# A singularity–free cosmological model with a conformally coupled scalar field

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#### Abstract

We explore the possibility of describing our universe with a singularity-free, closed, spatially homogeneous and isotropic cosmological model, using only general relativity and a suitable equation of state which produces an inflationary era. A phase transition to a radiation-dominated era occurs as a consequence of boundary conditions expressing the assumption that the temperature cannot exceed the Planck value. We find that over a broad range of initial conditions, the predicted value of the Hubble parameter is approximately 47 km s<sup>-1</sup> Mpc<sup>-1</sup>. Inflation is driven by a scalar field, which must be conformally coupled to the curvature if the Einstein equivalence principle has to be satisfied. The form of the scalar field potential is derived, instead of being assumed a priori.

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## 1 Introduction

Singularity-free cosmological models have received renewed attention in the past few years (Rosen 1985, Israelit & Rosen 1989 hereafter IR, Balbinot *et al.* 1990, Starkovich & Cooperstock 1992 hereafter SC, Brandenberger *et al.* 1993). In the paper by IR, the universe was modelled as a closed Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetime, bouncing from a Planck mass and radius state with inflation, emerging into the standard model of radiation and then matter dominance. The IR "prematter" heated with expansion as it emerged from the cold non-singular Planck state. This is an attractive simple picture, in contrast to the earlier prevailing theme among cosmologists that the universe emerged from a singular state in the big bang into a hot radiation-dominated universe, cooled to the point of vacuum energy dominance leading to the onset of inflation with further cooling. Then scenarios had to be constructed to re-heat the universe to the onset of the radiation-dominated phase of the standard model.

SC built upon the IR model, incorporating a scalar field to describe all phases of the universe evolution as an integrated, albeit simplified, field theory. Besides being a physically meaningful source of gravitation in the early universe, a scalar field provides a perfect fluid stress-energy tensor also in the later epochs of the universe's history. Moreover, it has been proposed as a dark matter candidate (Starkovich 1992; Kofman *et al.* 1993; Delgado 1993; McDonald 1993).

We adopt the Gliner (1966)–Markov (1982) picture of Planck limits to physical quantities and postulate that when the Planck temperature  $T_{pl}$  is reached, the phase change from prematter (with equation of state  $P = (-1 + \gamma)\rho$ , where  $\gamma$  is a small parameter) to radiation ( $P = \rho/3$ ) occurs.

In this paper, the work of SC is extended, refining the calculations of the evolution of the scale factor. We now determine the value of the present time as that corresponding to the evolution of the radiation temperature to 2.7 K. We then find that for a broad range of initial conditions (no "fine–tuning" requirement), the present value of the Hubble parameter is approximately 47 km s<sup>-1</sup> · Mpc<sup>-1</sup>.

The paper by SC is built upon the minimal coupling of the scalar field to the Ricci curvature. While minimal coupling is usually considered, it has been shown recently (Sonego & Faraoni 1993) that conformal coupling is the only acceptable possibility if the Einstein equivalence principle holds as applied to scalar wave propagation. Thus, it is of immediate interest to focus our cosmological theory on a conformally coupled scalar field.

Instead of adopting a particular form of the scalar field potential (such as, e.g.,  $\lambda \phi^4$ ) as is usually done, we start with the assumption that the equation of state is

inflationary, and derive the form of the potential. This approach reverses the procedure usually employed, but is perfectly legitimate, and has already been used in the literature (Lucchin & Matarrese 1985, Barrow 1990, Ellis & Madsen 1991, SC). Also, to be noted is a somewhat similar approach, deriving the scalar field potential from the observed spectrum of density perturbations (Hodges & Blumenthal 1990, Copeland *et al.* 1993, Turner 1993, Lidsey & Tavakol 1993).

We regard as acceptable those cosmological models that merge satisfactorily with the standard model without singularities and without the need for fine-tuning of the parameters. We are able to impose the conditions which are deemed necessary and deduce the potential which achieves these aims.

In sec. 2 we present the basic equations of the cosmological model with a scalar field. In sec. 3 we introduce boundary and initial conditions inspired by the Gliner–Markov ideas, and we solve analytically the equations for the dynamics of the universe. In sec. 4 we solve numerically for the evolution of the scalar field and we find the form of the scalar field potential. The results are discussed in sec. 5.

## **2** Basic equations

Our theory is based on a closed, spatially homogeneous and isotropic FLRW model. The metric is given by<sup>1</sup>

$$ds^{2} = dt^{2} - a^{2}(t) \left[ \frac{dr^{2}}{1 - r^{2}} + r^{2} \left( d\theta^{2} + \sin^{2} \theta \, d\varphi^{2} \right) \right] , \qquad (2.1)$$

where t is the comoving time,  $r \in (0, 1)$ ,  $\theta \in (0, \pi)$ , and  $\varphi \in (0, 2\pi)$ . We assume that the only source of gravitation is a scalar field  $\phi(t)$  during the entire history of the universe. In the contemporary literature it is customary to assume that the dynamics of the universe are dominated by a scalar field during an early inflationary epoch. We extend this assumption to the later radiation- and matter-dominated epochs, in which a scalar field provides the simplest field-theoretic realization of a perfect fluid stressenergy tensor. However, the use of the scalar field as a source of gravitation in these late eras may be more meaningful; in fact it has been pointed out (Starkovich 1992, Kofman *et al.* 1993, Delgado 1993; McDonald 1993) that the  $\phi$ -field could be a candidate for dark matter in the present era.

