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Iterative methods for tomography problems: implementation to a cross-well tomography problem

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Abstract. The velocity distribution between two boreholes is reconstructed by cross-well tomography, which is commonly used in geology. In this paper, iterative methods, Kaczmarz's algorithm, algebraic reconstruction technique (ART), and simultaneous iterative reconstruction technique (SIRT), are implemented to a specific cross-well tomography problem. Convergence to the solution of these methods and their CPU time for the cross-well tomography problem are compared. Furthermore, these three methods for this problem are compared for different tolerance values.

1. Introduction

Cross-well seismic tomography, which is used often in geology, mainly deals with reconstructing the velocity structure between two boreholes by measuring travel time for ray paths between them [1]. A cross-well tomography problem can be defined as an inverse problem because we have a set of measurements (travel time), then we try to reconstruct the model (velocity structure) from them [4] in the cross-well tomography problem. Enhanced oil recovery, imaging of geological structures in sediments and crystalline rock are some vital examples in which cross-well tomography is used [1].

Hydrologists should have information regarding the locations of hydraulically conductive fractures if they want to clean up contaminants which are in fractured bedrock as stated in [3]. Herewith, cross-well seismic tomography can be used to obtain the information. There are some methods [3] such as single-borehole hydraulic tests, cross-borehole hydraulic tests to get information about locations of these fractures. However, they have a weakness concerning the hydraulically conductive fractures which are far away from boreholes. So, cross-well seismic tomography is used to overcome this problem. As described in [3], P-waves are used to investigate the bedrock which is located between two wells. After P-waves are used in the bedrock, obtained data are processed by using tomographic methods. The result of this process is known as tomogram [3]; that is, the structure of P-wave velocity between two wells. Then, since P-wave velocity is reduced by fractures, the locations of fractures can be obtained from the low velocity anomalies in the tomogram. Thus, these fractures can be hydraulically conductive ones [3].

In this paper, we apply iterative methods, which are Kaczmarz's method, algebraic reconstruction technique (ART), and simultaneous iterative reconstruction technique (SIRT), to a specific cross-well tomography problem, which was taken from [2].

The problem [2] is explained as follows: there are two vertical wells and the distance between them is 1600 m. A seismic source is placed in one of wells at depths of 50, 150, . . . , 1550 m.

A set of receivers is placed in the other well at same depths. Then, a travel time with standard deviation of 0.5 ms is recorded for every source-receiver pair. Also, ray paths are assumed to be straight. Hence, this paper explains how we obtained the velocity distribution in the two-dimensional plane between the two wells by using three iterative methods for this problem.

This paper compares CPU time and convergence to the solution of Kaczmarz's method, ART, and SIRT for different tolerance values on the cross-well tomography problem.

2. Methods

In cross-well seismic tomography, the travel time for seismic energy [2], which travels through ray path ℓ, can be defined by

$$t = \int_{\ell} s(\mathbf{x}) d\ell, \tag{2.1}$$

if slowness at a point $\mathbf{x}$ and ℓ are known. Slowness is known as inverse of the velocity [9]. Since the model (slowness) and data (travel time) are continuous functions' variables, and we want to determine the model from the data in our cross-well tomography problem, this problem is defined as a continuous inverse problem. We can discretize Equation (2.1) and obtain a discrete inverse problem by using midpoint rule in the cross-well tomography problem. In addition, the reader can analyze [7, 8] to obtain detailed information regarding seismic tomography and slowness.

When we discretize Equation (2.1) into a 16 × 16 grid of 100 meter by 100 meter uniform blocks in the cross-well tomography problem, 256 model parameters are obtained. Therefore, approximated form of Equation (2.1) [2] can be written as

$$t = \int_{\ell} s(\mathbf{x}) d\ell \approx \sum_{\text{blocks}} s_{\text{block}} \cdot \Delta l_{\text{block}}, \tag{2.2}$$

where Δl_{block} represents lengths of ray paths within corresponding blocks.

Let $\mathbf{m}$ be a model vector with n components which are slowness, $\mathbf{d}$ be a data with m components which are measured travel time and $\mathbf{G}$ be a matrix whose components are lengths of ray paths within corresponding block in our cross-well tomography problem. Then, we can write Equation (2.2) as a linear system of equations [2] by

$$\mathbf{G}\mathbf{m} = \mathbf{d}. \tag{2.3}$$

As stated in [2], if we have a problem which has a large $\mathbf{G}$ matrix, methods such as singular value decomposition (SVD), Tikhonov regularization are not useful for this problem. The reason of this impracticability is that computers need too much memory to store such kind of $\mathbf{G}$ matrix. However, if $\mathbf{G}$ is a sparse matrix; that is, many of its components are zero, the memory problem can be solved by getting rid of zero components of the matrix $\mathbf{G}$. In this way, the nonzero components of the matrix and their locations can be kept [2]. Unlike factorization methods, iterative methods consist of products of vectors with $\mathbf{G}$ and $\mathbf{G}^T$ [5]. In the cross-well tomography problem, which we deal with, the data $\mathbf{d}$ and $\mathbf{G}$ matrix are given.

