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An Examination of Counterexamples in *Proofs and Refutations*

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..the first important notions in topology were acquired in the course of the study of polyhedra.
H. Lebesgue

Résumé : Dans son influent *Proofs and Refutations* (Preuves et Réfutations), Lakatos introduit les méthodes de preuves et de réfutations en discutant l'histoire et le développement de la formule $V-E+F=2$ d'Euler pour les polyèdres en 3 dimensions. Lakatos croyait, en effet, que l'histoire du polyèdre présentait un bon exemple pour sa philosophie et sa méthodologie des mathématiques, incluant la géométrie. Le présent travail met l'accent sur les propriétés mathématiques et topologiques qui sont incorporées dans l'approche méthodologique de Lakatos. Pour chaque exemple et contre-exemple utilisé par Lakatos, nous présenterons brièvement sa contrepartie topologique, ce qui nous permettra de présenter les fondations et les motivations mathématiques derrière sa philosophie de la méthodologie des mathématiques et finalement, par ce fait même, nous développerons certaines intuitions sur le fonctionnement de ses notions d'heuristique négative et d'heuristique positive.

Abstract: Lakatos's seminal work *Proofs and Refutations* introduced the methods of proofs and refutations by discussing the history and methodological development of Euler's formula $V - E + F = 2$ for three dimensional polyhedra. Lakatos considered the history of polyhedra illustrating a good example for his philosophy and methodology of mathematics and geometry. In this study, we focus on the mathematical and topological properties which play a role in Lakatos's methodological approach. For each example and

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counterexample given by Lakatos, we briefly outline its topological counterpart. We thus present the mathematical background and basis of Lakatos's philosophy of mathematical methodology in the case of Euler's formula, and thereby develop some intuitions about the function of his notions of positive and negative heuristics.

1 Introduction

Lakatos's influential work *Proofs and Refutations* (PR, afterwards), first published in the *British Journal of Philosophy of Science* in four parts between 1963 and 1964, introduced many new concepts to both philosophy and the methodology of mathematics. From these we focus here on the concept of *heuristics*. We analyze the heuristics employed in PR by focusing on the counterexamples discussed in the original work.

PR introduced the methods of proofs and refutations by discussing the history and methodological development of Euler's formula $V - E + F = 2$ for three dimensional polyhedra, where V, E and F are the number of vertices, edges and faces respectively. Lakatos considered the history of polyhedra to be a good example for illustrating his philosophy and methodology of mathematics and geometry. There are several reasons for this. The first is the fact that the history of Euler's formula spans the paradigm shift from the Cauchyian analytic school to the Poincaréian topological school, and thus is a natural illustration for theory change. The second such reason is that Lakatos's exposition of Euler's formula is a "rationally reconstructed" account of the subject matter, therefore diverging from the actual history in certain respects, as was pointed out by Lakatos in the *Introduction* to PR. However, this drawback does not detract from Lakatos's position as his focus was on theory change rather than precise historiography, evidenced by his remark that "informal, quasi-empirical mathematics does not grow through a monotonous increase of the number of indubitably established theorems, but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations" [Lakatos 1976, 5]. However, as Steiner emphasized, "this conclusion tends to undermine 'formalism' (...)" [Steiner 1983, 503]. In the present paper, our principal goal is to fill in the gap between Lakatosian heuristics and the formalism present in the counterexamples given in PR. Thus, the aforementioned divergence from the actual history of the formula is not relevant to the topic at hand.

Before beginning our exposition, we give a very short summary of the methods of proofs and refutations. However, we do not go into a detailed account of Lakatos's methodology; the interested reader is advised to refer to [Koetsier 1991] and [Kampis, Kvasz, Stöltzner 2002] for a more sophisticated treatment of Lakatos's methodology of proofs and refutations. Here, we use the exposition given by Corfield in [Corfield 2002]. The method of proofs and refutations is comprised of the following methodological steps:

1. Primitive conjecture.
2. Proof (a rough thought experiment or argument, decomposing the primitive conjecture into subconjectures and lemmas).
3. Global counterexamples.
4. Proof re-examined. The guilty lemma is spotted. The guilty lemma may have previously been hidden or misidentified.
5. Proofs of the other theorems are examined to see if the newly found lemma occurs in them.
6. Hitherto accepted consequences of the original and now refuted conjecture are checked.
7. Counterexamples are turned into new examples, and new fields of inquiry open up.

In this work, our aim is to show the heuristic role of the counterexamples. We identify how particular counterexamples helped to improve conjectures and how these counterexamples are identified. We employ certain geometrical and topological notions such as genus, connected component etc., in order to be as mathematically precise as possible.

