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Quadratic Pencil of Schrödinger Operators with Spectral Singularities:

Discrete Spectrum and Principal Functions

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In this article we investigated the spectrum of the quadratic pencil of Schrödinger operators $L(\lambda)$ generated in $L_2(\mathbb{R}_+)$ by the equation

$$-y'' + \left[V(x) + 2\lambda U(x) - \lambda^2\right]y = 0, \qquad x \in \mathbb{R}_+ = [0, \infty)$$

and the boundary condition

$$y(0) = 0,$$

where U, V are complex valued functions and U is absolutely continuous in each finite subinterval of $\mathbb{R}_+$. Discussing the spectrum, we proved that $L(\lambda)$ has a finite number of eigenvalues and spectral singularities with finite multiplicities, if the conditions

$$\sup_{0 \le x < \infty} \left\{\exp\left(\epsilon\sqrt{x}\right)\left[|V(x)| + |U'(x)|\right]\right\} < \infty, \qquad \epsilon > 0$$

and

$$\lim_{x \to \infty} U(x) = 0$$

hold. It is shown that the principal functions corresponding to eigenvalues of $L(\lambda)$ are in the space $L_2(\mathbb{R}_+)$, and the principal functions corresponding to spectral singularities are in another Hilbert space, which contains $L_2(\mathbb{R}_+)$. Some results about the spectrum of $L(\lambda)$ have also been applied to radial Klein–Gordon and one-dimensional Schrödinger equations. © 1997 Academic Press

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1. INTRODUCTION

Let us consider the non-self-adjoint one-dimensional Schrödinger operator L generated in $L_2(\mathbb{R}_+)$ by the differential expression

$$l(y) \equiv -y'' + V(x)y, \qquad x \in \mathbb{R}_+$$

and the boundary condition $y(0) = 0$, where V is a complex valued function. The spectral analysis of L has been investigated by Naimark [12]. In this article, he has proved that some of the poles of the resolvent's kernel of L are not the eigenvalues of the operator. He has also shown that those poles (which are called spectral singularities by Schwartz [17]) are on the continuous spectrum. Moreover, he has shown that the spectral singularities play an important role in the discussion of the spectral analysis of L, and if the condition

$$\int_0^\infty |V(x)|\exp(\epsilon x)\, dx < \infty, \qquad \epsilon > 0$$

holds, then the eigenvalues and the spectral singularities are of a finite number and each of them is of a finite multiplicity.

The effect of spectral singularities in the spectral expansion of the operator L in terms of principal functions has been investigated in [9]. In [14], the dependence of the structure of spectral singularities of L on the behavior of V at infinity has been considered. Some problems related to spectral analysis of differential and some other types of operators with spectral singularities have been discussed in [3–5, 10, 13, 15].

Let us consider the operator $L(\lambda)$ generated in $L_2(\mathbb{R}_+)$ by the equation

$$-y'' + \left[V(x) + 2\lambda U(x) - \lambda^2\right] y = 0, \qquad x \in \mathbb{R}_+, \tag{1.1}$$

and the boundary condition

$$y(0) = 0, \tag{1.2}$$

where U, V are complex-valued functions and U is absolutely continuous in each finite subinterval of $\mathbb{R}_+$. If $U \equiv 0$, then the operator $L(\lambda)$ reduces to the operator L. If we choose $V(x) = -U^2(x)$, then (1.1) will be reduced into the radial form of the Klein–Gordon equation. Some problems of the spectral theory of (1.1) and of the Klein–Gordon equation have been investigated by several authors [1, 2, 6, 7, 8, 11] with real functions U and V.

We also want to note that, the finiteness of the number of the eigenvalues of $L(\lambda)$ has been given by the following technique in [7]. First it is proved that the set of eigenvalues of $L(\lambda)$ is bounded in the complex plane and is of countable, and their limit points lie on the real axis. Later, assuming that there are no limit points of eigenvalues, it has been determined that the eigenvalues are of finite number. But in [7] the functions U and V for which the set of eigenvalues has no limit points have not been investigated. Using the technique of the uniqueness of analytic functions, we proved that the eigenvalues and the spectral singularities are of finite number.

In this paper, we discussed the spectrum of $L(\lambda)$ defined by (1.1) and (1.2), and proved that this operator has a finite number of eigenvalues and spectral singularities and that each of them is of a finite multiplicity, under the conditions

$$\sup_{0 \le x < \infty} \left\{ \exp\left(\epsilon\sqrt{x}\right)\left[|V(x)| + |U'(x)|\right]\right\} < \infty, \qquad \epsilon > 0,$$

and

$$\lim_{x \to \infty} U(x) = 0.$$

Afterward, the properties of the principal functions corresponding to the eigenvalues and the spectral singularities of $L(\lambda)$ are obtained. If U is absolutely continuous in every finite subinterval of $\mathbb{R}_+$ and satisfies

$$\sup_{0 \le x < \infty} \left\{ \exp\left(\epsilon\sqrt{x}\right)|U'(x)|\right\} < \infty, \qquad \epsilon > 0$$

and

$$\lim_{x \to \infty} U(x) = 0,$$

then the eigenvalues and the spectral singularities of the Klein–Gordon equation have similar properties.

In the particular case L of the operator $L(\lambda)$, the results we have obtained about the spectrum are better than the ones given by Lyance [9] and Naimark [12], and are the same as the ones obtained by Pavlov [15].

