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Equivariant Reduction of Gauge Theories over Fuzzy Extra Dimensions

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Abstract. In $SU(N)$ Yang-Mills theories on a manifold M, which are suitably coupled to a set of scalars, fuzzy spheres may be generated as extra dimensions by spontaneous symmetry breaking. This process results in gauge theories over the product space of the manifold M and the fuzzy spheres with smaller gauge groups. Here we present the $SU(2)-$ and $SU(2) \times SU(2)$-equivariant parametrization of $U(2)$ and $U(4)$ gauge fields on S_F^2 and $S_F^2 \times S_F^2$ respectively and outline the dimensional reduction of these theories over the fuzzy extra dimensions. The emerging dimensionally reduced theories are Higgs type models. Some vortex type solutions of these theories are briefly discussed.

1. Introduction

Recently, there has been significant advances in understanding the structure of gauge theories possessing fuzzy extra dimensions [1, 2] (for a review on fuzzy spaces see [3]). It is known that in certain $SU(N)$ Yang-Mills theories on a manifold M, which are suitably coupled to a set of scalars, fuzzy spheres may be generated as extra dimensions by spontaneous symmetry breaking. The vacuum expectation values (VEVs) of the scalar fields form the fuzzy sphere(s), while the fluctuations around the vacuum are interpreted as gauge fields over S_F^2 or $S_F^2 \times S_F^2$. The resulting theories can therefore be viewed as gauge theories over $M \times S_F^2$ and $M \times S_F^2 \times S_F^2$ with smaller gauge groups; which is further corroborated by the expansion of a tower of Kaluza-Klein modes of the gauge fields. Gauge theory on $M^4 \times S_F^2 \times S_F^2$ has recently been investigated in [4]. Inclusion of the fermions into these theories was considered in [4, 5]. For a review on these and related results [6] can be consulted.

It appears worthwhile to investigate the equivariant parametrization of gauge fields and perform dimensional reduction over the fuzzy extra dimensions to shed some further light into the structure of these theories. Essentially, It is possible to use the well known coset space dimensional reduction (CSDR) techniques to achive this task. To briefly recall the latter consider a Yang-Mills theory with a gauge group S over the product space $\mathcal{M} \times G/H$. G has a natural action on its coset, and requiring the Yang-Mills gauge fields to be invariant under the G action up to S gauge transformations leads to a G-equivariant parametrization of the gauge fields and subsequently to the dimensional reduction of the theory after integrating over the coset space G/H [7, 8].

Starting with the article [9], these ideas have been put under investigation. The most general $SU(2)$-equivariant $U(2)$ gauge field over $\mathcal{M} \times S_F^2$ have been found, and it was utilized to perform the dimensional reduction overS_F^2. It was shown that for $\mathcal{M} = \mathbb{R}^2$ the emergent theory is an Abelian Higgs type model which has non-BPS vortex solutions depending on the parameters in the model, corresponding to instantons in the original theory. This has been followed up by investigating the situation in which $\mathcal{M}$ is also a noncommutative space [10]. Performing SU(2)-equivariant dimensional reduction of this theory leads to a noncommutative $U(1)$ theory which couples adjointly to a set of scalar fields. On the Groenewald-Moyal plane $\mathcal{M} = \mathbb{R}_\theta^2$ the emergent models admit noncommutative, non-BPS vortex as well as fluxon solutions.

In this paper, we first give a brief account of these developments and then continue with outlining some of the new research results from an article in preparation [11]. Starting again from an $SU(\mathcal{N})$ gauge theory on M with a set of six scalars with the internal symmetry group $SO(3) \times SO(3)$ and identifying the VEV's of the scalars with $S_F^2 \times S_F^2$, fluctuations around this vacuum become the gauge fields on $S_F^2 \times S_F^2$ [4]. When M is identified to be the standard Minkowski space, this theory actually possess the same field content as the bosonic part of the $N = 4$ SUSY Yang-Mills and corresponds to a particular deformation of it with a potential for scalars breaking the $SO(6)$ R-symmetry down to $SO(3) \times SO(3)$ and the $N = 4$ supersymmetry. In [11] the ideas of our recent work is being applied to the $SU(2) \times SU(2)$-equivariant parametrization of $U(4)$ gauge fields on $S_F^2 \times S_F^2$. The reduction over the latter leads to $U(1)^3$ Higgs type models on $\mathbb{R}^2$ with more sophisticated vortex type solutions.