<sup>&</sup>lt;sup>1</sup>We adopt the notations and conventions of Birrell & Davies (1982).

Most of the contemporary literature is built upon the minimal coupling of the scalar field to the Ricci curvature R. The Klein–Gordon equation with more general coupling is

$$\Box \phi + \xi R \phi + \frac{dV}{d\phi} = 0 , \qquad (2.2)$$

where  $\Box \phi \equiv g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} \phi$ , which assumes the value  $\ddot{\phi} + 3\dot{\phi}\dot{a}/a$  for our metric,  $\xi$  is a numerical constant, and  $V(\phi)$  is the scalar field potential. While minimal coupling  $(\xi = 0)$  is usually considered, other values of  $\xi$  are possible. Recently, Sonego & Faraoni (1993) showed that the value  $\xi = 1/6$  ("conformal coupling") is the only acceptable possibility if scalar waves in a curved spacetime are to propagate *locally* as they do in Minkowski spacetime, as required by the Einstein equivalence principle. We stress that the derivation of this result is independent of the conformal structure of spacetime and of conformal transformations: in other words, conformal invariance of the Klein–Gordon equation is not required. In fact, eq. (2.2) with any non–constant potential (or a mass term) is not conformally invariant<sup>2</sup>.

Thus, we focus our cosmological theory on a conformally coupled scalar field. The stress–energy tensor is (see e.g. Birrell & Davies 1982, p. 87)

$$T_{\mu\nu} = \frac{2}{3} \nabla_{\mu} \phi \nabla_{\nu} \phi - \frac{1}{6} \nabla_{\alpha} \phi \nabla^{\alpha} \phi g_{\mu\nu} - \frac{1}{3} \phi \nabla_{\mu} \nabla_{\nu} \phi - \frac{1}{6} \phi^2 \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + V g_{\mu\nu} + \frac{1}{3} \phi \Box \phi g_{\mu\nu}$$

$$(2.3)$$

This can be written in the form of a perfect fluid stress-energy tensor:

$$T_{\mu\nu} = (\rho + P) u_{\mu} u_{\nu} - P g_{\mu\nu} , \qquad (2.4)$$

where

$$u^{\mu} = \frac{\nabla^{\mu}\phi}{\left(\nabla_{\alpha}\phi\nabla^{\alpha}\phi\right)^{1/2}}\tag{2.5}$$

is the four-velocity of comoving observers (we restrict ourselves to the consideration of a real scalar field which satisfies  $\nabla_{\alpha}\phi\nabla^{\alpha}\phi > 0$ ). The energy density and pressure are given by (a dot denotes differentiation with respect to t)

$$\rho = T_{00} = \frac{1}{2} \left( \dot{\phi} \right)^2 + \frac{1}{2} \phi^2 \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{1}{a^2} \right] + \frac{\dot{a}}{a} \phi \dot{\phi} + V , \qquad (2.6)$$

<sup>&</sup>lt;sup>2</sup>This can be understood physically by noting that the introduction of a mass, or a potential, sets a preferred length scale for the scalar field theory, which is then not invariant under a change of lengths and norm of vectors arising from a (position dependent) rescaling of the metric. Thus, conformal invariance follows from "conformal" coupling only in the absence of a potential.

$$P = \frac{1}{3} \left( \rho - T^{\mu}{}_{\mu} \right) = \frac{1}{6} \left( \dot{\phi} \right)^2 + \frac{1}{3} \frac{\dot{a}}{a} \phi \dot{\phi} + \frac{1}{3} \phi \frac{dV}{d\phi} + \frac{1}{6} \phi^2 \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{1}{a^2} \right] - V , \qquad (2.7)$$

where eq. (2.2) has been used. It has been shown by Madsen (1988) that the stressenergy tensor of a scalar field can be put in the form corresponding to a perfect fluid only under the assumptions of spatial homogeneity and isotropy. If spatially dependent perturbations  $\delta\phi(t, \vec{x})$  to the scalar field are considered,  $T_{\mu\nu}$  can still be put in the form corresponding to a fluid, but new terms appear, which can be interpreted as a heat flux and anisotropic stresses. These terms disappear in the case of minimal coupling (Madsen 1988). This has, as a consequence, the fact that, during a given era, there is no entropy production for the unperturbed field  $\phi(t)$ . This is also true for spatially dependent perturbations of a minimally coupled scalar field, but, interestingly, it no longer holds for conformal coupling, where entropy production does occur. To our knowledge, perturbations of a non-minimally coupled scalar field have not been considered in the literature. We will not deal with this in the present paper, leaving it for future analysis.

The Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi T_{\mu\nu}$$
(2.8)

give

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2} = \frac{8\pi}{3}\,\rho\,\,,\tag{2.9}$$

$$\frac{\ddot{a}}{a} + 4\pi \left(\gamma - \frac{2}{3}\right)\rho = 0 , \qquad (2.10)$$

where  $\gamma$  is defined by the equation of state

$$P = (\gamma - 1)\rho. \qquad (2.11)$$

For simplicity, we assume that  $\gamma$  is constant during each era in which we divide the history of the universe. We will consider three eras, corresponding to different values of  $\gamma$ . We consider first the evolution of the scale factor, and then we will turn to the dynamics of the scalar field.

