Distribution of Maximum Loss of Fractional Brownian Motion with Drift[1]

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Abstract

In this paper, we find bounds on the distribution of the maximum loss of fractional Brownian motion with $H \geq 1/2$ and derive estimates on its tail probability. Asymptotically, the tail of the distribution of maximum loss over $[0, t]$ behaves like the tail of the marginal distribution at time t.

Keywords: maximum drawdown, maximum loss, fractional Brownian motion, large deviation, Gaussian process

1. Introduction

The maximum loss can be defined as the maximum decrease from the higher values to the lower values of a process X, also called the maximum drawdown. It is expressed by $\sup_{0 \leq u \leq v \leq t}(X_u - X_v)$ for each $t \geq 0$. Previously, the exact distribution of the maximum loss has been studied for Brownian motion with no drift as given in [5], and with drift in [4, 8, 12]. Asymptotic expressions are observed by [8] for the distribution and the expected value as $t \to \infty$. One motivation to study maximum loss as a functional of X comes from mathematical finance; see e.g. [11, 12]. It is useful to quantify the risk and to measure the performance of a stock.

In this paper, we prove novel bounds and asymptotical results on the distribution of the maximum loss of fractional Brownian motion (fBm). We consider fBm with drift. Our main result is that the tail of the distribution of maximum loss over $[0, t]$ behaves like the tail of the marginal distribution at time t.

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Let $(\Omega, \Im, \mathbb{P})$ be a probability space. A (standard) fBm $\{B_t : t \geq 0\}$ with Hurst parameter $H \in (0, 1)$ is a continuous and centered Gaussian process with covariance function

$$E[B_t B_s] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}) \tag{1}$$

and $B_0 = 0$. It follows that B has stationary increments and the two increments of the form $B(t+h) - B(t)$ and $B(t+2h) - B(t+h)$ are positively correlated for $H > 1/2$, and negatively correlated for $H < 1/2$. For the price of a risky asset, fBm is used with $H > 1/2$ in order to capture long-range dependence observed in real data [2]. Since the covariance function of fBm is homogeneous of order $2H$, it possesses the self-similarity property. That is, for any constant $c > 0$, $(B_{ct})_{t \geq 0} \stackrel{d}{=} (c^H B_t)_{t \geq 0}$. The fractional Black-Scholes model for asset price P is given by

$$P_t = P_0 \exp\left((r + \mu)t + \sigma B_t\right), \qquad t \geq 0$$

where P_0 is the initial value, r is the constant interest rate, μ is the drift and σ is the diffusion coefficient of fBm denoted by B_t, $t \geq 0$. In view of fractional Black-Scholes model, the maximum possible loss in the log-price process corresponds to the maximum loss of fBm.

We consider B with Hurst parameter $H \in (1/2, 1)$. Let $\mu \in \mathbb{R}$ be the drift parameter and $\sigma > 0$ be the diffusion coefficient for fBm with drift defined by

$$Y_t := \mu t + \sigma B_t \; . \tag{2}$$

Let Φ denote the cumulative distribution function of standard Gaussian distribution and let $\overline{\Phi} = 1 - \Phi$. The following notation will be used in the sequel.

- Let $I_t^{H,\mu} := \inf_{0 \leq v \leq t} Y_v$ denote the *infimum* of fBm up to time t.
- Let $S_t^{H,\mu} := \sup_{0 \leq v \leq t} Y_v$ denote the *supremum* of fBm up to time t.
- Let $R_t^{H,\mu} := S_t^{H,\mu} - I_t^{H,\mu}$, called the *range* of fBm up to time t.
- The *maximum loss* of fBm before time t is defined as

  $$M_t^{H,\mu,-} := \sup_{0 \leq u \leq v \leq t} (Y_u - Y_v) = \sup_{0 \leq v \leq t} \left(\sup_{0 \leq u \leq v} (Y_u - Y_v) \right) =: \sup_{0 \leq v \leq t} L_v^{H,\mu}$$

  and $L^{H,\mu}$ is called the *loss* process.

