

On the Geometric Measures of Entanglement

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The geometric measure of entanglement, which expresses the minimum distance to product states, has been generalized to distances to sets that remain invariant under the stochastic reducibility relation. For each such set, an associated entanglement monotone can be defined. The explicit analytical forms of these measures are obtained for bipartite entangled states. Moreover, the three qubit case is discussed and argued that the distance to the W states is a new monotone.

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I. INTRODUCTION

Entanglement is a profound phenomenon which intrigued scientists with both its rich mathematical structure and its deep philosophical implications. In the last two decades it has also come into prominence as a key resource in quantum communication and quantum computation[1]. Quantifying entanglement is one of the main problems of quantum information theory. Although for bipartite pure states entanglement measures are well established, quantification of mixed state and multipartite pure state entanglement still contain many unresolved issues[1–3]. For multipartite states, not only the amount but also the flavor of entanglement becomes pertinent. For example, for three qubits there are two distinct flavors of genuine tripartite entanglement represented by the Greenberger-Horne-Zeilinger (GHZ) and W states which can never be converted into each other[4]. The essential property that is not present or lacking a proper analogue in these settings is the Schmidt decomposition. Still, a number of important applications require a proper measure for systems beyond the bipartite pure setting and the challenge continues.

A considerable number of multipartite entanglement measures are generalizations of bipartite measures. Geometric measure of entanglement[5] is one of them. It is introduced by Shimony [6] and generalized to multipartite states by Barnum and Linden[7]. Wei and Goldbart extended it to multipartite mixed states by employing the convex roof construction[5].

For a pure state $|\psi\rangle$, the geometric distance is defined as the minimum distance between $|\psi\rangle$ and the set of product states[5], i.e.,

$$d(\psi, S) = \inf_{|\phi\rangle \in S} \|\psi - |\phi\rangle\| \quad , \quad (1)$$

where S denotes the set of normalized product states. Even though the distance has a clear geometric meaning, an entanglement monotone[8], i.e., a quantity which

never increase on the average under local quantum operations assisted with classical communication (LOCC) is more useful in applications. The quantity

$$E(\psi, S) = 1 - \sup_{|\phi\rangle \in S} |\langle\psi|\phi\rangle|^2 \quad , \quad (2)$$

satisfies this property[5]. Moreover, it is related to the geometric distance by a monotone increasing function as

$$d(\psi, S) = \sqrt{2 \left(1 - \sqrt{1 - E(\psi, S)}\right)} \quad . \quad (3)$$

The geometric measure is used in a number of applications for expressing the degree of entanglement. It is related to the Groverian measure of entanglement[9], a quantity which measures the degree of success in Grover's search algorithm[10] as a function of the initial entangled state used. It is also involved, along with some other distancelike measures, in an upper bound expression on the maximum number of multipartite states that can be discriminated by LOCC[11]. Recently, it is shown that for measurement-based quantum computing, too much entanglement in the initial state, as measured by the geometric measure, is detrimental for the quantum speedup gained over classical algorithms[12]. Finally, the geometric measure has also found applications in many-body physics[13, 14].

It is interesting to investigate the distance of the state $|\psi\rangle$ to more general sets S other than the product states for the purpose of generalizing the geometric measure. In this way, it is possible to quantify the flavor of entanglement more directly; i.e., the degree of the difference of the entanglement in state $|\psi\rangle$ from those of the states in S can be measured by $E(\psi, S)$. In addition to this, when the state $|\psi\rangle$ is desired to be approximated by states in S , the maximum achievable fidelity is deficient by $E(\psi, S)$ from the absolute maximum 1. For example, there might be situations where the state $|\psi\rangle$ is required in a particular quantum communication or computation task; but it may be too difficult to establish this state between distant parties. Instead, states in the set S could be easily established with little cost. For such cases, smallness of the deficiency $E(\psi, S)$ can be used as a measure of replaceability of $|\psi\rangle$ by some state in S . An experiment is

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discussed in Ref. 15 where GHZ class states are used for obtaining approximate W states.

