DEFORMATION CLASSES OF REAL FOUR-DIMENSIONAL CUBIC HYPERSURFACES

S. Finashin, V. Kharlamov

Abstract. We study real nonsingular cubic hypersurfaces $X \subset P^5$ up to deformation equivalence combined with projective equivalence and prove that they are classified by the conjugacy classes of involutions induced by the complex conjugation in $H_4(X)$. Moreover, we provide a graph Γ_{K4} whose vertices represent the equivalence classes of such cubics and edges represent their adjacency. It turns out that the graph Γ_{K4} essentially coincides with the graph Γ_{K3} characterizing a certain adjacency of real non-polarized K3-surfaces.

The most familiar logics in the modal family are constructed from a weak logic called K (after Saul Kripke). Under the narrow reading, modal logic concerns necessity and possibility. A variety of different systems may be developed for such logics using K as a foundation.

Stanford Encyclopedia of Philosophy, Modal Logic.
http://plato.stanford.edu/

§1. Introduction

Hypersurfaces of degree d in the projective space P^n of dimension n naturally form a projective space P^N of dimension $N = \binom{n+d}{d} - 1$. This projective space has certain additional structures that are due to its origin. In particular, it is equipped with a special *discriminant hypersurface* Δ that consists of the polynomials $f \in P^N$ such that $\{f = 0\} \subset P^n$ is singular. In fact, Δ can be seen as the locus of critical values of the projection $\Gamma \to P^N$ of the universal (smooth) hypersurface

$$\Gamma \subset P^N \times P^n, \quad \Gamma = \{\, (f, x) \,|\, f(x) = 0, f \in P^N, x \in P^n\}.$$

As a consequence, over $P^N \setminus \Delta$ we get, in Kodaira-Spencer terminology, a deformation family of nonsingular varieties $\Gamma^0 \to P^N \setminus \Delta$, where $\Gamma^0 \subset \Gamma$ is the preimage of $P^N \setminus \Delta$.

Everything said before works equally well over $\mathbb{C}$ and $\mathbb{R}$. However, there is a principal difference: over $\mathbb{C}$ the space $P^N \setminus \Delta$ is connected and all hypersurfaces $\{f = 0\}$ with $f \in P^N \setminus \Delta$ are deformation equivalent, while over $\mathbb{R}$ this is no longer the case (except $d = 1$) and it is a natural (and classical) task to understand the nature

1991 *Mathematics Subject Classification*. 14P25, 14J10, 14N25, 14J35, 14J70.
The second author is supported by ANR-05-BLAN-0053-01.

of the connected components of $P^N(\mathbb{R}) \setminus \Delta(\mathbb{R})$. We will call two real nonsingular hypersurfaces in P^n *deformation equivalent* if they represent points in the same connected component of $P^N(\mathbb{R}) \setminus \Delta(\mathbb{R})$ (which is a natural deformation equivalence for nonsingular hypersurfaces $X \subset P^n$) and *coarse deformation equivalent* if one hypersurface is deformation equivalent to a projective transformation of the other (which is a natural deformation equivalence for embeddings $X \to P^n$). In the latter case we speak on hypersurfaces in the same *coarse deformation class*.

The group $PGL(n+1,\mathbb{R})$ of real projective transformations of P^n is connected if n is even, and has two connected components otherwise. Therefore, a coarse deformation equivalence coincides with the deformation equivalence if n is even, and otherwise a coarse deformation class contains at most two deformation classes.

In this paper we treat the case of cubic hypersurfaces in P^5 and respond to the following questions: *how many coarse deformation classes exist*; and *how to distinguish hypersurfaces of distinct coarse deformation classes* ?

Responses to similar questions for cubic hypersurfaces of lower dimension go up to Newton (1704) in the case of curves, and to Schläfli and Klein (1858–1873) in the case of surfaces (simple proofs going back to Newton [Ne] and, respectively, Klein [Kl] can be found in [DIK1].). The case of threefolds was treated recently by V. Krasnov [Kr1].

Two real nonsingular cubic curves in P^2 or surfaces in P^3 are deformation equivalent if and only if their real point sets are homeomorphic. Even stronger, "homeomorphic" can be replaced by "having the same homology groups". In fact, there are 2 classes of nonsingular cubic curves: S^1 and $S^1 \sqcup S^1$, and 5 classes of nonsingular cubic surfaces: $\mathbb{R}\mathrm{P}^2 \sqcup S^2$, $\mathbb{R}\mathrm{P}^2$, $\#_3\mathbb{R}\mathrm{P}^2$, $\#_5\mathbb{R}\mathrm{P}^2$, and $\#_7\mathbb{R}\mathrm{P}^2$. Here, $\#$ stands for the connected sum and $\sqcup$, for the disjoint sum.

As is shown by V. Krasnov, there are 9 classes of real nonsingular cubic threefolds in P^4, and they are distinguished by the homology of the real point set plus one additional bit of information: does the real point set realize a trivial or non-trivial $\mathbb{Z}/2$-homology class in the complex point set. In fact, his proof, similarly to Klein's, appeals to a relation between the cubics in P^n and complete intersections of bidegree $(2,3)$ in P^{n-1} (in Krasnov's case $n = 4$). Finally, the classification of real nonsingular cubic threefolds happens to be closely connected with the classification of real canonical curves of genus 4 in P^3, and to the fact that the latter curves form 8 deformation classes.

To solve the problem of coarse deformation classification of real nonsingular cubic fourfolds in P^5, we need to appeal to the complex point sets more systematically and somehow in a deeper way. Namely, considering a real cubic fourfold X we associate with it the involution $c\colon M \to M$ induced by the complex conjugation on $M = H_4(X(\mathbb{C}))$. Let us call the isomorphism class of this involution the *homological type* of X. Our first goal is to prove that this invariant is sufficient (in terminology of [DIK2] it means that the cubic fourfolds are homologically quasi-simple).

Theorem 1.1. *If two real nonsingular cubic hypersurfaces in P^5 have the same homological type, then they are coarse deformation equivalent.*

In fact, we show that the homological type can be replaced by the isomorphism class of the eigenlattice $M_- = \mathrm{Ker}(1+c)$. We also enumerate all the possible eigenlattices, which implies, in particular, that the number of coarse deformation classes of real nonsingular cubic hypersurfaces in P^5 is 75.

In addition to a description of the coarse deformation classes of real cubic fourfolds, our results provide the graph of their adjacency. It turns out that this graph essentially coincides with the adjacency graph for non-polarized real K3-surfaces, or equally with the graph determined by real 6-polarizations (in the latter description, outgoing edges of the graph are real 6-polarizations of the real K3-surfaces representing the vertex; see the precise definitions and formulations in section 3). Behind these strong relations between deformation classes of cubic fourfolds and K3-surfaces there are many reasons. One of them is a certain close relation between their lattices. To underline this important similarity, we decided to call *K*4*-lattice* the middle homology lattice of the cubic fourfolds.

The crucial role in our approach is played by Nikulin's coarse deformation classification of 6-polarized K3-surfaces in terms of involutions on the K3-lattice and his results on the arithmetics of such involutions, see [N1,N2].

The paper is organized as follows. In §2, we discuss the central projection correspondence for the nodal cubic hypersurfaces, which yields the relation between the cubic fourfolds and 6-polarized K3-surfaces. The adjacency graphs of projective hypersurfaces and the lattice graphs of involutions are introduced in §3, where we study the basic properties of these graphs, especially in the case of cubic hypersurfaces in P^5 (K4-graph) and real involutions in the K3-lattice (K3-graph). This section contains also a complete description of the K3-graph. In §4, we construct a morphism from the K4-graph to the K3-graph, which allows us to reconstruct the former from the latter, and therefore to prove the main theorem. In conclusion, we give a few remarks, which contain in particular some open questions. In the Appendix, we provide a list of the real involutions in the K3-lattice and their eigenlattices; this list is used in §3 and §4.

Acknowledgements. The authors thank for the hospitality IHES, where this research was initiated.

§2. The lattices of cubics

2.1. Cubics with a node or a cusp. The discriminant hypersurface $\Delta \subset P^N$ can be naturally stratified. Its principal stratum (the stratum of the highest dimension) is the smooth part of Δ, it is formed by those hypersurfaces which have a non-degenerate double point (so called A_1 singular point, or *node*) and no other singular points. In codimension one there are two strata: one is formed by the hypersurfaces which have two nodes and no other singular points; another is formed by the hypersurfaces which have a *cusp* (quadratic suspension over x^3, called also A_2) as their only singular point. We denote by Δ_0 the union of the principal and cuspidal strata and by $\Delta_0(\mathbb{R})$ its real part, $\Delta_0(\mathbb{R}) = \Delta_0 \cap \Delta(\mathbb{R})$.

The following statement is well known.

Proposition 2.1. $\Delta_0(\mathbb{R})$ *is a topological submanifold of* $P^N(\mathbb{R})$. □

The connected components of $\Delta_0(\mathbb{R})$ will be called *pseudo-deformation classes* in $\Delta_0(\mathbb{R})$ (the attribute "pseudo" reflects a possibility of turning a node into a cusp). The group $PGL(n+1,\mathbb{R})$ naturally acts on $\Delta_0(\mathbb{R})$ and we call *projective classes* in $\Delta_0(\mathbb{R})$ (in other words, projective classes of cubic hypersurfaces with a node or cusp) the orbits of this action. By *coarse pseudo-deformation classes* in $\Delta_0(\mathbb{R})$ we mean the orbits of the induced action on the set of connected components of $\Delta_0(\mathbb{R})$.

There is a natural *central projection correspondence*, between the projective classes of cubic hypersurfaces in P^n with a marked double point and the projective classes of complete intersections of bidegree $(2,3)$ in P^{n-1}.

Namely, start with a cubic hypersurface X in P^n which has a double point at $x \in X$, and consider an affine chart $A^n \subset P^n$ centered at x. Then, the degree 3 equation $f = 0$ of X written in such affine coordinates is defined up to a constant factor and it splits, $f = f_2 + f_3$, into two, quadratic and cubic, homogeneous components f_2, f_3 defined up to a common factor. A linear change of the affine coordinates leads to a simultaneous projective transformation of f_2, f_3 (note that a homothety with the center at x multiplies f_2, f_3 by λ^2, λ^3, where $\lambda \in \mathbb{R}$ in the case of real hypersurfaces). A change of the affine chart centered at x by a projective transformation preserving x leads, in addition to a projective transformation of f_2, f_3, to a transforming f_2, f_3 into $f_2, f_3 + l_1 f_2$, where $1 + l_1 = 0$ is a linear equation defining the infinity hyperplane of the new affine chart. Thus, the central projective correspondence yields a well defined map which associates to the projective class of a cubic hypersurface in P^n with a marked double point a projective class of complete intersections of bidegree $(2,3)$ in P^{n-1}.

In general one should be careful about the difference between the scheme-theoretic and the set-theoretic complete intersections (which both appear in the above central projection correspondence). For our purpose, *nonsingular complete intersections* are sufficient. They can be defined as set-theoretic intersections for which the Jacobian matrix of the defining equations has the rank equal to the codimension. So defined nonsingular complete intersections are scheme-theoretic intersections: if $F_1, \dots, F_k$ are homogeneous polynomials in n variables such that the rank of their Jacobian matrix is equal to k at each point of $X \subset P^{n-1}$ defined by $F_1 = \dots = F_k = 0$, then the ideal of X is generated by $F_1, \dots, F_k$.

Lemma 2.2. *The complete intersections which are obtained by central projection from a cubic hypersurface X in P^n with a marked double point are nonsingular complete intersections if and only if $X \in \Delta_0$.*

Proof. (cf., [Kr1]) Consider an affine chart $A^n \subset P^n$ centered at the double point and denote by $f = f_2 + f_3$ the degree 3 polynomial defining the cubic. A straightforward check shows that the singularity at $0 \in A^n$ of $f_2 + f_3 = 0$ is not of the type A_1 or A_2 if $f_2 = f_3 = 0$ has a singular point which is also a singular point of $f_2 = 0$ (that is a point with $df_2 = f_2 = f_3 = 0$), and that $f_2 + f_3 = 0$ has more than one singular point if $f_2 = f_3 = 0$ has a singular point which is a nonsingular point of $f_2 = 0$ (that is a point with $df_2 \neq 0$). Finally, if $f_2 + f_3 = 0$ has a singular point x different from 0, then $df_2 + df_3 = 0$ at x, hence, by the Euler relation, $2f_2 + 3f_3 = 0$ at x, which implies $f_2 = f_3 = 0$ at x, so that the intersection $f_2 = f_3 = 0$ is singular. □

The group $PGL(n, \mathbb{R})$ naturally acts on the set of real nonsingular complete intersections of bi-degree $(2,3)$ in P^{n-1} and we call *projective classes of real nonsingular complete intersections of bi-degree* $(2,3)$ the orbits of this action.