In this paper, three iterative methods, Kaczmarz, ART, and SIRT, are examined and they are applied to the cross-well tomography problem.

2.1. Kaczmarz's method

Kaczmarz's algorithm is one of the most commonly used methods for solving Equation (2.3) [2]. An essential property of this method is that its implementation is easy. Kaczmarz's method is based on "method of projections" [6]. Tomographic problems are frequently discretized as uniform blocks as we discretized in the cross-well tomography problem. Before explaining algorithm of Kaczmarz, let us write expanded form of Equation (2.3) [6]:

$$\begin{aligned} G_{11}m_1 + G_{12}m_2 + G_{13}m_3 + \ldots + G_{1n}m_n &= d_1 \\ G_{21}m_1 + G_{22}m_2 + G_{23}m_3 + \ldots + G_{2n}m_n &= d_1 \\ &\vdots \\ G_{m1}m_1 + G_{m2}m_2 + G_{m3}m_3 + \ldots + G_{mn}m_n &= d_m. \end{aligned} \tag{2.4}$$

According to [6], a model (or image), which has n degrees of freedom, can be obtained by a grid with n cells. Then, the model, $(m_1, m_2, \ldots, m_n)$ can be defined as a single point in n-dimensional space. So, each of the equations in Equation (2.4) is called a hyperplane in this n-dimensional space. Unique solution of Equation (2.4), if it exists, can be obtained by a single point which is the intersection of all hyperplanes [6]. In Kaczmarz's algorithm, each of m equations $\mathbf{G}_i \cdot \mathbf{m} = d_i$, as we described before, represents an n-dimensional hyperplane in $\mathbb{R}^m$ [2]. Algorithm of Kaczmarz for a system of equations $\mathbf{Gm} = \mathbf{d}$ for a tomography problem [2] is described in Algorithm 1.

Algorithm 1 Kaczmarz's Algorithm

```
1: $\mathbf{m}^{(0)} \leftarrow \mathbf{0}$
2: for $i \leftarrow 0, m-1$ do
       $\mathbf{m}^{(i+1)} \leftarrow \mathbf{m}^{(i)} - \frac{\mathbf{G}_{i+1,\cdot}\mathbf{m}^{(i)} - d_{i+1}}{\mathbf{G}_{i+1,\cdot}\mathbf{G}^T_{i+1,\cdot}}$
3: end for
4: If there is no convergence to the solution, go back to the second step.
```

The algorithm starts with an initial guess $\mathbf{m}^{(0)}$, then, $\mathbf{m}^{(1)}$ is obtained by projecting $\mathbf{m}^{(0)}$ onto the first equation of Equation (2.4), which is determined by the first row in $\mathbf{G}$ [2]. Similar operations are applied for the next solutions until the solution is projected onto all m hyperplanes, which are defined in Equation (2.4) [2].

2.2. Algebraic reconstruction technique

Algebraic reconstruction technique, also known as ART, is an updated form of Kaczmarz's method [2]. In ART, components in row $i+1$ of $\mathbf{G}$, which are not zero, are replaced with ones [2]. The purpose of this replacement is that we can easily determine the total number of cells traversed by ray path. The process of finding cell in the ray path is simplified by this replacement. Approximated travel time [2] through the ray path $i+1$ is defined as

$$q_{i+1} = \sum_{\text{cell } j \text{ in ray path } i+1} \mathbf{m}_j l, \tag{2.5}$$

where l is called cell dimension. Then, $q_{i+1} - d_{i+1}$ is the residual of the estimated travel time of ray $i+1$ [2]. Algorithm of ART [2] for a system of equations $\mathbf{Gm} = \mathbf{d}$ for a tomography problem is explained in Algorithm 2.