# 3 The dynamics of the universe

The evolution of the scale factor can be solved separately from the dynamics of the scalar field. The Einstein equations (2.9) and (2.10) give an equation involving only the

scale factor

$$\frac{\ddot{a}}{a} + \left(\frac{3}{2}\gamma - 1\right)\left(\frac{\dot{a}^2 + 1}{a^2}\right) = 0.$$
(3.1)

In order to solve this equation, we introduce the conformal time  $\eta$  defined by

$$dt = a(\eta)d\eta . (3.2)$$

In the following, we will solve for the dynamics of the universe and of the scalar field in terms of the conformal time. The transformation to comoving time is given explicitly in eq. (3.43), and the final results are also given in comoving time.

Equations (2.9), (2.10) and (3.1) become

$$\left(\frac{a'}{a^2}\right)^2 + \frac{1}{a^2} = \frac{8\pi}{3}\rho , \qquad (3.3)$$

$$\frac{a''}{a^3} - \left(\frac{a'}{a^2}\right)^2 + 4\pi \left(\gamma - \frac{2}{3}\right)\rho = 0 , \qquad (3.4)$$

$$\frac{a''}{a} + \left(\frac{3}{2}\gamma - 2\right)\left(\frac{a'}{a}\right)^2 + \frac{3}{2}\gamma - 1 = 0, \qquad (3.5)$$

where a prime denotes differentiation with respect to  $\eta$ . If we focus our attention on the variable

$$u \equiv \frac{a'}{a} = \frac{d\ln a}{d\eta} , \qquad (3.6)$$

we obtain, from eq. (3.5), the Riccati equation

$$u' + cu^2 + c = 0 , (3.7)$$

where

$$c = \frac{3}{2}\gamma - 1 . (3.8)$$

In the following, we will consider values of  $\gamma$  such that  $c \neq 0$ . Equation (3.7) is easily solved by setting

$$u = \frac{1}{c} \frac{w'}{w} = \left[\ln\left(w^{1/c}\right)\right]', \qquad (3.9)$$

which gives the real solution

$$a(\eta) = a_0 \left[\cos(\eta c + d)\right]^{1/c}$$
, (3.10)

where  $a_0$  and d are integration constants.

We consider three periods in the history of the universe:

- "prematter" (inflationary) era,  $0 \leq \eta \leq \eta_r$ , in which  $\gamma$  is small and positive (typical values are  $\gamma_p \sim 10^{-3}$ ). This gives an inflationary equation of state close to  $P = -\rho$  (the case  $\gamma_p = 0$  corresponding to eternal inflation is excluded). The smallness of the parameter  $\gamma_p$  determines the amount of inflation that the universe has experienced.
- radiation era,  $\eta_r \leq \eta \leq \eta_m$ , with  $\gamma_r = 4/3$ , corresponding to the equation of state  $P = \rho/3$ .
- matter era,  $\eta \ge \eta_m$ , with  $\gamma_m = 1$ , corresponding to a zero pressure dust

(in the following, subscripts or superscripts p, r, m denote the prematter, radiation and matter era, respectively). In our model, the transitions between different eras, which occur at the conformal times  $\eta_r$  and  $\eta_m$  are imposed by discontinuous changes in the equations of state. The phase change at  $\eta_r$  follows from the Gliner (1966)–Markov (1982) hypothesis of Planckian limits to physical quantities, in this case temperature, and the phase change at  $\eta_m$  occurs when the radiation density falls to the level of the nonrelativistic matter density. Although the imposition of discontinuous phase changes is not derived from an analysis of the microphysics, the microphysics relating to the scalar field is itself unknown (Ellis 1991). However, the present procedure does yield the form of the scalar field potential  $V(\phi)$ , in contrast to the usual procedure of assuming, a priori, a form for  $V(\phi)$ . This is a variation of the approach first proposed by Synge (1955) and which has since been followed by various authors (Lucchin & Matarrese 1985, Barrow 1990, Ellis & Madsen 1991, SC – see also Hodges & Blumenthal 1990 and Copeland etal. 1993). As applied to cosmology, this leads to the treatment of the early universe as a "hot laboratory for the particle physics relevant at these very early times" (Ellis & Madsen 1991).

### 3.1 Boundary conditions and solutions for the scale factor

Our initial and boundary conditions for the dynamics of the universe follow from the idea that the physical quantities are limited by the Planck values. This idea is present in the papers of Gliner (1966) and Markov (1982), and has been used by Rosen (1985), IR and SC. The closed cosmological model has a contracting phase preceding the turnaround, or "bounce", when the Planck density is reached. We set  $\eta = 0$  to be the time of the bounce, at which point the scale factor has a vanishing derivative<sup>3</sup>,

$$a'(0) = 0 , (3.11)$$

$$\rho(0) = \rho_{pl} , \qquad (3.12)$$

(where  $\rho_{pl} = 5.1566 \cdot 10^{93} \text{ g} \cdot \text{cm}^{-3}$  is the Planck density), which also imply

$$a(0) = a_0^{(p)} = \sqrt{\frac{3}{8\pi\rho_{pl}}}, \qquad (3.13)$$

$$\rho'(0) = 0. (3.14)$$

At this stage, the universe is said to be in the "prematter" phase. Equation (3.13) follows from eq. (3.3), and eq. (3.14) follows from

$$\rho' + 3\gamma \frac{a'}{a} \rho = 0 , \qquad (3.15)$$

which can be derived from the conservation equation  $\nabla^{\nu}T_{\mu\nu} = 0$  and the equation of state (2.11). The scale factor in the different eras is

$$a(\eta) = \begin{cases} a_0^{(p)} \left[\cos(\eta c_p)\right]^{1/c_p} & 0 \le \eta \le \eta_r ,\\ a_0^{(r)} \cos(\eta + d^{(r)}) & \eta_r \le \eta \le \eta_m ,\\ a_0^{(m)} \cos^2(\eta/2 + d^{(m)}) & \eta_m \le \eta . \end{cases}$$
(3.16)