Lemma 2.3. *The central projection correspondence defines a bijection between the set of projective classes in $\Delta_0(\mathbb{R})$ and the set of projective classes of real nonsingular complete intersections of bi-degree* $(2,3)$.

Proof. Performing a projective transformation of f_2, f_3 leads to a projective transformation of $f = f_2 + f_3$. Since f_2, f_3 generate the ideal of the intersection, to obtain bijectivity it remains to check the effect of independent rescaling of f_2, f_3 and also of replacing f_2, f_3 by $f_2, f_3 + l_1 f_2$, where l_1 is a degree 1 homogeneous polynomial. As we have seen above analyzing the central correspondence, the both modifications lead to a combination of rescaling and a projective transformation of $f = f_2 + f_3$. □

The central projection correspondence extends to real algebraic (or real analytic) families of cubics with a marked double point, where by a real algebraic (respectively, real analytic) family of cubics with a marked double point we mean a family which can be given by a real algebraic (respectively, real analytic) family of cubic equations equipped with a real algebraic (respectively, real analytic) family of their double points. Namely, a local real family of cubics can be calibrated by a real family of projective transformations to have the marked double points at the same point, and then the family of cubics can be defined by a real family of equations $f_2 + f_3$ so that we get in P^{n-1} a well defined, up to projective equivalence, real family of complete intersections, $f_2 = f_3 = 0$. In particular, in accordance with Lemma 2.2, there is a well defined map which associates to the coarse pseudo-deformation class of a real cubic hypersurface in P^n with a node or cusp a coarse deformation class of real nonsingular complete intersections of bidegree $(2, 3)$ in P^{n-1} (here, as usual, we call two real nonsingular complete intersections coarse deformation equivalent if one of them can be connected with a projective transformation of another by a real family of nonsingular complete intersections).

Proposition 2.4. *The coarse pseudo-deformation classes in $\Delta_0(\mathbb{R})$ are in one-to-one correspondence with the coarse deformation classes of nonsingular complete intersections of bi-degree $(2, 3)$ in $P^{n-1}(\mathbb{R})$.*

Proof. Any local real deformation family of nonsingular complete intersections of bi-degree $(2, 3)$ in P^{n-1} can be lifted to a family of (degree 2 and 3) homogeneous polynomials, f_2 and f_3, generating the ideal, and thus to a real family of cubics $f_2 + f_3$, which according to Lemma 2.2 have a node or a cusp as their unique singular point. Thus, the result follows from Lemma 2.3. □

2.2. Homology of singular cubics. Suppose that a cubic X_0 has a node or a cusp at $x \in X_0$ and that it is the only singular point. The central projection of X_0 from x to P^{n-1} is a birational isomorphism. The singularity of X_0 at x is resolved by a simple blowup of P^n at x, and the central projection $X_0 \to P^{n-1}$ lifts to a regular map $\pi : \widehat{X}_0 \to P^{n-1}$ which is the blowup of P^{n-1} at the (codimension two, nonsingular) complete intersection Y representing X_0.

The next proposition applied to $B = P^{n-1}$ and $X = \hat{X}_0$ shows how are related the middle dimensional lattices of $\hat{X}_0$ and Y.

In this proposition, and throughout the paper, by *a real structure* on a complex (algebraic or analytic) variety B we mean an antiholomorphic involution $B \to B$.

Proposition 2.5. *Let $p\colon X \to B$ denote a blowup of a compact complex variety B along a non-singular subvariety $Y \subset B$. Then there exists a short exact sequence*

$$0 \to H^*(B) \xrightarrow{p^* \oplus \mathrm{in}^*} H^*(X) \oplus H^*(Y) \xrightarrow{\mathrm{in}^* + p^*|_Y} H^*(p^{-1}(Y)) \to 0$$

If Y is of codimension 2 in B, this exact sequence implies that

$$H^*(B) \oplus H^*(Y) \xrightarrow{p^* + \mathrm{in}^! \, p^*|_Y} H^*(X)$$

is an isomorphism, and, in particular, for any k the group $H_k(X)$ splits as $H_{k-2}(Y) \oplus H_k(B)$. In the middle dimension, for $\dim B = 2m$, it gives a lattice isomorphism between $H_{2m}(X)$ and $-(H_{2m-2}(Y)) \oplus H_{2m}(B)$.

If Y is a real non-singular codimension 2 subvariety of a complex non-singular variety B with a real structure $c_B : B \to B$, then the induced involutions c_X and c_Y on the lattices are related as $c_X(x \oplus y) = -c_Y(x) \oplus c_B(y)$, for any $x \in H_{2m-2}(Y)$, $y \in H_{2m}(B)$.

Proof. (cf. [A]) The short exact sequence follows from the commutativity of the diagram of long exact sequences of the pairs (B, Y) and $(X, p^{-1}(Y))$ due to the injectivity of the pull back $H^*(B) \to H^*(X)$. To deduce the splitting $H^*(B) \oplus H^*(Y) = H^*(X)$ it is sufficient to use the Leray-Hirsch description of $H^*(p^{-1}(Y))$. The manifold $p^{-1}(Y)$ is the projectivization $\mathbb{P}(\mathcal{N})$ of the normal bundle $\mathcal{N}$ of Y in B, and the normal bundle of $p^{-1}(Y)$ in X is nothing but $O_{\mathbb{P}(\mathcal{N})}(-1)$ (that is the dual of the tautological line bundle of $\mathbb{P}(\mathcal{N})$). All these constructions respect the real structure. □

As it follows for example from Propositions 2.4 and 2.5, all cubics $X \in \Delta_0(\mathbb{R})$ from the same pseudo-deformation class yield the same real homology type of $\hat{X}$ and in particular, the same middle dimensional lattice and the same action of the real structure involution on the lattice.

2.3. Lattice twists. Given an integral quadratic form $q\colon L \to \mathbb{Z}$ on a finite rank free abelian group L, and $v \in L$ with $q(v) = \pm 2$, there is a unique integral quadratic form $q'\colon L \to \mathbb{Z}$ such that $q'(v) = -q(v)$ and $q'(w) = q(w)$ for all $w \in v^\perp$. Such a new quadratic form is given by $q'(x) = b(x, s_v x)$, where b is the bilinear form associated with q and s_v is the reflection $L \to L$ defined by $x \mapsto x - b(x,v)v$ if $q(v) = 2$ and by $x \mapsto x + b(x,v)v$ if $q(v) = -2$. We call q' *the v-twist of* q and denote by $t_v(L)$ the lattice (L, q').

Clearly, each automorphism $g : L \to L$ of q with $gv = \pm v$ is an automorphism of q' (and vice versa).

Lemma 2.6. *The quadratic forms q' and q have the same discriminant group. In particular, q' is non-degenerate (unimodular) if, and only if, q is non-degenerate (unimodular).*

Proof. According to $q'(x) = b(x, s_v x)$, the correlation homomorphisms $L \to L^*$ of q and q' are related by the reflection s_v and, hence, have the same image. It remains to notice that the discriminant group is the cokernel of the correlation homomorphism. □

In what follows we abbreviate $q(x) = x^2$ and $b(x,y) = \langle x, y\rangle = xy$ when it does not lead to a confusion.We denote the discriminant group of L by $\operatorname{discr} L$ or $\mathfrak{L}$. Recall that $\mathfrak{L}$ is a finite group, if q is non-degenerate, and then b induces a finite symmetric bilinear form $\mathfrak{b}\colon \mathfrak{L} \times \mathfrak{L} \to \mathbb{Q}/\mathbb{Z}$. If, in addition, L is even (that is $x^2 = 0 \mod 2$ for any $x \in L$), the discriminant group carries a canonical finite quadratic form $\mathfrak{q} : \mathfrak{L} \to \mathbb{Q}/2\mathbb{Z}$ defined via $\mathfrak{q}(x + L) = x^2 \mod 2\mathbb{Z}$. To extend this construction to odd lattices it is sufficient to select a characteristic element $w \in L \otimes \mathbb{Q}$ (that is an element w such that $b(w, l) = l^2 \mod 2\mathbb{Z}$ for any $l \in L$) and to put $\mathfrak{q}(x + L) = x^2 + b(x, w) \mod 2\mathbb{Z}$.

A finite rank free abelian group L equipped with a symmetric bilinear form

q is called *lattice*. If the form is non-degenerate, we say that the lattice is non-degenerate.

2.4. Homology of non-singular cubics. Our next goal is to relate the middle dimensional lattice of $\widehat{X}_0$ with that of a non-singular perturbation X of X_0. This relation holds for any hypersurface $X_0 \subset P^{2m+1}$ having a single node or a cusp. Since all the hypersurfaces from the same connected component of $\Delta_0(\mathbb{R})$ have the same middle dimensional lattice of $\hat{X}_0$ (and the same cohomology ring) with the same action of the real structure, we will restrict ourselves to the case of hypersurfaces with a node.

Recall that given a generic one-parameter perturbation, $t \mapsto X_t \in P^N$, $t \in \mathbb{C}$, $|t| < \epsilon$, of a hypersurface X_0 with a node, there is a well defined up to sign vanishing class $v_t \in H_{2m}(X_t)$ with $(-1)^m v_t^2 = 2$ (this class is realized by so called *vanishing spheres*, which are lagrangian spheres in X_t, and, hence, their normal bundle is isomorphic to the cotangent bundle). If X_0 is real and belongs to $\Delta_0(\mathbb{R})$, for any real $t \neq 0$, then $c_t(v_t) = \pm v_t$, where the sign $\pm$ depends only on the side of $\Delta_0(\mathbb{R})$, to where X_0 is shifted by the perturbation X_t. Moreover, there are canonical, well defined up to isotopy, diffeomorphisms between non-singular fibers X_t with $\operatorname{Im} t \geqslant 0$, $t \neq 0$, such that for any $t > 0$ it holds $c_t \circ c_{-t} = \mu$, where μ is the one-turn monodromy (also well defined up to isotopy). These diffeomorphisms provide an identification of the Poincaré duality lattices $M = H_{2m}(X_t), t \neq 0, t \in \mathbb{R}$, together with the vanishing cycles $v = v_t$, and in the case of a single node, the monodromy μ is the Picard-Lefschetz transformation $s_v : x \mapsto x - (-1)^m \langle x, v\rangle v$. In particular, in this case $c_+ = (c_{t>0})_* : M \to M$ commutes with $c_- = (c_{t<0})_* : M \to M$ and they coincide on the orthogonal complement $v^\perp$ of v, as it follows from $c_+ \circ c_- = s_v$. The vanishing cycle v belongs to one of the eigenlattices $M_\pm(c_+) = \operatorname{Ker}(1 \mp c_+)$ and jumps to the complementary sublattice $M_\mp(c_-) = \operatorname{Ker}(1 \pm c_-)$ of the other involution.

The comparison of the Euler characteristic of $X_t(\mathbb{R})$ with that of $X_{-t}(\mathbb{R})$ allows us to distinguish the two sides of $D_0(\mathbb{R})$, or in the other words, to coorient $D_0(\mathbb{R})$. Such a coorientation can be also characterized by the action of c_t on the vanishing class, as we formulate in the following Proposition.