2. Yang-Mills Theory on $\mathcal{M} \times S_F^2$

Our departure point is a $U(\mathcal{N})$ Yang-Mills theory over a suitable space $\mathcal{M}$, which may be commutative or noncommutative, with action give as

$$S = \int_{\mathcal{M}} \mathrm{Tr}_{\mathcal{N}}\Big(\frac{1}{4g^2} F_{\mu\nu}^\dagger F_{\mu\nu} + (D_\mu\phi_a)^\dagger (D_\mu\phi_a)\Big) + \frac{1}{\tilde{g}^2}\mathrm{Tr}_{\mathcal{N}}\big(F_{ab}^\dagger F_{ab}\big) + a^2 \mathrm{Tr}_{\mathcal{N}}\big((\phi_a\phi_a + \tilde{b})^2\big)\,. \quad (1)$$

Here, $\phi_a\,(a = 1, 2, 3)$ are anti-Hermitian scalars, transforming in the adjoint of SU($\mathcal{N}$) and in the vector representation of an additional global $SO(3)$ symmetry, $D_\mu\phi_a = \partial_\mu\phi_a + [A_\mu\,, \phi_a]$ are the covariant derivatives and A_μ are the $u(\mathcal{N})$ valued anti-Hermitian gauge fields associated to the curvature $F_{\mu\nu}$. F_{ab} is given as

$$F_{ab} := [\phi_a\,, \phi_b] - \varepsilon_{abc}\phi_c\,, \quad (2)$$

In above a, $\tilde{b}$, g and $\tilde{g}$ are constants and $\mathrm{Tr}_{\mathcal{N}} = \mathcal{N}^{-1}\mathrm{Tr}$ denotes a normalized trace.

This theory spontaneously develops extra dimensions in the form of fuzzy spheres [2]. The potential terms for the scalars are positive definite, and the solutions

$$F_{ab} = 0\,, \quad -\phi_a\phi_a = \tilde{b} \quad (3)$$

are evidently a global minima. Most general solution to this equation is not known. However depending on the values taken by the parameter $\tilde{b}$, a large class of solutions has been found in [2]. Here we restrict ourselves to the simplest situation and refer the reader to [2] for a general discussion and its physical consequences.

Taking the value of $\tilde{b}$ as the quadratic Casimir of an irreducible representation of SU(2) labeled by ℓ, $\tilde{b} = \ell(\ell+1)$ with $2\ell \in \mathbb{Z}$ and assuming further that the dimension $\mathcal{N}$ of the matrices ϕ_a is $(2\ell+1)n$, (3) is solved by the configurations of the form

$$\phi_a = X_a^{(2\ell+1)} \otimes \mathbf{1}_n\,, \quad (4)$$

where $X_a^{(2\ell+1)}$ are the (anti-Hermitian) generators of SU(2) in the irreducible representation ℓ, which has dimension $2\ell+1$. We observe that this vacuum configuration spontaneously breaks the U($\mathcal{N}$) down to U(n) which is the commutant of ϕ_a in (4).

Fluctuations about the vacuum (4) may be written as

$$\phi_a = X_a + A_a\,, \tag{5}$$

where $A_a \in u(2\ell+1)\otimes u(n)$ and we have used the short-hand notation $X_a^{(2\ell+1)} \otimes \mathbf{1}_n =: X_a$. Then A_a $(a = 1,2,3)$ may be interpreted as three components of a U(n) gauge field on the fuzzy sphere S_F^2. The latter is simply the algebra of $(2\ell+1)\times(2\ell+1)$ matrices $\mathrm{Mat}(2\ell+1)$, generated by the Hermitian "coordinate functions" $\hat{x}_a := \frac{i}{\sqrt{\ell(\ell+1)}} X_a^{(2\ell+1)}$ and their products. $\hat{x}_a$ fulfill the commutation relations

$$[\hat{x}_a\,, \hat{x}_b] = \frac{i}{\sqrt{\ell(\ell+1)}}\varepsilon_{abc}\hat{x}_c\,, \quad \hat{x}_a\hat{x}_a = 1\,, \tag{6}$$

ϕ_a are indeed the "covariant coordinates" on S_F^2 and F_{ab} is the field strength, which takes the form

$$F_{ab} = [X_a\,, A_b] - [X_b\,, A_a] + [A_a\,, A_b] - \varepsilon_{abc}A_c\,. \tag{7}$$

when expressed in terms of the gauge fields A_a.

To summarize, with (44) the action in (1) takes the form of a U(n) gauge theory on $\mathcal{M}\times S_F^2(2\ell+1)$ with the gauge field components $A_M(\hat{y}) = (A_\mu(\hat{y})\,, A_a(\hat{y})) \in u(n)\otimes u(2\ell+1)$ and field strength tensor ($\hat{y}$ are a set of coordinates for the noncommutative manifold $\mathcal{M}$)

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \\ F_{\mu a} &= D_\mu \phi_a = \partial_\mu \phi_a + [A_\mu, \phi_a] \\ F_{ab} &= [\phi_a, \phi_b] - \epsilon_{abc}\phi_c\,. \end{aligned} \tag{8}$$