The present paper is organized as follows. First, we briefly revisit the formula in question and its Cauchy proof. Then, in the second part, by following the train of thought followed by Lakatos himself in PR, we discuss the counterexamples one by one and point out their heuristic functions. We conclude by pointing out some possibilities for future work.

2 The Conjecture and the Proof

The main conjecture discussed in PR is the following:

$$V - E + F = 2,$$

where V, E and F denote the number of vertices, edges and faces respectively for all regular polyhedra. This conjecture is often called the *Descartes–Euler conjecture* for historical reasons. The integer which is the solution to the equation $V - E + F$ for some polyhedron P is called the *Euler characteristic* of P.

The proof given in PR was due to Cauchy. Let us briefly recall it step by step.

Step 1: Imagine that the polyhedron is hollow and made of a rubber sheet. Cut out one of the faces and stretch the remaining faces onto a flat surface (or board) without tearing. In this process, V and E will not alter. Thus, we have $V - E + F = 1$, since we have removed a face.

Step 2: The remaining map to be triangulated. Drawing diagonals for those curvilinear polygons which we obtained after stretching, will not alter $V - E + F$ since E and F increase simultaneously.

Step 3: Remove the triangles. This can be done in one of the two ways: either one edge and one face are removed simultaneously; or one face, one vertex and two edges are removed simultaneously.

At the end of this process, we end up with an ordinary triangle for which $V - E + F = 1$ holds trivially.

However, observe that there are three lemmas that have been used implicitly throughout the proof.

Lemma (i) Any polyhedron, from which a face has been removed, can be stretched flat on a flat surface.

Lemma (ii) In triangulating the map, a new face is obtained for every new edge.

Lemma (iii) There are only two alternatives for removing a triangle out of the triangulating map: the removal of one edge, or the removal of two edges and a vertex. Furthermore, the final result of this process will be a single triangle.

These three lemmas play a rather significant role in the proof. Thus, we focus more on them now.

3 Counterexamples and Their Heuristic Patterns

The first counterexamples are directed to the three lemmas which were seen to have been used in the above proof. We will categorize these counterexamples by following the classification which was employed in PR. The heuristic patterns of each counterexample and refutation played a significant role in PR, and we clarify them here from a topological/mathematical perspective.

3.1 Local but not Global Counterexamples

Local counterexamples contradict the specific lemmas or constructions which were used in the proof without being relevant to the main conjecture. We now consider the method of proof and the lemmas used in the proof.

Removing a triangle from the 'inside' of the triangulized map of the cube In removing a triangle from the 'inside' of the triangulized map, one is able to remove a triangle without removing any single edge or vertex. This, however, is not one of the two ways of removing a triangle given in Lemma (iii), and since this lemma claimed these were the only ways to remove a triangle, this proves the lemma to be false. But, the Euler conjecture clearly does hold for the cube. Thus, we see that this counterexample does not refute the main conjecture but only a lemma which comprised part of its proof. In order to handle this counterexample, a modification of Lemma (iii) is considered.

Modified lemma (iii) The new idea is to remove any boundary triangle by exactly following the pattern described in the third lemma. Yet, another counterexample was given which contradicts the modified Lemma (iii).

Counterexamples by disconnecting the network One can easily remove the triangles by disconnecting the network. In this way, $V - E + F$ is reduced by 1 by the removal of two edges and no vertices. So, Lemma (iii) again needs to be modifed to save the proof.

Second Modification of Lemma (iii) Remove the triangles in such a way that $V - E + F$ is not changed. Alternatively, we can modify the terminology: we call boundary triangles those whose removal does not disconnect the network. Equivalently, the triangles in our network can be numbered that in removing them in the right order $V - E + F$ will not change until we reach the last triangle.

The role of this counterexample was to describe a way to remove the triangles from the triangulized map of the polyhedron in such a way that the proof will succeed. As previously observed, one cannot remove the triangles from the inside of the flat mapping. Thus, the role of this counterexample is that of a positive heuristic as it helped us to prove what we set out to prove. The positive heuristic role of this particular counterexample played a significant role in emphasizing the topological character of Cauchy's proof. The underlying reason for this is the fact that the polyhedron was manipulated adopting a topological point of view, disregarding its metric qualities and quantities. Whatever the size of an edge or a face of the polyhedron, the proof still applies. But, what makes this specific proof applicable? The present counterexample underlines the topological "simple connectedness" property of the proof since, as soon as the planar network is simply connected, the removal of the triangles from outside does not alter the simply connected character of the network. Thus, the topological invariants of the network remain the same. Note that the "removal" operation which makes the proof go smoothly is a homeomorphism. Recall that a homeomorphism (or topological isomorphism) is an isomorphism which preserves topological properties. A homeomorphism, thus, is a bijective and bicontinuous function. As a result, homeomorphic topological objects have the same topological properties. For instance, a sphere and a torus are *not* homeomorphic, thus do not enjoy the same topological properties. In conclusion, this lemma shows that the *simple* topological object is preserved under homeomorphism, and in this particular case, the homeomorphism is the function of removing a triangle from outside of the planar network. Further, this mapping does not violate the simpleness of the topological object.