It will be shown that, if the set S remains invariant under the stochastic reducibility relation of Ref. 3, then the function $E(\psi, S)$ is also a monotone. In this case the distance d and the monotone E satisfy properties similar to those of the geometric measure of Wei and Goldbart. Hence, there appears to be a number of different *geometric measures*; the case for which S is the set of product states gives just one of those measures. These measures also include previous generalizations[7, 16], but the current generalization appears to exhaust all such pure-state measures that are based on the geometric distance to a set. The purpose of this article is to investigate these general geometric measures. It may be the case that some of these measures might be useful in the sense that they forbid some entanglement transformations allowed by all other known monotones. Indeed, it appears that the distance to the W class states of three qubits is an example of this.

The organization of the article is as follows. In section II, the precise definition of the key property of the sets S is presented and the monotonicity of $E(\psi, S)$ is proven. Section III contains the computation of all of the geometric measures for the bipartite entangled states. After that, the three qubit case is investigated and one of the monotones is shown to be a new monotone in Section IV. Finally a brief summary is given in section V.

II. DEFINITIONS AND PROOF OF MONOTONICITY

In the following, p -partite entangled pure states between p distant particles will be considered. Throughout this article we will be interested in pure states only. Let us briefly recall the following definitions. The entangled state $|\phi\rangle$ is said to be *stochastically reducible* to $|\phi'\rangle$, if, after starting with the initial state $|\phi\rangle$, it is possible to obtain $|\phi'\rangle$ with non-zero probability by LOCC operations[3]. This is equivalent to the existence of local operators A_i such that

$$|\phi'\rangle = N(A_1 \otimes A_2 \otimes \cdots \otimes A_p) |\phi\rangle \quad (4)$$

for some normalization constant N . We say that $|\phi\rangle$ and $|\phi'\rangle$ are *SLOCC equivalent* if they are stochastically reducible to each other. That statement is equivalent to the existence of invertible local operators A_i such that Eq. (4) is satisfied [4]. The equivalence classes obtained from this equivalence relation are called SLOCC classes. If $|\phi\rangle$ is stochastically reducible to $|\phi'\rangle$, then all states in the SLOCC class of $|\phi\rangle$ is stochastically reducible to any state in the class of $|\phi'\rangle$. In other words, stochastic reducibility is also a relation between SLOCC classes.

Let S be a set of states. If for any state $|\phi\rangle$ in S , all states $|\phi'\rangle$ that are stochastically reducible from $|\phi\rangle$ are in S , then we will say that S is a *stochastically invariant* (SI) or *SLOCC invariant* set. Obviously, if $|\phi\rangle$ is

in the SI set S , then the whole of the SLOCC class of $|\phi\rangle$ is a subset of S . This means that SI sets are unions of SLOCC classes. Moreover, if S contains a particular SLOCC class, then all classes that are stochastically reducible from this class are also contained in S . In particular, this implies that S contains (the class of) all product states. The central result of this article is the following theorem.

Theorem 1. *If S is a SI set of normalized states, then the function $E(\psi, S)$ of the normalized states $|\psi\rangle$ is an entanglement monotone.*

The requirements that S contain only normalized states and $|\psi\rangle$ is normalized are imposed for utilizing Eq. (2) as a definition of E . Before proving this theorem, we need the following lemma.

Lemma 2. *Let $|\alpha\rangle, |\beta_1\rangle, \dots, |\beta_n\rangle$ be vectors in a Hilbert space. Then, the operator inequality*

$$|\alpha\rangle\langle\alpha| \leq \sum_{i=1}^n |\beta_i\rangle\langle\beta_i| \quad (5)$$

is satisfied if and only if there exists numbers c_i such that

$$|\alpha\rangle = \sum_{i=1}^n c_i |\beta_i\rangle \quad \text{and} \quad \sum_{i=1}^n |c_i|^2 \leq 1 \quad . \quad (6)$$

Proof: We first show the necessity. Suppose that Eq. (5) is satisfied. Then, $B = \sum_i |\beta_i\rangle\langle\beta_i| - |\alpha\rangle\langle\alpha|$ is positive semidefinite. Let m be the number of non-zero eigenvalues of B . By using the spectral decomposition of B , we can find m vectors $|\gamma_1\rangle, \dots, |\gamma_m\rangle$ such that $B = \sum_i |\gamma_i\rangle\langle\gamma_i|$. Hence, we have the following relation

$$\sum_{i=1}^m |\gamma_i\rangle\langle\gamma_i| + |\alpha\rangle\langle\alpha| = \sum_{i=1}^n |\beta_i\rangle\langle\beta_i| \quad (7)$$