3. The SU(2)-Equivariant Gauge Field

Let us focus on the case of a U(2) gauge theory on $\mathcal{M}\times S_F^2$. The construction of the most general SU(2)-equivariant gauge field on S_F^2 can be performed as follows [9]:

We pick the symmetry generators ω_a which generate $SU(2)$ rotations upto $U(2)$ gauge transformations. Accordingly, we choose

$$\omega_a = X_a^{(2\ell+1)} \otimes \mathbf{1}_2 - \mathbf{1}_{2\ell+1}\otimes \frac{i\sigma^a}{2}\,, \quad \omega_a \in u(2)\otimes u(2\ell+1)\,, \text{for } a = 1,2,3 \tag{9}$$

These ω_a are the generators of the representation $\underline{1/2}\otimes\underline{\ell}$ of SU(2), where by $\underline{m}$ we denote the spin m representation of SU(2) of dimension $2m+1$. SU(2)-equivariance of the theory requires the fulfillment of the symmetry constraints,

$$[\omega_a\,, A_\mu] = 0\,, \quad [\omega_a, \phi_b] = \epsilon_{abc}\phi_c, \tag{10}$$

on the gauge field and a consistency condition on these constraints is $[\omega_a, \omega_b] = \varepsilon_{abc}\omega_c$ which is readily satisfied by our choice of ω_a.

The solutions to these constraints are obtained using the representation theory of $SU(2)$ and are presented in [9]. They are conveniently parametrized as

$$A_\mu = \frac{1}{2}Qa_\mu(\hat{y}) + \frac{1}{2}ib_\mu(\hat{y})\,, \tag{11}$$

$$A_a = \frac{1}{2}\varphi_1(\hat{y})[X_a, Q] + \frac{1}{2}(\varphi_2(\hat{y})-1)Q[X_a, Q] + i\frac{1}{2}\varphi_3(\hat{y})\frac{1}{2}\{\hat{X}_a, Q\} + \frac{1}{2}\varphi_4(\hat{y})\hat{\omega}_a,\,, \tag{12}$$

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with $\phi_a = X_a + A_a$ and a_μ, b_μ are Hermitian U(1) gauge fields, φ_i are Hermitian scalar fields over $\mathcal{M}$, the curly brackets denote anti-commutators throughout, and

$$\hat{X}_a := \frac{1}{\ell + 1/2} X_a \,, \quad \hat{\omega}_a := \frac{1}{\ell + 1/2} \omega_a . \tag{13}$$

They contain, in addition to the Mat2$(2\ell+1)$ identity matrix, the only non-trivial rotational invariant under ω, which is

$$Q := \frac{X_a \otimes \sigma^a - i/2}{\ell + 1/2} \,, \quad Q^\dagger = -Q \,, \quad Q^2 = -\mathbf{1}_{2(2\ell+1)} \,. \tag{14}$$

Indeed, Q is the fuzzy version of $q := i\sigma \cdot \mathbf{x}$ and converges to it in the $\ell \to \infty$ limit.

4. Reduced Action

Confining ourselves to a noncommutative $\mathcal{M}$ and using the SU(2)-equivariant gauge field in the noncommutative $U(2)$ Yang-Mills theory on $\mathcal{M} \otimes S_F^2$, we can explicitly trace it over the fuzzy sphere and reduce it to a theory on $\mathcal{M}$. The reduced action has the general form

$$S = \int_{\mathcal{M}} \mathcal{L}_F + \mathcal{L}_G + \frac{1}{\tilde{g}^2} \tilde{V}_1 + a^2 \tilde{V}_2 \tag{15}$$

where $\mathcal{L}_F$ stands for the curvature term, $\mathcal{L}_G$ for the gradient term and $\tilde{V}_1, \tilde{V}_2$ for the reduced forms of the potential terms in the original action. The explicit form of these terms and the related results are omitted here, they can be found in [10]. It turns out that the presence of extra degrees of freedom, namely $\varphi_3 \,, \varphi_4$, in the $SU(2)$-equivariant gauge field on S_F^2 leads to a further symmetry breaking in the reduced action and the reduced action is invariant only under a noncommutative $U(1)$ gauge group and it has the form [10],

$$\begin{aligned} S = \int_{\mathcal{M}} \frac{1}{4g^2} \left| F_{\mu\nu} \right|^2 + \frac{1}{2} \frac{\ell^2 + \ell}{(\ell + 1/2)^2} D_\mu \varphi D_\mu \varphi^\dagger + \frac{1}{8} \frac{\ell^2 + \ell}{(\ell + 1/2)^2} \left(\frac{(\ell + \frac{3}{2})(\ell - \frac{1}{2})}{(\ell + \frac{1}{2})^2} + 1 \right) (D_\mu \varphi_3)^2 \\ + \frac{\ell^2 + \ell + \frac{3}{4}}{4(\ell + \frac{1}{2})^2} (D_\mu \varphi_4)^2 + \frac{\ell^2 + \ell}{4(\ell + \frac{1}{2})^3} \{ D_\mu \varphi_3 \,, D_\mu \varphi_4 \} + \frac{1}{\tilde{g}^2} \tilde{V}_1 + a^2 \tilde{V}_2 \,. \end{aligned} \tag{16}$$