3.2 Global Counterexamples

Global counterexamples contradict the main conjecture. Essentially, global counterexamples, outside of any reference to the proof of the conjecture, demonstrate that it is false. However, the first global counterexample which was given in PR was also local.

A pair of nested cubes A nested cube is a complex object which is composed of two cubes one inside the other. Hence, it is a counterexample for the first lemma, since the nested cube cannot be stretched onto a plane after a face has been removed from the inner polyhedron. Also, for the nested cube, we have $V - E + F = 4$.

At this stage, the students in the fictional class of Lakatos, disagreed on whether a pair of nested cubes is a genuine polyhedron at all. Thus, they altered some definitions of polyhedra in order to save the conjecture, and suggested the following definitions.

Definition 1 A polyhedron is a solid whose surface consists of polygonal faces.

Definition 2 A polyhedron is a surface consisting of a system of polygons.

Let us now see how these attacks on the conjecture can be evaluated from a heuristic point of view. First, note that in order to deal with the case of the nested cube, the protective belt of the hard core of the theory of polyhedra was employed. This opened up a discussion on the definitions of polyhedron in order to save the conjecture. The underlying motivation was the following. In order to exclude the nested cube as a freak, or monster as Lakatos himself called it, it must be proven that the nested cube is not a polyhedron at all. To accomplish this, one can define the polyhedron as a proof generated concept. As long as the students stuck to the Cauchy proof, a rigorous and precise definition of polyhedra was needed and thus this definition was derived from the proof in such a way that the definition would not conflict with the proof and any of its lemmas. Essentially, this is a very important notion in Lakatos's methodology of scientific research programs (MSRP, henceforth), and this example is one of the examples Lakatos himself used in order to reject the deductivist and Euclidean methodology of mathematics which he attacked in Appendix 2 of PR, entitled "Deductivist Approach vs. Heuristic Approach". As Lakatos clearly and radically considered these concepts, we will not repeat them here. However, we note that, for Lakatos, the proof was in fact needed in order to shape the notion (or the definition) of the polyhedra.

Heuristically speaking, these definitions must be classified as negative heuristics, since they exclude some forms of geometric objects which lie beyond the given two definitions and thus restricted the class of objects to which they apply. Notice that this revision of definitions seems to be an *ad hoc*. After each counterexample, one might need to revise the definitions in order to exclude the monsters. As might be expected, new counterexamples emerged immediately after the new definitions were introduced.

Counterexamples to Definitions 1 and 2 A pair of two tetrahedra which have an edge in common and another pair of two tetrahedra which have a vertex in common serve as counterexamples to Definitions 1 and 2. In PR, they were labeled as Counterexamples 2a and 2b. Observe that both counterexamples fit both definitions. Furthermore, for both counterexamples, we observe that $V - E + F = 3$. In order to exclude these counterexamples, an immediate revision of the definitions took place in a rather ad hoc manner.

Definitions 1 and 2 were very intuitive, so it is not surprising that counterexamples to them were found. Counterexamples 2a and 2b were presented in a way as to show that those definitions were much too restrictive, i.e. their degree of negativity in terms of heuristic was too high. Being rather intuitive but very weak notions, Definitions 1 and 2 included too many objects as being regular polyhedra allowing some to serve as counterexamples.

We now try a new definition.

Definition 3 A polyhedron is a system of polygons arranged in such a way that (1) exactly two polygons meet at every edge and (2) it is possible to go from the inside of any polygon to the inside of any other polygon via a route which never crosses any edge at a vertex.

It is easy to see that in the first twin tetrahedron in our last counterexample, there was an edge at which four polygons meet and in the second twin tetrahedron it was impossible to get from the inside of a polygon of the upper tetrahedron to the inside of another polygon of the lower tetrahedron without traveling via a route which crosses some edge at a vertex.

As a consequence, Counterexamples 2a and 2b provided cause for a revision of the proof generated concept of polyhedra and Definition 3 arose as a result of the aforementioned revision. This was the protective belt of the theory which intervened and protected the hard core of the theory by introducing the Definition 3. The result was the Perfect Definition, which we denote by Definition P.

Definition P. A polyhedron is a system of polygons for which the equation $V - E + F = 2$ holds.