In the sequel, we use the notations

$$\mathbb{C}_+ = \{\lambda | \lambda \in \mathbb{C},\ Im\ \lambda > 0\}, \qquad \mathbb{C}_- = \{\lambda | \lambda \in \mathbb{C},\ Im\ \lambda < 0\}$$

$$\mathbb{R}^* = \mathbb{R} \setminus \{0\}, \qquad \overline{\mathbb{C}}_+ = \{\lambda | \lambda \in \mathbb{C},\ Im\ \lambda \ge 0\},$$

$$\overline{\mathbb{C}}_- = \{\lambda | \lambda \in \mathbb{C},\ Im\ \lambda \le 0\}$$

Furthermore, $\sigma_p(L(\lambda))$, $\sigma_{ss}(L(\lambda))$, $\rho(L(\lambda))$, and $R(L(\lambda))$ will denote the eigenvalues, the spectral singularities, the resolvent set, and the resolvent of the operator $L(\lambda)$, respectively.

2. PRELIMINARIES

Let us suppose that the functions U and V satisfy the following conditions:

$$\int_0^\infty |U(x)|\,dx < \infty, \qquad \int_0^\infty x\big[|V(x)| + |U'(x)|\big]\,dx < \infty. \tag{2.1}$$

Let us define the functions ω, α, and β by

$$\begin{aligned}\omega(x) &= \int_x^\infty U(t)\,dt\\ \alpha(x) &= \int_x^\infty \big[|V(t)| + |U'(t)|\big]\,dt\\ \beta(x) &= \int_x^\infty \big[t|V(t)| + 2|U(t)|\big]\,dt\end{aligned} \tag{2.2}$$

Jaulent and Jean [7] have obtained the following important result: if (2.1) holds, then (1.1) has the solution

$$f(x,\lambda) = e^{i\omega(x)}e^{i\lambda x} + \int_x^\infty A(x,t)e^{i\lambda t}\,dt \tag{2.3}$$

for $\lambda \in \overline{\mathbb{C}}_+$ and the solution

$$g(x,\lambda) = e^{-i\omega(x)}e^{-i\lambda x} + \int_x^\infty B(x,t)e^{-i\lambda t}\,dt \tag{2.4}$$

for $\lambda \in \overline{\mathbb{C}}_-$; moreover, the kernels $A(x,t)$ and $B(x,t)$ satisfy the inequality

$$|A(x,t)|, |B(x,t)| \le \frac{1}{2}\alpha\left(\frac{x+t}{2}\right)\exp\{\beta(x)\}. \tag{2.5}$$

Therefore, the solutions $f(x,\lambda)$ and $g(x,\lambda)$ are analytic with respect to λ, in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively, and are continuous up to the real axis. $f(x,\lambda)$ and $g(x,\lambda)$ also satisfy the following asymptotic equalities [7]:

$$\begin{aligned} f(x,\lambda) &= e^{i\lambda x}[1 + o(1)], && \lambda \in \overline{\mathbb{C}}_+, && x \to \infty\\ f_x(x,\lambda) &= e^{i\lambda x}[i\lambda + o(1)], && \lambda \in \overline{\mathbb{C}}_+, && x \to \infty\end{aligned} \tag{2.6}$$

$$\begin{aligned} g(x,\lambda) &= e^{-i\lambda x}[1 + o(1)], && \lambda \in \overline{\mathbb{C}}_-, && x \to \infty\\ g_x(x,\lambda) &= e^{-i\lambda x}[-i\lambda + o(1)], && \lambda \in \overline{\mathbb{C}}_-, && x \to \infty\end{aligned} \tag{2.7}$$

From (2.3) and (2.4) we easily find

$$\begin{aligned} f(x,\lambda) &= e^{i\omega(x)}e^{i\lambda x} + o(1), && \lambda \in \overline{\mathbb{C}}_+, && |\lambda| \to \infty\\ g(x,\lambda) &= e^{-i\omega(x)}e^{-i\lambda x} + o(1), && \lambda \in \overline{\mathbb{C}}_-, && |\lambda| \to \infty\end{aligned} \tag{2.8}$$

According to (2.6) and (2.7), the Wronskian of the solutions $f(x, \lambda)$ and $g(x, \lambda)$ is

$$W[f, g] = \lim_{x \to \infty} W[f, g] = -2i\lambda \tag{2.9}$$

for $\lambda \in \mathbb{R}$. So $f(x, \lambda)$ and $g(x, \lambda)$ provide the fundamental solutions of the equation (1.1) for $\lambda \in \mathbb{R}^*$.

Let $\varphi(x, \lambda)$ be the solution of (1.1) satisfying the initial conditions

$$\varphi(0, \lambda) = 0, \qquad \varphi_x(0, \lambda) = 1.$$

It is obvious that the solution $\varphi(x, \lambda)$ exists, is unique, and is an entire function of λ [7].