where we have

$$D_\mu \cdot = \partial_\mu \cdot + [c_\mu \,, \cdot] \,, \quad F_{\mu\nu} = \partial_\mu c_\nu - \partial_\nu c_\mu + i[c_\mu \,, c_\nu] \,, \quad c_\mu = \frac{1}{2} b_\mu \,. \tag{17}$$

5. Solutions of the Reduced Theory on $\mathbb{R}_\theta^2$

We look at the classical solutions of the system governed by the action given in (16) on the Groenewald-Moyal plane $\mathbb{R}_\theta^2$. We consider only one of the the two extreme cases of $a^2 = \infty$ and $a^2 = 0$ corresponding, respectively, to imposing the constraint $\phi_a \phi_a + \ell(\ell+1) = 0$ in full (i.e. "by hand") and imposing no constraint at all. In both cases, we consider the large ℓ limit; in the $a = \infty$ theory, we include only terms appearing at $O(\ell^{-2})$, whereas for the case $a = 0$, we assume $\ell = \infty$. Here we just give a brief account of the former, while a full account including the latter is given in [10].

$\mathbb{R}_\theta^2$ may be defined by two operators $\hat{y}_1 \,, \hat{y}_2$ acting on the standard Fock space $\mathcal{H}$. They fulfill the Heisenberg algebra commutation relation

$$[\hat{y}_1 \,, \hat{y}_2] = i\theta \,, \tag{18}$$

where θ is the noncommutativity parameter. Switching to the complex basis

$$z = \frac{1}{\sqrt{2}}(y_1 + iy_2)\,, \quad \bar{z} = \frac{1}{\sqrt{2}}(y_1 - iy_2)\,, \tag{19}$$

the commutation relations become

$$[z\,, \bar{z}] = \theta\,. \tag{20}$$

After performing the dimensional reduction, the fuzzy constraint $\phi_a\phi_a + \ell(\ell+1) = 0$ can be solved order by order in powers of the parameter $\frac{1}{\ell}$ to obtain φ_3 and φ_4 in terms of φ_1 and φ_2. Substituting back into the action yields an action involving only the scalar $\varphi = \varphi_1 + i\varphi_2$.

For large but finite ℓ, one can solve the constraint approximately by expanding it to leading order in powers of ℓ^{-1} around the $\ell = \infty$. Performing this to order $O\left(\ell^{-3}\right)$, we find [10]

$$\varphi_3 = -i\frac{4}{\ell}[\varphi_1\,, \varphi_2] + \frac{1}{2\ell^2}(\varphi_1^2 + \varphi_2^2 - 1) + O\left(\ell^{-3}\right)\,, \tag{21}$$

$$\varphi_4 = -\frac{1}{2\ell}(\varphi_1^2 + \varphi_2^2 - 1) + i\frac{3}{\ell^2}[\varphi_1\,, \varphi_2] + O\left(\ell^{-3}\right)\,. \tag{22}$$

Using these in (16), it is found in [10] that the action takes the form

$$S = 2\pi\theta \mathrm{Tr}_{\mathcal{H}}\Bigg[\frac{1}{4g^2}\left|F_{\mu\nu}\right|^2 + \frac{1}{2}\left(1 - \frac{1}{4\ell^2}\right)D_\mu\varphi D_\mu\varphi^\dagger + \frac{1}{\ell^2}\left(D_\mu[\varphi\,, \varphi^\dagger]\right)^2 + \frac{1}{32\ell^2}\left(D_\mu\{\varphi\,, \varphi^\dagger\}\right)^2$$
$$+ \frac{1}{\tilde{g}^2}\left(\left(\frac{1}{2} + \frac{1}{4\ell^2}\right)\left(\frac{1}{2}\{\varphi\,, \varphi^\dagger\} - 1\right)^2 + \frac{1}{8}\left(1 - \frac{1}{\ell} - \frac{3}{4\ell^2}\right)[\varphi\,, \varphi^\dagger]^2\right) + O\left(\ell^{-3}\right)\Bigg]\,. \tag{23}$$

