POLYOMINO CONVOLUTIONS AND TILING PROBLEMS

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ABSTRACT. We define a convolution operation on the set of polyominoes and use it to obtain a criterion for a given polyomino not to tile the plane (rotations and translations allowed). We apply the criterion to several families of polyominoes, and show that the criterion detects some cases that are not detectable by generalized coloring arguments.

1. INTRODUCTION:

Tiling properties of polyominoes have been studied by many authors using various methods (see for instance [1], [2], [5], [7]). We will assume that a polyomino f is a map from $\mathbf{Z} \times \mathbf{Z}$ to $\mathbf{Z}$ that takes the values 0 and 1. We associate a value 1 of f at (n, m) with the unit square $[n, n+1] \times [m, m+1]$. This allows us to envision polyominoes in the usual way as tiles in the plane. We assume f has finitely many occupied squares. (Several authors including S. Golomb refer to these objects as quasi-polyominoes ([4], pg. 85), and reserve the term polyomino for rookwise connected figures. A figure is rookwise connected if it can be constructed by placing squares in a way that each square except for the first shares an edge with a previously placed square. We will not assume rookwise connectedness or topological connectedness unless declared.)

Definition 1. *Let f be any map $\mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ (not necessarily a polyomino map). Suppose $f(n, m)$ is nonzero for finitely many pairs (n, m). Let $|f|_1 = \sum_{n,m} |f(n, m)|$, and let $|f|_\infty$ be the number of pairs (n, m) such that $f(n, m)$ is not zero. Both of these are norms in the usual sense. The assumption implies that both $|f|_1$ and $|f|_\infty$ are finite, and we say that f has finite area. If f is a polyomino, the two norms are equal, and we use the notation $|f|$.*

Definition 2. *For any map f as above, of finite area, let $diam(f)$ denote the maximum of the distances between pairs of points of support of f under the taxicab metric. The taxicab distance between two points*

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is the minimum number of grid steps from one point to the other (A grid step is a move from (x, y) to (u, v) where $|x - u| + |y - v| = 1$).

Definition 3. *Suppose f and g are two maps $\mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ such that at least one of them has finite area. Define $h = f \star g$ as:*

$$h(n, m) = \sum_{k,l} f(k, l) g(n - k, m - l) \tag{1}$$

$h(n, m)$ counts the number of intersections of f with an (n, m) translate of the reflection of g across the origin. It is clear that the sum in the definition is finite for a fixed (n, m), and h has finite area if both f and g do. We call h the convolution of f and g. It is easy to see that $\text{diam}(h) \leq \text{diam}(f) + \text{diam}(g)$.

If f and g are polyominoes, then their convolution $h = f \star g$ is not a polyomino in general since h may assume values other than 0 or 1. But one may obtain a polyomino from h by reducing each $h(n, m)$ to 0 or 1 depending on its congruence class modulo 2. In this way, we obtain a convolution operation on the set of polyominoes. This operation inherits the associativity and bilinearity properties of the usual convolution, since reduction modulo 2 commutes with the operations of addition and multiplication that constitute the convolution. Denote the composition of convolution and the reduction modulo 2 by $f \star_2 g$. Similarly, denote the composition of convolution and reduction modulo n for an arbitrary modulus n by $f \star_n g$.

Here is our main observation:

Theorem 1. *Suppose that f is a polyomino symmetric under rotations of 90 degrees. Suppose g is a polyomino. Then if $|f \star_n g|_1 < |\bar{f}||g|$, or if $|f \star_n g|_\infty < sgn(|\bar{f}|)|g|$, where $|\bar{f}|$ denotes the unique integer among $0, 1, ..., n - 1$ congruent to $|f|$ modulo n, then copies of g cannot tile the plane (i.e. cover it without overlaps) , translations and rotations being allowed.(Here we are thinking of g's as tiles)*

Proof: Say that copies of g tile the plane. This is another way to say that the full plane is a sum of non overlapping translates of copies of g and its rotations. Since f is rotationally symmetric, f convolved with g has the same norms as f convolved with a rotation of g. Consider a minimal pattern of g's in this tiling that contains a full N by N square. Call this figure G for reference. G is certainly contained in a $N + 2\text{diam}(g)$ by $N + 2\text{diam}(g)$ square, otherwise it wouldn't be minimal. Since G is obtained as a disjoint sum of g's, its norm is simply the sum of the norms of its constituents. We are going to estimate norms of $f \star_n G$ from two directions.