Then, by the Schrödinger-GHJW theorem[17], there is a $d \times d$ unitary matrix U (where $d = \max(m+1, n)$), such that

$$|\gamma_j\rangle = \sum_i U_{j,i} |\beta_i\rangle \quad (j = 1, \dots, m), \quad (8)$$

$$|\alpha\rangle = \sum_i U_{m+1,i} |\beta_i\rangle \quad . \quad (9)$$

Hence, we take $c_i = U_{m+1,i}$. Moreover,

$$\sum_{i=1}^n |c_i|^2 = \sum_{i=1}^n |U_{m+1,i}|^2 \leq \sum_{i=1}^d |U_{m+1,i}|^2 = 1 \quad . \quad (10)$$

Therefore, (6) is satisfied.

Now, for proving the sufficiency part, suppose that (6)

is satisfied. Let $|\gamma\rangle$ be any arbitrary vector. Then,

$$\langle\gamma|\alpha\rangle\langle\alpha|\gamma\rangle = |\langle\gamma|\alpha\rangle|^2 \quad (11)$$

$$= \left| \sum_i c_i \langle\gamma|\beta_i\rangle \right|^2 \quad (12)$$

$$\leq \left(\sum_i |c_i|^2 \right) \left(\sum_i |\langle\gamma|\beta_i\rangle|^2 \right) \quad (13)$$

$$\leq \sum_i |\langle\gamma|\beta_i\rangle|^2 = \sum_i \langle\gamma|\beta_i\rangle\langle\beta_i|\gamma\rangle. \quad (14)$$

Here, Schwarz inequality is used in passing from (12) to (13). Since the last inequality is valid for all vectors $|\gamma\rangle$, then the associated inequality for operators, i.e., (5) is satisfied. This completes the proof of the equivalence of (5) and (6). \square

Now, we can start with the proof of the theorem. It should be shown that, if by LOCC, $|\psi\rangle$ is transformed to states $|\psi_i\rangle$ with probability p_i , then

$$E(\psi, S) \geq \sum_i p_i E(\psi_i, S). \quad (15)$$

For this purpose, it is enough to prove this inequality for the local operations carried out by a single party only. Hence, without loss of generality, it will be assumed that the first party is carrying out a measurement. Let M_i be the local measurement operators associated with this operation. They satisfy $\sum_i M_i^\dagger M_i = \mathbb{1}_1$ where $\mathbb{1}_1$ is used for denoting the identity operator acting on the state space of the i th party's particle. Hence, we have

$$p_i = \langle\psi| (M_i^\dagger M_i \otimes \mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_p) |\psi\rangle, \quad (16)$$

$$|\psi_i\rangle = \frac{1}{\sqrt{p_i}} (M_i \otimes \mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_p) |\psi\rangle. \quad (17)$$

Let $P_i = M_i^\dagger M_i$ and U_i be the appropriate unitary operators satisfying $M_i = U_i \sqrt{P_i}$, whose existence is guaranteed by the polar decomposition of operators. For simplifying the notation, boldface letters will be used for denoting the 1st particle's local operators as an operator acting on the whole state space, i.e., $\mathbf{P}_i = P_i \otimes \mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_p$ etc.

Let $|\phi\rangle$ be an arbitrary vector in S ; it will be used in the maximization of the right-hand side of Eq. (2). Let

$$n_i = \langle\phi|\mathbf{P}_i|\phi\rangle \quad (18)$$

and note that these are non-negative numbers having the sum $\sum_i n_i = 1$. Let us define the vectors $|\phi_i\rangle$ in S as

$$|\phi_i\rangle = \begin{cases} n_i^{-1/2} \sqrt{\mathbf{P}_i} |\phi\rangle & \text{if } n_i \neq 0 \\ |\phi\rangle & \text{if } n_i = 0 \end{cases}. \quad (19)$$

Note that each vector $|\phi_i\rangle$ is normalized and stochastically reducible from $|\phi\rangle$. Therefore all of them are in S . For the case $n_i = 0$, the value of $|\phi_i\rangle$ is unimportant; it has just been assigned to a vector known to exist in S .

