) \\ e^{\eta/2} \sinh(\chi/2) & \cosh(\chi/2) \end{pmatrix}. \tag{16}$$

As we shall see in Sec. III, the expressions given in Eq.(14), Eq.(15), and Eq.(16) take the same mathematical forms as those of the representations of the $O(3)$, $E(2)$, and $O(2,1)$-like little groups. The transition from one to another form is a singular transformation. On the other hand, the core matrix of Eq.(11) is analytic in the x and ρ variables when both x_1 and x_2 are greater than 1. We are thus led to look for another set of Lorentz transformations with analytic parameters. This will enable us to write those transformation parameters in terms of the lens parameters of Eq.(11). In so doing, we can establish a correspondence between lens optics and the transformations of the little groups, and we can achieve transformations from one little group to another by adjusting focal conditions.

III. LITTLE GROUPS

In his 1939 paper on the Lorentz group [9], Wigner considered the maximum subgroup of the Lorentz group whose transformations leave the four-momentum of a given free particle invariant. This subgroup is called Wigner's little group. Wigner observed that there are three classes of the little group. In the Minkowskian space of the space-time coordinate (t, z, x, y), the four-vector

$$m(1,0,0,0) \tag{17}$$

corresponds to the four-momentum of a massive particle at rest. To this four-vector, we can apply three-dimensional rotation matrix, like the rotation matrix around the y axis:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{18}$$

without changing the four-momentum of Eq.(17). In optics, it is more convenient to use the two-by-two representation this matrix [4]. The rotation matrix then becomes

$$\begin{pmatrix} \cos(\phi/2) & -\sin(\phi/2) \\ \sin(\phi/2) & \cos(\phi/2) \end{pmatrix}. \tag{19}$$

In order to study the little group for a particle moving along the z direction, we can start with a particle with four-momentum [19]

$$m(\cosh\eta, -\sinh\eta, 0, 0). \tag{20}$$

This particle moves in the negative z direction with the speed of $c(\tanh\eta)$. To this four-vector, if we apply the boost matrix

$$\begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{21}$$

the four-vector returns to the form given in Eq.(17). Here again, it is more convenient to use the two-by-two representation of this boost matrix which takes the form [19]

$$\begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}. \tag{22}$$

Thus, in order to construct a representation of the little group for the four-momentum of Eq.(20), we boost it to that of Eq.(17) using the boost matrix of Eq.(21) or (22), perform the rotation of Eq.(18) or (19) which does not change the momentum, and then boost the momentum back to the original form of Eq.(20). In the two-by-two representation, this chain of matrices take the form

$$\begin{pmatrix} e^{-\eta/2} & 0 \\ 0 & e^{\eta/2} \end{pmatrix} \begin{pmatrix} \cos(\phi/2) & -\sin(\phi/2) \\ \sin(\phi/2) & \cos(\phi/2) \end{pmatrix} \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}. \tag{23}$$

After the multiplication, the result becomes

$$\begin{pmatrix} \cos(\phi/2) & -e^{-\eta}\sin(\phi/2) \\ e^{\eta}\sin(\phi/2) & \cos(\phi/2) \end{pmatrix}. \tag{24}$$

The mathematical form of this matrix is identical to that of Eq.(14).

This is the reason why the little groups can play a role in lens optics and vice versa. The group represented in this way is called the $O(3)$-like little group for a massive particle [19]. As was mentioned in Sec. II, this expression is also one of the starting formulas in laser optics [8,15,16,17].

What happens if the four-momentum is light-like? The four-momentum in this case is

$$\omega(1, 1, 0, 0). \tag{25}$$

The light-like particles cannot be brought to its rest frame, and thus cannot be brought to the form of Eq.(17). It is clear that this four-vector is invariant under rotations around the z axis. In addition, Wigner observed in his original paper that there are two additional transformations which leave this light-like four-momentum invariant. These matrices are extensively discussed in the literature, and the result is that they correspond to the form

$$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \tag{26}$$

where u is a complex parameter with two real independent parameters. Since, we will be dealing with real matrices in this paper, u represents only one real number. It is interesting to note that this form is identical to that of Eq.(15). This aspect of the little group also has been discussed in the literature [19].

The theory of the little group includes also the form

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \tag{27}$$

but it does not play a role in this paper. This form may be useful if we consider the case when the lower left element of the core matrix of Eq(11) vanishes. The group represented either in the form of Eq.(26) or Eq.(27) is called the $E(2)$-like little group for massless particles.