Our first result given in Section 2 is on the expectation of maximum loss and its probability distribution. The following bounds are derived:

$$\frac{\sqrt{2}\sigma t^H}{2\sqrt{\pi}} + (\mu \wedge 0)\, t \leq \mathbb{E}(M_t^{H,\mu,-}) \leq \frac{2\sqrt{2}\sigma t^H}{\sqrt{\pi}} + |\mu| t \ ,$$

$$\overline{\Phi}((x+\mu t)/(\sigma t^H)) \leq \mathbb{P}(M_t^{H,\mu,-} > x) \leq \frac{2\sqrt{2}\sigma t^H}{x\sqrt{\pi}} + \frac{|\mu| t}{x} \ .$$

In Section 3, we show that the lower bound for the distribution is attained asymptotically for large x. The maximum loss over $[0,t]$ behaves like the negative of the log-price at time t, which has the maximum variance in this time interval. Explicitly, we prove that

$$\lim_{x\to\infty} \frac{\mathbb{P}(M_t^{H,\mu,-} > x)}{\overline{\Phi}((x+\mu t)/(\sigma t^H))} = 1 \ .$$

2. Bounds on the distribution of maximum loss

In this section, we find bounds on the expected value of maximum loss of fBm and its distribution. As a stand alone result,we first show that the loss process has the same marginal distribution as supremum when there is no drift. In this case, let the loss process be denoted by $L^{H,0}$, namely, $L_v^{H,0} := \sup_{0\leq u\leq v}(B_u - B_v)$, and let supremum process be denoted by $S^{H,0}$.

Proposition 1. *The loss process $L^{H,0}$ is self-similar and $L_v^{H,0}$ has the same distribution as $S_v^{H,0}$ for each $v \geq 0$.*

Proof: The self-similarity of fBm corresponds to $\{B_{au} : u \geq 0\} \stackrel{d}{=} \{a^H B_u : u \geq 0\}$ for every $a > 0$. It follows that

$$\{B_{au} - B_{av} : 0 \leq u \leq v, v \geq 0\} \stackrel{d}{=} \{a^H(B_u - B_v) : 0 \leq u \leq v, v \geq 0\} \ .$$

Therefore, we get

$$\{L_{av}^{H,0} : v \geq 0\} \stackrel{d}{=} \{a^H L_v^{H,0} : v \geq 0\}$$

by the definition of $L_v^{H,0}$.

Because fractional Brownian motion has stationary increments, the collections $\{B_u - B_v : 0 \leq u \leq v\}$ and $\{-B_{v-u} : 0 \leq u \leq v\}$ have the same probability law for fixed v. Both are 0 mean Gaussian processes with covariance function

$$r(u,u') = 1/2\,[\,|v-u|^{2H} + |v-u'|^{2H} - |u-u'|^{2H}] \ .$$

Since supremum of the two collections will also have the same distribution and $\{-B_{v-u} : 0 \le u \le v\} \stackrel{d}{=} \{B_u : 0 \le u \le v\}$, we get $L_v^{H,0} \stackrel{d}{=} S_v^{H,0}$. □

Note that the result given above does not hold for fBm with drift. In Theorem 1 below, we find bounds on the maximum loss of fBm with drift as well as no drift, that is, $\mu = 0$. Towards that end, bounds on supremum of fBm are given in the following lemma. These bounds and the idea of the proof have been stated in [9, pg.162]. The upper bound is obtained in [16, pg.261] by considering fBm up to a random time which is exponentially distributed.

Lemma 1. *For fBm B with Hurst parameter $H \ge \frac{1}{2}$, the expected value of the supremum is bounded as*

$$\frac{\sqrt{2}t^H}{2\sqrt{\pi}} \le \mathbb{E}(\sup_{0\le s\le t} B_s) \le \frac{\sqrt{2}t^H}{\sqrt{\pi}} .$$

Proof: We show for $t = 1$, first. Since $\mathbb{E}(B_u - B_v)^2 = (u - v)^{2H}$, we have

$$\mathbb{E}(B_u^1 - B_v^1)^2 \le \mathbb{E}(B_u^H - B_v^H)^2 \le \mathbb{E}(B_u^{1/2} - B_v^{1/2})^2$$

for $u, v \in [0, 1]$, where we introduce notation B^H to denote fBm with parameter $H \in [1/2, 1]$. By Sudakov-Fernique inequality [1, Theorem II.2.9], we get

$$\mathbb{E} \sup_{0\le s\le 1} B_s^1 \le \mathbb{E} \sup_{0\le s\le 1} B_s^H \le \mathbb{E} \sup_{0\le s\le 1} B_s^{1/2} . \tag{3}$$