3. EIGENVALUES AND SPECTRAL SINGULARITIES

Let us define

$$\rho_1(\lambda) = \{\lambda | \lambda \in \mathbb{C}_+, F(\lambda) \neq 0\}, \qquad \rho_2(\lambda) = \{\lambda | \lambda \in \mathbb{C}_-, G(\lambda) \neq 0\}$$

where

$$F(\lambda) := f(0, \lambda), \qquad G(\lambda) := g(0, \lambda).$$

Using the standard techniques [13], we can show that

$$\rho(L(\lambda)) = \rho_1(\lambda) \cup \rho_2(\lambda)$$

and for $\lambda \in \rho(L(\lambda))$ the resolvent of $L(\lambda)$ is the integral operator defined as

$$R(L(\lambda))\psi(x) = \int_0^\infty R(x, t; \lambda)\psi(t)\, dt$$

for $\psi \in L_2(\mathbb{R}_+)$, where the kernel $R(x, t; \lambda)$ (i.e., the Green's function for $L(\lambda)$) is given by

$$R(x, t; \lambda) = \begin{cases} R_1(x, t; \lambda), & \lambda \in \rho_1(\lambda) \\ R_2(x, t; \lambda), & \lambda \in \rho_2(\lambda) \end{cases} \tag{3.1}$$

in which

$$R_1(x, t; \lambda) = \frac{1}{F(\lambda)} \begin{cases} f(x, \lambda)\varphi(t, \lambda) & 0 \le t < x \\ f(t, \lambda)\varphi(x, \lambda) & x \le t < \infty \end{cases} \tag{3.2}$$

$$R_2(x, t; \lambda) = \frac{1}{G(\lambda)} \begin{cases} g(x, \lambda)\varphi(t, \lambda) & 0 \le t < x \\ g(t, \lambda)\varphi(x, \lambda) & x \le t < \infty. \end{cases} \tag{3.3}$$

From the asymptotic equalities (2.6) and (2.7), we have

$$f(\cdot,\lambda), f_x(\cdot,\lambda) \in L_2(\mathbb{R}_+) \tag{3.4}$$

for each $\lambda \in \mathbb{C}_+$ and

$$g(\cdot,\lambda), g_x(\cdot,\lambda) \in L_2(\mathbb{R}_+) \tag{3.5}$$

for each $\lambda \in \mathbb{C}_-$.

LEMMA 3.1.

(a) $\sigma_p(L(\lambda)) = \{\lambda | \lambda \in \mathbb{C}_+, F(\lambda) = 0\} \cup \{\lambda | \lambda \in \mathbb{C}_-, G(\lambda) = 0\}$,

(b) $\sigma_{ss}(L(\lambda)) = \{\lambda | \lambda \in \mathbb{R}^*, F(\lambda) = 0\} \cup \{\lambda | \lambda \in \mathbb{R}^*, G(\lambda) = 0\}$.

Proof. (a) Obviously,

$$\{\lambda | \lambda \in \mathbb{C}_+, F(\lambda) = 0\} \cup \{\lambda | \lambda \in \mathbb{C}_-, G(\lambda) = 0\} \subset \sigma_p(L(\lambda)).$$

On the other hand, let us suppose that $\lambda_0 \in \sigma_p(L(\lambda))$ and examine the following cases:

(1) Let $\lambda_0 \in \mathbb{C}_+$. Since $\lambda_0 \in \sigma_p(L(\lambda))$, then (1.1) has a solution $y(x,\lambda_0)$ in $L_2(\mathbb{R}_+)$ for $\lambda = \lambda_0$ that is nontrivial and $y(0,\lambda_0) = 0$. Since

$$W[y(x,\lambda_0), \varphi(x,\lambda_0)] = 0,$$

there exists a constant $c \neq 0$ such that $y(x,\lambda_0) = c\varphi(x,\lambda_0)$. Then we have

$$W[f(x,\lambda_0), y(x,\lambda_0)] = cF(\lambda_0). \tag{3.6}$$

But it is evident from (3.4) that

$$W[f(x,\lambda_0), y(x,\lambda_0)] = \lim_{x\to\infty} W[f(x,\lambda_0), y(x,\lambda_0)] = 0. \tag{3.7}$$

From (3.6) and (3.7) we find

$$F(\lambda_0) = 0.$$

(2) Let $\lambda_0 \in \mathbb{C}_-$. In a similar way, we can prove that

$$G(\lambda_0) = 0.$$

(3) Let $\lambda_0 \in \mathbb{R}$. In this case the general solution of (1.1) is

$$y(x,\lambda_0) = c_1 f(x,\lambda_0) + c_2 g(x,\lambda_0)$$

for $\lambda = \lambda_0$. From (2.6) and (2.7) we get

$$y(x,\lambda_0) = c_1 e^{i\lambda_0 x} + c_2 e^{-i\lambda_0 x} + o(1) \qquad \text{as } x \to \infty.$$

That is, $y(\cdot, \lambda_0) \notin L_2(\mathbb{R}_+)$. Hence

$$\sigma_p(L(\lambda)) \cap \mathbb{R} = \phi.$$

Combining (1), (2), and (3), we find

$$\sigma_p(L(\lambda)) \subset \{\lambda | \lambda \in \mathbb{C}_+, F(\lambda) = 0\} \cup \{\lambda | \lambda \in \mathbb{C}_-, G(\lambda) = 0\}.$$

This completes the proof of (a).