By imposing that the scale factor and its derivative (and hence, also the Hubble parameter) are continuous at the transitions between the different eras ( $\eta_r$  and  $\eta_m$ ), we derive appropriate values for the integration constants:

$$d^{(p)} = 0 (3.17)$$

$$a_0^{(r)} = a_0^{(p)} \left[ \cos(\eta_r c_p) \right]^{\frac{1}{c_p} - 1} , \qquad (3.18)$$

$$d^{(r)} = \eta_r (c_p - 1) , \qquad (3.19)$$

$$a_0^{(m)} = \frac{a_0^{(r)}}{\cos(\eta_m + d^{(r)})} , \qquad (3.20)$$

$$d^{(m)} = \frac{\eta_m}{2} + d^{(r)} . aga{3.21}$$

<sup>&</sup>lt;sup>3</sup>The possibility of imposing the initially static condition (3.11) is by no means trivial, and is guaranteed by the inflationary equation of state that we assume, which circumvents the Hawking–Penrose singularity theorem. An equation of state satisfying the strong energy condition would imply a singular derivative of the scale factor, and an initial singularity. The inflationary equation of state violates the strong energy condition.

### 3.2 Other quantities

We will now derive expressions for the conformal times  $\eta_r$  and  $\eta_m$  marking the duration of the prematter and radiation eras, respectively. In our model, as in IR and SC, the universe starts in a very cold state and heats during expansion in the inflationary era. This follows from the equation of state and, as a result, there is no requirement for "re-heating" as there is in other models of the early universe. We assume, according to the above mentioned idea of limiting values for the physical quantities, that inflation stops when the temperature reaches the Planck value  $T_{pl} = 1.4169 \cdot 10^{32}$  K. The energy density at this time ( $\eta_r$ ) is

$$\rho(\eta_r) = \frac{\pi^2}{15} \left( kT_{pl} \right)^4 \,, \tag{3.22}$$

where  $k = 1.38 \cdot 10^{-16} \text{ K}^{-1}$  is the Boltzmann constant.

Equations (3.3), (3.16) and (3.22), together with (3.13), (3.18) and (3.19), give

$$\eta_r = \frac{1}{c_p} \arccos\left[ \left( \frac{15}{\pi^2} \frac{\rho_{pl}}{(kT_{pl})^4} \right)^{c_p/3\gamma_p} \right] . \tag{3.23}$$

The dimensionless quantity in square brackets in eq. (3.23) will be used in the following, and its numerical value, with the velocity of light and the (reduced) Planck constant restored, is

$$\frac{15}{8\pi^5} \frac{\rho_{pl}}{(kT_{pl})^4} c^5 h^3 = 1.5201 . \tag{3.24}$$

We can derive an expression for  $\eta_m$  by considering the evolution of the temperature T, which is described by the equation

$$\gamma' - \gamma \frac{T'}{T} - 3 \frac{a'}{a} \gamma (\gamma - 1) = 0 \tag{3.25}$$

(corresponding to eq. (2.20) in SC). The solution during the radiation era ( $\gamma = 4/3$ ) is

$$a(\eta) T(\eta) = \text{const.} = a(\eta_*) T(\eta_*) , \qquad (3.26)$$

where  $\eta_*$  is any given value of  $\eta$  in the radiation era. In particular, for  $(\eta, \eta_*) = (\eta_m, \eta_r)$ we get, from Eqs. (3.16) and (3.19),

$$\eta_m = \arccos\left[\frac{T_{pl}}{T_m}\cos(\eta_r c_p)\right] + \eta_r (1 - c_p) , \qquad (3.27)$$

where  $T_m$  is the temperature at the last phase transition  $\eta_m$ . Kolb and Turner (1990, p. 77) give

$$T_m = T_{now} \cdot 2.32 \cdot 10^4 \ \Omega_0 h^2 \ \mathrm{K} , \qquad (3.28)$$

where  $\Omega_0 = \rho/\rho_c$ ,  $\rho_c = 3H^2/(8\pi)$  is the critical density,  $h = H_0/(100 \text{ Km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1})$ , and  $T_{now} = 2.7 \text{ K}$ . We use the numerical value  $\Omega_0 h^2 = 0.396$  (which is compatible with a rather wide range of acceptable values for  $\Omega_0$  and h), giving  $T_m = 2.4805 \cdot 10^4 \text{ K}$ . Actually, it will turn out that our final results depend only weakly on the value of  $T_m$ , and are much more strongly affected by the choice of the parameter  $\gamma_p$  in the prematter era.