First of all, $f \star G$ has at least $(N - 2\text{diam}(f))^2$ points of value $|f|$, since at least that many translates of f fall completely into the N by N square. Reducing modulo n, we obtain $|f \star_n G|_1 \geq (N - 2\text{diam}(f))^2 |\bar{f}|$ and $|f \star_n G|_\infty \geq (N - 2\text{diam}(f))^2 \text{sgn}(|\bar{f}|)$. On the other hand G is made up of at most $\frac{(N + 2\text{diam}(g))^2}{|g|}$ copies of g. If we assume to the contrary that the inequalities in the hypothesis may hold, by the triangle inequality we obtain that

$$(2) \quad (N + 2\text{diam}(g))^2 \frac{|\bar{f}||g| - 1}{|g|} \geq |f \star_n G|_1 \geq (N - 2\text{diam}(f))^2 |\bar{f}|$$

or

(3)

$$(N + 2\text{diam}(g))^2 \frac{\text{sgn}(|\bar{f}|)|g| - 1}{|g|} \geq |f \star_n G|_\infty \geq (N - 2\text{diam}(f))^2 \text{sgn}(|\bar{f}|)$$

Both inequalities fail to hold asymptotically for large values of N, since the coefficients of N^2 on the left hand sides of the equations are strictly less than those on the right hand sides. This contradiction finishes the proof. $\square$

We remark that the theorem remains valid if we replace g by a finite collection of prototiles $g_1, ..., g_k$ such that the inequalities hold for each of them separately.

2. Some Applications

We would like to show some applications of the criterion. Our first example is a certain sequence of disconnected polyominoes.

We define a sequence D_n of polyominoes as follows: D_n is obtained by aligning n dominoes horizontally along their longer sides, leaving a spacing of one square between any two consecutive dominoes (see figure 1). For instance, an accordingly positioned D_n would occupy the squares $(0,0), (1,0), (3,0), (4,0), (6,0), (7,0), ..., (3n-3,0), (3n-2,0)$. D_1 is a domino itself. Therefore it tiles the plane in many ways. The question for $n \geq 2$ has the following answer:

Proposition 1. D_n *tiles the plane iff $n \leq 3$, translations and rotations allowed.*

Proof: Examples of tilings for D_2 and D_3 are shown in figures 3 and 4 respectively. Both tilings are doubly periodic, thus only one fundamental region is shown in either case. We must remark that a tiling pattern for D_2 or D_3 needs to obey severe restrictions, and our guess is that the D_3 tiling is essentially unique.

Next we prove the impossibility part of the assertion. Suppose that $S_{3\times3}$ represents the 3 by 3 square polyomino. It is not hard to check that $|S_{3\times3}\star_2 D_n| = 6$ for any value of n (see figure 2). This happens since all but 6 translates of $S_{3\times3}$ meet D_n in an even number of squares. The 6 are those where $S_{3\times3}$ meets the first or last square of D_n. Therefore, by Theorem 1 , D_n cannot tile the plane if $|D_n| = 2n > 6$, i.e. if $n > 3$. $\square$

There are many ways that one can seek generalizations of this example. A similar argument works for the negative part of the corresponding assertion on higher dimensional analogues. We show another generalization since it uses the other norm $|f|_\infty$:

Proposition 2. *Let $D_{n,a,b}$ represent the polyomino obtained by aligning n horizontal bars of length a, leaving a spacing of b blank squares between any two consecutive bars (Therefore, the D_n above are $D_{n,2,1}$ with this notation). Then, if b^2 is not divisible by a, $D_{n,a,b}$ cannot tile the plane if $n > \frac{2(a+b)(a-1)}{a}$.*

Proof: Let $S_{(a+b)\times(a+b)}$ represent the square polyomino of side length $(a+b)$. Then $|S_{(a+b)\times(a+b)}\star_a D_{n,a,b}|_\infty = 2(a-1)(a+b)$, and $|D_{n,a,b}| = na$. Thus the inequality follows from the theorem unless $(a+b)^2$ is 0 modulo a. This is equivalent to $a|b^2$. $\square$

Next, we consider some rookwise connected polyominoes. All such polyominoes of area 6 or less tile the plane ([6]), so we have to consider larger polyominoes. The first polyomino in figure 5 is a 9-omino that clearly doesn't tile. This is provable by our criterion as well, as demonstrated in the same figure. Figure 8 shows another non-tiler , and this is also easy to prove directly.