Proposition 2.7. *Given a real t, $0 < |t| < \epsilon$, the equality $c_t(v_t) = v_t$ holds if and only if $\chi(X_t(\mathbb{R})) > \chi(X_{-t}(\mathbb{R}))$ (which is equivalent to $\chi(X_{-t}(\mathbb{R})) = \chi(\hat{X}_0(\mathbb{R}))$).*

Proof. All the statements follow from the direct inspection of the *local Euler characteristic* of the real loci. By the latter we mean the Euler characteristic of the real part of the Milnor fiber, $X_t^{loc}(\mathbb{R}) \subset X_t(\mathbb{R})$, in the case of $t \neq 0, t \in \mathbb{R}$, and the Euler characteristic of the real part of a regular neighborhood of the exceptional divisor, $\hat{X}_0^{loc}(\mathbb{R}) \subset \hat{X}_0(\mathbb{R})$, in the case of $t = 0$. The Morse lemma implies that $\chi(X_t(\mathbb{R})) - \chi(X_{-t}(\mathbb{R})) = \chi(X_t^{loc}(\mathbb{R})) - \chi(X_{-t}^{loc}(\mathbb{R})) = 2$ if $c_t(v_t) = v_t$. □

Corollary 2.8. *Increase of $\chi(X_t(\mathbb{R}))$ defines a co-orientation of $D_0(\mathbb{R})$. For the increasing Euler characteristic perturbations X_t, $t \neq 0$, it holds $\chi(\mathbb{R}X_t) = \chi(\mathbb{R}X_{-t}) + 2$.* □

It will be convenient for us to use such an "increasing" coorientation in case of odd m and the opposite, "decreasing" coorientation in case of even m. The side of $D_0(\mathbb{R})$ and the corresponding perturbation X_t for which $c_t(v_t) = (-1)^m v_t$ will be called *the ascendant side* and respectively *the ascendant perturbation*.

Recall next that the exceptional divisor $Q = p^{-1}(x_0) \subset \widehat{X}_0$ is a $(2m-1)$-dimensional quadric in P^{2m} and the normal bundle of Q in $\widehat{X}_0$ is isomorphic to the line bundle induced from $O_{P^{2m}}(-1)$. Thus, the generator $w_Q \in H_{2m}(Q)$ realized by intersection of Q with a codimension $m-1$ plane in P^{2m} provides an element $w = \mathrm{in}_* w_Q \in H_{2m}(\widehat{X}_0)$ with $w^2 = -2$.

Proposition 2.9. *If m is even, then the lattices $M = H_{2m}(X_t), t \neq 0$, and $\widehat{M} = H_{2m}(\widehat{X}_0)$ are related by a v-twist about a vanishing cycle $v \in M$.*

Proof. It is sufficient to compare the exact sequences

$$\begin{aligned} H^{2m-1}(V) =& 0 \to H^{2m}(X_t, V) \to H^{2m}(X_t) \to H^{2m}(V) \cong \mathbb{Z}, \\ H^{2m-1}(Q) =& 0 \to H^{2m}(\widehat{X}_0, Q) \to H^{2m}(\widehat{X}_0) \to H^{2m}(Q) \cong \mathbb{Z}, \end{aligned}$$

where V stands for a vanishing sphere, and to observe that the both homomorphisms $H^{2m}(V) \to H^{2m+1}(X_t, V)$ and $H^{2m}(Q) \to H^{2m+1}(\widehat{X}_0, Q)$ vanish, since $v = [V] \in H_{2m}(X_t)$ and $w \in H_{2m}(\widehat{X}_0)$ are primitive (which follows from $v^2 = 2$ and $w^2 = -2$). □

In what follows we fix a vanishing class v (which is well-defined only up to sign) and select that of the two group isomorphisms $M = \widehat{M}$ given by Proposition 2.9 for which $v = w$.

Corollary 2.10. *Let X_0 be a cubic with a single node, $Y \subset P^{2m}$ be the complete intersection corresponding to X_0, and $X = X_t$ with $t \neq 0$. If m is even, then the lattices of $X, \hat{X}_0, Y$, and P^{2m} are related as follows:*

$$\begin{aligned} H_{2m}(\widehat{X}_0) &= (-H_{2m-2}(Y)) \oplus H_{2m}(P^{2m}(\mathbb{C})), \\ H_{2m}(X) &= t_w(H_{2m}(\widehat{X}_0)), \quad w = h + 2e, \end{aligned}$$

where $h \in H_{2m-2}(Y)$ is the hyperplane-section class and e the canonical generator $[P^m(\mathbb{C})] \in H_{2m}(P^{2m}(\mathbb{C})) = \mathbb{Z}$. All the relations respect the real structure, and, in particular, if X is obtained by an ascendant real perturbation of $\hat{X}_0$ then the involutions $c_X : H_{2m}(X) \to H_{2m}(X)$ and $c_{\hat{X}_0} : H_{2m}(\widehat{X}_0) \to H_{2m}(\widehat{X}_0)$ induced by the real structures coincide as we identify $H_{2m}(X)$ and $H_{2m}(\widehat{X}_0)$ as groups.

Proof. All the statements, except the formula $w = h + 2e$, follow directly from Propositions 2.5 and 2.9. This formula follows from $H_{2m}(\widehat{X}) = (-H_{2m-2}(Y)) \oplus H_{2m}(P^{2m}(\mathbb{C}))$, since $w = \mathrm{in}_* w_Q \in H_{2m}(\widehat{X})$, where $w_Q \in H_{2m}(Q)$ is realized by intersection of Q with a codimension $m-1$ plane in P^{2m}. □

2.5. The lattices K3 and K4. Let X_0 be a cubic hypersurface in P^5 with a unique singular point which is a node, and $Y \subset P^4$ be associated with X_0 (so that $m = 2$ in notation of the previous sections). Denote by $\hat{M}$ and M the lattices $\hat{M} = H_4(\hat{X})$ and $M = H_4(X)$, where X is a perturbation of X_0.

According to Lemma 2.2, Y is non-singular. Therefore, Y is a $K3$-surface which is 6-polarized by a very ample hyperplane section divisor $P, P^2 = 6$ (it may be worth to mention here that conversely any K3-surface of degree 6 in P^4 is a complete intersection of a quadric and a cubic, see, e.g., [SD]). In particular, $L = H_2(Y)$ is the K3-lattice $3U \oplus 2E_8$ (U states for the unique even unimodular lattice of rang

2 (and signature $(1,1)$), and E_8 for the unique even unimodular negative definite lattice of rang 8) with a marked element $h=[P]$, $h^2=6$.

By Corollary 2.10,

$$\hat M = (-L)\oplus\mathbb{Z},$$
$$M = t_w(\hat M),\quad w=h+2e,$$

where e is the generator of $\mathbb{Z}$ (corresponding to $[\mathbb{C}\mathrm{P}^2]\in H_4(\mathbb{C}\mathrm{P}^4)$) and $e^2=1$. Note that $h^2=-6$ in $\hat M$.

If X_0 and X are real, then $\hat X_0$ and Y are real as well. By Corollary 2.10, the action of the real structures on $\hat M$ and L are related as $c_{\hat M}(x\oplus ne)=-c_L(x)\oplus ne$, and if the vanishing class $v\in M$ of a real deformation X is c_M-invariant (the case of ascendant perturbations), then the involutions $c_M(x)$ and $c_{\hat M}(x)$ coincide as we identify M and $\hat M$ as groups.

Let us denote by $\hat M_\pm$, $M_\pm$ and $L_\pm$ the (±1)-eigenlattices of the involutions induced on $\hat M$, M and L by the real structures. The following relations are also immediate consequences of Corollary 2.10.

Lemma 2.11. *If X_0 is real, then*

$$\hat M_+ =(-L_-)\oplus\mathbb{Z}.$$
$$\hat M_- =-\,L_+,$$

and if, an addition, M corresponds to an ascendant real perturbation X of X_0, then

$$M_+ =t_w(-L_-\oplus\mathbb{Z}),$$
$$M_- =-\,L_+\,.\quad\square$$

The lattices $\hat M$ and M are 3-polarized by the fundamental class H of the intersection of $\hat X_0$ and X respectively with a generic projective subspace of codimension 2 (in other words, by the square of the hyperplane section).

Lemma 2.12. *The vanishing class $v\in M$ and polarization class $H\in M$ can be expressed as $v=h+2e$, $H=h+3e$ under the group identification $M=\hat M=L\oplus\mathbb{Z}$.*

Proof. The first relation follows from $v=w$ and $w=h+2e$.

To prove the other one it is sufficient to check that $\langle H,\xi\rangle=\langle h+3e,\xi\rangle$ for $\xi=e$ and also for any $\xi\in -L$. Note that under identification $\hat M=M$ the polarization class $H\in M$ becomes $D\circ D$, where $D\in H_6(\hat X_0)$ is the fundamental class of the proper transform $C\subset\hat X_0$ of the cubic threefold given in P^4 by the equation $f_3=0$. Since e is realized by a generic 2-plane in P^4, we have $\langle H,e\rangle=3=\langle h+3e,e\rangle$. If $\xi\in -L$, then $\langle e,\xi\rangle=0$ and under the group identification $-L=H_2(Y)$ we have $\langle h+3e,\xi\rangle=\langle -H_Y,\xi\rangle$, where $H_Y\in H_2(Y)$ is the fundamental class of a hyperplane section of Y. Therefore, it remains to notice that the normal bundle of $C\cap p^{-1}(Y)$ in $p^{-1}(Y)=\mathrm{Proj}(O(3)\oplus O(2))$ is $O(-1)$, so that $\langle H,\xi\rangle=\langle D\circ D,p^*\xi\rangle=\langle -H_Y,\xi\rangle$. $\square$

Finally, let us observe that the both lattices M and $\hat M$ are odd, with the natural (Chern) representatives of the characteristic classes belonging to M_+ and, respectively, to $\hat M_+$, so that the lattices M_- and $\hat M_-$ are even.

2.6. Bi-nodal 4-cubics and flips of 6-polarized $K3$. Assume that a cubic X in P^5 has two non-degenerate nodes, s_1 and s_2, and that it has no other singular points. Pick projective coordinates $(x_0, x_1, \dots, x_5)$ such that $s_1 = (1, 0, 0, \dots, 0)$ and $s_2 = (0, 1, 0, \dots, 0)$. Then, in the affine coordinates centered at s_1 (here we let $x_0 = 1$) X is given by equation

$$x_1 L + A + (x_1 B + C),$$

and in the affine coordinates centered at s_2 (here we let $x_1 = 1$) it is given by equation

$$x_0 L + B + (x_0 A + C),$$

where L, A, B, and C depend only on $x_2, \dots, x_5$ and have degree $1, 2, 2$, and 3 respectively.

As usual, we associate with a node s_i of X a 6-polarized K3-surface Y_i which is given in P^4 by system of equations $x_1 L + A = x_1 B + C = 0$, if $i = 1$, and $x_0 L + B = x_0 A + C = 0$, if $i = 2$. Each of Y_i has a single singular point, it is a nondegenerate double point and is located at the center of the affine chart. The surfaces Y_i are birationally equivalent and a birational isomorphism between them is given by $x_1 = x_0 \frac{A}{B}$.

Consider the induced biregular isomorphism $f : \hat{Y}_1 \to \hat{Y}_2$ between the minimal models of Y_1, Y_2 and note that as any birational isomorphism between minimal $K3$-surfaces f is biregular. In our case, each $\hat{Y}_i \to Y_i$ contracts a unique (-2)-curve. Denote their homology classes by v_i, and the hyperplane section homology classes by h_i.

Lemma 2.13.

$$\begin{aligned} h_2 &= f_*(2h_1 - 3v_1), \\ v_2 &= f_*(h_1 - 2v_1). \end{aligned}$$

Proof. The classes $h_i - v_i$ are given by the proper transform of hyperplane sections passing through s_i. Since f preserves this condition, it implies $f_*(h_1 - v_1) = h_2 - v_2$. The class $2h_1 - 3v_1$ is given by the proper transform of the section of Y_1 by $A = 0$. Since f maps this section of Y_1 into the section of Y_2 by $x_1 = 0$, it implies $f_*(2h_1 - 3v_1) = h_2$.

§3. The adjacency graphs

3.1. Involutions on unimodular lattices. Let L be an unimodular lattice and $c : L \to L$ its involution (that is an isometry with $c^2 = \mathrm{id}$). Denote by $L_\pm = \mathrm{Ker}(1 \mp c)$ the eigenlattices and by $\mathfrak{L}_\pm$ their discriminant groups. If L is odd, then suppose that there exists a characteristic element w_L belonging either to L_+ or to L_- (recall that *characteristic* are the elements $w \in L$ such that $x^2 = \langle w, x \rangle$ mod 2 for any $x \in L$). This hypothesis allows us to extend the usual definition of the discriminant quadratic forms $\mathfrak{q}_\pm : \mathfrak{L}_\pm \to \mathbb{Q}/2\mathbb{Z}$ to the case of odd L: we put $\mathfrak{q}_\pm(x_\pm + L_\pm) = x_\pm^2 + \langle w_L, x_\pm \rangle \mod 2\mathbb{Z}$ for any $x_\pm + L_\pm \in \mathfrak{L}_\pm$, where $w_L \in L_\pm$ is a characteristic element of L ($w_L \in L_\pm$ is well-defined modulo $2L_\pm$, thus, $\mathfrak{q}_\pm$ is independent of its choice).