First note that

$$|\phi\rangle = \sum_i \sqrt{n_i} \sqrt{\mathbf{P}_i} |\phi_i\rangle. \quad (20)$$

and $\sum_i n_i = 1$. Hence, by applying the lemma, we see that

$$|\phi\rangle\langle\phi| \leq \sum_i \sqrt{\mathbf{P}_i} |\phi_i\rangle\langle\phi_i| \sqrt{\mathbf{P}_i} \quad (21)$$

holds as an operator inequality. If the expectation value in the state $|\psi\rangle$ is taken, we get

$$|\langle\phi|\psi\rangle|^2 \leq \sum_i \left| \langle\phi_i|\sqrt{\mathbf{P}_i}|\psi\rangle \right|^2 \quad (22)$$

$$= \sum_i p_i \left| \langle\phi_i|\mathbf{U}_i^\dagger|\psi_i\rangle \right|^2 \quad (23)$$

$$= \sum_i p_i |\langle\phi'_i|\psi_i\rangle|^2 \quad (24)$$

where $|\phi'_i\rangle = \mathbf{U}_i |\phi_i\rangle$, which are also in S . Using the obvious fact that a particular value of a function is smaller than its maximum, i.e., $|\langle\phi'_i|\psi_i\rangle|^2 \leq 1 - E(\psi_i, S)$, we get

$$|\langle\phi|\psi\rangle|^2 \leq \sum_i p_i (1 - E(\psi_i, S)). \quad (25)$$

If the left-hand side is maximized over $|\phi\rangle$, then the inequality (15) is obtained. Finally, we note that S contains product states and hence $E(\psi, S) = 0$ vanishes if $|\psi\rangle$ is a product state. This completes the proof of the monotonicity of $E(\psi, S)$. \square

Let us make a few remarks on the dependence of the measure E in Eq. (2) on the set S . It can be readily observed that for any arbitrary set S , the topological closure \overline{S} yields the same measure, i.e., $E(\psi, S) = E(\psi, \overline{S})$. In addition to this, two different sets yield the same measure E (and therefore the same distance d) if and only if they have the same closure. Keeping this in mind, the converse of theorem 1 also holds as follows: if $E(\psi, S)$ is an entanglement monotone, then the closure \overline{S} is a SI set. This can be shown easily using the following observation: a state $|\psi\rangle$ is in the closure \overline{S} if and only if $E(\psi, S) = 0$. Hence, if $E(\psi, S)$ is a monotone and $|\psi\rangle$ is a state in \overline{S} , then by the non-negativity and monotonicity of the measure, $E(\psi', S) = 0$ for all states $|\psi'\rangle$ that are stochastically reducible from $|\psi\rangle$. This shows that all such $|\psi'\rangle$ are also in \overline{S} and hence \overline{S} is SI. In conclusion, SI sets are the most general sets that can be used in the definition (2) in order to make E a monotone.

Let us also make a few remarks on SI sets. Note that the union of two SI sets is also SI. Moreover, if S_1 and S_2 are SI, then the monotone associated with the union $S_1 \cup S_2$ is given by

$$E(\psi, S_1 \cup S_2) = \min(E(\psi, S_1), E(\psi, S_2)) \quad (26)$$

This basically follows from the fact that the optimum state in $\overline{S_1 \cup S_2}$ which is closest to $|\psi\rangle$ [i.e., the state

that optimizes Eqs. (1) and (2)] is equal to one of the corresponding optimum states in \overline{S}_1 and \overline{S}_2 . Because of Eq. (26), $E(\psi, S_1 \cup S_2)$ is less useful in applications since any entanglement transformation allowed by the monotones of S_1 and S_2 is also allowed by the monotone of their union. Hence, when computing geometric distances, it is sufficient to consider only “minimal” SI sets, which are sets that cannot be expressed as the union of two SI sets that do not contain each other. There is a one-to-one relation between these minimal SI sets and the SLOCC classes. A minimal SI set essentially contains one SLOCC-class C at the top and includes only the classes that can be stochastically reduced from C . Note also that if $S_1 \subset S_2$, then

$$E(\psi, S_1) \geq E(\psi, S_2) \quad . \quad (27)$$

The set of product states S_P , is contained in all SI sets, and therefore $E(\psi, S_P)$ is the largest monotone among the monotones investigated here.

III. GEOMETRIC MEASURES FOR BIPARTITE ENTANGLEMENT

For bipartite entanglement, only the Schmidt rank of the states, i.e., the number of terms in the Schmidt decomposition, is sufficient for describing the SLOCC classes and SI sets. For an integer $n \geq 1$, the SI set S_n is composed of states having Schmidt rank at most n . Hence, S_1 is the set of product states and we have the inclusion chain

$$S_1 \subset S_2 \subset \dots \subset S_n \subset S_{n+1} \subset \dots \quad . \quad (28)$$

These are all possible SI sets for the bipartite case.