We will call a polyomino L a "log" if it is an a by b rectangle with $a > 1$ and $b > 1$. Let L' be the $a + 2$ by $b + 2$ rectangle containing L in the middle. We define a "log with spikes" (or a "spiky log") to be a polyomino obtainable from such an L by adjoining a number of 1 by 1 squares directly to L (each sharing an edge with a square of L) so that there are at least two blank squares between any two of them (Around corners, count along the squares of $L' - L$). We call the 1 by 1 squares "spikes".

Proposition 3. *No log with more than four spikes (moreover, no finite collection of such prototiles) can tile the plane.*

Proof: Convolve the polyomino with the X pentomino modulo 2. One may verify that convolving an a by b rectangle with the X pentomino gives a polyomino of norm $ab + 8$, and each spike placed on the log reduces the norm of the result by 1, while increasing the norm of the

initial object by 1. Therefore, if 5 or more spikes are placed, the convolution is norm decreasing. (See the example in figure 7)□

We call a polyomino a "snake" if no subset of its squares is a T tetromino or a square tetromino. We say that the snake makes n "U-turns" if it has n distinct subsets forming U-pentominoes.

Proposition 4. *No snake making 3 or more U-turns (moreover, no finite collection of such prototiles) can tile the plane.*

Proof: Convolve the polyomino with the X pentomino modulo 3 and look at the $| \ |_1$ norm. Except for the two squares at the ends, each square of the snake has three neighbors (counting the square itself), so these do not contribute. The squares at the ends may contribute 4 in total at most. Any square not on the snake makes a contribution only if it shares an edge with the snake. If the snake has n squares, the maximum possible total contribution of this type is the number of edges, $2n + 2$. Every U-turn costs 3, therefore if there are more than 2 U-turns, the norm of the convolution is less than $2n$, and the criterion gives the result. □

Golomb defines a "reptile" to be a polyomino that tiles a larger copy of itself [4]. All reptiles tile the plane. Therefore snakes making 3 U-turns (actually even 2 U-turns) cannot be reptiles!

## 3. A Comparison to Coloring Arguments

There are several other sufficient criteria to prove the impossibility of tiling a given figure by another. Of these, perhaps the best known are coloring and generalized coloring arguments. One may ask where our criterion stands. We show that there exist tiling problems such that the impossibility is detected by our criterion whereas no generalized coloring argument can do so. We follow the method in [2]. If a generalized coloring argument proves impossibility of tiling R with f, then it also proves impossibility of a "signed tiling" of R with f. A signed tiling permits using the map $-f$ as well as f, and of course, overlaps allowed.

Theorem 2. *(i) It is not possible to tile a torus by D_4's.*
(ii) There exist signed tilings of a 24 by 12 torus by D_4's.

Proof: (i) is clear. If D_4 tiled a torus, it would tile the plane in a doubly periodic way. But this was shown not to happen.

(ii) Notice that f superposed with $-f$ shifted 3 squares to the right gives a map g such that $g(0,0) = g(1,0) = 1, g(12,0) = g(13,0) = -1$, and 0 otherwise. Horizontally, stack 6 g's, with a shift of two squares between consecutive g's. We obtain a new map h such that $h(k,0) = 1$

FIGURE 1. D_n
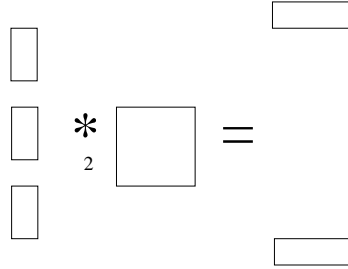
FIGURE 2. $D_3 \star s_{3\times3}$
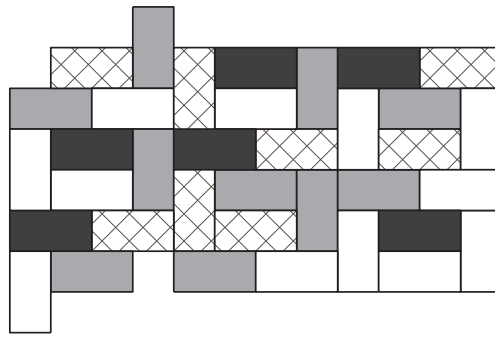
FIGURE 3. Tiling a 12×6 torus by D_2's

for $k = 0, ..., 11$, $h(k, 0) = -1$ for $k = 12, ..., 23$, and 0 otherwise. Rotate h 90 degrees clockwise.