For Wiener process $B^{1/2}$, it is well known that $\mathbb{E}\sup_{0\le s\le 1} B_s^{1/2} = \sqrt{2/\pi}$ [3, pg.399]. On the other hand, B^1 is a degenerate process satisfying $B_t^1 = tZ$ for all $t \ge 0$, where Z is a standard normal random variable [13, Ex.7.2.5]. For $x \ge 0$, we find that

$$P(\sup_{0\le s\le 1} B_s^1 > x) = P(\sup_{0\le s\le 1} sZ > x) \;=\; P(Z > x) \;=\; \overline{\Phi}(x)$$

as $sZ \le Z$, for $0 \le s \le 1$. It follows that $\mathbb{E}\sup_{0\le s\le 1} B_s^1 = \int_0^\infty \overline{\Phi}(x)dx = 1/\sqrt{2\pi}$. Therefore, we get $1/\sqrt{2\pi} \le \mathbb{E}\sup_{0\le s\le 1} B_s^H \le \sqrt{2/\pi}$ from (3). Since $\sup_{0\le s\le 1} t^H B_s^H \stackrel{d}{=} \sup_{0\le s\le t} B_s^H$ due to self-similarity of fBm, the result follows. □

Theorem 1. *For fBm with Hurst parameter $H \ge 1\frac{1}{2}$, drift $\mu \in \mathbb{R}$ and $\sigma > 0$, we have*

$$\frac{\sqrt{2}\sigma t^H}{2\sqrt{\pi}} + (\mu \wedge 0)\, t \le \mathbb{E}(M_t^{H,\mu,-}) \le \mathbb{E}(R_t^{H,\mu}) \le \frac{2\sqrt{2}\sigma t^H}{\sqrt{\pi}} + |\mu| t$$

and

$$\overline{\Phi}((x+\mu t)/(\sigma t^H)) \le \mathbb{P}(M_t^{H,\mu,-} > x) \le \mathbb{P}(R_t^{H,\mu} \ge x) \le \frac{2\sqrt{2}\sigma t^H}{x\sqrt{\pi}} + \frac{|\mu| t}{x}$$

for $x > 0$ *and* $t > 0$.

Proof: From Lemma 1, the expectation of supremum of standard fBm is bounded from above by $\frac{\sqrt{2}}{\sqrt{\pi}} t^H$, and by symmetry, the expectation of its infimum is bounded from below by $-\frac{\sqrt{2}}{\sqrt{\pi}} t^H$. It follows that $\mathbb{E}(S_t^{H,\mu}) \le \frac{\sqrt{2}}{\sqrt{\pi}} \sigma t^H + \mu t$ and $\mathbb{E}(I_t^{H,\mu}) \ge -\frac{\sqrt{2}}{\sqrt{\pi}} \sigma t^H$ when $\mu > 0$; and $\mathbb{E}(S_t^{H,\mu}) \le \frac{\sqrt{2}}{\sqrt{\pi}} \sigma t^H$ and $\mathbb{E}(I_t^{H,\mu}) \ge \mu t - \frac{\sqrt{2}}{\sqrt{\pi}} \sigma t^H$ when $\mu < 0$. Combining the results given above, we find an upper bound for the expected value of range $R_t^{H,\mu}$, that is, $\mathbb{E}(R_t^{H,\mu}) \le \frac{2\sqrt{2}}{\sqrt{\pi}} \sigma t^H + |\mu| t$. Clearly, we have $M_t^{H,\mu,-} \le R_t^{H,\mu}$ and the upper bound for $\mathbb{E}(M_t^{H,\mu,-})$ follows. By Markov's inequality, we get

$$\mathbb{P}(M_t^{H,\mu,-} > x) \le \mathbb{P}(R_t^{H,\mu} \ge x) \le \frac{2\sqrt{2}\sigma t^H}{x\sqrt{\pi}} + \frac{|\mu| t}{x} .$$