The little-group matrix of Eq.(26) is invariant under the Lorentz boost along the z direction, as can be seen from

$$\begin{pmatrix} e^{-\eta/2} & 0 \\ 0 & e^{\eta/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}. \tag{28}$$

There are no particles in nature with space-like four-momentum, whose four-vector may be written as [20]

$$m(0, 1, 0, 0), \tag{29}$$

but it occupies an important position in group theory [9]. It will become more important as it finds its place in optical sciences. This four-vector is also invariant under rotations around the z axis. In addition, it remains invariant under boosts along the x and y directions. The boost matrix along the x direction takes the form

$$\begin{pmatrix} \cosh(\chi/2) & \sinh(\chi/2) \\ \sinh(\chi/2) & \cosh(\chi/2) \end{pmatrix}. \tag{30}$$

If we apply the same Lorentz boosts as we did in two previous little groups, the little group matrix should become

$$\begin{pmatrix} \cosh(\chi/2) & e^{-\eta}\sinh(\chi/2) \\ e^{\eta}\sinh(\chi/2) & \cosh(\chi/2) \end{pmatrix}. \tag{31}$$

This form is identical to Eq.(16).

We have thus far shown that the transformation matrices of the little groups and the one-lens system take the same form. However, there is one crucial problem. In the case of the core matrix of Eq(9), the sign change of the upper-right element can be done analytically, but this is not true for the little group representation. The transition to Eq.(26) either from Eq.(24) and from Eq.(31) is possible and is known as the group contraction in the literature. However, in both cases, the two independent parameters collapse into one independent parameter. Thus, the inverse transformation is not possible. This keeps us from continuing analytically from Eq.(24) to Eq.(30). What should we do?

IV. CONTRACTIONS OF THE LITTLE GROUPS

In order to circumvent the singularity problem mentioned in the preceding section, we are interested in finding a set of Lorentz transformations which will remain analytic as we go through the transition point where the upper-right element vanishes. Let us restate the problem.

If x is smaller than $2\cosh\rho$, the upper-right element of the core matrix of Eq.(11) is negative while the remaining three are positive, and it can be written in the form of Eq.(14). If it is greater than $2\cosh\rho$, all the elements are positive, and the core matrix should be written as Eq.(16). There is a value zero between these two values, which corresponds to the focal condition. This is precisely the point where the expressions Eq.(14) and Eq.(16) become singular. The purpose of this section is to establish the connection between the little groups and the one-lens system without this singularity. In the computer language, this singularity means a memory loss.

We are thus interested in a different set of Lorentz transformations for the little groups. We note here again that the little group consists of transformations which leave the four momentum of a given particle invariant [9]. In order to find the set of transformations which will bring back the four-momentum of Eq.(20) to itself [18], let us first rotate the four-momentum by θ, using the rotation matrix

$$\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \tag{32}$$

to Eq.(20). Then the four-momentum becomes

$$m(\cosh\eta, (\sinh\eta)\cos\theta, -(\sinh\eta)\cos\theta, 0). \tag{33}$$

This four-momentum can be boosted along the x direction, which then becomes

$$m(\cosh\eta, (\sinh\eta)\cos\theta, (\sinh\eta)\cos\theta, 0), \tag{34}$$

with the boost matrix

$$\begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix}. \tag{35}$$

We can return to the four-momentum of Eq.(20), by applying again the rotation matrix of Eq.(32). The net effect is

$$\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \tag{36}$$

which becomes

$$\begin{pmatrix} \cosh\lambda\cos\theta & -\cosh\lambda\sin\theta + \sinh\lambda \\ \cosh\lambda\sin\theta + \sinh\lambda & \cosh\lambda\cos\theta \end{pmatrix}. \tag{37}$$

Indeed, these two different ways of returning to the same four-momentum should give the same effect. Thus, the effect of Eq.(24) and and that of Eq.(37) are the same, and

$$\begin{pmatrix} \cos(\phi/2) & -e^{-\eta}\sin(\phi/2) \\ e^{\eta}\sin(\phi/2) & \cos(\phi/2) \end{pmatrix} = \begin{pmatrix} \cosh\lambda\cos\theta & -\cosh\lambda\sin\theta + \sinh\lambda \\ \cosh\lambda\sin\theta + \sinh\lambda & \cosh\lambda\cos\theta \end{pmatrix}, \tag{38}$$

with

$$\cos(\phi/2) = \cosh\lambda\cos\theta,$$
$$e^{-2\eta} = \frac{\cosh\lambda\sin\theta - \sinh\lambda}{\cosh\lambda\sin\theta + \sinh\lambda}. \tag{39}$$

Conversely, λ and θ can be written in terms of ϕ and η as

$$\cosh\lambda = (\cosh\eta)\sqrt{1 - \cos^2(\phi/2)\tanh^2\eta},$$
$$\cos\theta = \frac{\cos(\phi/2)}{(\cosh\eta)\sqrt{1 - \cos^2(\phi/2)\tanh^2\eta}}. \tag{40}$$

This leads to

$$\cosh\lambda = \frac{\cosh\eta}{\sqrt{1 + (\sinh^2\eta)\cos^2\theta}}, \tag{41}$$

which means that the boost parameter λ is determined from the rotation angle θ for a given value of the boost parameter η.

