The language of Einstein spoken by optical instruments

Sibel Başkal [1]

Department of Physics, Middle East Technical University,
06531 Ankara, Turkey

Y. S. Kim[2]

Department of Physics, University of Maryland,
College Park, Maryland 20742, U.S.A.

Abstract

Einstein had to learn the mathematics of Lorentz transformations in order to complete his covariant formulation of Maxwell's equations. The mathematics of Lorentz transformations, called the Lorentz group, continues playing its important role in optical sciences. It is the basic mathematical language for coherent and squeezed states. It is noted that the six-parameter Lorentz group can be represented by two-by-two matrices. Since the beam transfer matrices in ray optics is largely based on two-by-two matrices or $ABCD$ matrices, the Lorentz group is bound to be the basic language for ray optics, including polarization optics, interferometers, lens optics, multilayer optics, and the Poincaré sphere. Because the group of Lorentz transformations and ray optics are based on the same two-by-two matrix formalism, ray optics can perform mathematical operations which correspond to transformations in special relativity. It is shown, in particular, that one-lens optics provides a mathematical basis for unifying the internal space-time symmetries of massive and massless particles in the Lorentz-covariant world.

1   Introduction

Before formulating his special relativity in 1905, Einstein studied Maxwell's equations and concluded that classical electromagnetic theory is consistent with the Lorentz-covariant space and time, instead of the Galilean world on which Newton's mechanics is based [1]. When he wrote down Newton's $\vec{f} = m\vec{a}$ with the Lorentz force for $\vec{f}$, which is covariant under Lorentz transformations, Einstein to had to make $m\vec{a}$ Lorentz-covariant.

Indeed, Einstein introduced to physics the mathematics of Lorentz transformations formulated earlier by Henri Poincaré, which is known today as the Lorentz group. In addition, as is well known, Einstein established the relation between the

---

[1]electronic address:baskal@newton.physics.metu.edu.tr

[2]electronic address: yskim@physics.umd.edu

photon energy and the frequency of the associated light wave. In so doing, Einstein showed that photons are massless particles in the Lorentz-covariant world.

Newton had to invent a new mathematics now called calculus to formulate his physical laws. Newton's mathematical influence is not restricted to physics. Calculus is now an important scientific language even economics, biology, and behavioral science. Likewise, the Lorentz group, which Einstein used for his relativity, serves as the underlying scientific language for many different fields of physics, including quantum field theory, the phase-space picture of classical and quantum mechanics, and theories of superconductors.

Recently Einstein's influence is becoming more prominent in optical sciences. It is by now well established that coherent and squeezed states are harmonic-oscillator representations of the Lorentz group [2]. More recently, the Lorentz group has been shown to be the underlying scientific language for classical ray optics. The group of Lorentz transformations consists of four-by-four matrices applicable to the four-dimensional Minkowskian space-time. However, it is mathematically possible to represent the same Lorentz group using much simpler two-by-two matrices.

Since classical ray optics is mostly based on two-by-two matrices, it is essentially the physics of the Lorentz group, as special relativity is. It is remarkable that this physics of two-by-two matrices embraces two completely separate branches of physics. It is straight-forward to rewrite the Jones-matrix formalism in terms of the Lorentz group [3]. Since it is possible to construct mathematically the four-by-four representation from the two-by-two representation, the four-parameter Stokes parameters form a Minkowskian four-vector, on which Einstein's special relativity is based [4].

Para-axial lens optics is also based on two-by-two matrices, so are the optical rays in laser cavities. Thus, both can be regarded as the physics of the Lorentz group. This group allows us to derive some powerful results in lens optics and laser cavities [5, 6, 7]. Multilayer optics involving reflections and transmissions is also the physics of two-by-two matrices [8, 9]. Here also, the Lorentz group can play a fundamental role [10, 11]. These latest developments in ray optics have been summarized in a recent review paper [12].