As is well-known, $\mathfrak{L}_\pm$ are 2-periodic finite groups with anti-isometric quadratic forms. The group isomorphisms between $\mathfrak{L}_\pm$ and $\mathfrak{L}(c) = L/(L_+ + L_-)$ providing such an anti-isometry are induced by the orthogonal projection $p_\pm\colon L \to L_\pm \otimes \mathbb{Q}$. If, for example, $w_L \in L_-$ then L_+ is even and the above isomorphism between $\mathfrak{L}_+$ and $\mathfrak{L}(c)$ converts $\mathfrak{q}_+$ into $\mathfrak{q}_c : \mathfrak{L}(c) \to \mathbb{Q}/2\mathbb{Z}$ given by $\mathfrak{q}_c(x+L_+ +L_-) = \frac{1}{2}(x^2 + \langle x, cx\rangle)$, while $\mathfrak{q}_-$ is converted into $\frac{1}{2}(x^2 - \langle x, cx\rangle) + \langle x, w_L\rangle = -\mathfrak{q}_c(x + L_+ + L_-) \mod 2\mathbb{Z}$.

The rank of the (isomorphic) 2-periodic groups $\mathfrak{L}_\pm$ and $\mathfrak{L}(c)$ will be denoted by $d(c)$.

We distinguish three types of elements in $L_\pm$. An element $h \in L_\pm$ is called *even* if (h, l) is even for any $l \in L_\pm$, otherwise, h is called *odd*. If h is even, then $\frac{1}{2}h$ defines an element of $\mathfrak{L}_\pm$ and we call h a *Wu element* if $\frac{1}{2}h \in \mathfrak{L}_\pm$ is the characteristic element of the bilinear form $\mathfrak{b}_\pm : \mathfrak{L}_\pm \times \mathfrak{L}_\pm \to \mathbb{Q}/\mathbb{Z}$. As is known (and easy to check), $h \in L_\pm$ is even if and only if $h \in (1 \pm c)L$. In addition, an even $h \in L_\pm$ is a Wu element if and only if $x^2 + \langle x, cx\rangle = \langle x, h\rangle \mod 2$ for any $x \in L$.

A lattice involution $c\colon L \to L$ is called *even*, or *of type* I, if $\langle x, cx\rangle + x^2 = 0 \mod 2\mathbb{Z}$ for any $x \in L$. Otherwise, it is called *odd*, or *of type* II. Note, that c is even if and only if the (isomorphic) bilinear forms $\mathfrak{b}_\pm : \mathfrak{L}_\pm \times \mathfrak{L}_\pm \to \mathbb{Q}/\mathbb{Z}$ are even (that is $\mathfrak{b}_\pm(x, x)$ vanishes, or in the other words, the characteristic element of $\mathfrak{b}_\pm$ vanishes, or equivalently, $\mathfrak{q}_\pm(x)$ is integral for any $x \in \mathfrak{L}_\pm$).

3.2. Adjacency graphs $\Gamma_{d,n}$ and lattice graphs Γ_L. For any $d, n \geqslant 1$, one can consider the graph $\Gamma_{d,n}$ whose vertices are the coarse deformation classes of real nonsingular hypersurfaces of degree d in P^{n+1} and the edges are the coarse pseudo-deformation classes in $\Delta_0(\mathbb{R})$: according to Lemma 2.1, connected components of $\Delta_0(\mathbb{R})$ are topological manifolds; therefore, such a component and its orbit under the action of $PGL(n+1, \mathbb{R})$ is adjacent to at most two coarse deformation classes of nonsingular hypersurfaces, which are, by definition, the vertices joined by the edge. As it follows from Corollary 2.8, if $n = 2m$, then this graph has no *graph-loops*, that is edges joining a vertex with itself, and moreover, since connected components of $\Delta_0(\mathbb{R})$ are co-oriented and this co-orientation is preserved by the actions of $PGL(n + 1, \mathbb{R})$, we can orient the graph pointing the edge from the ascendant side vertex to the opposite side vertex. For any $d, n \geqslant 1$, the graph $\Gamma_{d,n}$ is connected as a non-oriented graph.

Replacing the vertices of $\Gamma_{d,n}$ by the middle homology lattices of the corresponding hypersurfaces and the edges by Picard-Lefschetz transformations as in the beginning of Section 2.4 we come naturally to the following general notion of the lattice graph, Γ_L, for an arbitrary lattice L.

The automorphism group $\mathrm{Aut}(L)$ acts by conjugation on the set of involutions $c : L \to L$ and on the set of pairs (c, x), where $c\colon L \to L$ is an involution and $x \in L_-(c)$. We denote by V_L the set of the orbits of the first action, and by $[c] \in V_L$ the orbit of an involution $c : L \to L$. We let $[c, x]$ denote the orbit of (c, x) under the second action, and by E_L the set of orbits of the pairs (c, x) with $x^2 = -2, x \in L_-(c)$.

The *lattice graph* Γ_L is an oriented graph with the vertex set V_L and the edge set E_L, where an edge $[c, v] \in E_L$ has $[c]$ as its initial vertex and $[c_v]$ with $c_v = c\, s_v = s_v c$ as its terminal vertex. The elements $[c] \in V_L$ and $[c_v] \in V_L$ (respectively, the involutions c, c_v) are called *adjacent*, if they are joined by an edge in Γ_L.

The map which associates with a real nonsingular hypersurface X of degree d in P^{2m+1} the lattice $L = (-1)^{m+1}H_{2m}(X)$ and the involution $(-1)^{m+1}c : L \to L$, where c is induced by the complex conjugation, defines a morphism of oriented graphs $\Phi_{d,2m}\colon \Gamma_{d,2m} \to \Gamma_L$. (It may be worth to notice that Γ_L splits as a disjoint union of subgraphs $\Gamma_{L,p}$, corresponding to a fixed value $p = \sigma_+(L_+(c))$, and the image of $\Phi_{d,2m}$ is contained in the subgraph $\Gamma_{L,p}$, where $p = \frac{1}{2}(\sigma_+(L) - 1)$ for odd m and $p = \frac{1}{2}\sigma_+(L)$ for even.)

An orthogonal pair $v_1, v_2 \in L_-(c)$, $v_i^2 = -2$, defines two paths in Γ_L which connect the vertices $[c]$ and $[c\, s_{v_1} s_{v_2}] = [c\, s_{v_2} s_{v_1}]$: one paths consists of the edge $[c, v_1]$ followed by the edge $[c_{v_1}, v_2]$, and the other one consists of $[c, v_2]$ followed by $[c_{v_2}, v_1]$. The group $\mathrm{Aut}(L)$ acts on the set of such triples (c, v_1, v_2); we denote by $[c, v_1, v_2]$ the orbit of (c, v_1, v_2), and by C_L the set of all such orbits. The graph-cycle in Γ_L that we obtain by following the first path, from $[c]$ along edges $[c, v_1]$, $[c_{v_1}, v_2]$ and then returning along the other path will be called *an elementary cycle*; it depends obviously only on the orbit $[c, v_1, v_2]$, which will be used as a label for this cycle. The vertex $[c]$ will be called *the origin of the elementary cycle* $[c, v_1, v_2]$.

3.3. Characterization of the edges in Γ_L. For $x, y \in L_-(c)$ let us write $x \sim_c y$ if either

(1) x and y are odd,
(2) x and y are Wu elements (in particular the both are even),
(3) x and y are even and both are not Wu elements.

Proposition 3.1. *If $[c, v]$ is an edge of Γ_L, then the ranks $d(c)$ and $d(c_v)$ at the endpoints, $[c]$ and $[c_v]$, of $[c, v]$ are related as follows*

(1) *if v is odd, then $d(c_v) = d(c) + 1$*
(2) *if v is even, then $d(c_v) = d(c) - 1$,*
(3) *v is a Wu element with respect to c if and only if $[c_v]$ is even.*

In particular, if the involution c_v is even, then $d(c_v) = d(c) - 1$.

Proof. $L_-(c_v) \oplus \mathbb{Z}v = L_-(c)$ if $v \in L_-(c)$ is even; otherwise $L_-(c_v) \oplus \mathbb{Z}v$ is a subgroup of index 2 in $L_-(c)$. This implies (1) and (2) and allows to consider $\mathfrak{L}(c_v)$ as an index 2 subgroup of $\mathfrak{L}(c)$ if v is even.

Assume that v is even. Since $(\langle x, c_v x\rangle = \langle x, cx + \langle cx, v\rangle v\rangle = \langle x, cx\rangle + \langle cx, v\rangle\langle x, v\rangle = \langle x, cx\rangle - \langle x, v\rangle^2$ has the same parity as $\langle x, cx\rangle + \langle x, v\rangle$, the characteristic classes $\mathrm{wu}(c) \in \mathfrak{L}(c)$ and $\mathrm{wu}(c_v) \in \mathfrak{L}(c_v) \subset \mathfrak{L}(c)$ of involutions c and c_v satisfy the relation $\mathrm{wu}(c_v) = \mathrm{wu}(c) + v$. □

The graph Γ_L has obviously no graph-loops but in general it may have *multiple edges* (two or more edges connecting the same pair of vertices). In our case of interest, that is for the K3-lattice $L = 3U \oplus 2E_8$, as it follows from Proposition 3.3 below, the graph Γ_L does not contain multiple edges.

The proof of Proposition 3.3 and, moreover, the classification of edges in the case of K3-lattice makes use of the following result that can be found in [N1].

Theorem 3.2. *Assume that even non-degenerate lattices L_i, $i = 1, 2$, with 2-periodic discriminants $\mathfrak{L}_i$ have the same inertia indices $\sigma_\pm(L_1) = \sigma_\pm(L_2)$ and isomorphic finite quadratic forms $\mathfrak{q}_i : \mathfrak{L}_i \to \mathbb{Q}/2\mathbb{Z}$. Then L_i are isomorphic, and moreover, any isomorphism $\mathfrak{L}_1 \to \mathfrak{L}_2$ of quadratic forms $\mathfrak{q}_i$, $i = 1, 2$, is induced by an isomorphism $L_1 \to L_2$ if any of the following conditions is satisfied*

(1) *L_i are indefinite lattices;*
(2) *L_i are definite lattices of the rank $\leqslant 2$.* □

Remark. The case (1) in Theorem 3.2 is non-trivial and it is found in [N1], whereas (2) is straightforward, because the only definite lattices of ranks 1 and 2, with 2-periodic discriminants are $\langle \pm 2\rangle$ and $2\langle \pm 2\rangle$ respectively.

Proposition 3.3. *Assume that $L = 3U \oplus 2E_8$ and that the both eigenlattices $L_\pm = L_\pm(c)$ of a lattice involution $c : L \to L$ are indefinite. Then for any $v_i \in L_-$, $v_i^2 = -2$, $i = 1, 2$, the following conditions are equivalent*

(1) $[c, v_1] = [c, v_2]$;
(2) $[c, v_1]$ *and* $[c, v_2]$ *have the same endpoints,* $[c_{v_1}] = [c_{v_2}] \in V_L$;
(3) $v_1 \sim_c v_2$.

Proof. (1) trivially implies (2), and by Proposition 3.1, (2) implies (3).

Assume (3), consider the orthogonal complements of v_i, $L_-^{v_i} = \{x \in L_- \mid \langle x, v_i\rangle = 0\}$, and put $\mathfrak{L}^i = \operatorname{discr}(L_-^{v_i})$. By definition of $\sim_c$ the elements $v_1, v_2 \in L_-$ are either both even or both odd.

First, restrict our attention to the case of even v_i. Under this assumption, there are orthogonal direct sum decompositions, $L_- = L_-^{v_i} \oplus \mathbb{Z}v_i$ which give orthogonal direct sum decompositions of the discriminants, $\mathfrak{L}_- = \mathfrak{L}^i \oplus \operatorname{discr}\langle -2\rangle$. It implies that $\mathfrak{L}^i$ have the same rank and Brown invariant (it is sufficient to apply the additivity properties of these invariants). They have also the same parity: indeed, it is sufficient to notice that $\mathfrak{L}^i = \mathfrak{L}(c_{v_i})$ and apply Proposition 3.1. Therefore, $\mathfrak{L}^i$ are isomorphic as quadratic spaces (see, for example, [W] and [GM]). Furthermore, due to Theorem 3.2 the lattices $L_-^{v_i}$ are isomorphic. The established isometry $L_-^{v_1} \to L_-^{v_2}$ extends to an automorphism ϕ_- of $L_- = L_-^{v_i} \oplus \mathbb{Z}v_i$ sending v_1 to v_2. Applying Theorem 3.2 to L_+ we can extend the induced by ϕ_- automorphism of $\mathfrak{L}_+ = -\mathfrak{L}_-$ to an automorphism ϕ_+ of L_+. The pair (ϕ_+, ϕ_-) yields an automorphism of (L, c) sending v_1 to v_2.