Next, stack 12 D_4's vertically. We get a figure of 4 rectangles of dimensions 12 by 2, longer sides vertical, with one horizontal separation between neighboring rectangles. Using copies of h, we can shift the 2nd and 4th rectangles up by 12 spaces while leaving the other two untouched. A horizontal stack of three copies of this final figure gives a torus tiling. □

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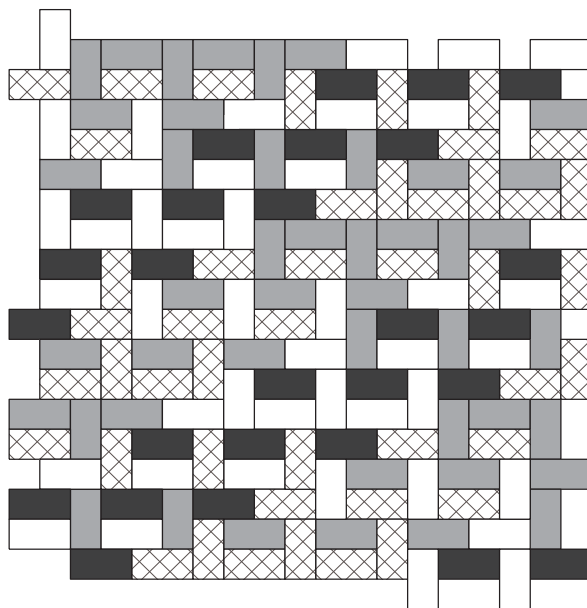
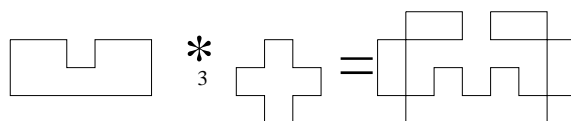FIGURE 4. Tiling an 18×18 torus by D_3's

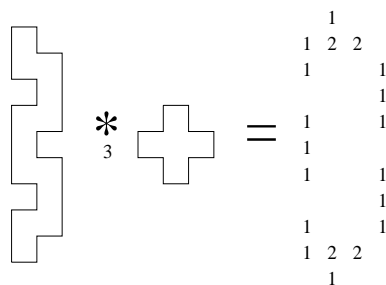FIGURE 5. A 9-omino which doesn't tile

FIGURE 6. A snake making 3 U-turns can't tile

REFERENCES

[1] N.G. de Bruijn, Filling boxes with bricks, *Amer. Math. Monthly* **76** (1969), 37—40

[2] J.H. Conway and J.C. Lagarias, Tiling with polyominoes and combinatorial group theory, *J. of Combin. Theory Ser. A* **53** (1990), 183—208
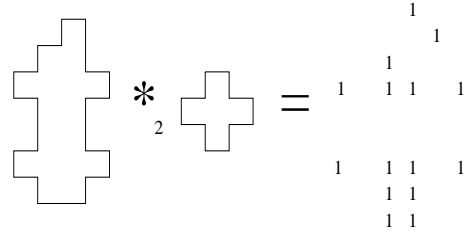
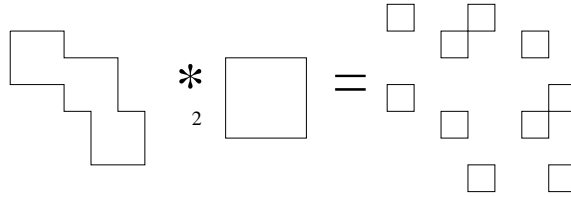FIGURE 7. A spiky log with 5 spikes can't tile

FIGURE 8. A 12-omino which doesn't tile

[3] S.W. Golomb, Tiling rectangles with polyominoes, *Math. Intelligencer* **18** no. 2 (1996), 38—47

[4] S.W. Golomb, "Polyominoes", Princeton University Press, 1994

[5] S.W. Golomb, Tiling with polyominoes, *J. of Combin. Theory* **1** (1966), 280—296

[6] K. Keating and A. Vince, Isohedral polyomino tiling of the plane, *Discrete and Computational Geometry* **21** (1999), 615—630

[7] D.W. Walkup, Covering a rectangle with T-tetrominoes, *Amer. Math. Monthly* **72** no. 9 (1965) 986—988