(b) Spectral singularities are the poles of the kernel of the resolvent, but not the eigenvalues of the operator $L(\lambda)$. From (3.1)–(3.3) and part (a) we obtain that the spectral singularities of $L(\lambda)$ are the zeros of F and G on the real axis. We can easily show that $0 \notin \sigma_{ss}(L(\lambda))$, which completes the proof of part (b). ∎

Note that from

$$W[f(x, \lambda), g(x, \lambda)] = F(\lambda) g_x(0, \lambda) - G(\lambda) f_x(0, \lambda) = -2i\lambda, \qquad \lambda \in \mathbb{R},$$

we immediately get

$$\{\lambda | \lambda \in \mathbb{R}^*, F(\lambda) = 0\} \cap \{\lambda | \lambda \in \mathbb{R}^*, G(\lambda) = 0\} = \phi. \tag{3.8}$$

To investigate the quantitative properties of the eigenvalues and the spectral singularities of $L(\lambda)$, we need to discuss the quantitative properties of the zeros of F and G in $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$, respectively. For the sake of simplicity, we will consider only the zeros of F in $\overline{\mathbb{C}}_+$. A similar procedure may also be employed for the zeros of G in $\overline{\mathbb{C}}_-$.

Let us define

$$Q_1 = \{\lambda | \lambda \in \mathbb{C}_+, F(\lambda) = 0\}, \qquad Q_2 = \{\lambda | \lambda \in \mathbb{R}, F(\lambda) = 0\}.$$

LEMMA 3.2. (a) *The set Q_1 is bounded and has at most a countable number of elements, and its limit points can lie only in a bounded subinterval of the real axis.*

(b) *The set Q_2 is compact and its linear Lebesque measure is zero.*

Proof. From (2.8) we get

$$F(\lambda) = e^{i\omega(0)} + o(1), \lambda \in \overline{\mathbb{C}}_+, |\lambda| \to \infty. \tag{3.9}$$

The asymptotic equality (3.9) shows the boundedness of the sets Q_1 and Q_2. From the analyticity of the function F in $\mathbb{C}_+$, we get that Q_1 has at most countable numbers of elements. By the uniqueness of analytic functions we find that the limit points of Q_1 can lie only in a bounded subinterval of the real axis. The closedness and the property of having linear Lebesgue measure zero of the set Q_2 can be obtained from the uniqueness theorem of the analytic functions [16]. ∎

From Lemmas 3.1 and 3.2 we get the following.

THEOREM 3.3. *The sets of eigenvalues and spectral singularities of* $L(\lambda)$ *are bounded and are, at most, countable, and their limit points can lie only in a bounded subinterval of the real axis if the condition* (2.1) *holds.*

Note. For the moment we want to start with "*Assumption III*: *F and G has no real zeros*" in [7], that is,

$$\sigma_{ss}(L(\lambda)) = \phi.$$

In this case, from Lemma 3.2 and Theorem 3.3 we may derive that the number of eigenvalues is finite, which corresponds to the technique given in [7]. But it has not been clarified for which functions U and V Assumption III holds. So Assumption III does not seem to be natural. Hence we will prove by another method that the number of the eigenvalues and the spectral singularities of $L(\lambda)$ are finite, without employing Assumption III.

Up to now we have assumed that condition (2.1) holds. In the rest of the article the assumptions we will use are

$$\lim_{x\to\infty} U(x) = 0, \qquad \sup_{0\le x<\infty} \left\{\exp(\epsilon\sqrt{x})\left[|V(x)| + |U'(x)|\right]\right\} < \infty, \qquad \epsilon > 0. \tag{3.10}$$

From (2.2) and (3.10) we have

$$\alpha(x) \le c_0 \exp\left(-\frac{\epsilon}{2}\sqrt{x}\right), \qquad \beta(x) \le c_0 \exp\left(-\frac{\epsilon}{2}\sqrt{x}\right), \tag{3.11}$$

where $c_0 > 0$ is constant. By (2.5) and (3.11) we get

$$|A(x,t)|, |B(x,t)| \le c \exp\left(-\frac{\epsilon}{2}\sqrt{\frac{x+t}{2}}\right), \tag{3.12}$$

where $c > 0$.

We have seen that whenever condition (2.1) holds, the function F is analytic in $\mathbb{C}_+$ and continuous up to the real axis. It is obvious from (3.12) that if the conditions (3.10) hold, then all derivatives of F are continuous up to the real axis. So the inequalities

$$|F^{(r)}(\lambda)| \le M_r, \qquad \lambda \in \overline{\mathbb{C}}_+, \qquad r = 0, 1, 2, \dots \tag{3.13}$$

hold, where

$$M_r = 2^{r+1}c\int_0^{\infty} t^r \exp\left(-\frac{\epsilon}{2}\sqrt{t}\right) dt, \qquad r = 1, 2, \dots \tag{3.14}$$

and

$$M_0 = \exp\left\{-\int_0^{\infty} ImU(t)\, dt\right\} + \int_0^{\infty} |A(0,t)|\, dt,$$

provided that $c > 0$ is a constant.

Let us indicate the set of all limit points of Q_1 and Q_2 by Q_3 and Q_4, respectively, and the set of all zeros of F with infinite multiplicity in $\overline{\mathbb{C}}_+$ by Q_5.

From the uniqueness theorem of analytic functions, it is obvious that

$$Q_3 \subset Q_2, \qquad Q_4 \subset Q_2, \qquad Q_5 \subset Q_2.$$

LEMMA 3.4. $Q_3 \subset Q_5$, $Q_4 \subset Q_5$.

The proof of this lemma can be obtained by use of the continuity of all derivatives of F up to the real axis.

We will use the following uniqueness theorem for the analytic functions on the upper half-plane to prove the next result.