The simplest matrices, next to one-by-one, are two-by-two matrices. What more is there to learn? Yes, they are simple to deal with if there are two or three two-by-two matrices. If there are more, calculations become tedious and uncontrollable. We need group theory to deal with systematically those complicated matrix multiplications. In addition, those matrices speak Einstein's language for special relativity. By arranging optical instruments, we can perform the mathematics corresponding to Lorentz transformations. Compared with those transformations performed high-energy laboratories, optics experiments are very inexpensive.

In Sec. 2, we illustrate how polarization optics naturally accommodates the language of the Lorentz group. In Sec. 3, we illustrate how one-lens optics, with three two-by-two matrices, can perform the calculation of group contractions which corresponds to unification of internal space-time symmetries of massive and massless

particles.

2 Polarization Optics

Let us consider two optical beams propagating along the z axis. We are then led to the column vector:

$$\begin{pmatrix} A_1 \ \exp\left(-i\left(kz - \omega t + \phi_1\right)\right) \\ A_2 \ \exp\left(-i\left(kz - \omega t + \phi_2\right)\right) \end{pmatrix}. \tag{1}$$

We can then achieve a phase shift between the beams by applying the two-by-two matrix:

$$\begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}. \tag{2}$$

If we are interested in mixing up the two beams, we can apply

$$\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \tag{3}$$

to the column vector.

If the amplitudes become changed by either by attenuation or reflection, we can use the matrix

$$\begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} \tag{4}$$

for the change. In this paper, we are dealing only with the relative amplitudes, or the ratio of the amplitudes.

Repeated applications of these matrices lead to the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{5}$$

where the elements are in general complex numbers. The determinant of this matrix is one. Thus, the matrix can have six independent parameters.

Indeed, this matrix is the most general form of the matrices in the $SL(2, C)$ group, which is known to be the universal covering group for the six-parameter Lorentz group. This means that, to each two-by-two matrix of $SL(2, C)$, there corresponds one four-by-four matrix of the group of Lorentz transformations applicable to the four-dimensional Minkowski space.[13] It is possible to construct explicitly the four-by-four Lorentz transformation matrix from the parameters α, β, γ, and δ. This expression is available in the literature,[13] and we consider here only special cases.

We can translate the above two-by-two matrices into their four-by-four counterparts applicable to the four-dimensional Minkowskian space-time (ct, z, x, y). The phase shift matrix of Eq.(2) corresponds to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}, \tag{6}$$

3

and the rotation matrix of Eq.(3) to

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos\theta & -\sin\theta & 0 \\
0 & \sin\theta & \cos\theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\tag{7}
$$

Repeated applications of these two matrices with different angle parameters will lead to the most general form of the three-dimensional rotation matrix applicable to the three-dimensional space of (z, x, y) [14].

As for the attenuation matrix of Eq.(4), the corresponding four-by-four matrix is

$$
\begin{pmatrix}
\cosh\eta & \sinh\eta & 0 & 0 \\
\sinh\eta & \cosh\eta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\tag{8}
$$

which performs a Lorentz boost along the z direction. Repeated applications of the above three four-by-four matrices lead to the most general form for the Lorentz-transformation matrix.

If the Jones matrix contains all the parameters for the polarized light beam, why do we need the mathematics in the four-dimensional space? The answer to this question is well known. In addition to the basic parameter given by the Jones vector, the Stokes parameters give the degree of coherence between the two rays.

Let us write Eq.(1) as a Jones spinor of the form

$$
\begin{pmatrix}
\psi_1(z, t) \\
\psi_w(z, t)
\end{pmatrix},
\tag{9}
$$

Then the Stokes vector consists of

$$
S_0 = <\psi_1^*\psi_1> + <\psi_2^*\psi_2>, \quad S_1 = <\psi_1^*\psi_1> - <\psi_2^*\psi_2>,
$$

$$
S_2 = <\psi_1^*\psi_2> + <\psi_2^*\psi_1>, \quad S_3 = -i\left(\psi_1^*\psi_2> - <\psi_2^*\psi_1>\right).
\tag{10}
$$

The four-component vector (S_0, S_1, S_2, S_3) transforms like the four-vector (t, z, x, y) under Lorentz transformations. The Mueller matrix is therefore like the Lorentz-transformation matrix.