If $|\psi\rangle$ has the Schmidt rank n , then only $E(\psi, S_k)$ for $k = 1, \dots, n-1$ are nonzero; and $E(\psi, S_n) = E(\psi, S_{n+1}) = \dots = 0$. Let $|\psi\rangle$ have the Schmidt decomposition

$$|\psi\rangle = \sum_{i=1}^n \sqrt{\lambda_i} |i\rangle_1 \otimes |i\rangle_2 \quad , \quad (29)$$

and let λ_i^\downarrow denote the Schmidt coefficients arranged in decreasing order, i.e., $\lambda_1^\downarrow \geq \lambda_2^\downarrow \geq \dots \geq \lambda_n^\downarrow$. Then, it will be shown below that

$$E(\psi, S_k) = 1 - \left(\lambda_1^\downarrow + \dots + \lambda_k^\downarrow \right) \quad (30)$$

for any $k \leq n$.

In order to show Eq. (30), the following inequality will be used. If A and B are arbitrary $n \times n$ square matrices, then

$$|\text{tr } AB| \leq \mathbf{s}^\downarrow(A) \cdot \mathbf{s}^\downarrow(B) = \sum_{i=1}^n s_i^\downarrow(A) s_i^\downarrow(B) \quad , \quad (31)$$

where $\mathbf{s}(A)$ represents the vector of singular values of the matrix A (i.e., $s_i(A)$ is the i th eigenvalue of $\sqrt{A^\dagger A}$) and

similarly for B . This inequality can be deduced from the corresponding inequality for hermitian matrices: if A' and B' are $n \times n$ hermitian matrices, then

$$\text{tr } A' B' \leq \boldsymbol{\lambda}^\downarrow(A') \cdot \boldsymbol{\lambda}^\downarrow(B') = \sum_{i=1}^n \lambda_i^\downarrow(A') \lambda_i^\downarrow(B') \quad , \quad (32)$$

where $\boldsymbol{\lambda}(A')$ and $\boldsymbol{\lambda}(B')$ represent the vector of eigenvalues of A' and B' respectively[18]. The inequality (31) can be proved easily by using (32) as follows. Let

$$A' = \begin{bmatrix} 0 & A \\ A^\dagger & 0 \end{bmatrix} \quad , \quad B' = \begin{bmatrix} 0 & e^{i\theta} B^\dagger \\ e^{-i\theta} B & 0 \end{bmatrix} \quad . \quad (33)$$

Note that A' is a hermitian matrix having eigenvalues $\pm s_i(A)$; similarly for the matrix B' . Using the inequality (32) we get

$$2\text{Re}(e^{-i\theta} \text{tr } AB) \leq 2\mathbf{s}^\downarrow(A) \cdot \mathbf{s}^\downarrow(B) \quad . \quad (34)$$

Choosing θ to be the phase angle of $\text{tr } AB$ produces the desired inequality (31).

Now, consider any state $|\phi\rangle$ having Schmidt rank at most k , i.e., $|\phi\rangle$ is any state in the SI set S_k . Such a state can be expressed as

$$|\phi\rangle = \frac{1}{\sqrt{\text{tr } A^\dagger A}} \sum_{i,j=1}^n A_{ij} |i\rangle_1 |j\rangle_2 \quad , \quad (35)$$

where A is any $n \times n$ matrix having matrix rank at most k . Hence,

$$\langle \psi | \phi \rangle = \frac{\text{tr } A \sqrt{\Lambda}}{\sqrt{\text{tr } A^\dagger A}} \quad (36)$$

where Λ represents the diagonal matrix formed from the Schmidt coefficients of $|\psi\rangle$, i.e., $\Lambda_{ij} = \lambda_i \delta_{ij}$. Now, using the inequality (31) we get

$$|\langle \psi | \phi \rangle| \leq \frac{\mathbf{s}^\downarrow(A) \cdot \mathbf{s}^\downarrow(\sqrt{\Lambda})}{\sqrt{\text{tr } A^\dagger A}} \quad (37)$$

$$= \frac{\sum_{i=1}^k s_i^\downarrow(A) \sqrt{\lambda_i^\downarrow}}{\sqrt{\sum_{i=1}^k s_i^\downarrow(A)^2}} \quad (38)$$