THEOREM 3.5 ([14]). *Let us assume that the function g is analytic in $\mathbb{C}_+$, all of its derivatives are continuous up to the real axis, and there exist $N > 0$ such that*

$$|g^{(r)}(\lambda)| \le M_r, \qquad r = 0, 1, 2, \dots, \qquad \lambda \in \overline{\mathbb{C}}_+, \qquad |\lambda| < 2N, \tag{3.15}$$

and

$$\left|\int_{-\infty}^{-N} \frac{\ln|g(x)|}{1+x^2}\, dx\right| < \infty, \qquad \left|\int_N^{\infty} \frac{\ln|g(x)|}{1+x^2}\, dx\right| < \infty \tag{3.16}$$

hold. If the set P with linear Lebesque measure zero is the set of all zeros of the function g with infinity multiplicity and if

$$\int_0^h \ln E(s)\, d\mu(P_s) = -\infty$$

holds, then $g(\lambda) \equiv 0$, where $E(s) = \inf_r M_r s^r / r!$, and $r = 0, 1, 2, \dots$, $\mu(P_s)$ is the Lebesque measure of the s-neighborhood of P and h is an arbitrary positive constant.

LEMMA 3.6. $Q_5 = \phi$.

Proof. It is trivial from Lemma 3.2 and (3.13) that F satisfies the conditions (3.15) and (3.16). Since the function F is not identically equal to zero, then by Theorem 3.5, Q_5 satisfies the condition

$$\int_0^h \ln E(s)\, d\mu(Q_{5,s}) > -\infty, \tag{3.17}$$

where $E(s) = \inf_r M_r s^r / r!$, $\mu(Q_{5,s})$ is the Lebesque measure of the s-neighborhood of Q_5, and the constant M_r is defined by (3.14).

Now we will obtain the following estimates for M_r:

$$\begin{aligned} M_r &= 2^{r+1} c \int_0^\infty t^r \exp\left(-\frac{\epsilon}{2}\sqrt{t}\right) dt \\ &\leq B b^r r^r r!, \end{aligned} \tag{3.18}$$

where B and b are constants depending on c and ϵ. Substituting (3.18) in the definition of $E(s)$, we arrive at

$$E(s) = \inf_r \frac{M_r s^r}{r!} \leq B \inf_r \{b^r s^r r^r\} \leq B \exp\{-b^{-1} e^{-1} s^{-1}\},$$

or by (3.17),

$$\int_0^h \frac{1}{s}\, d\mu(Q_{5,s}) < \infty. \tag{3.19}$$

The inequality (3.19) holds for an arbitrary s, if and only if $\mu(Q_{5,s}) = 0$ or $Q_5 = \phi$. ∎

LEMMA 3.7. *The function F has a finite number of zeros with finite multiplicity in $\overline{\mathbb{C}}_+$.*

Proof. Lemmas 3.4 and 3.6 give $Q_3 = Q_4 = \phi$. So the bounded sets Q_1 and Q_2 have no limit points (see Lemma 3.2), i.e., the function F has only a finite number of zeros in $\overline{\mathbb{C}}_+$. Since $Q_5 = \phi$, these zeros are of finite multiplicity. ∎

The discussions given above for F may be repeated for G. Hence we obtain the following.

LEMMA 3.8. *The function G has a finite number of zeros with finite multiplicity in $\overline{\mathbb{C}}_-$.*

DEFINITION 3.9. The multiplicity of a zero of F (or G) in $\overline{\mathbb{C}}_+$ (or $\overline{\mathbb{C}}_-$) is called the multiplicity of the corresponding eigenvalue or spectral singularity of $L(\lambda)$.

Lemmas 3.1, 3.7, and 3.8 give the following.

THEOREM 3.10. *The operator $L(\lambda)$ has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity if the condition* (3.10) *holds.*

4. PRINCIPAL FUNCTIONS

In this section we assume that (3.10) holds. Let $\lambda_1^+, \ldots, \lambda_j^+$ and $\mu_1^-, \ldots, \mu_k^-$ denote the zeros of F in $\mathbb{C}_+$ and G in $\mathbb{C}_-$ with multiplicities $m_1^+, \ldots, m_j^+$ and $m_1^-, \ldots, m_k^-$, respectively. Similarly, let $\lambda_1, \ldots, \lambda_p$ and $\mu_1, \ldots, \mu_q$ be the zeros of F and G on the real axis with multiplicities $m_1, \ldots, m_p$ and $n_1, \ldots, n_q$, respectively. It is trivial that

$$\left\{\frac{\partial^n}{\partial\lambda^n} W[f(x,\lambda), \varphi(x,\lambda)]\right\}_{\lambda=\lambda_i^+} = \left\{\frac{d^n}{d\lambda^n} F(\lambda)\right\}_{\lambda=\lambda_i^+} = 0 \tag{4.1}$$

for $n = 0, 1, \ldots, m_i^+ - 1$, $i = 1, 2, \ldots, j$, and

$$\left\{\frac{\partial^\nu}{\partial\lambda^\nu} W[g(x,\lambda), \varphi(x,\lambda)]\right\}_{\lambda=\mu_l^-} = \left\{\frac{d^\nu}{d\lambda^\nu} G(\lambda)\right\}_{\lambda=\mu_l^-} = 0 \tag{4.2}$$

for $\nu = 0, 1, \ldots, m_l^- - 1$, $l = 1, 2, \ldots, k$. If $n = \nu = 0$, we get

$$\varphi(x, \lambda_i^+) = a_0(\lambda_i^+) f(x, \lambda_i^+), \qquad i = 1, 2, \ldots, j, \tag{4.3}$$

$$\varphi(x, \mu_l^-) = b_0(\mu_l^-) g(x, \mu_l^-), \qquad l = 1, 2, \ldots, k. \tag{4.4}$$

So $a_0(\lambda_i^+) \neq 0$, $b_0(\mu_l^-) \neq 0$.