Algorithm 2 ART Algorithm

```
1: $\mathbf{m}^{(0)} \leftarrow \mathbf{0}$
2: for $i \leftarrow 0, m$ do
       $N_i :=$ # of cells traversed by ray path $i$,
       $L_i :=$ the length of ray path $i$.
3: end for
4: for $i \leftarrow 0, m-1, \quad j \leftarrow 1, n$ do
       $m_j^{(i+1)} \leftarrow \begin{cases} m_j^{(i)} + \frac{d_{i+1}}{L_{i+1}} - \frac{q_{i+1}}{lN_{i+1}}, & \text{if cell } j \text{ in ray path } i+1, \\ m_j^{(i)}, & \text{otherwise.} \end{cases}$
5: end for
6: If convergence of the solution is not obtained yet, let $\mathbf{m}^{(0)} \leftarrow \mathbf{m}^{(m)}$ and turn back to
   fourth step. Otherwise, the solution $\mathbf{m} \leftarrow \mathbf{m}^{(m)}$ is returned.
```

2.3. Simultaneous iterative reconstruction technique

Simultaneous iterative reconstruction technique, also known as SIRT, is a variation of ART. SIRT gives better tomographic images than ART, but its convergence is slower than ART [6]. Algorithm of SIRT [2] for a system of equations $\mathbf{Gm} = \mathbf{d}$ is given in Algorithm 3.

Algorithm 3 SIRT Algorithm

```
1: $\mathbf{m}^{(0)} \leftarrow \mathbf{0}$
2: for $j \leftarrow 0, n$ do
       $K_j :=$ # of paths which pass through cell $j$.
3: end for
4: for $i \leftarrow 0, m$ do
       $N_i :=$ # of cells traversed by ray path $i$,
       $L_i :=$ the length of ray path $i$.
5: end for
6: $\Delta\mathbf{m}^{(i+1)} \leftarrow \mathbf{0}$
7: for $i \leftarrow 0, m-1, \quad j \leftarrow 1, n$ do
       $\Delta m_j^{(i+1)} \leftarrow \begin{cases} \Delta m_j^{(i+1)} + \frac{d_{i+1}}{L_{i+1}} - \frac{q_{i+1}}{lN_{i+1}}, & \text{if cell } j \text{ in ray path } i+1, \\ 0, & \text{otherwise.} \end{cases}$
8: end for
9: for $j \leftarrow 1, n$ do
       $m_j^{(i+1)} \leftarrow m_j^{(i+1)} + \frac{\Delta m_j^{(i+1)}}{K_j}$
10: end for
11: If there is no convergence of the solution, go back to line 6.
    Otherwise, the current solution is returned.
```

In simultaneous iterative reconstruction technique, firstly, models are calculated if cell j is in ray path $i + 1$ [2]. Then, average of these models are computed before the model is updated [10].

3. Results and discussion

This paper focuses on the cross-well tomography problem, which is described in Section 1. In this problem, components of $\mathbf{G}$ are lengths of ray paths within corresponding blocks. In addition, noisy data, **d**, represents measured travel time, and the model, **m**, represents slowness in grid in which row-by-row indexing is used. Thus, the velocity structure can be obtained from this model. Our aim is to find velocity distribution in 2-D plane between two wells by using Kaczmarz's method, ART, and SIRT in this paper. We used MATLAB functions for ART and SIRT, which were produced by [2], by modifying and arranging them for the implementation to the problem.

Our problem is called ill-conditioned and rank deficient since rank($\mathbf{G}$)= 243 is smaller than $n = 256$ (number of model parameters) and cond($\mathbf{G}$)= 1.87^{18} where rank($\mathbf{G}$) is rank of $\mathbf{G}$ matrix and cond($\mathbf{G}$) is condition number of $\mathbf{G}$ matrix.

When we implemented Kaczmarz's algorithm to the cross-well tomography problem, we obtained a model vector

$$\mathbf{m} = [0.0003430, 0.0003424, 0.003419, \ldots, 0.0003404, 0.0003406, 0.0003409]^T. \qquad (3.1)$$

In Equation (3.1), there are 256 components which denote slowness in the grid of 100 meter by 100 meter block. For example, first component of **m** in Equation (3.1) is the block which is at first row and first column. Similarly, second component of the same model is the block which is at first row and second column and so on. Therefore, first 16 components (first row) are grids at depth of 50 m, second 16 components (second row) are grids at depth of 150m and so forth. As a result, we obtained the figure of slowness (s/m) vs. depth (m), as shown in Figure 1, for the cross-well tomography problem by using Kaczmarz's algorithm.