It is possible to employ the solution generating techniques introduced in [12] to find noncommutative vortex type solutions of (23). To this end we proceed with defining the covariant coordinates

$$X = -\frac{1}{\theta}\bar{z} + ic_z\,, \quad X^\dagger = -\frac{1}{\theta}z - ic_{\bar{z}}\,, \tag{24}$$

where the complex combinations $c_z = \frac{1}{\sqrt{2}}(c_1 - ic_2)$, $c_{\bar{z}} = \frac{1}{\sqrt{2}}(c_1 + ic_2)$ are introduced. The covariant derivatives and the field strength may be expressed as

$$D_z\varphi = [X\,, \varphi]\,, \quad D_{\bar{z}}\varphi = -[X^\dagger\,, \varphi]\,. \tag{25}$$

$$F_{z\bar{z}} = \partial_z c_{\bar{z}} - \partial_{\bar{z}} c_z + i[c_z\,, c_{\bar{z}}]\,, \quad = i[X\,, X^\dagger] + \frac{i}{\theta}\,, \tag{26}$$

All the basic constituents of the action (23) transform covarianty under the gauge symmetry

$$X \longrightarrow UXU^\dagger\,, \quad \varphi \longrightarrow U\varphi U^\dagger\,, \quad D_z\varphi \longrightarrow UD_z\varphi U^\dagger\,, \quad F_{z\bar{z}} \longrightarrow UF_{z\bar{z}}U^\dagger\,. \tag{27}$$

It follows that the equations of motion will transform covariantly, that is,

$$\frac{\delta S}{\delta X} \longrightarrow U\frac{\delta S}{\delta X}U^\dagger\,, \quad \frac{\delta S}{\delta\varphi} \longrightarrow U\frac{\delta S}{\delta\varphi}U^\dagger\,, \tag{28}$$

under a partial isometry U satisfying

$$U^\dagger U = 1\,, \quad UU^\dagger = P\,, \tag{29}$$

where P is a projection operator [12]. Thus, the partial isometries (29) generate solutions from a known solution.

A trivial solution to the equations of motion of (23) may easily found to be $X = -\frac{1}{\theta}\bar{z}\,, \varphi = 1$. Taking $U = S^m$, where S is the usual shift operator $S = \sum_{k=0}^{\infty} |k+1\rangle\langle k|$, we can write a set of non-trivial solutions for the theory governed by (23) as

$$\begin{aligned} \varphi &= S^m S^{\dagger m} = 1 - P_m\,, \\ X &= -\frac{1}{\theta} S^m \bar{z} S^{\dagger m}\,, \end{aligned} \tag{30}$$

where

$$P_n = \sum_{k=0}^{n-1} |k\rangle\langle k|\,, \tag{31}$$

is the projection operator of rank m. The corresponding field strength is $F_{12} = -iF_{z\bar{z}} = \frac{1}{\theta}P_m$. We can view these solutions as noncommutative vortices [12, 13, 14] carrying m units of flux:

$$2\pi\theta\, \mathrm{Tr} F_{12} = 2\pi m\,. \tag{32}$$

It is useful to evaluate the value of the action (23) on these solutions; we find

$$S = \pi\theta m \left(\frac{1}{g^2\theta^2} + \frac{1}{\tilde{g}^2}\left(1 + \frac{1}{2\ell^2}\right)\right) + O\left(\ell^{-3}\right)\,. \tag{33}$$

This corresponds to the energy of the static vortices in $2+1$ dimensions, $\mathbb{R}^2_\theta \times \mathbb{R}^1$ with $\mathbb{R}^1$ standing for time. We observe that to leading order in ℓ^{-1} there is a ℓ^{-2} contribution adding to the energy, which is a residue of the fact that the present model has descended from a model with a fuzzy sphere of order ℓ, $S_F^2(\ell)$ as extra dimensions.

Two limiting cases may also be easily recorded from (33). For $\tilde{g} \to \infty$, our solutions collapse to the fluxon solutions discussed in [15, 16]; whereas, for $\theta \to \infty$, the action gets a contribution only from the potential term, and our vortex solution collapses to a noncommutative soliton solution of the type first discussed in [17].

6. $U(4)$ Gauge Theory over $\mathcal{M} \times S_F^2 \times S_F^2$

i. Gauge theory on $\mathcal{M} \times S_F^2 \times S_F^2$:

Starting again with an $SU(\mathcal{N})$ gauge theory, which is now coupled adjointly to six scalar fields $\Phi_i\,, (i = 1\,, \cdots, 6)$, the relevant action is given in the form [4]

$$S = \int_{\mathcal{M}} \mathrm{Tr}_{\mathcal{N}}\Big(\frac{1}{4g^2} F_{\mu\nu}^\dagger F_{\mu\nu} + \frac{1}{2}(D_\mu\Phi_i)^\dagger (D_\mu \Phi_i)\Big) + V(\Phi)\,. \tag{34}$$

In this expression, A_μ are $u(\mathcal{N})$ valued anti-Hermitian gauge fields, $\Phi_i\,(i = 1, \cdots 6)$ are six anti-Hermitian scalars transforming in the adjoint of SU($\mathcal{N}$) and $D_\mu\Phi_i = \partial_\mu\Phi_i + [A_\mu\,, \Phi_i]$ is the covariant derivative.