In the case of odd v_i, we have $\mathfrak{L}^i = -\operatorname{discr}(L_+ \oplus \mathbb{Z}v_i) = \mathfrak{L}_- \oplus \operatorname{discr}\langle 2\rangle$. The underlying isomorphism $\mathfrak{L}^i \to -\operatorname{discr}(L_+ \oplus \mathbb{Z}v_i)$ transforms into v_i an element $w_i \in L_-^{v_i}$ such that $\frac{1}{2}(v_i + w_i)$ belongs to L_-. Thus, there is an isometry $\mathfrak{L}^1 \to \mathfrak{L}^2$ which transforms such a $\frac{1}{2}w_1 + L_-^{v_1}$ into $\frac{1}{2}w_2 + L_-^{v_2}$. Therefore, its lift $\phi : L_-^{v_1} \to L_-^{v_2}$ given by Theorem 3.2 extends to an isometry $\phi_1 : L_- \to L_-$ sending v_1 to v_2. Applying once more Theorem 3.2 like in the previous case we extend ϕ_1 to an isometry of L. □

3.4. The K3-graph. It is convenient to understand by a K3-surface any (not necessarily projective and not necessarily algebraic) nonsingular compact simply-connected complex surface with the trivial first Chern class and to speak on real deformations of real K3-surfaces in a sense of Kodaira-Spencer. Since K3-surfaces have $h^{0,2} = 1$ and the complex conjugation involution maps $H^{0,2}$ to $H^{2,0}$, the involutions induced by the complex conjugation on the K3-lattice $L = 3U \oplus 2E_8$ have $\sigma_+(L_+) = 1$ and $\sigma_+(L_-) = 2$, so that these involutions belong to the component $\Gamma_{L,1}$ of Γ_L. We call K3-*graph* the graph $\Gamma_{L,1}$ with $L = 3U \oplus 2E_8$ and denote this graph, its vertex, edge and elementary cycle sets by, respectively, Γ_{K3}, V_{K3}, E_{K3}, and C_{K3}. Involutions c with $[c] \in V_{K3}$ will be called *real K3-involutions.*

Theorem 3.4.

(1) *A vertex $[c] \in V_{K3}$ is determined by the isomorphism type of $L_+(c)$.*
(2) *The isomorphism type of $L_+(c)$ for $[c] \in V_{K3}$ is determined by the rank of $L_+(c)$ and the rank and parity of its discriminant.*

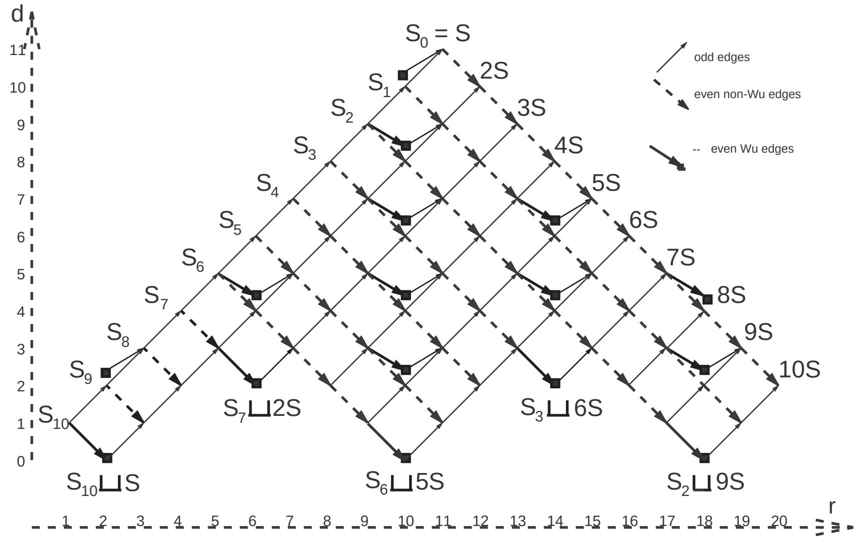


FIGURE 1. THE K3-GRAPH Γ_{K3}

Vertices denoted by ■ represent K3-surfaces of type I. Other vertices represent K3-surfaces of type II. The coordinates are $r = 10 + \frac{1}{2}\chi(X(\mathbb{R}))$ and $d = 12 - \frac{1}{2}\dim H_*(X(\mathbb{R}); \mathbb{Z}/2\mathbb{Z})$ for $X(\mathbb{R}) \neq \varnothing$. The case $(r,d) = (10,10)$ corresponds to $X(\mathbb{R}) = \varnothing$ (type I) and $X(\mathbb{R}) = S_1$ (type II). The case $(r,d) = (10,9)$ corresponds to $X(\mathbb{R}) = 2S_1$ (type I) and $X(\mathbb{R}) = S_2 \sqcup S$ (type II). Other surfaces have $X(\mathbb{R}) = S_p \sqcup qS$, where $p \geqslant 0$ and $q \geqslant 0$ are uniquely determined by (r,d).

(3) *The graph Γ_{K3} is like is shown on Figure 1.*

Proof. Assertions (1) and (2), as well as a list of necessary and sufficient conditions on the rank of $L_+(c)$ and the rank and parity of its discriminant, are contained in [N1]. This gives the set of vertices of Γ_{K3}. All the restrictions on the edges follow from Propositions 3.1 and 3.3, and their existence follows from Propositions 4.2–4.3 applied to the list of the real K3-involutions given in Tables 2–3 of the Appendix. (Note that the existence of all but one edge can be deduced also from [I], which contains a classification of adjacency-edges in the case of 2-polarized K3-surfaces.) □

One can observe furthermore that the cycles $[c, v_1, v_2] \in C_{K3}$, where v_1 is an odd and v_2 is an even nodal class, $v_i \in L_-(c)$, form a basis for 1-homology of Γ_{K3}. We will call them *basic cycles*.

It may be worth to mention (although not really essential for us) that the topological type of the real locus, $X(\mathbb{R})$, of a K3-surface X determines the rank r of L_+ and the rank d of its discriminant (see Figure 1; more details and the references can be found, for example, in [DIK1]). In a number of cases the topological type of $X(\mathbb{R})$ determines as well the parity of c, then it defines a certain vertex $[c] \in V_{K3}$, and, moreover, the topological type is uniquely defined by such a vertex. This is the first kind of vertices. To identify a vertex $[c] \in V_{K3}$ of the second kind we need to determine the parity of the involution. Therefore, when it is appropriate, we will use notation $[T]$, for the vertices of the first kind and for the odd vertices of the second kind. For the even vertices of the second kind we use notation $[T]_I$. Here T is a topological type from the list of real K3-surfaces (see Figure 1), which can be $S_p \sqcup qS$ (disjoint union of an orientable surface of genus p and q spheres), or $2S_1$ (disjoint pair of tori), or $\varnothing$.

3.5. The K4-graph. The graph $\Gamma_{3,4}$ generated by cubic fourfolds is our principal object of interest. We call it *K4-graph* and denote this graph, its vertex set, and its edge set by, respectively, Γ_{K4}, V_{K4}, and E_{K4}.

Proposition 2.3 yields a bijection between E_{K4} and the set of coarse deformation classes of bi-degree $(2,3)$ real K3-surfaces in P^4. On the other hand, as is shown in [N1] the latter set is in one-to-one correspondence with the set of isomorphism classes $[c,h]$ such that $[c] \in V_{K3}$, $h \in L_-(c)$, and $h^2 = 6$. We will use $[c,h]$ as a label for the corresponding K4-edge and denote by $w[c,h]$ and $w_+[c,h]$ its initial and terminal vertices.

The classification of isomorphism classes $[c,h]$ such that $[c] \in V_{K3}$, $h \in L_-(c)$, and $h^2 = 6$ which is given in [N2] can be reformulated as the following classification of K4-edges.

Theorem 3.5. *Edges $[c,h_1],[c,h_2] \in E_{K4}$ coincide if and only if $h_1 \sim_c h_2$. In particular, for any K3-vertex $[c]$, there exist at most three K4-edges $[c,h]$, namely, with $h \in L_-$ being (1) an odd element, (2) a Wu element, or (3) even but not Wu element.* □

These three edges are characterized by the following relation between the discriminant ranks $d(w)$ and $d(w_+)$ of the K4-lattice involutions $c_{M(w)} : M(w) \to M(w)$ and $c_{M(w_+)} \colon M(w_+) \to M(w_+)$ of the cubic fourfolds corresponding to the adjacent pair of vertices $w = w[c,h]$ and $w_+ = w_+[c,h]$.

Corollary 3.6. *For any K4-edge $[c,h]$*

1. *if h is odd, then $d(w_+) = d(w) + 1$,*
2. *if h is even, then $d(w_+) = d(w) - 1$,*
3. *the involution in the lattice $M(w_+)$ is even if and only if h is a Wu element with respect to c.*

Proof. According to Corollary 2.10, $M(w)$ is canonically identified as a group with $L \oplus \mathbb{Z}$, and under this identification the involution $c_{M(w)}$ coincides with $-c \oplus \mathrm{id}$. Furthermore, by Lemma 2.12 the vanishing class $v \in M_+(w)$ is of the same parity as h, namely $v = h + 2e$ where e is a generator of $\mathbb{Z}$. To prove assertions (1) and (2) it remains to notice that the involution $c_{M(w_+)}$ in $M(w_+)$ is equal to $s_v \circ c_{M(w)}$, to use Lemma 2.6 and to apply Proposition 3.1. To prove assertion (3) one needs in addition to check that $c_{M(w_+)}$ is even if and only if h is a Wu element of c. The latter equivalence follows from a v-twist relation $\langle x+ne, s_v c_{M(w_+)}(x+ne)\rangle + \langle x+ne, s_v(x+ne)\rangle = x^2 + \langle x, cx\rangle + \langle x, h\rangle \mod 2\mathbb{Z}$ for any $x \in L$ and $n \in \mathbb{Z}$, which is straightforward. □

3.6. K4-cycles. The automorphism group $Aut(L)$ of the K3-lattice L naturally acts on the set of triples (c,h,v), where $c \in V_{K3}$, $h, v \in L_-(c)$, $h^2 = 6$, $v^2 = -2$, and $v \perp h$. Let $[c,h,v]$ denote the orbit of a triple (c,h,v) and C_{K4} the set of all such orbits.

The following assertion is well known. We give its proof, since did not find a straightforward reference.

Theorem 3.7. *Any* $[c, h_1, v_1] \in C_{K4}$ *can be represented by the complex conjugation, hyperplane section, and exceptional divisor of the nonsingular model of a real bi-degree* $(2, 3)$ *complete intersection K3-surface in* P^4 *with a single node.*

Proof. Pick a generic $\omega = \omega_+ + i\omega_-$ with $\omega_+ \in L_+ \otimes \mathbb{R}$, $\omega_- \in L_- \otimes \mathbb{R}$, $\omega_+^2 = \omega_-^2 > 0$, and $(\omega, h) = (\omega, v) = 0$ (generic in a sense that there is no $l \in L$ orthogonal to ω other than linear combinations of h and v). Due to the surjectivity of the period map, there exists a marked K3-surface Y equipped with an anti-holomorphic involution $\tau : Y \to Y$ and an isometry $\phi : H^2(Y; \mathbb{Z}) \to L$ such that $c \circ \phi = \phi \circ \tau_*$ and $\phi^{-1}\omega \otimes \mathbb{C} = H^{2,0}(Y)$ (to take into account the anti-holomorphic involution, one can apply, for example, [DIK1], Theorem 13.4.3). Due to the prescribed isomorphism type of the lattice generated by h and v and the generic choice of ω, the Picard group $\operatorname{Pic} Y \subset H^2(Y; \mathbb{Z})$ is generated by the fundamental classes v_1, v_2 of two (-2)-curves with $(v_1, v_2) = 4$. Therefore, one can adjust the marking ϕ to have $\phi(v_2) = v$ and $\phi(v_1 + 2v_2) = h$. Then, $H = v_1 + 2v_2$ is a big nef divisor, and since, in addition, $E^2 \neq 0$ for any $E \in \operatorname{Pic}(Y)$, it follows from [SD], Propositions 2.6 and Theorem 5.2, that the linear system $|H|$ has no fixed components or fixed points and, moreover, defines a degree one map onto a normal surface with simple singularities (in our case, the only singularity is the node representing v_2).