The lower bound is obtained simply by definition of the maximum loss:

$$\begin{aligned} \mathbb{P}(M_t^{H,\mu,-} > x) &= \mathbb{P}(\sup_{0 \le u \le v \le t} (Y_u - Y_v) > x) \\ &\ge \mathbb{P}(-\sigma B_t - \mu t > x) = \overline{\Phi}((x+\mu t)/(\sigma t^H)) . \end{aligned}$$

Now, we have the inequality

$$-I_t^{H,\mu} = -\inf_{0\le v \le t} Y_v = \sup_{0 \le v \le t} (-Y_v) \le \sup_{0\le v\le t} (\sup_{0 \le u \le v} (Y_u - Y_v)) = M_t^{H,\mu,-}$$

for every $t > 0$. Hence,

$$-\mathbb{E}(I_t^{H,\mu}) \le \mathbb{E}(M_t^{H,\mu,-}) \tag{4}$$

is obtained. It follows from Lemma 1 that

$$\mathbb{E}(S_t^{H,\mu}) \ge \mathbb{E}(\sup_{0\le s\le t} \sigma B_s) + (\mu \wedge 0)\, t \ge \frac{\sqrt{2}\sigma t^H}{2\sqrt{\pi}} + (\mu \wedge 0)\, t .$$

Because $-\mathbb{E}(I_t^{H,\mu}) = \mathbb{E}(S_t^{H,\mu})$, we obtain the lower bound for $\mathbb{E}(M_t^{H,\mu,-})$ using (4). □

3. Asymptotic distribution of maximum loss

In this section, we show that the asymptotic form of the tail probability of maximum loss is the same as the tail probability of the difference with maximum variance over $[0, t]$. The proof is based on the result of [14] which provides two technical conditions equivalent to the result that the asymptotic distribution of supremum of a centered Gaussian process with continuous covariance, indexed by a compact metric space, is asymptotically equal to the marginal distribution with the maximal variance. Another proof of this characterization is given in [15, Prop.2.9] where the method is described as concentration of measure phenomenon on the marginal distribution with the maximal variance in the form of Gaussian isoperimetric inequality.

The theorem in [14] for centered Gaussian processes extends to Gaussian processes with drift by the following modifications in the characterization conditions. In comparison with the original result, the first condition remains the same and requires that there is a unique point in T for which the variance is maximized. The second condition, which is on the expectation of supremum of a subcollection of the process, needs to be modified to include the drift. These are given in Lemma 2. Its proof follows along the same lines of [14] for the centered case.

Lemma 2. *Suppose that $(X_t)_{t\in T}$ is a real-valued separable Gaussian process with continuous drift function $u_t : T \to \mathbb{R}$ and continuous covariance, where T is a compact metric space, and that $(X_t)_{t\in T}$ has almost surely bounded sample paths. Let $\alpha^2 = \sup_{t\in T} Var(X_t)$. Then, the statement*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\sup_{t\in T} X_t > x)}{\overline{\Phi}((x-u_\tau)/\alpha)} = 1$$

is equivalent to the following two conditions

- *(i) There exists a unique $\tau \in T$ such that $\alpha^2 = Var(X_\tau)$.*
- *(ii) For $h > 0$, if we define $T_h = \{t \in T : Cov(X_t, X_\tau) \geq \alpha^2 - h^2\}$, then we have*

$$\lim_{h\to 0} h^{-1}\, \mathbb{E} \sup_{t\in T_h} (X_t - X_\tau + u_t - u_\tau) = 0 \,.$$

Note that we have left the form of the drift function u unspecified in Lemma 2. We consider a linear drift in the next theorem, which is our main result.

Theorem 2. *For fBm with Hurst parameter $H \geq \frac{1}{2}$, drift $\mu \in \mathbb{R}$ and $\sigma > 0$, we have*

$$\lim_{x\to\infty} \frac{\mathbb{P}(M_t^{H,\mu,-} > x)}{\overline{\Phi}((x+\mu t)/(\sigma t^H))} = 1 \ .$$

Proof: The maximum loss $M_t^{H,\mu,-}$ is defined as the supremum of $(Y_u - Y_v)_{(u,v)\in T}$ where $T = \{(u,v) : 0 \leq u \leq v \leq t\}$. As T is compact and fBm has continuous sample paths, we will show that the two conditions in Lemma 2 hold for the Gaussian process $\{Y_u - Y_v : 0 \leq u \leq v \leq t\}$ to complete the proof.