Product of two fuzzy spheres emerges as extra dimensions from this theory as a consequence of spontaneous breaking of the original gauge symmetry. Following [4], we consider a potential of the form

$$V(\Phi) = \frac{1}{g_L^2} V_1(\Phi^L) + \frac{1}{g_R^2} V_1(\Phi^R) + \frac{1}{g_{LR}^2} V_1(\Phi^{L,R}) + a_L^2 V_2^L(\Phi_L) + a_R^2 V_2^R(\Phi_R)\,, \tag{35}$$

where

$$\Phi_a^L = \Phi_a\,, \quad \Phi_a^R = \Phi_{a+3}\,, \quad (a = 1,2,3)\,, \tag{36}$$

and

$$\begin{aligned} V_1(\Phi^L) &= \mathrm{Tr}_{\mathcal{N}} F_{ab}^{L\dagger} F_{ab}^L\,, \quad F_{ab}^L = [\Phi_a^L\,, \Phi_b^L] - \varepsilon_{abc}\Phi_c^L \\ V_1(\Phi^R) &= \mathrm{Tr}_{\mathcal{N}} F_{ab}^{R\dagger} F_{ab}^R\,, \quad F_{ab}^R = [\Phi_a^R\,, \Phi_b^R] - \varepsilon_{abc}\Phi_c^R \\ V_2(\Phi^L) &= \mathrm{Tr}_{\mathcal{N}}(\Phi_a^L\Phi_a^L + \tilde{b}_L)^2\,, \quad V_2(\Phi^R) = \mathrm{Tr}_{\mathcal{N}}(\Phi_a^R\Phi_a^R + \tilde{b}_R)^2 \\ V_1(\Phi^{L,R}) &= \mathrm{Tr}_{\mathcal{N}} F_{ab}^{(L,R)\dagger} F_{ab}^{(L,R)}\,, \quad F_{ab}^{(L,R)} = [\Phi_a^L\,, \Phi_b^R]\,. \end{aligned} \tag{37}$$

We observe that the potential $V(\Phi)$ is positive definite, and it is possible to pick $\tilde{b}_L$ and $\tilde{b}_R$ as the quadratic Casimirs of respectively $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ with IRR's labeled by ℓ_L and ℓ_R

$$\tilde{b}_L = \ell_L(\ell_L+1)\,, \quad \tilde{b}_R = \ell_R(\ell_R+1)\,, \quad 2\ell_L\,, 2\ell_R \in \mathbb{Z}\,. \tag{38}$$

If it is further assumed that $\mathcal{N} = (2\ell_L+1)(2\ell_R+1)n$, $(n \in \mathbb{Z})$ then the configuration

$$\begin{aligned} \Phi_a^L &= X_a^{(2\ell_L+1)} \otimes \mathbf{1}_{(2\ell_R+1)} \otimes \mathbf{1}_n\,, \\ \Phi_a^R &= \mathbf{1}_{(2\ell_L+1)} \otimes X_a^{(2\ell_R+1)} \otimes \mathbf{1}_n \end{aligned} \tag{39}$$

$$[\Phi_a^L\,, \Phi_b^R] = 0\,, \tag{40}$$

is a global minimum of the potential $V(\Phi)$ where $X_a^{(2\ell_L+1)}$ and $X_a^{(2\ell_R+1)}$ are the anti-Hermitian generators of $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ respectively in the IRR's ℓ_L and ℓ_R, with the commutation relations

$$[X_a^{(2\ell_L+1)}\,, X_b^{(2\ell_L+1)}] = \varepsilon_{abc}X_c^{(2\ell_L+1)}\,, \quad [X_a^{(2\ell_R+1)}\,, X_b^{(2\ell_R+1)}] = \varepsilon_{abc}X_c^{(2\ell_R+1)}\,. \tag{41}$$

This vacuum configuration spontaneously breaks the $\mathrm{U}(\mathcal{N})$ down to $\mathrm{U}(n)$ which is the commutant of $\Phi_a^L\,, \Phi_a^R$ in (39).

Defining

$$\hat{x}_a^L = \frac{i}{\sqrt{\ell_L(\ell_L+1)}} X_a^{(2\ell_L+1)} \otimes 1_{(2\ell_R+1)}\,, \quad \hat{x}_a^R = 1_{(2\ell_L+1)} \otimes \frac{i}{\sqrt{\ell_R(\ell_R+1)}} X_a^{(2\ell_R+1)}\,, \tag{42}$$

$$\hat{x}_a^L\hat{x}_a^L = 1\,, \quad \hat{x}_a^R\hat{x}_a^R = 1\,. \tag{43}$$

the vacuum is a product of two fuzzy spheres $S_F^2 \times S_F^2$ generated by $\hat{x}_a^L$ and $\hat{x}_a^R$.