$$\leq \sqrt{\sum_{i=1}^k \lambda_i^\downarrow} \quad (39)$$

where we have used the fact that A has at most k non-zero singular values and then invoked the Schwarz inequality in the last step. This places an upper bound on the inner product $|\langle \psi | \phi \rangle|$ for all $|\phi\rangle$ in S_k . This bound is tight and can be reached by the following state in S_k ,

$$|\phi_{\max}\rangle = \frac{\sum_{i=1}^k \sqrt{\lambda_i^\downarrow} |i'\rangle_1 \otimes |i'\rangle_2}{\sqrt{\sum_{i=1}^k \lambda_i^\downarrow}} \quad , \quad (40)$$

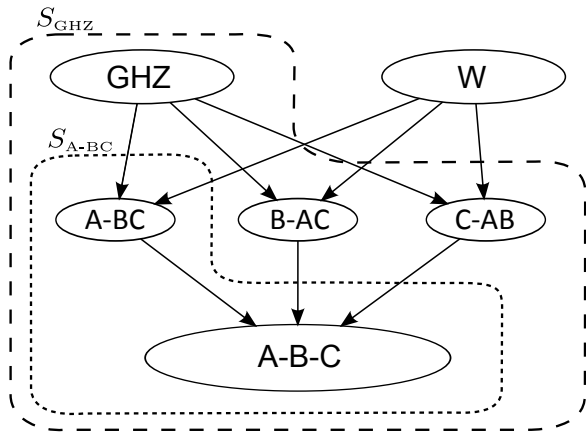


FIG. 1: SI sets generated by the GHZ and the A-BC classes

where i' denotes the state label that has the i th largest Schmidt coefficient (i.e., $\lambda_{i'}^\downarrow = \lambda_i$). This shows that $E(\psi, S_k)$ is given by Eq. (30).

The LOCC transformation rules for bipartite pure states have been determined for both deterministic[19] and probabilistic[20] transformations. It should be noted that the necessary and sufficient conditions for the possibility of a given entanglement transformation are exactly the condition that all of the geometric distance monotones are non-decreasing. In other words, an initial state $|\psi\rangle$ can be converted into the final states $|\psi_i\rangle$ with respective probabilities p_i , if and only if

$$E(\psi, S_k) \geq \sum_i p_i E(\psi_i, S_k) \quad \text{for all } k \quad . \quad (41)$$

Hence, LOCC transformation rules for the bipartite states can be expressed in terms of monotones which have a simple geometric meaning. Note that a similar, but different, projection-operator based expression has been given for these monotones by Barnum and Linden[7].

IV. GEOMETRIC MEASURES FOR THREE QUBITS

The simplest multipartite configuration is the three qubits which has been extensively investigated. There are six SLOCC classes for three qubits: the multipartite classes GHZ and W; the classes for bipartite entanglement A-BC, AB-C, C-AB; and the completely separable states A-B-C where A, B and C denote local parties[4]. The stochastic reducibility relation between these classes are as follows: Any bipartite entangled state can be obtained by SLOCC operations either from GHZ class or W class and a product state can be obtained from all the other classes.

This hierarchy allows us to construct ten non-trivial SI sets. Six of those are minimal, containing a top SLOCC class and all the other classes that can be obtained from

it. Hence, as shown in Fig. 1

$$\begin{aligned} S_{GHZ} &= \text{GHZ} \cup \text{A-BC} \cup \text{AB-C} \cup \text{C-AB} \cup \text{A-B-C} \\ S_{\text{A-BC}} &= \text{A-BC} \cup \text{A-B-C} \end{aligned}$$

and similarly for the other classes. There are four non-trivial unions of any two or all three of $S_{\text{A-BC}}$, $S_{\text{AB-C}}$ and $S_{\text{B-AC}}$, which are non-minimal SI sets; but these are not worth investigating for discovering new useful monotones.