Why do we need this Stokes four-vector, in addition to the Jones spinor? The Stokes parameters can deal with coherence between the two independent beams. As in the case of special relativity, let us consider the quantity

$$
M^2 = S_0^2 - S_1^2 - S_2^2 - S_3^2.
\tag{11}
$$

Then M is like the mass of the particle while the Stokes four-vector is like the four-momentum.

4

If $M = 0$, the two-beams are in a purely state. As M increases, the system becomes mixed, and the entropy increases. If it reaches the value of S_0, the system becomes completely random. It is gratifying to note that this mechanism can be formulated in terms of the four-momentum in particle physics.[4]

3 One-lens System

In analyzing optical rays in para-axial lens optics, we start with the lens matrix and the translation matrix written as

$$L = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & z \\ -0 & 1 \end{pmatrix}, \tag{12}$$

respectively. Then the one-lens system consists of

$$\begin{pmatrix} 1 & z_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & z_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - z_2/f & z_1 + z_2 - z_1 z_2/f \\ -1/f & 1 - z_1/f \end{pmatrix}. \tag{13}$$

If we assert that the upper-right element be zero, then

$$\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{f}, \tag{14}$$

and the image is focussed, where z_1 and z_2 are the distance between the lens and object and between the lens and image respectively. They are in general different, but we shall assume for simplicity that they are the same: $z_1 = z_2 = z$. We are doing this because this simplicity does not destroy the main point of our discussion, and because the case with two different values has been dealt with in the literature [11]. Under this assumption, we are left with

$$\begin{pmatrix} 1 - z/f & 2z - z^2/f \\ -1/f & z/f - 1 \end{pmatrix}, \tag{15}$$

which can be renormalized to

$$C = \begin{pmatrix} x - 1 & x - 2 \\ x & x - 1 \end{pmatrix}, \tag{16}$$

with $x = z/f$.

Here, the important point is that the above matrices can be written in terms of transformations in the Lorentz group. In the two-by-two matrix representation, the Lorentz boost along the z direction takes the form of Eq.(4), and the rotation along the y axis can be written as Eq.(3). The boost along the x axis takes the form

$$X(\chi) = \begin{pmatrix} \cosh(\chi/2) & \sinh(\chi/2) \\ \sinh(\chi/2) & \cosh(\chi/2) \end{pmatrix}. \tag{17}$$

Then the core matrix of Eq.(16) can be written as

$$Z(\eta)R(\phi)Z(-\eta), \tag{18}$$

or

$$\begin{pmatrix} \cos(\phi/2) & -e^{-\eta}\sin(\phi/2) \\ e^{+\eta}\sin(\phi/2) & \cos(\phi/2) \end{pmatrix}, \tag{19}$$

if $1 < x < 2$, where $Z(\eta)$ corresponds to a boost matrix along the z direction.

If x is greater than 2, the upper-right element of the core is positive and it can take the form

$$Z(\eta)X(\chi)Z(-\eta), \tag{20}$$

or

$$\begin{pmatrix} \cosh(\chi/2) & e^{-\eta}\sinh(\chi/2) \\ e^{+\eta}\sinh(\chi/2) & \cosh(\chi/2) \end{pmatrix}. \tag{21}$$

The expressions of Eq.(18) and Eq.(20) are a Lorentz boosted rotation and a Lorentz-boosted boost matrix along the x direction respectively. These expressions play the key role in understanding Wigner's little groups for relativistic particles [15].

Wigner's little group is the maximal subgroup of the Lorentz group which leaves the four-momentum of a given particle invariant. If the particle is massive, the little group is in the form of the three-dimensional rotation group, as given in Eq.(18) or Eq.(19). If the particle has a space-like momentum, the little group is like Eq.(20) or Eq.(21).

Let us look at their explicit matrix representations given in Eq.(19) and Eq.(21). The transition from Eq.(19) to Eq.(21) requires the upper right element going through zero. This can only be achieved through η going to infinity. If we like to keep the lower-left element finite during this process, the angle ϕ and the boost parameter χ have to approach zero. The process of approaching the vanishing upper-right element is necessarily a singular transformation.