It remains to notice that, according, for example, to [SD] Theorem 6.1, any degree 6 normal K3-surface in P^4 is a complete intersection of a quadric with a cubic hypersurface. □

Recall that the central projection correspondence associates to a uni-nodal K3-surface Y_1, which is a degree $(2, 3)$ complete intersection in P^4, a bi-nodal cubic fourfold X_0 with one of the nodes, $x_1 \in X_0$, being marked. Marking the other node, $x_2 \in X_0$, yields another uni-nodal K3-surface, Y_2, related similarly to $[c, h_2, v_2]$, where $h_2 = 2h_1 - 3v_1$, $v_2 = h_1 - 2v_1$ (see Proposition 2.13). The involution $C_{K4} \to C_{K4}$, $[c, h, v] \mapsto [c, 2h - 3v, h - 2v]$, will be called *the flip involution.*

In the space P^N of all cubic fourfolds, the bi-nodal cubics represent the points of transversal self-intersection of the discriminant hypersurface $\Delta \subset P^N$ (cf. 2.1) and the two (intersecting transversally) branches of Δ correspond to the two nodes. The branches are real if the nodes are real. If the nodes are real, then each of these real branches is divided by their intersection into a pair of "halves", which are the adjacent pseudo-deformation classes representing K4-edges $[c, h_i]$ and $[c_{v_i}, h_i]$ (see Figure 2). These edges form a cycle connecting the four vertices of Γ_{K4} presented by the coarse deformation components in $P^N(\mathbb{R})$ which are locally separated by the pair of branches of $\Delta(\mathbb{R})$. Being oriented in accordance with the orientation of the edge $[c, h_i]$, such cycle will be labelled by $[c, h_i, v_i]$. Note that the flip $[c, h_1, v_1] \mapsto [c, h_2, v_2]$ just changes the orientation of the cycle, like the involution $C_{K3} \to C_{K3}$, $[c, v_1, v_2] \mapsto [c, v_2, v_1]$ changes the orientation of the elementary K3-cycles.

Summarizing, we obtain the following result.

Proposition 3.8. *For any real K3-lattice involution* $c \colon L \to L$ *and classes* $h_1, v_1 \in L_-(c)$, $h_1^2 = 6$, $v_1^2 = -2$, $h_1 \perp v_1$, *there exists a cycle in* Γ_{K4} *formed by four edges* $[c, h_1]$, $[c, h_2]$, $[c_{v_1}, h_1]$, *and* $[c_{v_2}, h_2]$, *where* $h_2 = 2h_1 - 3v_1$, $v_2 = h_1 - 2v_1$.

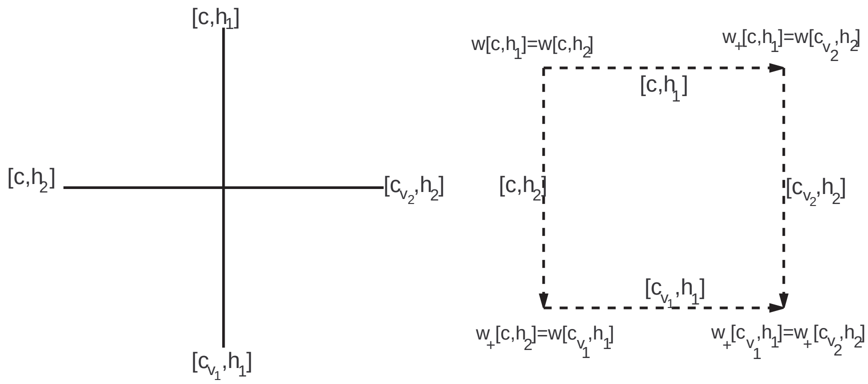


FIGURE 2. Self-intersection of the discriminant $\Delta(\mathbb{R})$ and the corresponding K4-cycle

In the other words, the following endpoints of these edges coincide

$$\begin{aligned} w[c,h_1] &= w[c,h_2] \\ w_+[c,h_1] &= w[c_{v_2},h_2] \\ w_+[c,h_2] &= w[c_{v_1},h_1] \\ w_+[c_{v_1},h_1] &= w_+[c_{v_2},h_2] \quad \square \end{aligned}$$

§4. THE GRAPH ISOMORPHISM $\Gamma^*_{K4} \cong \Gamma^*_{K3}$

4.1. The graph morphism $\Gamma^*_{K4} \to \Gamma^*_{K3}$. We say that vertices $[c] \in V_{K3}$ and $w \in V_{K4}$ correspond to each other if there is an anti-isometry between their eigen-lattices, $M_-(w) = -L_+(c)$. We say also that there is correspondence between edges $[c,v] \in E_{K3}$, $[c,h] \in E_{K4}$ if $h \sim_c v$. Cycles $[c,v_1,v_2] \in C_{K3}$ and $[c,h,v] \in C_{K4}$ correspond to each other if $h \sim_c v_1$ and $v \sim_c v_2$. Note that if cycles $[c,v_1,v_2]$ and $[c,h,v]$ correspond to each other, then after their orientation is reversed they also correspond to each other, i.e., $[c,v_2,v_1]$ corresponds to $[c,2h-3v,h-2v]$. This is because $2h-3v \sim_c v \sim_c v_2$ and $h-2v \sim_c h \sim_c v_1$.

A vertex, an edge, or a cycle (in Γ_{K3} or Γ_{K4}) will be called *regular* if it has a corresponding one (in the other graph), otherwise it will be called *irregular*. The sets of regular vertices, edges, and cycles in Γ_{K3} (or Γ_{K4}) will be denoted V^*_{K3}, E^*_{K3}, and C^*_{K3} (respectively, V^*_{K4}, E^*_{K4}, and C^*_{K4}).

Theorem 3.4(1-2) implies the uniqueness of K3-vertex $[c]$ that corresponds to a vertex $w \in V^*_{K4}$ and Proposition 3.3 implies similarly the uniqueness of $[c,v] \in V^*_{K3}$ that corresponds to $[c,h] \in V^*_{K4}$. Let $F_V : V^*_{K4} \to V^*_{K3}$ and $F_E : E^*_{K4} \to E^*_{K3}$ denote the maps defined by this correspondence.

Proposition 4.1. *$\Gamma^*_{K3} = (V^*_{K3}, E^*_{K3})$ and $\Gamma^*_{K4} = (V^*_{K4}, E^*_{K4})$ are subgraphs of Γ_{K3} and Γ_{K4} spanned by the vertex sets V^*_{K3} and V^*_{K4}, and the pair of maps $F = (F_V, F_E)$ is a morphism of oriented graphs, $F\colon \Gamma^*_{K4} \to \Gamma^*_{K3}$. More precisely,*

1. *the initial vertex $w[c,h]$ is regular for any edge $[c,h] \in E_{K4}$; the terminal vertex $w_+[c,h]$ is regular if and only if the edge $[c,h]$ is regular;*

(2) *an edge* $[c,v]\in E_{K3}$ *is regular if and only if its both endpoints are regular;*
(3) F *preserves incidences of edges and vertices and the order of vertices of an edge; that is, if* $[c,h]\in E^*_{K4}$ *and* $F_E([c,h])=[c,v]$*, then* $F_V(w[c,h])=[c]$ *and* $F_V(w_+[c,h])=[c_v]$*;*
(4) F_E *is bijective;*
(5) F *sends a regular cycle* $[c,h,v]\in C^*_{K4}$ *into the corresponding one,* $[c,v_1,v_2]\in C^*_{K3}$*,* $h\sim_c v_1$*,* $v\sim_c v_2$ *(in particular, all vertices and edges of the corresponding to each other cycles are regular).*

Proof. By Proposition 2.11, $L_+(c)=-M_-(w)$ for any $[c,h]\in E_{K4}$, so $w=w[c,h]$ is regular and $F(w)=[c]$.

Assume that $[c,h]\in E_{K4}$ is a regular edge and consider $v\in L_-(c)$ such that $v^2=-2$ and $h\sim_c v$. If $h,v\in L_-(c)$ are odd, then $L_+(c_v)=L_+(c)\oplus\langle -2\rangle$ and $M_-(w_+)=M_-(w)\oplus\langle 2\rangle$, which implies $F(w_+)=[c_v]$. If $h,v\in L_-(c)$ are even, then a similar decomposition for the complementary lattices, $L_-(c)=L_-(c_v)\oplus\langle -2\rangle$ and $M_+(w)=M_+(w_+)\oplus\langle 2\rangle$, implies that the discriminants of $M_\pm(w_+)$ and $-L_\mp(c_v)$ have the same rank and the same Brown invariant. By Proposition 3.1 and Corollary 3.6, their discriminant forms have the same parity. Thus, Theorem 3.2 applied to the even lattices $M_-(w_+)$ and $-L_+(c_v)$ shows that they are isometric.

Conversely, if $w_+[c,h]$ is regular and corresponds to $[c']$, then $L_+(c')=-M_-(w_+)$, where the latter, on the other hand, is either isomorphic to the lattice $-(M_-(w)\oplus\langle 2\rangle)$, or contains it as a subgroup of index 2. Therefore, $c=s_v\circ c'$, where $v\in L_+(c')$ corresponds to the second summand in $-(M_-(w)\oplus\langle 2\rangle)\subset L_+(c')$. Moreover, according to Proposition 3.1 and Corollary 3.6, $h\sim_c v$. This completes the proof of (1) and (3). Furthermore, (2) is their straightforward corollary.

Surjectivity of F_E holds by its definition and injectivity is due to Theorem 3.5, which yields (4).

For (5), note that $[c,h]$ corresponds to $[c,v_1]$ because $h\sim_c v_1$, $[c,2h-3v]$ corresponds to $[c,v_2]$ because $2h-3v\sim_c v$, $[c_v,h]$ corresponds to $[c_{v_2},v_1]$ because $[c_v]=[c_{v_2}]$ by Proposition 3.3, and similarly, $[c_{h-2v},2h-3v]$ corresponds to $[c_{v_1},v_2]$. □

4.2. Regularity of K3 and K4 edges. A simultaneous description of K3 and K4 edges in the following two lemmas implies their regularity with two exceptions: one K3-edge and one K4-edge.

We refer to the Appendix for the notation of the standard lattices which appear in the direct sum decomposition of $L_\pm(c)$. By the *diagonal component* of $L_-(c)$ for $[c]\in V_{K3}$ we mean the component $s\langle 2\rangle\oplus t\langle -2\rangle$ in the decomposition of $L_-(c)$, as presented in Tables 2–3 in the Appendix. Note that c is an even involution if and only if $t=s=0$. In particular, the diagonal component vanishes for all the involutions in Table 3. An opposite kind of extremal case is $L_-(c)=s\langle 2\rangle\oplus t\langle -2\rangle$, which corresponds to $[c]=[kS]$, $1\leqslant k\leqslant 10$, $s=2$ and $t=10-k$, as is seen from Table 2. It is also obvious from this table that K3-vertices $[10S]$ and $[8S]_I$ are terminal, i.e., do not have outgoing edges, since $L_-([10S])=2\langle 2\rangle$ is positive, and all $x\in L_-([8S]_I)=U(2)$, have x^2 divisible by 4.

Proposition 4.2. *For any given* $[c]\in V_{K3}$ *and* $n\in\{0,1\}$*, the following conditions are equivalent:*

(1) *there exists an odd element* $x\in L_-(c)$*,* $x^2=8n-2$*;*
(2) $[c]$ *differs from* $[kS]$*,* $k\geqslant 1$*, and* $[8S]_I$*.*

In particular, all the odd edges, $[c,v]\in E_{K3}$ and $[c,h]\in E_{K4}$, are regular.