We note that eq. (3.27) sets a limit on the acceptable values of  $\gamma_p$ . Using eq. (3.23), the requirement that the argument of the arccos function in eq. (3.27) is not larger than 1 gives

$$\gamma_p \le \left\{ 3 \left[ \frac{1}{2} + \frac{\ln(T_{pl}/T_m)}{\ln 1.5201} \right] \right\}^{-1} . \tag{3.29}$$

At this stage, it is useful to deduce the order of magnitude of the quantities involved, and make some approximations. We first note that the expression (3.16) for the scale factor in the prematter era becomes singular at  $\eta |c_p| = \pi/2$ . Typical values of  $\gamma_p$  that we consider are of order  $10^{-3}$ , which gives

$$N \equiv -\frac{|c_p|}{3\gamma_p} \simeq -300 \tag{3.30}$$

and, from eq. (3.23),

$$\eta_r |c_p| = \arccos(1.5201^N) \equiv \frac{\pi}{2} + \delta = \frac{\pi}{2} - |\delta| , \qquad (3.31)$$

with  $|\delta| \sim 10^{-55}$ . Therefore,  $\eta_r |c_p| < \pi/2$ , and inflation stops before the singularity in eq. (3.16) is reached, but  $\eta_r |c_p|$  is numerically very close to  $\pi/2$ .

We now introduce

$$x \equiv \frac{T_{pl}}{T_m} \left| \delta \right| \tag{3.32}$$

which, for  $T_m \sim 10^4$  K and  $\gamma_p \sim 10^{-3}$ , is of order  $10^{-27}$ . These numbers justify the following expansions: We have

$$\cos(\eta_r c_p) = \cos(\pi/2 + \delta) = -\sin\delta \simeq |\delta|$$
(3.33)

and eq. (3.27) gives

$$\eta_m = \arccos x + \eta_r (1 - c_p) = -\pi/2 + x + \eta_r (1 - c_p) + O(x^2) , \qquad (3.34)$$

where we invert the function  $\cos s$  in the interval  $-\pi \leq s \leq 0$ . Any other branch of  $\arccos x$  does not give the desired (increasing) behavior of the scale factor after the last phase transition.

We now compute the age of the universe,  $\eta_{now}$ , in conformal time. To this end, we note that after the transition between the radiation and matter eras, radiation is decoupled from matter, and behaves like a perfect fluid with  $\gamma = 4/3$ . Equation (3.26) gives, for  $(\eta, \eta_*) = (\eta_m, \eta_{now})$ 

$$a(\eta_m) T_m = a(\eta_{now}) T_{now}$$
(3.35)

and using eq. (3.16)

$$\cos^{2}\left(\frac{\eta_{now}}{2} + d^{(m)}\right) = \frac{T_{m}}{T_{now}} \cos^{2}\left(\frac{\eta_{m}}{2} + d^{(m)}\right) .$$
(3.36)

Now, Eqs. (3.34), (3.19) and (3.21) give

$$\frac{\eta_m}{2} + d^{(m)} \simeq \arccos x \tag{3.37}$$

and therefore

$$\cos\left(\frac{\eta_{now}}{2} + d^{(m)}\right) = \frac{T_{pl}}{\sqrt{T_m T_{now}}} \left|\delta\right| , \qquad (3.38)$$

where we have chosen the positive sign. We must now invert the function  $\cos s$  in the interval  $(-\pi, 0)$  as before. This branch (which we call "branch 2") is the opposite of that obtained by inverting  $\cos s$  in the interval  $(0,\pi)$  (denoted by "branch 1"). Any branch other than branch 2 will not give the desired behavior of the scale factor (i.e. increasing for  $\eta_m \leq \eta \leq \eta_{now}$ , and for  $\eta_{now} \leq \eta \leq \eta_{max}$ , where  $\eta_{max}$  is the time of maximal expansion of the universe before it starts recollapsing). Therefore, we write

$$\eta_{now} = -2 \arccos\left[\left(\frac{T_m}{2.7 K}\right)^{1/2} \cdot x\right] \bigg|_{\text{branch } 1} - \eta_m - 2d^{(r)} .$$
(3.39)

Equation (3.39) gives an upper limit on  $\gamma_p$ . The requirement that the argument of the arccos function in eq. (3.39) is not larger than 1 gives, using eqs. (3.30)–(3.33),

$$\gamma_p \le \frac{\ln 1.5201}{3\ln\left(\sqrt{1.5201} T_{pl}/\sqrt{T_m T_{now}}\right)}$$
 (3.40)

The conformal time of maximum expansion of the universe is trivially given by eq. (3.16) as

$$\eta_{max} = -2d^{(m)} ; (3.41)$$

it is reached when

$$a(\eta_{max}) = a_0^{(m)} . (3.42)$$

It is useful to translate conformal times into comoving times. Conformal time is defined by eq. (3.2). Choosing t = 0 at  $\eta = 0$  we get, from eq. (3.10),

$$t(\eta) = a_0 \int_0^{\eta} d\eta' \left[ \cos(\eta' c + d) \right]^{1/c} .$$
 (3.43)

This integral has to be computed numerically in the prematter era, while in the radiation  $(c_r = 1)$  and matter  $(c_m = 1/2)$  eras, the integration is trivial. The expressions for the comoving times corresponding to  $\eta_r$ ,  $\eta_m$ ,  $\eta_{now}$  and  $\eta_{max}$  are:

$$t_r = a_0^{(p)} \int_0^{\eta_r} d\eta \, \frac{1}{\left[\cos(\eta c_p)\right]^{1/|c_p|}} \,, \tag{3.44}$$

$$t_m = t_r + a_0^{(r)} \left[ \sin(\eta_m + d^{(r)}) - \sin(\eta_r c_p) \right] , \qquad (3.45)$$

$$t_{now,max} = t_m + \frac{a_0^{(m)}}{2} \left[ \eta_{now,max} - \eta_m + \sin(\eta_{now,max} + 2d^{(m)}) - \sin(\eta_m + 2d^{(m)}) \right] .$$
(3.46)