The above relations are valid only when $(\cosh\lambda\sin\theta)$ is greater than $\sinh\lambda$. Otherwise, instead of Eq.(23), we have to start from

$$\begin{pmatrix} e^{-\eta/2} & 0 \\ 0 & e^{\eta/2} \end{pmatrix} \begin{pmatrix} \cosh(\chi/2) & \sinh(\chi/2) \\ \sinh(\chi/2) & \cosh(\chi/2) \end{pmatrix} \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}, \tag{42}$$

which leads to

$$\begin{pmatrix} \cosh(\chi/2) & e^{-\eta}\sinh(\chi/2) \\ e^{\eta}\sinh(\chi/2) & \cosh(\chi/2) \end{pmatrix}. \tag{43}$$

This form is identical to Eq.(16), and should also be equal to Eq.(37). We write this as

$$
\begin{pmatrix} \cosh(\chi/2) & e^{-\eta}\sinh(\chi/2) \\ e^{\eta}\sinh(\chi/2) & \cosh(\chi/2) \end{pmatrix} = \begin{pmatrix} \cosh\lambda\cos\theta & -\cosh\lambda\sin\theta + \sinh\lambda \\ \cosh\lambda\sin\theta + \sinh\lambda & \cosh\lambda\cos\theta \end{pmatrix}, \tag{44}
$$

which leads to the identities

$$
\cosh(\chi/2) = \cosh\lambda\cos\theta,
$$

$$
e^{-2\eta} = -\left(\frac{\cosh\lambda\sin\theta - \sinh\lambda}{\cosh\lambda\sin\theta + \sinh\lambda}\right). \tag{45}
$$

Conversely,

$$
\cosh\lambda = (\cosh\eta)\sqrt{\cosh^2(\chi/2) - \tanh^2\eta}\,,
$$

$$
\cos\theta = \frac{\cosh(\chi/2)}{(\cosh\eta)\sqrt{\cosh^2(\chi/2) - \tanh^2\eta}}. \tag{46}
$$

In this case, the boost parameter λ takes the form

$$
\cosh\lambda = \frac{\sinh\eta}{\sqrt{\cosh^2\eta\cos^2\theta - 1}}. \tag{47}
$$

Here, the boost parameter λ is determined by the little group parameter θ for a given value of η.

While the quantity

$$
\frac{\cosh\lambda\sin\theta - \sinh\lambda}{\cosh\lambda\sin\theta + \sinh\lambda} \tag{48}
$$

changes the sign from (*plus*) to (*minus*), it has to go through zero. With the parameters λ and θ, this process is quite analytic. On the other hand, the exponential factor $\exp(-2\eta)$ is always positive. Thus, changing $\exp(-2\eta)$ to $-\exp(-2\eta)$ cannot be achieved analytically. This is necessarily a singular transformation. However, this exponential factor becomes vanishingly small when η becomes very large. Perhaps we are allowed to change the sign when it is vanishingly small, but this is still a non-analytic continuation. Furthermore, let us look at the expressions given in Eq.(24) and Eq.(43). This sign change is accompanied by the transition of a rotation matrix of the form of Eq.(19) to a boost matrix of the form given in Eq.(30).

Indeed, by changing the parameters from ϕ and η to θ and λ, we can analytically navigate through the vanishing value of the upper-right element of matrices of Eq.(38). The process of approaching this zero value either from the positive or negative side is called the group contraction in the literature. In this paper, however, we are eventually interested in how these parameters operate in lens optics. We shall come back to this issue in Sec. V.

V. LENS OPTICS AND GROUP CONTRACTIONS

In Sec. II, we started with a camera-like one-lens system, and derived

$$\begin{pmatrix} z-1 & x-2\cosh\rho \\ x & z-1 \end{pmatrix} = \begin{pmatrix} \cos(\phi/2) & -e^{-\eta}\sin(\phi/2) \\ e^{\eta}\sin(\phi/2) & \cos(\phi/2) \end{pmatrix}, \tag{49}$$

for x smaller than $2(\cosh\rho)$, and x is positive. Here all the parameters are determined from d_1, d_2, and f of the lens optics. If we gradually increase the value of x, the upper-right element becomes zero and then positive. The right-hand side of the above expression cannot accommodate this transition.

The right-hand side is a familiar expression both in optics and the Lorentz group. In Sec. III, we started with Wigner's little groups, and noted that the expressions given in Eq.(14) and Eq.(24) are identical to each other. The parameters in Eq.(24) are the Lorentz-transformation parameters for Wigner's $O(3)$-like little group for massive particles.