Definition P. is a rather deductivist definition which Lakatos would never agree. We believe that this is the reason why PR contained further material.

Even though, it was the perfect definition, the discussions on polyhedra did not finish with its treatment. The question then becomes, why did Lakatos include this ad hoc definition which was indeed very narrow, restricted and heuristically weak? In our opinion, the underlying reason for the introduction of Definition P. was for showing the fallacies of the Euclidean mathematical tradition which Lakatos attacked in the Appendices of PR. Obviously, in the very narrow domain described by Definition P., the Descartes–Euler conjecture was trivially satisfied as the Definition P. reduced the Eulerness of the polyhedra to a single formula, i.e. $V-E+F=2$, which did nothing to explain the global properties of Eulerness. Moreover, it should also be underlined that this definition was proof-generated, and thus came after the proof. At the first glance, Definition P. seems to be a deductive concept. However, when considering the fact that the Definition P. is proof generated, it is easy to see that it cannot be a deductive notion. We believe that this move was a very significant accomplishment of Lakatos: he turned a deductive-looking concept into a proof generated concept by revising the heuristic presentation.

The Urchin of Kepler The Urchin of Kepler is the original name for the small stellated dodecahedron, and it was a counterexample due to the fact that for this "polyhedron" $V - E + F = -6$. Yet, it satisfied Definition 3.

One of the most interesting counterexamples of PR was definitely the urchin. It was proposed to refute Definition 3. So, by the rules of negative heuristics, Definition 3 was revised to exclude the urchin.

It was clear that Definition 3 needed to be revised after the urchin was introduced. In order to exclude the urchin, another new definition, Definition 4, is proposed.

Definition 4 A polygon is a system of edges arranged in such a way that (1) exactly two edges meet at every vertex, and (2) the edges have no points in common except the vertices.

With Definition 4, the problem of the urchin was avoided since, in the urchin, the edges have common points beyond the vertices. Consequently, Definition 4 was modified again in order to "save" the urchin.

Definition 4' A polygon is a system of edges arranged in such a way that exactly two edges meet at every vertex.

In order to save the urchin, a new version of the definition of the polyhedra appeared, which was Definition 4'. This definition was again a product of negative heuristics, since it excluded a clause of the previous definition and restricted the concept of a polygon even further. The proponent of Definition 4' gave a brilliant explanation in order to justify Definition 4, viz. that one should consider the polygons in space not in the plane. In that way, he thought, the second clause of Definition 4 becomes useless, since "what you think to be a point in common is not really one point, but two different points lying one above the other" [Lakatos 1976, 17].

Clearly, the increase in the spatial dimensions reflects a topological intuition. The lines which intersect in two dimensions may not intersect in three dimensions. One can approach this naive observation from low-dimensional topology or even from linear algebra of vector spaces.

Area of the urchin An attempt at solving the problem of the urchin was made by pointing out the null areas of some polygons inside it. Urchin was tried to be refuted by indicating the null areas of the some polygons in it. However, this attack was parried by a disregard for the connection between the "idea of area" and the "idea of polygon".

This refutation was again via a negative heuristic following the Popperian tradition. However, mathematically speaking this metric approach to geometric objects is far from being a topological approach. As soon as the concept of area is introduced as a quantity, one might ignore, for instance, the degenerate polygons. In other words, two objects might be homeomorphic but might have different areas. For example, consider two cubes, one having area a, the other having area $2a$. Clearly, these two cubes are homeomorphic, but their areas are different.

The problem of null areas can also be considered from a topological point of view. Null areas appear when we have a dimension lacking such that some regions are still covered. It is essentially similar to the crossing lines problem which we discussed above.

The next counterexample we will discuss is the picture frame.

The Picture frame The picture frame was a counterexample to Definition 4 and Definition 4'. It satisfies all the definitions but for the picture frame we have $V - E + F = 0$. Immediately after this observation, the picture frame was handled by a new definition of polyhedra.

Definition 5 In the case of a genuine polyhedron, through any arbitrary points in space, there is at least one plane whose cross-section with the polyhedron consists of a single polygon.

The picture frame satisfied all of the definitions hitherto proposed. However, it had an Euler characteristic of 0. It should be recalled that the picture frame had a "hole", and can only be inflated onto a torus. The reason for this is that a torus has only one hole, or more properly speaking, a torus is of genus one. More precisely, one can always find a homeomorphism between a picture frame and a torus. Thus, the topological properties of a torus and a picture frame are identical. Therefore, it would not be too strange to expect that the picture frame might be handled in some way via these properties. Definition 5 served for this purpose. It emphasized the simple connectedness requirement for Descartes–Euler polyhedra. Observe that if a point is taken in the picture frame, a plane whose cross-section with the picture frame consists of a single polygon does not exist.