The Hubble parameter at the present time is

$$H(\eta_{now}) = \frac{a'(\eta_{now})}{a^2(\eta_{now})} = -\frac{9.2503 \cdot 10^{29}}{a_0^{(m)}} \frac{\sin(\eta_{now}/2 + d^{(m)})}{\cos^3(\eta_{now}/2 + d^{(m)})} \quad \text{Km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}.$$
 (3.47)

The evolution of the energy density is governed by eq. (3.15), which has the solution

$$\rho a^{3\gamma} = \text{constant} = \rho(\eta_*) a^{3\gamma}(\eta_*) , \qquad (3.48)$$

where  $\eta_*$  is any fixed conformal instant during a given era. By choosing  $(\eta_*, \eta) = (0, \eta_r)$ ,  $(\eta_r, \eta_m)$ ,  $(\eta_m, \eta_{now})$ ,  $(\eta_m, \eta_{max})$  we get, respectively,

$$\rho(\eta_r) = \rho_{pl} \left[ \frac{a_0^{(p)}}{a(\eta_r)} \right]^{3\gamma_p} , \qquad (3.49)$$

$$\rho(\eta_m) = \rho(\eta_r) \left[ \frac{a(\eta_r)}{a(\eta_m)} \right]^4 , \qquad (3.50)$$

$$\rho(\eta_{now}) = \rho(\eta_m) \left[ \frac{a(\eta_m)}{a(\eta_{now})} \right]^3 , \qquad (3.51)$$

$$\rho(\eta_{max}) = \rho(\eta_m) \left[ \frac{a(\eta_m)}{a_0^{(m)}} \right]^3 .$$
(3.52)

### **3.3** Numerical values

We present tables with numerical values for the conformal and comoving times marking the transitions between the prematter, radiation and matter eras, for the present age of the universe and for the time of its maximal expansion when it just starts recollapsing. We also give the corresponding values for the scale factor (in cm) and energy density (in g· cm<sup>-3</sup>). The present value of the Hubble parameter (in Km·s<sup>-1</sup>·Mpc<sup>-1</sup>) is also computed. The tables correspond to the following choices of parameters:  $T_m = 2.4805 \cdot 10^4$  K and  $\gamma_p = 1.8500 \cdot 10^{-3}$ ,  $1.9000 \cdot 10^{-3}$ ,  $1.9500 \cdot 10^{-3}$ ,  $2.0000 \cdot 10^{-3}$ ,  $2.0100 \cdot 10^{-3}$ ,  $2.0153 \cdot 10^{-3}$ , which are allowed by the limits in eqs. (3.29) and (3.40). It is to be noted that for a broad range of initial conditions corresponding to values of  $\gamma_p$  between  $1.75 \cdot 10^{-3}$  and  $1.95 \cdot 10^{-3}$ , the value of the Hubble parameter hardly varies, remaining close to 46.7km· s<sup>-1</sup>· Mpc<sup>-1</sup>. If the preferred model of the universe is one which does not demand the fine–tuning of parameters, it could be argued that this is the proper value of  $H_0$ . While this value is relatively low, it is compatible with the currently accepted range.

It is to be noted that in the narrow range of allowed values of  $\gamma_p$  near the critical point from  $2.01 \cdot 10^{-3}$  to  $2.0323 \cdot 10^{-3}$ , the predicted  $H_0$  drops rapidly beyond the acceptable range. Moreover, while the value of  $H_0$  is substantively governed by the choice of the parameter  $\gamma_p$ , it depends only weakly on the value of the parameter  $T_m$ .

### 4 The scalar field dynamics

We turn now to the dynamics of the conformally coupled scalar field driving inflation. As discussed above, we do not assume a priori a form for the scalar field potential, but rather we derive it as a consequence of the assumption that the equation of state changes in the different eras of the universe. The Klein–Gordon equation (2.2) with  $\xi = 1/6$  is

$$\phi'' + 2\frac{a'}{a}\phi' + \left(\frac{a''}{a} + 1\right)\phi + a^2\frac{dV}{d\phi} = 0.$$
(4.1)

Equations (2.11), (2.7) and (2.6) give

$$\gamma = \frac{P}{\rho} + 1 = 2 \left\{ \frac{2}{3} \left(\phi'\right)^2 + \frac{4}{3} \frac{a'}{a} \phi \phi' + \frac{1}{3} a^2 \phi \frac{dV}{d\phi} + \frac{2}{3} \phi^2 \left[ 1 + \left(\frac{a'}{a}\right)^2 \right] \right\} \\ \cdot \left\{ \left(\phi'\right)^2 + 2 \frac{a'}{a} \phi \phi' + 2a^2 V + \phi^2 \left[ 1 + \left(\frac{a'}{a}\right)^2 \right] \right\}^{-1}, \quad (4.2)$$

from which we get

$$V = \frac{1}{2\gamma a^2} \left(\frac{4}{3} - \gamma\right) \left\{ \left(\phi'\right)^2 + 2\frac{a'}{a}\phi\phi' + \phi^2 \left[1 + \left(\frac{a'}{a}\right)^2\right] \right\} + \frac{1}{3\gamma}\phi\frac{dV}{d\phi}.$$
 (4.3)