This limiting process, called group contraction [17], plays the key role in unifying the internal space-time symmetries of massive and massless particles. This is like Einstein's $E = \sqrt{(pc)^2 + m^2c^4}$ becoming $E = pc$ in the limit of large momentum or zero mass, as illustrated in Table 1. This aspect of internal space-time symmetry has been discussed extensively in the literature. Table 1 represents a further content of Einstein's energy-momentum relation [16].

On the other hand, the core matrix of Eq.(16) is an analytic function of the variable x. Thus, the lens matrix allows a parametrization which allows the transition from massive particle to massless particle analytically. The lens optics indeed serves as the analogue computer for this important transition in particle physics.

From the mathematical point of view, Eq.(19) and Eq.(21) represent circular and hyperbolic geometries, respectively. The transition from one to the other is not a trivial mathematical procedure. It requires a further investigation.

Let us go back to the core matrix of Eq.(16). The x parameter does not appear to be a parameter of Lorentz transformations. However, the matrix can be written

Table 1: Massive and massless particles in one package. Wigner's little group unifies the internal space-time symmetries for massive and massless particles.

	Massive, Slow	COVARIANCE	Massless, Fast
Energy-Momentum	$E = p^2/2m$	Einstein's $E = [p^2 + m^2]^{1/2}$	$E = p$
Internal Space-time Symmetry	S_3 S_1, S_2	Wigner's Little Group	S_3 Gauge Trans.

in terms of another set of Lorentz transformations. This aspect has been discussed in the literature.[7]

Concluding Remarks

The Lorentz group was introduced to physics by Einstein. He was initially interested in understanding Maxwell's equations. The Lorentz group, the language of Einstein, is the scientific language applicable to all aspects of optical sciences, starting from Maxwell's equations.

References

[1] A. Sommerfeld, *Electrodynamics, Lectures on Theoretical Physics, Vol. III*, translated into English by E. G. Ramberg (Academic Press, New York, 1952).

[2] Y. S. Kim and M. E. Noz, *Phase Space Picture of Quantum Mechanics* (World Scientific, Singapore, 1991).

[3] D. Han, Y. S. Kim, and M. E. Noz, Phys. Rev. E **56**, 6065 (1977).

[4] D. Han, Y. S. Kim, and M. E. Noz, Phys. Rev. E **60**, 1036 (1999).

[5] S. Baskal and Y. S. Kim, Phys Rev. E **63**, 056606 (2001).

[6] S. Baskal and Y. S. Kim, Phys. Rev. **E 66**, 026604 (2002), and the references contained in this paper.

[7] S. Baskal and Y. S. Kim, Phys. Rev. E **67**, 056601 (2003).

[8] R. A. M. Azzam and I. Bashara, *Ellipsometry and Polarized Light* (North-Holland, Amsterdam, 1977).

[9] J. J. Monzón and L. L. Sánchez-Soto, J. Opt. Soc. Am. A, **17**, 1475 (2000); J. J. Monzón, T. Yonte, L. L. Sánchez-Soto, and J. Carinena, J. Opt. Soc. Am. A, **19**, 985 (2002).

[10] E. Georgieva and Y. S. Kim, Phys. Rev. E **64**, 026602 (2001).

[11] E. Georgieva and Y. S. Kim, Phys. Rev. E **68**, 026606 (2003).

[12] S. Baskal, E. Georgieva, Y. S. Kim, and M. E. Noz, J. Opt. B: Quantum Semiclass. Opt. **6** S455 (2004).

[13] Y. S. Kim and M. E. Noz, *Theory and Applications of the Poincaré Group* (Reidel, Dordrecht, 1986).

[14] H. Goldstein, *Classical Mechanics,* 2nd Ed. (Addison-Wesley, Reading, Mass., 1980).

[15] E. Wigner, Ann. Math. **40**, 149 (1939).

[16] Y. S. Kim and E. P. Wigner, J. Math. Phys. **31**, 55 (1990).

[17] E. Inonu and E. P. Wigner, Proc. Natl. Acad. Sci. (U.S.) **39**, 510 (1953).