THEOREM 4.1. *The formulas*

$$\left\{\frac{\partial^n}{\partial\lambda^n} \varphi(x,\lambda)\right\}_{\lambda=\lambda_i^+} = \sum_{k=0}^{n} \binom{n}{k} a_{n-k} \left\{\frac{\partial^k}{\partial\lambda^k} f(x,\lambda)\right\}_{\lambda=\lambda_i^+} \tag{4.5}$$

for $n = 0, 1, \ldots, m_i^+ - 1$, $i = 1, 2, \ldots, j$, and

$$\left\{\frac{\partial^\nu}{\partial\lambda^\nu} \varphi(x,\lambda)\right\}_{\lambda=\mu_l^-} = \sum_{j=0}^{\nu} \binom{\nu}{j} b_{\nu-j} \left\{\frac{\partial^j}{\partial\lambda^j} g(x,\lambda)\right\}_{\lambda=\mu_l^-}, \tag{4.6}$$

for $\nu = 0, 1, \ldots, m_l^- - 1$, $l = 1, 2, \ldots, k$ hold, where the constants $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_\nu$ depend on λ_i^+ and μ_l^-, respectively.

Proof. Let us start with equality (4.5). We will utilize mathematical induction. For $n = 0$, the proof is trivial from (4.3). Let us assume that for $1 \le n_0 \le m_i^+ - 2$ the equality (4.5) holds, i.e.,

$$\left\{ \frac{\partial^{n_0}}{\partial \lambda^{n_0}} \varphi(x, \lambda) \right\}_{\lambda = \lambda_i^+} = \sum_{k=0}^{n_0} \binom{n_0}{k} a_{n_0 - k}(\lambda_i^+) \left\{ \frac{\partial^k}{\partial \lambda^k} f(x, \lambda) \right\}_{\lambda = \lambda_i^+} \tag{4.7}$$

Now we will prove that (4.5) also holds for $n_0 + 1$. If $y(x, \lambda)$ is a solution of the equation (1.1), then $\partial^n / \partial \lambda^n \; y(x, \lambda)$ satisfies

$$\begin{aligned} &\left\{ -\frac{d^2}{dx^2} + V(x) + 2\lambda U(x) - \lambda^2 \right\} \frac{\partial^n}{\partial \lambda^n} y(x, \lambda) \\ &\quad = 2\lambda n \frac{\partial^{n-1}}{\partial \lambda^{n-1}} y(x, \lambda) + n(n-1) \frac{\partial^{n-2}}{\partial \lambda^{n-2}} y(x, \lambda) \\ &\qquad - 2nU(x) \frac{\partial^{n-1}}{\partial \lambda^{n-1}} y(x, \lambda) \end{aligned} \tag{4.8}$$

Writing (4.8) for $\varphi(x, \lambda_i^+)$ and $f(x, \lambda_i^+)$, and using (4.7), we find

$$\left\{ -\frac{d^2}{dx^2} + V(x) + 2\lambda_i^+ U(x) - (\lambda_i^+)^2 \right\} \Phi_{n_0+1}(x, \lambda_i^+) = 0,$$

where

$$\begin{aligned} \Phi_{n_0+1}(x, \lambda_i^+) = &\left\{ \frac{\partial^{n_0+1}}{\partial \lambda^{n_0+1}} \varphi(x, \lambda) \right\}_{\lambda = \lambda_i^+} \\ &- \sum_{k=1}^{n_0+1} \binom{n_0+1}{k} a_{n_0+1-k}(\lambda_i^+) \left\{ \frac{\partial^k}{\partial \lambda^k} f(x, \lambda) \right\}_{\lambda = \lambda_i^+}. \end{aligned}$$

From (4.1) we have

$$W\left[f(x, \lambda_i^+), \Phi_{n_0+1}(x, \lambda_i^+) \right] = \left\{ \frac{\partial^{n_0+1}}{\partial \lambda^{n_0+1}} W[f(x, \lambda), \varphi(x, \lambda)] \right\}_{\lambda = \lambda_i^+} = 0.$$

Hence there exists a constant $a_{n_0+1}(\lambda_i^+)$ such that

$$\Phi_{n_0+1}(x, \lambda_i^+) = a_{n_0+1}(\lambda_i^+) f(x, \lambda_i^+).$$

This shows that (4.5) holds for $n = n_0 + 1$. In a similar way, we can prove that (4.6) also holds. ∎

THEOREM 4.2.

$$\left\{\frac{\partial^n}{\partial\lambda^n}\varphi(\cdot,\lambda)\right\}_{\lambda=\lambda_i^+} \in L_2(\mathbb{R}_+), \qquad n = 0,1,\dots,m_i^+ - 1, \quad i = 1,2,\dots,j, \tag{4.9}$$

$$\left\{\frac{\partial^\nu}{\partial\lambda^\nu}\varphi(\cdot,\lambda)\right\}_{\lambda=\mu_l^-} \in L_2(\mathbb{R}_+), \qquad \nu = 0,1,\dots,m_l^- - 1, \quad l = 1,2,\dots,k. \tag{4.10}$$