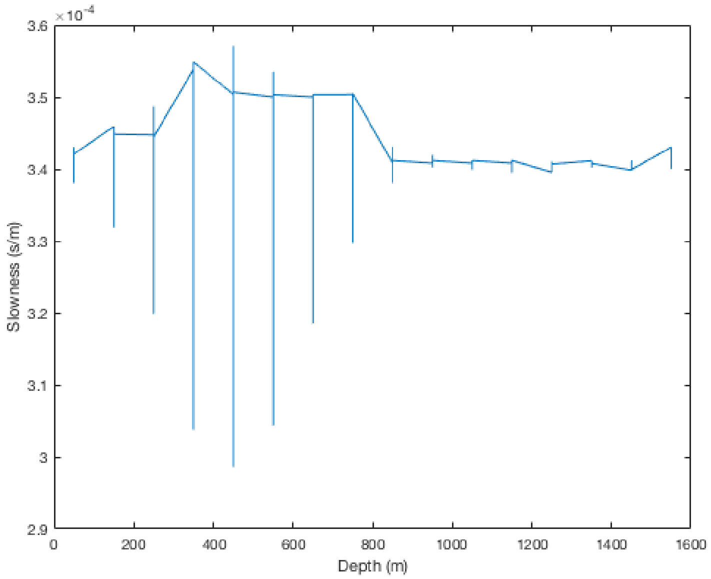

Figure 1: Kaczmarz's method solution for the cross-well tomography problem.

When we applied ART to the cross-well tomography problem, we obtained a slowness model vector

$$\mathbf{m} = [0.0003713, 0.0003519, 0.003444, \ldots, 0.0003148, 0.0003119, 0.0003075]^T, \qquad (3.2)$$

which has 256 components as in Equation (3.1). Figure 2, which we obtained, represents slowness (s/m) vs. depth (m) for the cross-well tomography problem by using ART.

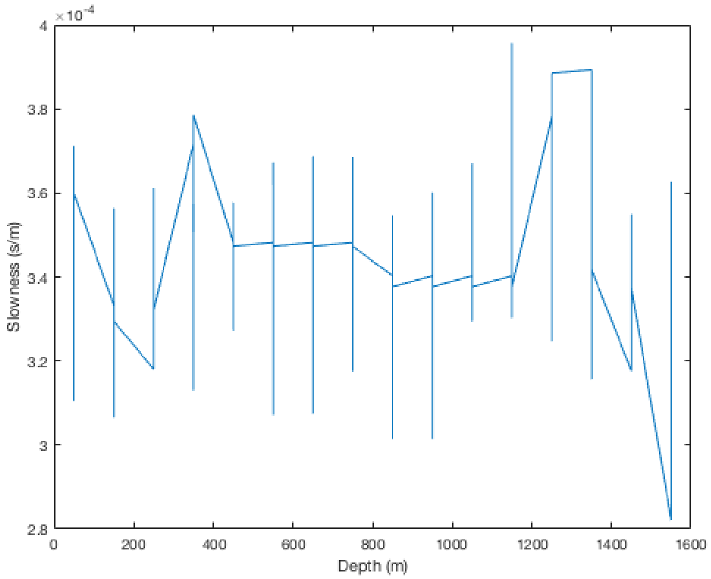

Figure 2: ART solution for the cross-well tomography problem.

Lastly, when we implemented SIRT to the cross-well tomography problem, we obtained a model vector

$$\mathbf{m} = [0.0003747, 0.0003562, 0.003464, \ldots, 0.0003177, 0.0003152, 0.0003097]^T, \qquad (3.3)$$

whose 256 components are slowness at corresponding depths. Then, Figure 3, which we produced, indicates slowness (s/m) vs. depth (m) for the cross-well tomography problem by using SIRT.

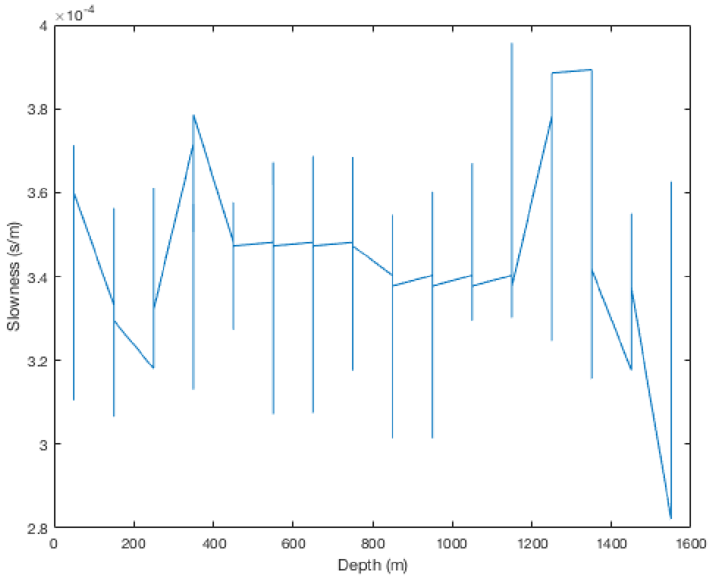

Figure 3: SIRT solution for the cross-well tomography problem.