Nonetheless, Definition 5 is not the final definition. The following counterexamples challenge Definition 5.

Cylinder The cylinder was a counterexample to Definition 5 and all other definitions presented hitherto. The cylinder was labeled Counterexample 5 since $V - E + F = 1$ for it. The "unusual structure" of the cylinder was given attention in the following discussion on the concept of an edge. It was then claimed that the cylinder cannot be a counterexample to the definitions of a polyhedron, since it was inconsistent with the proper definition of an edge. As a result, a new definition of an edge was proposed.

Definition 6 An edge has two vertices.

The cylinder was addressed in the discussion of the proof-generated concept of an edge. We observe that by defining an edge, the cylinder was excluded from being a polyhedron by the methods of negative heuristics. Nevertheless, this is not the final step. The consequent definition of an edge in this respect can be considered a product of positive heuristics as well. By subsequently discussing the definition of an edge, Lakatos demonstrated precisely how his MSRP worked. One negative heuristic can indeed be followed by a positive heuristic move in such a way that, similar to the Hegelian dialectics, a new concept would be generated.

Having discussed the *defining terms* such as an edge, the discourse now proceeded onto one of the most significant aspects of Lakatosian heuristics. Let us now briefly review one such aspect, monster-barring.

The Method of Monster-Barring "Using this method one can eliminate any counterexample to the original conjecture by a sometimes deft but always ad hoc redefinition of the polyhedron, [or] of its defining terms, or of the defining terms of its defining terms" [Lakatos 1976, 23].

As Lakatos indicated in PR, the method of monster-barring is "always ad hoc". In monster-barring, we may theoretically redefine the terms recursively, i.e. we first start with the terms and define them, and then define the terms that defined the original term and so on. Therefore, the process of definining becomes ad hoc, and we may need another tool to get ourselves out of this vicious circle of ad hocness. This is precisely what Lakatos wanted us to do.

Nonetheless, the method of monster-barring does not necessarily yield positive heuristics. This is because, it can be used to avoid the counterexamples or similarly to improve the conjecture by polishing the defining terms.

A New Statement of the Theorem For all polyhedra that have no cavities, no tunnels and no "multiple structures", we have $V - E + F = 2$.

This new statement excludes all monsters, i.e. exceptions. Yet, this process, as we pointed out, is ad hoc. Although it did exclude all the exceptions, we might not be able to determine whether this is true. In fact, as was remarked in PR, the urchin and the cylinder are counterexamples to this new statement of the theorem.

It is also worthwhile to underline that the method of monster-barring *forgets about* the proof. Instead, it focuses on the definitions, its domain and its truth sets—the objects that satisfy the conjecture. As was said in PR, in monster-barring, "by suitably restricting both conjecture and the proof to a proper domain, the conjecture, which is now true, will be perfected, and the basically sound proof, which is now rigorous, will be perfected and obviously will contain no more false lemmas" [Lakatos 1976, 29].

Let us now briefly review an application of monster-adjustment which "adjusted" the urchin. In other words, the defining terms of the monster (urchin) were redefined in such a way that the monster became a regular.

Method of Monster-Adjustment on the Urchin It was claimed that there were no star-polygons in the urchin, but only triangular faces. 60 faces, 90 edges and 32 vertices give an Euler characteristic of 2. It was then concluded that the urchin is a polyhedron and is Eulerian as well. However, its star-polyhedral interpretation was faulty.

Recall that on the first interpretation of the urchin, it was claimed that its faces were star-polygon. In order to be able to utilize positive heuristics, it was then claimed that the urchin had triangular faces. Thus, this interpretation yielded the observation that the urchin was in fact Eulerian with the appropriate calculations. Yet, some objections were given against the monster-barring interpretation of the urchin. Since all of its triangles lie on the same plane in the groups of five and thus surround a regular pentagon behind the solid angle, it was claimed that the triangular interpretation was false. In this case, these objections played the role of positive heuristics as all of the interpretations or misinterpretations exposed what the urchin really was and how it should have been truly understand. Positive heuristics again helped us how and in which direction we should proceed. In this specific example, we were told how to analyse the urchin, and how to apply it to the Euler conjecture by these positive heuristics.

Discussions on Geometrical Topology Having discussed the monster adjustment, a naive geometrical topological approach is laid out. The picture frame was considered first in this regard. It was concluded that the picture frame cannot be inflated into a sphere or a plane. The reason for this is that the *genus* of the sphere is zero, and this is true even under the planar interpretation of stretching. Namely, it was topologically the "same" to stretch the polyhedra onto the [Euclidean] plane or onto the sphere. In other words, a homeomorphism between these two manifolds exists. From this, it was concluded that the picture frame could be inflated into a torus. It should now be recalled that the torus has genus one. Therefore, the general formula of Euler characteristics in manifolds is:

$$v - e + f = 2 - 2.g(S)$$

where S is the surface onto which the polyhedron is being inflated.