Now, using eqs. (2.6) and (3.2) in eq. (2.9) we find

$$V(\eta) = -\frac{1}{2a^2} (\phi')^2 - \frac{3}{2a^2} \left(\frac{\phi^2}{3} - \frac{1}{4\pi}\right) \left[\left(\frac{a'}{a}\right)^2 + 1\right] - \frac{a'}{a^3} \phi \phi' .$$
(4.4)

Substituting the value of  $dV/d\phi$  given by eq. (4.1) into eq. (4.3), and the result into eq. (4.4), we find

$$\phi\phi'' - 2\frac{a'}{a}\phi\phi' - 2(\phi')^2 + \phi^2 \left[\frac{a''}{a} - 2\left(\frac{a'}{a}\right)^2 - 1\right] + \frac{9\gamma}{8\pi} \left[\left(\frac{a'}{a}\right)^2 + 1\right] = 0.$$
(4.5)

The last equation can be solved numerically in the prematter era, once the parameter  $\gamma_p$  is fixed, and initial conditions  $\phi_0$ ,  $\phi'_0$  are specified. Then, one can solve numerically for the values of  $\phi'(\eta)$  and  $V(\eta)$  from eq. (4.4) and, by eliminating the conformal time, one obtains a numerical form for the scalar field potential  $V(\phi)$ . Using the MAPLE computer program, we have done this for  $\gamma_p = 2.0153 \cdot 10^{-3}$  (corresponding to Table 6). The result is plotted in two different ranges of values of  $\phi$ . The resulting  $V(\phi)$  does not depend on the particular solution  $\phi(\eta)$  coming from specific initial data ( $\phi_0, \phi'_0$ ), as must be the case. Integration gives only narrow ranges of  $\phi(\eta)$  for given initial conditions, so the relation  $\phi - V$  is plotted in narrow windows spanning only a part of the possible range of values of  $\phi$ . We treat the initial conditions as arbitrary. The question if there are preferred initial conditions, and then a corresponding particular range for  $\phi$  remains an open problem.

The features of the resulting  $V(\phi)$  are as follows: the scalar field potential has the symmetry  $V(\phi) = V(-\phi)$ , as to be expected from an examination of eq. (4.4), and for

 $\phi > 0$  the overall shape is plotted in fig. 1. V is slightly positive for small values of  $\phi$ , and becomes negative as  $\phi$  increases. Superposed upon the basic overall shape are variations of  $V(\phi)$  on a small scale, which occur in narrow ranges of  $\phi$ ; some of these variations, with local maxima and minima are plotted in fig. 2.

If one accepts the above mentioned idea that the scalar field is the dominant component of dark matter also in the post-inflationary eras of the universe, it is immediate to derive the form of the potential V in the radiation era. In fact, the trace of the energy-momentum tensor is

$$T^{\alpha}{}_{\alpha} = \rho - 3P = 4V - \phi \frac{dV}{d\phi} , \qquad (4.6)$$

which vanishes when  $\gamma = 4/3$ , giving

$$V(\phi) = \lambda \phi^4 , \qquad (4.7)$$

where  $\lambda$  is a constant. This form of the scalar field potential has been used widely in the literature. It should be noted that this result holds only in the case of conformal coupling. For example, when  $\xi = 0$ , SC obtained a different form of V.

### 5 Discussion and conclusion

We have constructed a closed cosmological model which has no singularities. In contrast to other works (e. g. Balbinot *et al.* 1990, Brandenberger *et al.* 1993), we use only general relativity, and an inflationary equation of state in the early universe. We impose discontinuous changes in the equation of state, and solve for the dynamics of the universe. The key ingredient to obtain a singularity-free model is the Gliner-Markov idea that the physical quantities are limited by the Planck values. Our boundary and initial conditions on the scale factor reflect this idea. The evolution equations for the scale factor and the energy density can be solved exactly by using conformal time instead of the more commonly used comoving time. However, the physically significant conformal times have been converted to comoving times. We considered three different eras for the history of the universe, corresponding to different values of the constant  $\gamma$  in the equation of state. The early inflationary era is dominated by prematter, then a radiation-dominated and a matter-dominated era follow. We determined the times of transition between the different eras, the present age of the universe, the time of its maximal expansion, and the Hubble parameter, for various values of the parameter  $\gamma_p$ . We obtained values of  $H_0$  and  $t_{now}$  in a range which agrees with the observations and the current theoretical expectations.

The only source of gravitation is provided by a scalar field, which is chosen to be conformally coupled to the Ricci curvature, for the above mentioned reasons. We proceeded to the numerical solution of the equations for the scalar field and the potential as functions of conformal time. From these, we derived the form of the potential  $V(\phi)$ . It is interesting to note that the derived potential differs from that computed in SC for a minimally coupled field. Moreover, the greater complexity of the non-minimal coupling prevents one from deriving a differential equation, and then an analytical expression for  $V(\phi)$ , as was possible in the SC paper where the coupling was minimal.

The potential  $V(\phi)$  exhibits variations on a small scale, with local maxima and minima. One could ask which range of values of  $\phi$  is relevant to the physics of the early universe, and hence what is the shape of the potential for the appropriate range of the scalar field. The answer to this problem is not an easy one, since the actual values of  $\phi$  are determined by the arbitrary initial data set  $(\phi_0, \phi'_0)$ . Although one can try to constrain the value of  $\phi_0$  with arguments based on mass scales (a very speculative approach, but one which has received some attention in the literature – see e.g. Hosotani 1985 and Futamase & Maeda 1989), there are no indications on how to constrain the initial derivative  $\phi'_0$ . It turns out that the minimally coupled case requires only the consideration of one initial value,  $\phi_0$ .