Fluctuations about this vacuum give a $U(n)$ gauge theory over $S_F^2 \times S_F^2$. We can write

$$\Phi_a^L = X_a^L + A_a^L\,, \quad \Phi_a^R = X_a^R + A_a^R \tag{44}$$

where $A_a^L\,, A_a^R \in u(2\ell_L+1) \otimes u(2\ell_R+1) \otimes u(n)$ with the short-hand notation $X_a^{(2\ell_L+1)} \otimes \mathbf{1}_{(2\ell_R+1)} \otimes \mathbf{1}_n =: X_a^L$ and $\mathbf{1}_{(2\ell_L+1)} \otimes X_a^{(2\ell_R+1)} \otimes \mathbf{1}_n =: X_a^R$.

Thus, $\Phi_a^L\,, \Phi_a^R$ are the "covariant coordinates" on $S_F^2 \times S_F^2$ and the associated curvatures F_{ab}^L, F_{ab}^R, $F_{ab}^{L,R}$ take their familiar form after expanding

$$\begin{aligned} F_{ab}^L &= [X_a^L\,, A_b^L] - [X_b^L\,, A_a^L] + [A_a^L\,, A_b^L] - \varepsilon_{abc}A_c^L\,. \\ F_{ab}^R &= [X_a^R\,, A_b^R] - [X_b^R\,, A_a^R] + [A_a^R\,, A_b^R] - \varepsilon_{abc}A_c^R\,. \\ F_{ab}^{L,R} &= [X_a^L\,, A_b^R] - [X_b^R\,, A_a^L] + [A_a^L\,, A_b^R]\,. \end{aligned} \tag{45}$$

ii. The SU(2) × SU(2)*-Equivariant Gauge Field*

We formulate the $SU(2)_L \times SU(2)_R \cong SO(4)$-equivariant $U(4)$ gauge theory on $S_F^2 \times S_F^2$. For this purpose we need to introduce $SO(4)$ symmetry generators under which A_μ is a scalar up to a $U(4)$ gauge transformation, that is carrying the $SO(4)$ IRR $(0,0)$ and A_a^L and A_a^R are $SO(4)$ tensors carrying the IRRs $(1,0)$ and $(0,1)$, respectively. In other words, A_a^L is a vector under the left $SU(2)$ and a scalar under the right $SU(2)$, whereas A_a^R is an $SU(2)_R$ vector and an $SU(2)_L$ scalar.

On $S_F^2 \times S_F^2$ the $SU(2) \times SU(2) \cong SO(4)$ rotational symmetry is implemented by the adjoint actions $\mathrm{a}dX_a^L$ and $\mathrm{a}dX_a^R$:

$$\mathrm{a}dX_a^L \cdot = [X_a^L\,,\cdot]\,,\quad \mathrm{a}dX_a^R \cdot = [X_a^R\,,\cdot]\,,\quad [\mathrm{a}dX_a^L\,,\mathrm{a}dX_a^R] = 0\,. \tag{46}$$

We introduce the anti-Hermitian symmetry generators

$$\begin{aligned} \omega_a^L &= X_a^{(2\ell_L+1)} \otimes \mathbf{1}_{(2\ell_R+1)} \otimes \mathbf{1}_4 - 1_{(2\ell_L+1)} \otimes \mathbf{1}_{(2\ell_R+1)} \otimes i\frac{L_a^L}{2}\,, \\ \omega_a^R &= \mathbf{1}_{(2\ell_L+1)} \otimes X_a^{(2\ell_R+1)} \otimes \mathbf{1}_4 - 1_{(2\ell_L+1)} \otimes \mathbf{1}_{(2\ell_R+1)} \otimes i\frac{L_a^R}{2}\,. \end{aligned} \tag{47}$$

fulfilling the consistency conditions

$$\begin{aligned} [\omega_a^L\,,\omega_b^L] &= i\varepsilon_{abc}\omega_c^L\,, \\ [\omega_a^R\,,\omega_b^R] &= i\varepsilon_{abc}\omega_c^R\,, \end{aligned} \tag{48}$$

$$[\omega_a^L\,,\omega_b^R] = 0\,. \tag{49}$$

Here L_a^L and L_a^R are 4×4 matrices which fulfill

$$\begin{aligned} [L_a^L\,,L_b^L] &= 2i\varepsilon_{abc}L_c^L\,, \\ [L_a^R\,,L_b^R] &= 2i\varepsilon_{abc}L_c^R\,, \end{aligned} \tag{50}$$

$$[L_a^L\,,L_b^R] = 0\,. \tag{51}$$

It is not very hard to see that there are four invariants under the action of ω_a^L and ω_a^R. These are the three "idempotents"