When we compared the results of the implementation of these three methods to the cross-well problem, we observed that Kaczmarz's algorithm has the fastest convergence to the solution among the methods. When we chose the value of tolerance as 10^{-9}, the solution was converged after 7964, 21597, 61034 iterations by using Kaczmarz's method, ART, and SIRT respectively.

Moreover, CPU time of the implementation of Kaczmarz's algorithm, ART, and SIRT for the cross-well tomography problem was measured as 14.17 seconds, 32.65 seconds, 96.98 seconds respectively.

We also found that although ART is slower than SIRT until a specific tolerance value (5×10^{-9}), for values greater than that specific value, SIRT is slower than ART for this problem. When we chose tolerance value as 5×10^{-9}, the solution was converged after 10271 iterations with 16.83 seconds and 9432 iterations with 15.38 seconds by using SIRT and ART respectively. When we chose tolerance value as 10^{-8}, which is greater than 5×10^{-9}, the solution was converged after 3563 iterations with 5.94 seconds and 5180 iterations with 8.49 seconds by using SIRT and ART respectively.

Table 1: Convergence to the solution of the iterative methods, when the value of tolerance is 10^{-9}.

	Kaczmarz's algorithm	ART	SIRT
Number of Iterations	7964	21597	61034
CPU Time (seconds)	14.17	32.65	96.98

Table 2: Convergence to the solution of the iterative methods, when the value of tolerance is 5×10^{-9}.

	Kaczmarz's algorithm	ART	SIRT
Number of Iterations	2655	9432	10271
CPU Time (seconds)	4.92	15.38	16.83

Table 3: Convergence to the solution of the iterative methods, when the value of tolerance is 10^{-8}.

	Kaczmarz's algorithm	ART	SIRT
Number of Iterations	1715	5180	3563
CPU Time (seconds)	3.21	8.49	5.94

According to [2], Kaczmarz's algorithm converges quickly, if the hyperplanes are almost orthogonal. If more than two hyperplanes are nearly parallel to each other, then, convergence of the solution could have been immensely slow. Consequently, we can say that hyperplanes in our cross-well tomography problem are almost orthogonal. Moreover, we observed that the solutions of ART and SIRT are noisier than Kaczmarz's method as we expected.

4. Conclusion and outlook

Our major purpose, in the present paper, is to implement Kaczmarz's algorithm, ART, and SIRT to the cross-well tomography problem and compare their convergence to the solution, and

CPU time. As stated in [2], there are some methods, which are based on matrix factorizations, to solve linear system of equations. QR factorization and singular value decomposition (SVD) can be given as examples for these methods. However, they are not enough to solve problems which have sparse matrices [2]. Matrices which are created by using these factorizations are denser than the **G** matrix, which is factorized [2]. Herewith, usage of iterative methods such as Kaczmarz's method, ART, and SIRT for large problems are more appropriate than usage of factorization methods.

The cross-well tomography problem, which was analyzed in this paper, was discretized into a 16×16 grid of 100 meter by 100 meter uniform blocks. This discretization provided 256 model parameters, which are slowness. Then, the velocity distribution in two-dimensional plane between two wells was obtained as the model. Figures of Slowness vs. Depth, which we produced, represent slowness distribution according to the corresponding depths in the cross-well tomography problem. The results we had show that Kaczmarz's method is the method which converges fastest to the solution for the cross-well tomography problem when it is compared to ART and SIRT. Moreover, we found that ART has slower convergence than SIRT when we chose the value of tolerance greater than 5×10^{-9} in the problem.

In the present paper, ray paths were assumed to be straight in the problem, which we examined. Then, we used iterative methods for the linear inverse problem to get velocity structure in the two-dimensional plane between the two wells. If there is ray-path bending because of changing seismic velocity in a cross-well tomography problem, the problem becomes a nonlinear inverse problem [2]. Hence, the implementation of methods for nonlinear inverse problems such as Tikhonov regularization and Occams inversion [2] to the cross-well tomography problem will be future work. Furthermore, in synthetic aperture radar (SAR) imaging [11], a plane or satellite with an antenna moves along a flight track, then, electromagnetic radiation waves are emitted by the antenna. These electromagnetic waves scatter off the land, and the antenna, which emitted the waves, identifies the scattered waves [11]. Herewith, the terrain's image is generated by the received signals. Therefore, SAR imaging can be used in cross-well tomography problems, and this radar imaging application will be our future work.

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