For a torus S, we have the Euler characteristic $V - E + F = 2 - 2 \times 1 = 0$ as mentioned earlier. Hence, it can be concluded that the picture frame is not planar. This is the exact reason which made the picture frame a global counterexample. Since the torus is not simply connected, it was argued that the original conjecture holds only for simple polyhedra, namely for those which, once one of their faces has been removed can be stretched onto a plane.

In this way, the domain of the conjecture and Lemma (i) gets restricted. However, the proof still remains the same.

This interpretation of the picture frame in this context is a positive heuristic. Apparently, by the introduction of the concepts of torus and that of genus, we were shown how to interpret the picture frame correctly. Thus, we observe that the positive heuristics introduced us to a new concept of genus in order to allow us to interpret the non-simple polyhedra precisely. The conjecture was improved and thus we ended up with a new formulation.

Yet, this does not mark the end of the counterexamples. The next such counterexample is called a crested cube.

Crested Cube Counterexample 6 was the crested cube. It agreed with all the definitions hitherto proposed, in that it had no cavities, tunnels or multiple structures. Yet, the Euler characteristic of the crested cube was 3.

The crested cube was handled by modifying Lemma (ii). Consequently, the original conjecture was modified by incorporating the new version of Lemma (ii) as follows: For a simple polyhedron, with all its faces simply connected, $V - E + F = 2$. Thus, in this case, "even though [proofs] may not prove, [they] help [us] to improve the conjecture" [Lakatos 1976, 37]. From this observation of Lakatos, we conclude that the restriction on Lemma (ii) was a positive heuristic since this restriction let us understand the proof better, and thus improve it with a modified lemma. In this specific case, the sides of the crested cube were interpreted as ring-shaped faces. Therefore, the simply-connectedness condition excluded the crested cube as it had sides which were not simply-connected. However, it was then claimed that the ring-shaped face interpretation was misleading. So, we were told by negative heuristic that, this was a misinterpretation and we should reject it, and that the crested cube is a genuine counterexample to the Euler conjecture. "The misinterpretation interpretation" again utilizes the observation that in lower dimensional manifolds, faces which would not intersect in higher dimensions might intersect.

All these discussions resulted in the following formulation of the theorem.

A New Formulation of the Theorem All polyhedra are Eulerian, which (a) simple, (b) have each face simply connected, and (c) are such that the triangles in the planar triangular network, which results from the processes of stretching and triangulating, can be so numbered that, in removing them in the right order, $V - E + F$ will not change until we reach the last triangle. In this formulation the lemmas have been turned into conditions.

The ultimate application of the method of lemma incorporation yielded a new formulation of the theorem. This formulation was, without doubt, a positive heuristic, since it explicitly and clearly revealed what a polyhedron was and to which objects the Euler conjecture applied. Note that, in this new formulation, all the lemmas we discussed previously were turned into conditions. In other words, the "facts" we used in the proof were discussed and found suspicious, and then, in order to alleviate the suspicion, they were pushed to the meta-level as conditions of the theorem.

However, the method of the lemma incorporation suffers from an old problem, that is the problem of induction. How can one ensure that we have incorporated all the lemmas into the formula as conditions?

Hidden Lemmas It was claimed that there were some hidden lemmas in some of the mathematical formulations such as "all triangles have three edges and vertices". In the method of lemma incorporation, one must include all lemmas. However, in this way, the problem of infinite regress appears: how can one determine where to stop turning lemmas into conditions?

From the heuristic point of view, we observe that positive heuristics told us that we must include and incorporate the lemmas into the formula in order to properly apply this method. However, in practice, we included only the lemmas for which we had counterexamples thus far. Ad hocness thus still remains although we adopted a new method.

3.3 Global but not Local Counterexamples

Cylinder Again The cylinder, it was observed, served as a counterexample for both the naive conjecture and the theorem. Therefore, it is a global but not a local counterexample since it refutes the theorem, but not the lemmas. The problem appears when one tries to inflate the cylinder onto a plane by removing the side face (the jacket). At this point, it seems that the cylinder should satisfy another lemma: the resulting planar network needs to be connected. But it turned out that this was a hidden assumption.

Moreover, it was claimed that the cylinder did not satisfy lemma (ii), viz. that is "any face dissected by a diagonal falls into two pieces". The counter argument that followed immediately was the observation that, since one cannot draw a diagonal on circular faces, the cylinder should satisfy the lemma. It was then observed that this interpretation of diagonalization did not follow from the proof due to the fact that it was impossible to arrive at a triangular network and conclude the proof.