$$Q_L = \frac{X_a^{\ell_L} \otimes \mathbf{1}_{(2\ell_R+1)} \otimes L_a^L - \frac{i}{2}\mathbf{1}}{\ell_L + 1/2}\,,\quad Q_L^\dagger = -Q_L\,,\quad Q_L^2 = -\mathbf{1}_{4(2\ell_L+1)(2\ell_R+1)}\,, \tag{52}$$

$$Q_R = \frac{\mathbf{1}_{(2\ell_L+1)} \otimes X_a^{\ell_R} \otimes L_a^R - \frac{i}{2}\mathbf{1}}{\ell_R + 1/2}\,,\quad Q_R^\dagger = -Q_R\,,\quad Q_R^2 = -\mathbf{1}_{4(2\ell_L+1)(2\ell_R+1)}\,, \tag{53}$$

$$Q_L Q_R$$

and the identity matrix $-\mathbf{1}_{4(2\ell_L+1)(2\ell_R+1)}$.

These lead to the parametrization

$$A_\mu = \frac{1}{2}a_\mu^L Q^L + \frac{1}{2}a_\mu^R Q^R + \frac{i}{2}b_\mu \mathbf{1} + \frac{1}{2}ic_\mu Q^L Q^R\,, \tag{54}$$

where a_μ, b_μ, c_μ and d_μ are all Hermitian $U(1)$ gauge fields, and to the parametrizations

$$\begin{aligned} A_a^L = {}& \frac{1}{2}(\chi_1 + \chi_1')[X_a^L\,,Q^L] + \frac{1}{2}(\chi_2 + \chi_2' - 1)Q^L[X_a^L\,,Q^L] + i\frac{1}{2}\chi_3\frac{1}{2}\{\widehat{X}_a^L\,,Q^L\} + \frac{1}{2}\chi_4\widehat{\omega}_a^L \\ & + \frac{1}{2}(\chi_1 - \chi_1')iQ^R[X_a^L\,,Q^L] + \frac{1}{2}(\chi_2 - \chi_2')iQ^R Q^L[X_a^L\,,Q^L] + i\frac{1}{2}\chi_3'\frac{1}{2}iQ^R\{\widehat{X}_a^L\,,Q^L\} + \frac{1}{2}\chi_4' iQ^R\widehat{\omega}_a^L\,. \end{aligned} \tag{55}$$

$$A_a^R = \frac{1}{2}(\lambda_1+\lambda_1')[X_a^R, Q^R] + \frac{1}{2}(\lambda_2+\lambda_2'-1)Q^R[X_a^R, Q^R] + i\frac{1}{2}\lambda_3\frac{1}{2}\{\widehat{X}_a^R, Q^R\} + \frac{1}{2}\lambda_4\widehat{\omega}_a^R$$
$$+\frac{1}{2}(\lambda_1-\lambda_1')iQ^L[X_a^R, Q^R] + \frac{1}{2}(\lambda_2-\lambda_2')iQ^LQ^R[X_a^R, Q^R] + i\frac{1}{2}\lambda_3'\frac{1}{2}iQ^L\{\widehat{X}_a^R, Q^R\} + \frac{1}{2}\lambda_4' iQ^L\widehat{\omega}_a^R\,. \tag{56}$$

Here χ_i, χ_i', λ_i and λ_i' $i = (1,2,3,4)$ are Hermitian scalar fields over $\mathcal{M}$, the curly brackets denote anti-commutators throughout, and we have used

$$\widehat{X}_a := \frac{1}{\ell+1/2}X_a\,, \quad \widehat{\omega}_a := \frac{1}{\ell+1/2}\omega_a\,, \tag{57}$$

for both the left and the right quantities.

Dimensional reduction of the gauge theory over $S_F^2 \times S_F^2$ leads essentially to a $U(1)^3$ Abelian Higgs type model coupled to four complex and eight real scalar fields. In the large ℓ limit, it appears that the part of the potential governing the vortex solutions has the following form

$$(|\chi|^2 - \frac{1}{4})^2 + (|\chi'|^2 - \frac{1}{4})^2 + (|\lambda|^2 - \frac{1}{4})^2 + (|\lambda'|^2 - \frac{1}{4})^2 + 2(|\chi\lambda' - \chi'\lambda|^2 + |\bar{\lambda}\chi - \chi'\bar{\lambda}'|^2)\,. \tag{58}$$

where $\chi := \chi_1 + i\chi_2$ and other complex fields are likewise defined. The vacuum manifold here is $T^3 = S^1 \times S^1 \times S^1$ and $\pi_1(T^3) = \mathbb{Z}^3$, which clearly indicates the existence of vortex solutions in this model. Complete results on this work will be given in [11].

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