Proof. From (2.3) and (3.12) we obtain

$$\left|\left\{\frac{\partial^n}{\partial\lambda^n}f(x,\lambda)\right\}_{\lambda=\lambda_i^+}\right| \le x^n \exp\left\{\int_0^\infty |U(t)|\,dt\right\}\exp\{-xIm\,\lambda_i^+\} + c\exp\left(-\frac{\epsilon}{2}\sqrt{x}\right)\int_0^\infty t^n \exp\{-tIm\,\lambda_i^+\}\,dt$$

for $n = 0,1,\dots,m_i^+ - 1$, $i = 1,2,\dots,j$, which gives (4.9) using (4.5). Equation (4.10) may be derived analogously. ∎

$$\varphi(x,\lambda_i^+), \left\{\frac{\partial}{\partial\lambda}\varphi(x,\lambda)\right\}_{\lambda=\lambda_i^+}, \dots, \left\{\frac{\partial^{m_i^+-1}}{\partial\lambda^{m_i^+-1}}\varphi(x,\lambda)\right\}_{\lambda=\lambda_i^+}$$

and

$$\varphi(x,\mu_l^-), \left\{\frac{\partial}{\partial\lambda}\varphi(x,\lambda)\right\}_{\lambda=\mu_l^-}, \dots, \left\{\frac{\partial^{m_l^- -1}}{\partial\lambda^{m_l^- -1}}\varphi(x,\lambda)\right\}_{\lambda=\mu_l^-}$$

are called the principal functions corresponding to eigenvalues $\lambda = \lambda_i^+$, $i = 1,2,\dots,j$, and $\lambda = \mu_l^-$, $l = 1,2,\dots,k$ of $L(\lambda)$, respectively.

If $\lambda_1,\dots,\lambda_p$ and $\mu_1,\dots,\mu_q$ are spectral singularities of $L(\lambda)$ (i.e., the real zeros of F and G), we then obtain

$$\left\{\frac{\partial^n}{\partial\lambda^n}W[f(x,\lambda),\varphi(x,\lambda)]\right\}_{\lambda=\lambda_i} = \left\{\frac{d^n}{d\lambda^n}F(\lambda)\right\}_{\lambda=\lambda_i} = 0$$

for $n = 0,1,\dots,m_i - 1$, $i = 1,2,\dots,p$, and

$$\left\{\frac{\partial^\nu}{\partial\lambda^\nu}W[g(x,\lambda),\varphi(x,\lambda)]\right\}_{\lambda=\mu_l} = \left\{\frac{d^\nu}{d\lambda^\nu}G(\lambda)\right\}_{\lambda=\mu_l} = 0,$$

for $\nu = 0,1,\dots,n_l - 1$, $l = 1,2,\dots,q$. In a way similar to that of the proof of Theorem 4.1, we have the following.

Remark 4.3. The formulas

$$\left\{\frac{\partial^n}{\partial\lambda^n}\varphi(x,\lambda)\right\}_{\lambda=\lambda_i} = \sum_{k=0}^{n}\binom{n}{k}d_{n-k}(\lambda_i)\left\{\frac{\partial^k}{\partial\lambda^k}f(x,\lambda)\right\}_{\lambda=\lambda_i} \tag{4.11}$$

for $n = 0,1,\dots,m_i - 1$, $i = 1,2,\dots,p$ and

$$\left\{\frac{\partial^\nu}{\partial\lambda^\nu}\varphi(x,\lambda)\right\}_{\lambda=\mu_l} = \sum_{k=0}^{\nu}\binom{\nu}{j}h_{\nu-j}(\mu_l)\left\{\frac{\partial^j}{\partial\lambda^j}g(x,\lambda)\right\}_{\lambda=\mu_l}, \tag{4.12}$$

for $\nu = 0,1,\dots,n_l - 1$, $l = 1,2,\dots,q$ hold.

LEMMA 4.4.

$$\left\{\frac{\partial^n}{\partial\lambda^n}\varphi(\cdot,\lambda)\right\}_{\lambda=\lambda_i} \notin L_2(\mathbb{R}_+), \qquad n = 0,1,\dots, \qquad m_i - 1,$$
$$i = 1,2,\dots,p, \tag{4.13}$$

and

$$\left\{\frac{\partial^\nu}{\partial\lambda^\nu}\varphi(\cdot,\lambda)\right\}_{\lambda=\mu_l} \notin L_2(\mathbb{R}_+), \qquad \nu = 0,1,\dots, \qquad n_l - 1,$$
$$l = 1,2,\dots,q. \tag{4.14}$$

The proof of this lemma may easily be obtained from (2.3), (2.4), (4.11), and (4.12). Now let us introduce the Hilbert spaces:

$$H_n = \left\{f : \int_0^\infty (1+x)^{2n}|f(x)|^2\,dx < \infty\right\}, \qquad n = 0,1,\dots$$
$$H_{-n} = \left\{g : \int_0^\infty (1+x)^{-2n}|g(x)|^2\,dx < \infty\right\}, \qquad n = 0,1,\dots$$

with

$$\|f\|_n^2 = \int_0^\infty (1+x)^{2n}|f(x)|^2\,dx; \qquad \|g\|_{-n}^2 = \int_0^\infty (1+x)^{-2n}|g(x)|^2\,dx,$$

respectively. It is evident that

$$H_0 = L_2(\mathbb{R}_+), H_n \subsetneqq L_2(\mathbb{R}_+) \subsetneqq H_{-n}, \qquad n = 1,2,\dots.$$

Obviously H_{-n} is isomorphic to the dual of H_n.