We first observe that the problem of cylinder was avoided by a positive heuristic rule which says that, in the proof, the resulting network should be connected. On the other hand, a negative heuristic told that we should not remove the side face to obtain a planar network. So, in the case of the cylinder, both heuristic patterns are used. A similar procedure goes for the applicability of Lemma (ii) since we cannot draw a diagonal inside a circle. This discussion might lead to the proof generated concept of diagonal addressing the question of whether it can be drawn on a circle or not.

3.4 Returning to the Local but not Global Counterexamples

Problem of Content "Proof-analysis, when increasing certainty, decreases content. Each new lemma in the proof analysis, corresponding to a new condition in the theorem, reduces its domain." [Lakatos 1976, 57] Another student then introduced the following terminology: Quasi-convex polyhedra are polyhedra which have at least one face from which we can photograph inside of the polyhedron. One should notice that this interpretation is quite similar to the notion of inflating the polyhedra onto the sphere. In response to this, the theorem next suggested by the student was "All quasi-convex polyhedra with simply-connected faces are Eulerian". Consider the great stellated dodecahedron. For this polyhedron, we have $V - E + F = 2$. It was then proposed that a proof must explain the phenomenon of Eulerianness in its entirety. This is because it was not possible to imagine a proof that would explain the Eulerian character of both convex and concave polyhedra, e.g. the cube and the great stellated dodecahedron respectively, by one single idea. This is to say, the conditions of the theorem were not only sufficient, but also necessary. There must not be any counterexamples in its domain and likewise there must not be any examples outside its domain.

One student thought that the problem was to discover the domain of truth of $V-E+F = 2$. But another disagreed, as for him the problem was to discover the relations between V, E, and F; not only in the case where $V - E + F = 2$, but also when $V - E + F = 0$, $V - E + F = -6$, etc. They then observed that the picture frame with ring-shaped faces both in the front and in the back is Eulerian, but is not a Cauchy polyhedron.

Therefore, in discussing both the inductive and the deductive basis of the method of proofs and refutations, they set out to discuss as well the relations between edges and vertices in polygons.

The idea of deductive reasoning was considered next. The student with this idea performed a pasting on polyhedra which is called *Connected Sum* in topology. He explains [Lakatos 1976, 76]:

> Take two closed normal polyhedra and paste them together along a polygonal circuit so that two faces that meet disappear. Since for the two polyhedra $V - E + F = 4$, the disappearance of two faces in the united poyhedron will just restore the Euler formula. (...) [Then], let us now try a double-pasting test: let us paste the two polyhedra together along two polygonal circuits. Now 4 faces will disappear and for the new polyhedron $V - E + F = 0$.

$$V - E + F = 2 - 2 \times (n - 1) + \sum_{k=1}^{F} e_k$$

Going on in this way he concluded that:[1]

> For a monospheroid polyhedron $V - E + F = 2$, for a dispheroid polyhedron $V - E + F = 0$, (...) for a n-spheroid polyhedron $V - E + F = 2 - 2 \times (n - 1)$.

The immediate counterexample is the crested cube with $V - E + F = 1$, recalled by Omega. But Zeta thinks that his method may not be applied to all poyhedra, but only to all n-spheroidal polyhedra built up according to his construction. Now, Sigma argues about the polyhedra with ringshaped faces. For him, it is possible to construct a ringshaped polygon by deleting in a suitable proof-generated system of polygons an edge without reducing the number of faces. Then, he concludes with the formula:

$$V - E + F = 2 - 2 \times (n - 1) + \sum_{j=1}^{K} \{2 - 2 \times (n_j - 1) + \sum_{k=1}^{F} e_{kj}\}$$

From a heuristic point of view, we remark that the distinctions and the interaction between positive and negative heuristics are used to determine the necessary and sufficient conditions of a mathematical theorem, the Descartes–Euler conjecture. However it would be too hasty to draw the conclusion that *positive heuristics corresponds to necessary conditions whereas negative heuristics correspond to sufficient conditions*, or vice versa. Thus, we avoid making this inference for the time being.

The problem of content is concerned with the task of containing every genuine polyhedron in its domain and excluding every counterexample. As we already pointed out, the tools that were used were primarily positive and negative heuristics. Occasionally, they helped to enlarge and restrict the domain. It is not clear for Lakatos whether they have pre-defined and predetermined tasks.

However, since Lakatos disagrees with Euclidean methodology considerably, we can not expect Lakatos to give definite tasks for both heuristic rules.