Proof. All elements of $L_-([kS])=2\langle 2\rangle\oplus(10-k)\langle -2\rangle$ and $L_-([8S]_I)=U(2)$ are even, so (1) implies (2). Assuming (2), we can observe in Tables 2–3 that $L_-(c)$ contains as a direct summand one of the following three lattices: U, or $\langle 2\rangle\oplus E_8$, or $U(2)\oplus D_4$. In the first case, $x=(1,k)\in U$ is an odd element for any $k\in\mathbb{Z}$, and in particular for $k=4n-1$, which gives $x^2=2k=8n-2$. In the second case, $x=x_1\oplus x_2\in\langle 2\rangle\oplus E_8$ is an odd element, if $x_1=2n\,e_+$, $x_2=(2n-1)e_-$, where $e_\pm^2=\pm 2$, and $n\in\mathbb{Z}$. It gives $x^2=8n-2$. In the third case, element $x=x_1\oplus x_2\in U(2)\oplus D_4$ is odd with $x^2=4k-2$ if $x_1=(1,k)$ in $U(2)$ and $x_2\in D_4$, $x_2^2=-2$. $\square$

Proposition 4.3. *Let $s\langle 2\rangle\oplus t\langle -2\rangle$ be the diagonal component of $L_-(c)$ for $[c]\in V_{K3}$, and $n\in\{0,1\}$. Then*

(1) *there exists an even element $x\in L_-(c)$, with $x^2=8n-2$ if and only if $t\geqslant 1$;*
(2) *in the case of $[c]\neq[kS]$, existence of a Wu element $x\in L_-(c)$, with $x^2=8n-2$ is equivalent to $s-t=-1 \mod 4$;*
(3) *in the case of $[c]=[kS]$, existence of a Wu element $x\in L_-(c)$, with $x^2=8n-2$ is equivalent to $s-t=4n-1 \mod 8$;*
(4) *there exists a non-Wu even element $x\in L_-(c)$ with $x^2=8n-2$ if and only if one of the following two conditions is satisfied: (a) $t>1$, or (b) $t=1$ and $s-t\neq -1 \mod 4$.*

Proof. Proving (1)–(3), we consider the decomposition of elements $x\in L_-(c)$ as $x=x_1\oplus x_2$, where $x_1\in s\langle 2\rangle\oplus t\langle -2\rangle$ and x_2 is from the complementary non-diagonal component of $L_-(c)$. Note that x is even in $L_-(c)$ if and only if x_2 is even in its non-diagonal component (because any element $x_1\in s\langle 2\rangle\oplus t\langle -2\rangle$ is even). It is straightforward to observe also that x_2^2 is divisible by 4 if x_2 is even (just by checking such a divisibility for the direct summands U, E_8, D_4, $U(2)$ and $E_8(2)$ that appear in the non-diagonal component of $L_-(c)$). Thus, if $s=t=0$ then $x^2=x_2^2\neq 8n-2$. If $s>t=0$, then from Table 2 we can see that the non-diagonal component of $L_-(c)$ contains only summands U and E_8. Observing that even elements of unimodular lattices are obviously divisible by two, we conclude that x_2^2 is divisible by 8. Thus, $x^2=x_1^2 \mod 8$, which cannot be $-2 \mod 8$ for $s\leqslant 2$, because x_1^2 is just the doubled square of an integer if $s=1$ and the doubled sum of squares of two integers if $s=2$. This proves that the condition in (1) is necessary. To show its sufficiency, we observe from Table 2 that if $t\geqslant 1$, then either $s\geqslant 1$, or $L_-(c)$ contains a summand U orthogonal to $\langle -2\rangle$. In the first case, $x=(k+1,k)\in\langle 2\rangle\oplus\langle -2\rangle$, is an even element, $x^2=(4k+2)=8n-2$, if we let $k=2n-1$. If $L_-(c)$ contains $\langle -2\rangle\oplus U$, then the sum of a generator of $\langle -2\rangle$ with $y=(2,2n)\in U$ is an even element of square $8n-2$. This completes the proof of (1).

Since a direct sum decomposition of a lattice yields a direct sum decomposition of the discriminant form, x is a Wu element of $L_-(c)$ if and only if x_1 and x_2 are Wu elements in the corresponding components of $L_-(c)$. It is straightforward to observe that if x_1 is a Wu element, then $x_1^2=2(s-t) \mod 16$, and if x_2 is a Wu element, then x_2^2 is divisible by 8 (indeed, the discriminant form of $s\langle 2\rangle\oplus t\langle -2\rangle$ is $s\langle\frac{1}{2}\rangle\oplus t\langle -\frac{1}{2}\rangle$, the discriminant forms of D_4, $U(2)$, $E_8(2)$ are even, and those of U

and E_8 are trivial). This shows that the conditions $s-t=-1 \mod 4$ in (2) and $s-t=4n-1$ in (3) are necessary for existence of a Wu element x with $x^2=8n-2$.

If $s-t=4n-1$ in (3), then for the existence part of the statement, we need only to observe that $L_-([7S])=2\langle 2\rangle\oplus 3\langle -2\rangle$ contains a Wu class v, $v^2=-2$, and that $L_-([3S])=2\langle 2\rangle\oplus 7\langle -2\rangle$ contains a Wu class h, $h^2=6$.

If $s-t=-1 \mod 4$ in (2), then we note from Tables 2–3 that either $L_-(c)$ contains U, or it contains $s\langle 2\rangle\oplus t\langle -2\rangle\oplus E_8$ with $s=2$. If there is a summand U, then letting $x_2=(2,2k)\subset U$ we obtain $x^2=x_1^2+x_2^2=2(s-t)+8k$, where $x_1=(1,1,\dots,1)\in s\langle 2\rangle\oplus t\langle -2\rangle$. Choosing a suitable k, we obtain a Wu element x with $x^2=8n-2$, if $(s-t)=-1 \mod 4$. Similarly, we obtain a Wu element x with $x^2=8n-2$ in the second case. For example, one can choose $x_1=(3,1,\dots,1)\in s\langle 2\rangle\oplus t\langle -2\rangle$, so that $x_1^2=2(s-t)+16$ is either 6 or 14 (since $1\leqslant s\leqslant 2$ and $0\leqslant t\leqslant 9$), and then let $x_2=2(\kappa_1e_1+\kappa_2e_2)$, for any $e_i\in E_8$, $e_i^2=-2$, $e_1\perp e_2$ and appropriate $\kappa_i\in\{0,1\}$ to obtain $x^2=8n-2$.

The conditions (a) or (b) in (4) are necessary because in the case of $t=1$, $s=0$ the discriminant form is either $\langle -\frac12\rangle$, or $\langle -\frac12\rangle\oplus \mathrm{discr}(D_4)$, which implies that non-Wu even elements $x\in L_-(c)$ have x^2 divisible by 4, in contradiction to $x^2=8n-2$. Sufficiency of (b) follows from (1)–(3). In case of (a), the construction of x in (1) gives a non-Wu even element, because it involves only one $\langle -2\rangle$ summand. □

Corollary 4.4. *There exists only one irregular K3-edge, namely,* $[c,v]$ *with* $[c]=[7S]$*, and a Wu element* $v\in L_-(c)=2\langle 2\rangle\oplus 3\langle -2\rangle$*,* $v^2=-2$*, and only one irregular K4-edge, namely,* $[c',h]$*, with* $[c']=[3S]$ *and a Wu element* $h\in L_-(c')=2\langle 2\rangle\oplus 7\langle -2\rangle$*,* $h^2=6$*.*

The terminal vertices of these edges, $[c_v]=[8S]_I\in V_{K3}$ *and* $w_+[c',h]\in V_{K4}$ *are the only irregular vertices in* V_{K3} *and* V_{K4}*.*

Proof. The criteria of existence for v and h (odd, Wu, and even non-Wu) are all the same except the one in Proposition 4.3(3). The conclusion about the irregular vertices follows from Proposition 4.1(1)-(2). □

4.3. Graph Γ_{K4} and proof of Theorem 1.1.

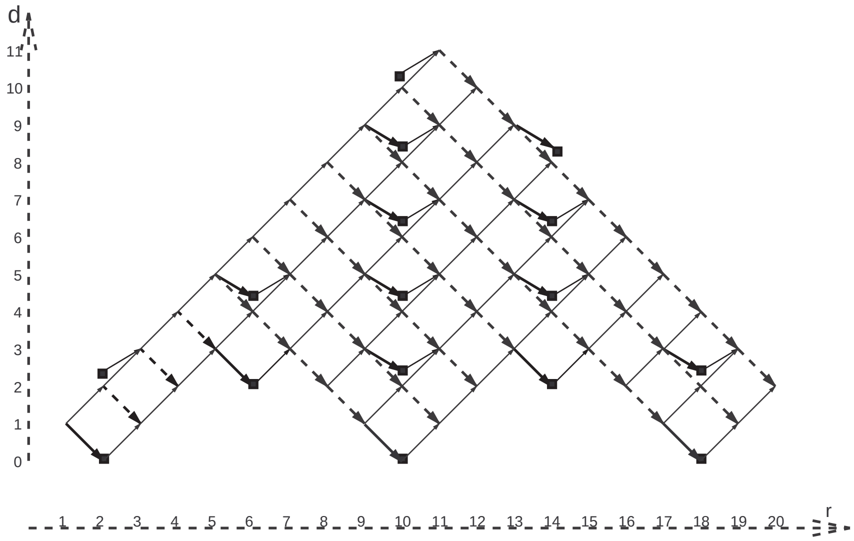
Proposition 4.5. *The basic K3-cycles are all regular. More precisely, for any elementary cycle* $[c,v_1,v_2]\in C_{K3}$ *such that* v_1 *is odd and* v_2 *is even there exists an odd element* $h\in L_-(c)$ *with* $h^2=6$ *and* $h\perp v_2$*, so that* $[c,h,v_2]\in C_{K4}$ *corresponds to* $[c,v_1,v_2]$*.*

Proof. Note that v_1 is an odd element of $L_-(c_{v_2})$ because of the direct sum decomposition $L_-(c)=L_-(c_{v_2})\oplus\langle -2\rangle$, where $\langle -2\rangle$ is generated by v_2. Proposition 4.2 applied to the involution c_{v_2} implies the existence of an odd element $h\in L_-(c_{v_2})$, $h^2=6$, and, thus, $[c,h,v_2]\in C_{K4}$ is such as required. □

Proposition 4.6. *The mapping* $F_V\colon V^*_{K4}\to V^*_{K3}$ *is bijective, and thus* $F\colon \Gamma^*_{K4}\to\Gamma^*_{K3}$ *is an isomorphism of graphs.*

Proof. The both graphs, Γ^*_{K4} and Γ^*_{K3}, are connected, since they are obtained from connected graphs Γ_{K4} and Γ_{K3} by removing irregular vertices and edges, that is, by Corollary 4.4, just one vertex of valency one with the adjacent edge from each graph. On the other hand, F_V is surjective, by its definition, F_E is bijective by Proposition 4.1(4), and F induces an epimorphism at the level of the first homology by Proposition 4.5. All this together implies that F is an isomorphism of graphs. □

FIGURE 3. THE K4-GRAPH Γ_{K4}



The two types of vertices (I and II) and the three types of edges (odd, Wu, and even non-Wu) on Figure 3 are marked in the same way as on Figure 1.

Corollary 4.7. *The adjacency graph Γ_{K4} for real nonsingular cubic hypersurfaces in P^5 is the graph shown on Figure 3. In particular, the number of coarse deformation classes for such hypersurfaces is* 75*.*

Proof. By Proposition 4.6 and Corollary 4.4, to obtain Γ_{K4} one needs just to remove from Γ_{K3} the irregular K3-edge together with its endpoint $[8S]_I$ and to add instead an irregular K4-edge with the origin at the K4-vertex corresponding to $[3S]$. □

Finally, we will prove Theorem 1.1, which can be reformulated as follows.

Theorem 4.8. *If the eigenlattices $M_- = \mathrm{Ker}(1+c)$ of two real nonsingular cubic hypersurfaces in P^5 are isomorphic, then these two hypersurfaces are coarse deformation equivalent. In particular, the coarse deformation classes of real nonsingular cubic hypersurfaces in P^5 are distinguished by the homological type of these hypersurfaces.*

Proof. Injectivity of F_V implies that regular K4-vertices w can be distinguished by their eigenlattices $M_-(w) = -L_+(F_V(w))$, due to Theorem 3.4(1). The irregular vertex, $w \in V_{K4} \setminus V^*_{K4}$, can be also distinguished, because its lattice $M_-(w)$ is not isomorphic to $-L_+(c)$ for all real K3-involutions c.