Letting $\alpha^2 = \sup_{(u,v)\in T} \mathrm{Var}(Y_u - Y_v)$, we get $\alpha^2 = \sigma^2 t^{2H}$. Since we have $\mathrm{Var}(Y_{u_0} - Y_{v_0}) = \alpha^2$, only for $(u_0, v_0) = (0,t)$ in T, Condition (i) of Lemma 2 holds. Note that the Gaussian random variable $Y_0 - Y_t$ has drift $-\mu t$ and diffusion parameter σ. Therefore, the limiting distribution is $\overline{\Phi}((x+\mu t)/(\sigma t^H))$.

For showing Condition (ii) of Lemma 2, let $T_h = \{(u,v) \in T : \mathrm{Cov}\,(Y_u - Y_v, Y_t) \geq \alpha^2 - h^2\}$ for $h > 0$. In view of definition (2) and the value of α^2, this is simplified as

$$T_h = \{(u,v) \in T : \sigma^2 \mathbb{E}(B_t(B_u - B_v)) \geq \sigma^2 t^{2H} - h^2\} \ .$$

Then, Condition (ii) is explicitly

$$\lim_{h\to 0} h^{-1}\, \mathbb{E} \sup_{(u,v)\in T_h} \sigma(B_u - B_v + B_t) + \mu[(t - (v-u)] = 0 \ . \tag{5}$$

Note that the expectation in (5) is bounded above by

$$\mathbb{E} \sup_{(u,v)\in T_h} \sigma(B_u - B_v + B_t) + \sup_{(u,v)\in T_h} \mu[t - (v-u)] \ . \tag{6}$$

In order to identify T_h, we find that

$$\mathbb{E}(B_t(B_u - B_v)) = \frac{1}{2}(v^{2H} - u^{2H} + (t-u)^{2H} - (t-v)^{2H}) \ .$$

Therefore, $(u,v) \in T_h$ satisfy

$$v^{2H} - u^{2H} + (t-u)^{2H} - (t-v)^{2H} \geq 2t^{2H} - 2\sigma^{-2}h^2 \ . \tag{7}$$

Clearly, (7) is satisfied by $(\bar{u}, v) \in T_h$, for fixed $\bar{u} \in [0,t]$ as well. Let us now consider

$$T_h^{\bar{u}} := \{\bar{u} \leq v \leq t : \mathbb{E}(B_t(B_{\bar{u}} - B_v)) \geq t^{2H} - \sigma^{-2}h^2\} \ . \tag{8}$$

Now, for $\bar{u} > t/2$, we have $(t-\bar{u})^{2H} - \bar{u}^{2H} < 0$, and $(t-v)^{2H} > 0$ for all $v \in [0,t]$. Then, we get

$$v^{2H} \geq 2t^{2H} - 2\sigma^{-2}h^2 \geq t^{2H} - 2\sigma^{-2}h^2$$

from (7). On the other hand, for $\bar{u} \leq t/2$, we have $(t-\bar{u})^{2H} \leq t^{2H}$. Therefore, we get

$$v^{2H} + t^{2H} \geq v^{2H} - u^{2H} + (t-u)^{2H} - (t-v)^{2H} \geq 2t^{2H} - 2\sigma^{-2}h^2$$

from (7), which again implies

$$v^{2H} \geq t^{2H} - 2\sigma^{-2}h^2 \tag{9}$$

for $v \in T_h^{\bar{u}}$. Considering the same arguments in [14, pg.309] and [1, pg.122], we deduce that

$$v \geq t - Kh^2$$

for some constant K and $v \in T_h^{\bar{u}}$. To show this, we use the assumption $H \geq 1/2$, which implies that $f(v) := v^{2H}$ is convex. It follows that v satisfying (9) also satisfy $v \geq t - 2\sigma^{-2}h^2/K_{\bar{u}}$ where $K_{\bar{u}} = (t^{2H} - \bar{u}^{2H})/(t-\bar{u})$, uniformly for $h > 0$. Taking $K := 2/(\sigma^2 K_{\bar{u}})$, we have

$$\mathbb{E} \sup_{v \in T_h^{\bar{u}}} B_{\bar{u}} - B_v + B_t \leq \mathbb{E} \sup_{v \geq \bar{u},\, t-v \leq Kh^2} B_{\bar{u}} - B_v + B_t \, . \tag{10}$$