1. Omega, Sigma and Zeta are the names of the students in Lakatos's imaginary classroom.

This is precisely the reason why we avoid drawing conclusions too quickly about the correspondence between “sufficient and necessary conditions” and “negative and positive heuristics”.

One of the significant ideas that Lakatos used in PR was the interpretation of the Euler characteristic as a function. This idea was stated by one of the students in Lakatos’s imaginary classroom. This positive heuristic approach enabled us to understand the domain of the Euler characteristic better. In this way, not only some specific integers that the Euler characteristic function returned are considered, but in addition, any integer that can be the output of the function is considered. Hence, in hindsight, it is not surprising that such long and sophisticated formulas were present in the preceding pages. They were the products of positive heuristics and led us to all polyhedral objects with our formulation.

Lakatos concluded PR by a rather optimistic motto which we treat in the following section.

3.5 How Criticism may Turn Mathematical Truth into Logical Truth

In the last part of PR, Lakatos’s optimism reveals itself via his conclusion with an observation about “stretching”.

New Version of the Theorem At this stage, a new version of the theorem was put forward: “For all simple objects with simply-connected faces such that the edges of the faces terminate in vertices $V-E+F=2$”. However, an immediate refutation, namely the twin tetrahedron, was given. In inflating the twin tetrahedron, the critical edge is split into two edges.

Definition 7 Stretching is a bicontinuous one-to-one mapping. Disagreements about its definition were centered on stretching a square along its boundaries. After a few discussions on the concept of simply connectedness the session ended. One student remarked “(...) now I have nothing but problems” [Lakatos 1976, 80].

3.6 Heuristics and Necessitation

Thus far, we have not given a precise definition of the notions of positive and negative heuristics. We will briefly recall their definitions now in order to confirm that our exposition agreed with them.

Lakatos informally defined negative heuristics as “the methodological rule (...) [which] tells us what paths of research to avoid”; and similarly stated that positive heuristics “tells us what paths [of research] to pursue” [Lakatos 1978, 47]. However, in the present paper, our goal is not to elaborate on the concepts of negative and positive heuristics nor on their relations with the hard core of the theory. Thus, we refer the interested reader to the original paper of Lakatos for detailed discussion and exposition [Lakatos 1978].

Kiss remarked that "the aim of heuristic investigations is to find the *method of thinking*, the *rules* by which one can receive results more easily and surely" [emphasis is hers] [Kiss 2002, 243]. Our goal in this respect is not much different from that of Kiss. However, based on our present case study, we refrain ourselves from giving a formal account of positive and negative heuristics which depends on the mathematical necessitation. As was noted by one of Lakatos's colleagues, PR "removes the last Aristotelian element, the element of necessitation, from modern science" [Feyerabend 1975, 14]. In this paper, we have attempted to document why and how PR removed necessitation from topological reasoning.

One of the reasons for this is the fact that some thought experiments and counterexamples had the function both of negative and of positive heuristics which prevented us from concluding the role of these thought experiments and counterexamples with respect to the Aristotelian notion of necessitation. This point also agrees with Lakatos's thesis that mathematics is a quasi-empirical science. In other words "*nothing* in mathematics is self-evident. Self-evidence in mathematics is an illusion" [emphasis is his] [Koetsier 1991, 24].

However, we can pursue Lakatosian methodology further in order to formalize the logic of heuristics by diverging a bit from the actual philosophical notions. Recent work in epistemic logic seems to give hints regarding the logic of heuristics by utilizing modal operators for necessitation [Başkent 2007]. However, it is far from being precise in terms of Lakatosian MSRP. Therefore, the modal logical tool at hand—assuming that modal logic is an appropriate tool for formalizing necessitation—is still far from giving a precise and formal account of Lakatosian heuristics.

4 Conclusion and Future Work

The main aim of this paper was to identify the counterexamples in PR and to discuss whether they play the role of positive heuristics or negative heuristics or both by using some simple and intuitive ideas of geometric topology.

While we were treating Lakatos's philosophy and methodology of mathematics (as he outlined in the Appendices of PR), we refrained from making clear-cut distinctions between the two types of heuristics. This is because, we observed that this distinction would mistakenly lead us to make a similar type of distinction between the correspondence of positive heuristic to the necessary conditions of a theorem and the correspondence of negative heuristic to the sufficient conditions of a theorem. The main reason in our view that Lakatos did not make that kind of distinction lies in the fact that Lakatos did not disagree with the rule of double negation and hence the method of *Reductio*

Ad Absordum [Lakatos 1976]. Therefore, any utilization of intuitionistic and thus modal logical tools to formalize Lakatosian heuristics will be doomed to failure since they cannot be perfectly consistent with these heuristic notions.

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