It must be noted that various inflationary scenarios with a nonminimally coupled scalar field and an *a priori* chosen form of the potential encounter problems (Abbott 1981; Starobinsky 1981; Futamase & Maeda 1989; Futamase *et al.* 1989; Bruni 1993). On the other hand, conformal coupling is required if the Einstein equivalence principle is to hold. In our approach, because of the fact that we do not fix a form of the potential, but rather derive it, the previous difficulties are avoided, and we can have both conformal coupling and a viable inflation (at least to the extent explored in this paper).

Our model universe starts very cold and initially static from Planck size and density, and undergoes an inflationary expansion, during which it heats. It must be noted that the possibility of imposing the initial condition a'(0) = 0, and the absence of an initial singularity are by no means trivial, in the light of the Hawking–Penrose singularity theorems (see the formulations in Hawking & Ellis 1973; Wald 1984; and references therein). In fact, the strong energy condition required by these theorems, which can be written as

$$\rho + P \ge 0 \quad \text{and} \quad \rho + 3P \ge 0 , \tag{5.1}$$

is violated for  $\gamma < 2/3$ , which is our case, since  $\gamma_p \sim 10^{-3}$ . On the contrary, in the

standard model starting from the big bang in a radiation-dominated epoch,  $\gamma = 4/3$ . The weak ( $\rho \ge 0$  and  $\rho + P \ge 0$ ) and dominant ( $\rho \ge |P|$ ) energy conditions are satisfied. The inflationary equation of state permits the avoidance of singularities, and in general relativity, inflation (defined as positive acceleration of the scale factor) occurs if and only if the condition  $\rho + 3P \ge 0$  is violated.

Regarding the regularity of the functions of physical interest, the continuity of the scale factor  $a(\eta)$  and its derivative were assured by the proper choice of integration constants in eqs. (3.17)–(3.21). Thus, the Lichnerowicz (1955) conditions on the metric are satisfied and from eq. (2.9), the density  $\rho$  is continuous at the phase transitions. Since we choose different constant values of  $\gamma$  at the three different universe epochs  $(\gamma_p \sim 10^{-3}, \gamma_r = 4/3, \gamma_m = 1 \text{ respectively})$ , the pressure from eq. (2.11) is discontinuous at the phase transitions.

Since the idea of treating the scalar field as the primary source of gravitation in the radiation- and matter-dominated epochs is a speculation at this stage, we only found  $V(\phi)$  in the radiation era (eq. (4.7)) and we did not present a numerical solution for  $\phi$  and  $V(\phi)$  otherwise for those epochs. We have treated the initial data  $(\phi_0, \phi'_0)$  for the scalar field as arbitrary and we can use the values of  $\phi$  and  $\phi'$  at the end of a given epoch as the initial values for the succeeding epoch. Thus, it is always possible to evolve a continuous solution in  $\phi$ ,  $\phi'$ . However, since the potential  $V(\phi)$  has a different functional form in the different epochs, corresponding to the different equations of state, it is not continuous. The difference in form is evident in comparing the numerical solution in figs. 1, 2 with the analytical form in eq. (4.7). However, if  $\phi(\eta)$ ,  $\phi'(\eta)$  are continuous, then the potential as a function of the conformal time,  $V(\eta)$ , will be continuous as can be seen in eq. (2.6). However, this is not inconsistent with the discontinuity of V as expressed as a function of  $\phi$ .

After the universe reaches the point of maximal expansion, it experiences its past history again in reversed order, until it reaches a point, in a prematter era, in which the derivative of the scale factor vanishes, and then the past history is repeated. The vanishing of a' makes it possible to match the two FLRW metrics before and after the bounce in such a way that the scale factor and its derivative are continuous, thus satisfying the Lichnerowicz (1955) conditions.

We did not address the problem of density perturbations and the formation of structures in our model universe. Perturbations of a non-minimally coupled scalar field suffer from the problems pointed out in sec. 2, and will be the subject of future analysis. Possible generalizations of this paper include the study of open models and of inhomogeneous metrics.

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#### Table and figure captions:

**Table 1:** results for  $\gamma_p = 1.8500 \cdot 10^{-3}$ .

**Table 2:** results for  $\gamma_p = 1.9000 \cdot 10^{-3}$ .

**Table 3:** results for  $\gamma_p = 1.9500 \cdot 10^{-3}$ .

**Table 4:** results for  $\gamma_p = 2.0000 \cdot 10^{-3}$ .

**Table 5:** results for  $\gamma_p = 2.0100 \cdot 10^{-3}$ .

**Table 6:** results for  $\gamma_p = 2.0153 \cdot 10^{-3}$ .

**Figure 1:** the overall shape of the scalar field potential  $V(\phi)$  (rescaled by  $10^{68}$  cm<sup>-2</sup>, in units such that G = c = 1) for  $\phi > 0$ . The plot is obtained by superposing different windows like that in fig. 2, each spanning a small range of values of  $\phi$ . The plots in fig. 1 and Fig. 2 correspond to  $T_m = 2.4805 \cdot 10^4$  K and  $\gamma_p = 2.0153 \cdot 10^{-3}$ .

**Figure 2:** an example of the variations of  $V(\phi)$  on a small scale, with local maxima and minima.

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