THEOREM 4.5.

$$\left\{\frac{\partial^n}{\partial\lambda^n}\varphi(\cdot,\lambda)\right\}_{\lambda=\lambda_i}\in H_{-(n+1)},\qquad n=0,1,\ldots,\qquad m_i-1,$$
$$i=1,2,\ldots,p,\quad (4.15)$$

and

$$\left\{\frac{\partial^\nu}{\partial\lambda^\nu}\varphi(\cdot,\lambda)\right\}_{\lambda=\mu_l}\in H_{-(\nu+1)},\qquad \nu=0,1,\ldots,\qquad n_l-1,$$
$$l=1,2,\ldots,q.\quad (4.16)$$

Proof. From (2.3) we obtain

$$\left|\left\{\frac{\partial^n}{\partial\lambda^n}f(x,\lambda)\right\}_{\lambda=\lambda_i}\right|\leq x^n\exp\left\{-\int_0^\infty ImU(t)\right\}+\int_x^\infty |t|^n|A(x,t)|\,dt.$$

By the definition of the space $H_{-(n+1)}$ and using (3.12) and (4.11), we arrive at (4.15). In the same manner, we can show that (4.16) also holds. ∎

$$\varphi(x,\lambda_i),\left\{\frac{\partial}{\partial\lambda}\varphi(x,\lambda)\right\}_{\lambda=\lambda_i},\ldots,\left\{\frac{\partial^{m_i-1}}{\partial\lambda^{m_i-1}}\varphi(x,\lambda)\right\}_{\lambda=\lambda_i}$$

and

$$\varphi(x,\mu_l),\left\{\frac{\partial}{\partial\lambda}\varphi(x,\lambda)\right\}_{\lambda=\mu_l},\ldots,\left\{\frac{\partial^{n_l-1}}{\partial\lambda^{n_l-1}}\varphi(x,\lambda)\right\}_{\lambda=\mu_l}$$

are called the principal functions corresponding to the spectral singularities $\lambda=\lambda_i$, $i=1,2,\ldots,p$, and $\lambda=\mu_l$, $l=1,2,\ldots,q$ of $L(\lambda)$, respectively.

Let us choose n_0 so that

$$n_0=\max\{m_1,\ldots,m_p,n_1,\ldots,n_q\}.$$

Then

$$H_{n_0+1}\subsetneqq L_2(\mathbb{R}_+)\subsetneqq H_{-(n_0+1)}.$$

By Theorem 4.5 we have the following.

Remark 4.6.

$$\left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(\cdot, \lambda) \right\}_{\lambda = \lambda_i} \in H_{-(n_0+1)}, \qquad n = 0, 1, \ldots, \qquad m_i - 1,$$

$$i = 1, 2, \ldots, p,$$

$$\left\{ \frac{\partial^\nu}{\partial \lambda^\nu} \varphi(\cdot, \lambda) \right\}_{\lambda = \mu_l} \in H_{-(n_0+1)}, \qquad \nu = 0, 1, \ldots, \qquad n_l - 1,$$

$$l = 1, 2, \ldots, q.$$

In a forthcoming article we will show that the spectral expansion of $L(\lambda)$ converges in the sense of $H_{-(n_0+1)}$ for every function in H_{n_0+1}.

5. KLEIN–GORDON EQUATION

As we have pointed out previously, if we choose $V(x) = -U^2(x)$, then the boundary value problem (1.1–1.2) is reduced to

$$y'' + [\lambda - U(x)]^2 y = 0, \qquad x \in \mathbb{R}_+ \tag{5.1}$$

$$y(0) = 0, \tag{5.2}$$

where U is a complex-valued and absolutely continuous function in each finite subinterval of $\mathbb{R}_+$. Equation (5.1) is called the Klein–Gordon s-wave equation for a particle of zero mass with static potential $U(x)$. It is trivial that the condition (3.10) assumes the form

$$\sup_{0 \le x < \infty} \left\{ \exp(\epsilon \sqrt{x}) |U'(x)| \right\} < \infty, \qquad \epsilon > 0$$

$$\lim_{x \to \infty} U(x) = 0 \tag{5.3}$$

for (5.1). We can deduce the following from Theorem 3.10.

THEOREM 5.1. *The boundary value problem* (5.1), (5.2) *has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity if the conditions* (5.3) *hold.*

Let us also note that if the function U is real and analytic and vanishes rapidly as $x \to \infty$, then the eigenvalues of (5.1), (5.2) have been discussed previously [2].

6. ONE-DIMENSIONAL SCHRÖDINGER OPERATOR

If $U(x) \equiv 0$ in (1.1), the operator $L(\lambda)$ is reduced to L, which was given in the Introduction. In this case the condition (3.10) will assume the form

$$\sup_{0 \le x < \infty} \left\{ \exp\left(\epsilon \sqrt{x} \right) |V(x)| \right\} < \infty, \qquad \epsilon > 0. \tag{6.1}$$

Hence from Theorem 3.10 we get the following.

THEOREM 6.1. *Under the condition (6.1), the operator L has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

The same result has been obtained by Lyance [9] and Naimark [12] under the stronger assumption

$$\int_0^\infty |V(x)| \exp(\epsilon x)\, dx < \infty, \qquad \epsilon > 0.$$

The condition (6.1) has been obtained in Pavlov [15].

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