In order to circumvent the above-mentioned singularity problem, we have chosen a different set of Lorentz-transformation parameters, and the result was the expression given in Eq.(38). In terms of these parameters, the core matrix can be written as

$$\begin{pmatrix} z-1 & x-2\cosh\rho \\ x & z-1 \end{pmatrix} = \begin{pmatrix} \cosh\lambda\cos\theta & -\cosh\lambda\sin\theta + \sinh\lambda \\ \cosh\lambda\sin\theta + \sinh\lambda & \cosh\lambda\cos\theta \end{pmatrix}. \tag{50}$$

Here, both sides have the upper-right their upper-right elements which are analytic as they go through zero.

The parameters are now related by

$$x - 2\cosh\rho = \sinh\lambda - \cosh\lambda\sin\theta,$$
$$x = \sinh\lambda + \cosh\lambda\sin\theta, \tag{51}$$

and therefore to

$$\sinh\lambda = x - \cosh\rho,$$
$$\sin\theta = \frac{\cosh\rho}{\sqrt{1 + (x-\cosh\rho)^2}}. \tag{52}$$

We are thus able to write the Lorentz-transformation parameters λ and θ in terms of the parameters of the one-lens system.

Thus, by adjusting the lens parameters, we can now perform transformations in Wigner's little groups. It is interesting to note that we perform group contractions whenever we try to focus the object before taking a camera photo. Unlike the traditional procedures, the contraction presented this paper is an analytic transformation, which provides a reversible process from Eq.(14) to Eq.(16) through Eq.(15). What significance does this have? We shall return to this question in Sec. VII.

VI. CAVITY OPTICS

In our previous paper [8], we studied light beams in laser cavities. One cavity cycle there consists of two lenses with the same image and object distances. We are thus led to consider the one-lens system with $d_1 = d_2 = d$, and thus

$$x_1 = x_2 = x. \tag{53}$$

The core matrix of Eq.(11) becomes

$$\begin{aligned} x - 2 &= \sinh\lambda - \cosh\lambda \sin\theta, \\ x &= \sinh\lambda + \cosh\lambda \sin\theta. \end{aligned} \tag{54}$$

Therefore, $\cosh\lambda \sin\theta = 1$, or

$$\sin\theta = \frac{1}{\cosh\lambda}, \tag{55}$$

which is satisfied by the physical values of θ and λ. Furthermore, this relation reduces Eq.(50) to

$$\begin{pmatrix} x-1 & x-2 \\ x & x-1 \end{pmatrix} = \begin{pmatrix} \sinh\lambda & -1+\sinh\lambda \\ 1+\sinh\lambda & \sinh\lambda \end{pmatrix}. \tag{56}$$

From this expression, we can compute both λ and θ in terms of the x variable, as they can be written as

$$\sinh\lambda = x - 1, \qquad \sin\theta = \frac{1}{\sqrt{1 + (1+x)^2}}. \tag{57}$$

Indeed, this is an oversimplified example, but it is interesting to note that it contains all the ingredients of the group contractions discussed in this paper.

VII. LORENTZ GROUP AND OPTICAL COMPUTING

Each individual is equipped with a natural computer. He/she has ten fingers. With them, we can do additions and subtractions of numbers smaller than ten. This is how our decimal system was developed. Then Chinese came up with the abacus which is an extension of the ten-finger computer. About 150 years ago, French artillery men invented the slide rule which converts multiplication into addition. In the 1940s, von Neumann observed that vacuum tubes can perform the yes-or-no logic, and started building electronic computers.

In building computers, it is not enough to develop computer mathematics. In the final stage, we have to adjust those mathematical tools to the language spoken by devices. As we noted in Sec. I, the Lorentz group is the standard language for classical and quantum optics. The Lorentz group is also the natural language for light beams and for the materials through which the beams propagate. Thus, if we intend to build optical computers, we have to translate all mathematical algorithms into the language of the Lorentz group. In fact, it has been shown that some optical systems have a slide-rule-like property [3].

In this paper, we noted first that a camera-like single-lens system can perform the algebra of Wigner's little groups and their contractions. While discussing group contractions, we observed the difference between the rotation matrix

$$\begin{pmatrix} \cos(\phi/2) & -\sin(\phi/2) \\ \sin(\phi/2) & \cos(\phi/2) \end{pmatrix} \tag{58}$$

of Eq.(19), and the boost matrix

$$\begin{pmatrix} \cosh(\lambda/2) & \sinh(\lambda/2) \\ \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix} \tag{59}$$

of Eq.(30). These matrices operate in two different spaces, namely the rotation matrix on a circle and the boost matrix on a hyperbola. Since we now have a procedure which makes an analytic continuation from one to the other, we can perform computations in the hyperbolic world and carry it to the circular world.

Indeed, the circle versus hyperbola is a very old problem known as the conic sections. It is a geometrical as well as a topological problem, but these issues are beyond the scope of this paper. It is interesting to see that the single-lens system can tell us a story about this fundamental problem.

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