Namely, $-M_-(w) = U(2) + 3D_4$, since due to Theorem 3.2 it is the only even lattice having the required signature, $(1, 13)$, and an even discriminant form of the required rank $d = 8$. These values follows from Corollary 3.6 applied to the irregular K4-edge. □

4.4. A few related remarks.

Deformations of cubics. [1] In the case of nonsingular cubic hypersurfaces of dimen-

[1] We are grateful to J.Kollár for pointing out to us Fujita's results we are using in this remark.

sion $\geqslant 3$ the coarse deformation equivalence can be reformulated as a deformation equivalence of polarized varieties.

Namely, nonsingular cubic hypersurfaces $X \subset P^n$ come with a natural, projective, polarization L, which is of degree 3 and of Δ-genus $\Delta(X, L) = n - 1 + L^{n-1} - h^0(X, L) = 1$. And vice versa, according to T. Fujita [F] any polarized manifold of degree 3 and $\Delta = 1$ is a cubic hypersurface. In addition, any antiholomoprhic involution $X \to X$ lifts to an involutive anti-isomorphism of L if $n \geqslant 4$ (recall that $X(\mathbb{R}) \neq \varnothing$ for any cubic and $\operatorname{Pic} X = \mathbb{Z}$ for nonsingular cubics in P^n with $n \geqslant 4$), and therefore any real structure $c : X \to X$ is induced by the complex conjugation under a suitable realization of X as a projective cubic if $n \geqslant 4$.

On the other hand, for nonsingular cubic hypersurfaces it holds $K + (n-1)L = L$, which implies that $H^i(X; L) = 0$ for any $i > 0$. By semi-continuity, it implies the invariance of Δ-genus under deformations preserving the polarization. As a result, coarse deformation classes of real nonsingular cubic hypersurfaces of dimension $n - 1 \geqslant 3$ coincide with equivariant deformation classes of real polarized manifolds with polarization of degree 3 and Δ-genus 1.

Some open problems. When the dimension of the projective space P^n is even, the group $PGL(n+1, \mathbb{R})$ is connected, and hence in this case there is no difference between coarse deformation and deformation equivalences as they are defined in Introduction. If this dimension is odd, the group has two connected components, and therefore there may be a certain difference between the two classifications. For the moment, we do not know does it really happen or not in the case of cubics in P^5.

Let us mention a few other open problems concerning real nonsingular cubics in P^5. First, it would be interesting to give an explicit description of the topology of the real part $X_{\mathbb{R}}$ of the cubics in each deformation class. It is not difficult to observe, for instance, that K3-vertex $[\varnothing]$ corresponds to the cubics with $X_{\mathbb{R}} = \mathbb{R}\mathrm{P}^4 \sqcup S^4$, and $[S]$ corresponds to $X_{\mathbb{R}} = \mathbb{R}\mathrm{P}^4$. The simplest candidate to be proposed in the case of $[S_p \sqcup qS]$ is $X_{\mathbb{R}} = \mathbb{R}\mathrm{P}^4 \# p(S^2 \times S^2) \# q(S^1 \times S^3)$.

The second question is to relate the topology of $X_{\mathbb{R}}$ with the topology of the real part of the corresponding Fano varieties, which are, as is known, are nonsingular for any nonsingular cubic and hence also depend only on the coarse deformation class of the cubic. (The case of Fano surfaces of three-dimensional cubics is worked out in [Kr2].)

Some other relations between cubic fourfolds and K3-surfaces. In this paper we related nodal cubic hypersurfaces of dimension 4 with K3-surfaces via a central projection from a double point. As is known, there exist many other connections between them. For example, if (instead of a double point) a cubic hypersurfaces of dimension 4 contains a plane, then the critical locus of the projection from this plane to a complementary plane is a curve of degree 6, so that taking the double covering of the plane ramified in such a curve one obtains once more a K3-surface. Another option is to consider the Fano variety of the cubic. In dimension 4, this variety (which is always nonsingular) is deformation equivalent to the symmetric square of a K3-surface. (It may be worth to mention that these deformation equivalent varieties are hyperkähler and the Fano varieties in question provide a hypersurface in the corresponding moduli space.)

In fact, the main reason of relating the cubic hypersurfaces with the K3-surfaces is that the moduli of K3-surfaces are much better understood than those of cubic hypersurfaces them-selves. In particular, in the case of K3-surfaces one disposes the surjectivity of the period map. One can expect that a similar phenomenon holds for the cubic hypersurfaces of dimension 4. Would such a result be available, the proofs of the classification developed in this paper could mimic the proofs of the corresponding results for K3-surfaces, which would be more simple and more natural. (The injectivity of the period map and the Torelli theorem established by C. Voisin [V] are not sufficient for our purpose.)

Appendix. The list of the eigenlattices of the real $K3$-involutions.

Our list of the eigenlattices $L_\pm(c)$ of the real K3-involutions $c\colon L\to L$ splits into 3 tables. In the first column of each table, K3-vertices $[c]$ are characterized by the real locus $S_p\sqcup qS$, or $2S_1$, or $\varnothing$, of the corresponding K3-surfaces, as explained in 3.4 (cf. also the caption for Figure 1). The first two tables describe the so called *principal series* of lattices $L_\pm$, characterizing the K3-vertices $[S_p\sqcup qS]$ (on Figure 1, these vertices have integer coordinates). The third table covers the remaining cases and describe $L_\pm$ for K3-vertices of type $[S_p\sqcup qS]_I$, and in addition $[\varnothing]$ and $[2S_1]$ (on Figure 1, these vertices are presented by black squares which are slightly shifted above the points matching their values of (d,r)). In the first table, we list the lattices L_+ fixing in each row the value of q in $[S_p\sqcup qS]$, while in the second table we list the lattices L_- and fix p.

By $\langle\pm2\rangle$ we mean the lattice of rank 1 generated by an element with square ±2, and by U the even unimodular lattice of rank 2. Standard notation E_7, E_8, D_4 is used for the corresponding negative definite root lattices. By $L(2)$ (in our tables $L=U$ or E_8) we denote the lattice whose matrix is the matrix of L multiplied by 2. A coefficient, k, before a lattice stands for the direct sum of k copies, for instance, $2U=U\oplus U$. The direct sum $s\langle2\rangle\oplus t\langle-2\rangle$ in the first two tables is called *the diagonal component* of the corresponding eigenlattice $L_\pm$.

Table 1. Principal series of L_+

$K3$-vertex	diagonal component	non-diagonal component
$[S_{10-t}]$	$\langle2\rangle+t\langle-2\rangle$	
$[S_{10-t}\sqcup S]$	$t\langle-2\rangle$	$+U$
$[S_{7-t}\sqcup 2S]$	$t\langle-2\rangle$	$+U+D_4$
$[S_{6-t}\sqcup 3S]$	$\langle2\rangle+t\langle-2\rangle$	$+E_7$
$[S_{6-t}\sqcup 4S]$	$\langle2\rangle+t\langle-2\rangle$	$+E_8$
$[S_{6-t}\sqcup 5S]$	$t\langle-2\rangle$	$+U+E_8$
$[S_{3-t}\sqcup 6S]$	$t\langle-2\rangle$	$+U+D_4+E_8$
$[S_{2-t}\sqcup 7S]$	$\langle2\rangle+t\langle-2\rangle$	$+E_7+E_8$
$[S_{2-t}\sqcup 8S]$	$\langle2\rangle+t\langle-2\rangle$	$+2E_8$
$[S_{2-t}\sqcup 9S]$	$t\langle-2\rangle$	$+U+2E_8$

Table 2. Principal series of L_-

K3-vertex	diagonal component	non-diagonal component
$[(10-t)S]$	$2\langle 2\rangle + t\langle -2\rangle$	
$[S_1 \sqcup (9-t)S]$	$\langle 2\rangle + t\langle -2\rangle$	$+U$
$[S_2 \sqcup (9-t)S]$	$t\langle -2\rangle$	$+2U$
$[S_3 \sqcup (6-t)S]$	$t\langle -2\rangle$	$+2U + D_4$
$[S_4 \sqcup (5-t)S]$	$2\langle 2\rangle + t\langle -2\rangle$	$+E_8$
$[S_5 \sqcup (5-t)S]$	$\langle 2\rangle + t\langle -2\rangle$	$+U + E_8$
$[S_6 \sqcup (5-t)S]$	$t\langle -2\rangle$	$+2U + E_8$
$[S_7 \sqcup (2-t)S]$	$t\langle -2\rangle$	$+2U + D_4 + E_8$
$[S_8 \sqcup (1-t)S]$	$2\langle 2\rangle + t\langle -2\rangle$	$+2E_8$
$[S_9 \sqcup (1-t)S]$	$\langle 2\rangle + t\langle -2\rangle$	$+U + 2E_8$
$[S_{10} \sqcup (1-t)S]$	$t\langle -2\rangle$	$+2U + 2E_8$

Table 3. Other types of $L_\pm$

K3-vertex	L_+	L_-
$[8S]_I$	$U + 2D_4 + E_8$	$2U(2)$
$[S_1 \sqcup 8S]_I$	$U(2) + 2E_8$	$U + U(2)$
$[S_1 \sqcup 4S]_I$	$U + 3D_4$	$2U(2) + D_4$
$[S_2 \sqcup 5S]_I$	$U(2) + D_4 + E_8$	$U + U(2) + D_4$
$[S_3 \sqcup 2S]_I$	$U(2) + 2D_4$	$U + U(2) + 2D_4$
$[S_4 \sqcup 3S]_I$	$U + 2D_4$	$2U + 2D_4$
$[S_5 \sqcup 4S]_I$	$U(2) + E_8$	$U + U(2) + E_8$
$[S_6 \sqcup S]_I$	$U(2) + D_4$	$U + U(2) + D_4 + E_8$
$[S_9]_I$	$2U$	$U + U(2) + 2E_8$
$[2S_1]$	$U + E_8(2)$	$2U + E_8(2)$
$[\varnothing]$	$U(2) + E_8(2)$	$U + U(2) + E_8(2)$

References

[A] Aeppli, Alfred, *Modifikation von reellen und komplexen Mannigfaltigkeiten*, Comment. Math. Helv. **31** (1957), 219-301.

[DIK1] A. Degtyarev, I.Itenberg, V.Kharlamov, *Real Enriques surfaces*, Lecture Notes Math., Springer **1746** (2000), 259 pages.

[DIK2] A. Degtyarev, I. Itenberg, V. Kharlamov, *Finiteness and Quasi-Simplicity for Symmetric K3-Surfaces*, Duke Math. J. **122** (2004), 1–49.

[F] T. Fujita, *Classification Theorems of Polarized Varieties*, Lecture Notes Series, London Math. Society, vol. 155, 1990.

[GM] L. Guillou, A. Marin, *Generalization of Rokhlin theorem on the signature*, A la recherche de la topologie perdue (Birkhäser, ed.), 1986, pp. 97–118.

[I] I. Itenberg, *Plane projective real curves of degree* 6 *with one non-degenerate double point*, PhD Thesis (Leningrad, 1991).

[Kl] F. Klein, *Ueber Flächen dritter Ordnung*, Math. Ann. **6** (1873), 551–581.

[Kr1] V.A. Krasnov, *Rigid isotopic classification of real three-dimensional cubics* **to appear**.

[Kr2] V.A. Krasnov, *Topological classification of Fano surfaces of real three-dimensional cubics* **to appear**.

[Ne] I. Newton, *Enumeratio linearum tertii ordinis* (1704).
[N1] V. V. Nikulin, *Integer quadratic forms and some of their geometrical applications*, Math. USSR – Izv. **43** (1979), 103–167.
[N2] V. V. Nikulin, *Remarks on connected components of moduli of real polarized K3 surfaces*, ArXiV:Math. AG/0507197 (2006).
[SD] B. Saint-Donat, *Projective models of K-3 surfaces*, Amer. J. Math **96** (1974), no. 4, 602–639.
[V] C. Voisin, *Théorème de Torelli pour les cubiques de P^5*, Invent. Math. **86** (1986), 577–601.
[W] C. T. C. Wall, *Quadratic forms in finite groups and related topics*, Topology **2** (1964), 281–298.

Middle East Technical University, Department of Mathematics
Ankara 06531 Turkey

Université Louis Pasteur et IRMA (CNRS)
7 rue René Descartes 67084 Strasbourg Cedex, France