Since fBm has stationary increments and $\mathbb{E}B_{\bar{u}} = 0$, we observe that

$$\lim_{h \to 0} h^{-1}\, \mathbb{E} \sup_{v \geq \bar{u},\, t-v \leq Kh^2} B_{\bar{u}} - B_v + B_t = \lim_{h \to 0} h^{-1}\, \mathbb{E} \sup_{t-v \leq Kh^2} B_{t-v} \tag{11}$$

for fixed $\bar{u} \in [0,t]$. Under the assumption $H \geq 1/2$, we have

$$\mathbb{E} \sup_{t-v \leq Kh^2} B_{t-v} \leq \frac{\sqrt{2}K^H h^{2H}}{\sqrt{\pi}}$$

by Lemma 1. Hence, we get the limit in (11) to be 0, which implies that left hand side of (10) is $O(h)$. Since T is separable, a monotone convergence argument extends the result for fixed $\bar{u}$ to all T_h proving that the expectation in (6) is also $O(h)$ [1, pg.47].

Now, $(u,v) \in T_H$ satisfy

$$\sigma^2[v^{2H} - u^{2H} + (t-u)^{2H} - (t-v)^{2H}] \geq 2\sigma^2 t^{2H} - 2h^2 \, . \tag{12}$$

By reverse triangle inequality, the left hand side of (12) is bounded above by $2\sigma^2(v-u)^{2H}$. Therefore, $(u,v) \in T_H$ also satisfy

$$(v-u)^{2H} \geq t^{2H} - \sigma^{-2}h^2 \ .$$

By fixing $\bar{u}$ and replacing the role of v by $v-\bar{u}$ in (9), we now get

$$v - \bar{u} \geq t - Kh^2$$

for v in $T_h^{\bar{u}}$ of (8), and $K = 2/(\sigma^2 K_{\bar{u}})$ with $K_{\bar{u}} = (t-\bar{u})^{2H}/(t-\bar{u})$ as $0 \leq v - \bar{u} \leq t - \bar{u}$. Therefore, we have

$$\lim_{h\to 0} h^{-1} \sup_{v\in T_h^{\bar{u}}} \mu[t-(v-\bar{u})] = 0$$

which concludes the proof of (5) by monotone convergence. □

Note that Theorem 2 implies that the tail probability decays exponentially as

$$\lim_{x\to\infty} \frac{1}{x^2} \log \mathbb{P}(M_t^{H,\mu,-} > x) = -\frac{1}{2\sigma^2 t^{2H}}$$

since this holds for $\overline{\Phi}$ [1, pg.42]. The special case with no drift follows by substitution of $\mu = 0$ and $\sigma = 1$.

As related work, the method used in [6, Thm.1] for approximating the tail distribution of the supremum of fractional Brownian motion with a negative drift function is comparable to our approach for proving Theorem 2. The process which is defined over the index set $\mathbb{R}_+$ in [6] is first transformed to a related process by self-similarity. The supremum of the latter process is found to occur in the neighborhood of a unique index $\tau > 0$ for the marginal distribution with maximal variance. Then, the method of proof is based on [10, Thm.D.3] which specifies the asymptotic distribution of the process confined to an interval containing τ. The main idea is to obtain a neighborhood of τ with the largest contribution to the asymptotic behavior, and to compare the process with a related stationary Gaussian process by Slepian's inequality. The result is generalized to multidimensional index set in [10, Sec.8] . In Theorem 2, we have used the relatively more structured approach of [14, 15] which is interpreted as the concentration of measure and has been of further interest in various fields [7]. Our approach is based on a subcollection of the process which has sufficiently high covariance with the random variable of maximal variance, whereas the method of [6] focuses on a neighborhood of τ. In spite of this contrast, we have identified the index set of this subcollection as a neighborhood of τ in the present work as well. This can be attributed to the closeness of the marginal distribution with maximal variance and the distribution of the supremum in the tail.

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