Among the minimal SI sets, the product states $S_{\text{A-B-C}}$ produce the original geometric distance that has been investigated before. The SI sets generated by bipartite entangled states, i.e., $S_{\text{A-BC}}$, $S_{\text{AB-C}}$ and $S_{\text{B-AC}}$, give rise to monotones that can be computed analytically by the expression derived in section III. The SI set S_{GHZ} is dense in the complete set of states. In other words, all states outside S_{GHZ} , namely the states in the W class, are the limit of some sequence of states in S_{GHZ} . In fact, the generic W-class state

$$|\psi\rangle = |\beta_1\alpha_2\alpha_3\rangle + |\alpha_1\beta_2\alpha_3\rangle + |\alpha_1\alpha_2\beta_3\rangle \quad (42)$$

is the limit of $|\phi(\epsilon)\rangle = (|\gamma_1^\epsilon\gamma_2^\epsilon\gamma_3^\epsilon\rangle - |\alpha_1\alpha_2\alpha_3\rangle)/\epsilon$ as ϵ tends to zero, where $|\gamma_i^\epsilon\rangle = |\alpha_i\rangle + \epsilon|\beta_i\rangle$ for $i = 1, 2, 3$. For this reason, the geometric distance and the associated monotone for this SI set is identically zero, i.e.,

$$d(\psi, S_{GHZ}) = E(\psi, S_{GHZ}) = 0 \quad , \quad (43)$$

for all $|\psi\rangle$.

Finally, the SI set S_W generated by the W class gives rise to a new monotone. This monotone is non-zero only for states in the GHZ class. It may appear that $E(\psi, S_W)$ is similar to the three-tangle[21], which is also zero on S_W and non-zero only for the GHZ states. However, it turns out that $E(\psi, S_W)$ and the three-tangle are independently useful as it will be argued below.

First, note that if e_1, \dots, e_n are entanglement monotones and $f(t_1, t_2, \dots, t_n)$ is a concave function which is increasing in each argument t_i , then e' defined by

$$e'(\psi) = f(e_1(\psi), e_2(\psi), \dots, e_n(\psi)) \quad , \quad (44)$$

is also a monotone[22]. In this case, we will say that the new monotone e' can be generated from e_1, e_2, \dots, e_n . For applications, e' has no use whatsoever (if all e_i can be computed) since any entanglement transformation allowed by all e_i is also allowed by e' . It might be of interest to investigate which of the known monotones can be generated from the others. By using only a few numerical evidences, it is possible to show that a given set of monotones cannot be generated from each other. This can be done either by finding an example against the increasing property or an example against the concavity property of the function f .

For the current case of three qubits, there are five nontrivial monotones based on the geometric distance, namely $E(\psi, S_{\text{A-B-C}})$, $E(\psi, S_{\text{A-BC}})$, $E(\psi, S_{\text{AB-C}})$, $E(\psi, S_{\text{B-AC}})$ and $E(\psi, S_W)$, and another monotone, the

three-tangle τ . It can be shown that none of these are generated from the others. Numerical calculations carried out indicate that for any of these six functions, it is possible to find a pair of states where the transformation between them is allowed by the other five functions while forbidden by the selected function. This means that all of these monotones are independently useful in analyzing entanglement transformations.

The following is a simple example that shows that $E(\psi, S_W)$ cannot be generated from the other geometric measures and the three tangle. Let

$$|\psi\rangle = \frac{1}{\sqrt{10}}(3|000\rangle + |111\rangle) , \quad (45)$$

$$|\varphi\rangle = \frac{1}{\sqrt{N}}(|000\rangle - |\beta\beta\beta\rangle) , \quad (46)$$

where $|\beta\rangle = (|0\rangle + 2|1\rangle)/\sqrt{5}$ and N is the normalization factor. The function $E(\cdot, S_W)$ indicates a transformation ordering different than those of the tangle and the other monotones as shown in Table I. This shows that $E(\cdot, S_W)$ is a new entanglement monotone.

	$ \psi\rangle$	$ \varphi\rangle$
$E(\cdot, S_{A-B-C})$	0.1	< 0.5143
$E(\cdot, S_{AB-C})$	0.1	< 0.3643
$E(\cdot, S_{AC-B})$	0.1	< 0.3643
$E(\cdot, S_{A-BC})$	0.1	< 0.3643
$E(\cdot, S_W)$	0.09	> 0.0464
τ	0.36	< 0.6175

TABLE I: The values of the selected entanglement measures for two states that cannot be converted into each other. The geometric monotones are computed numerically by an iterative algorithm that converge to local extrema. Algorithm is repeated for several initial random configuration for finding the global extremum. This example shows that the monotonicity of $E(\psi, S_W)$ does not follow from the other measures shown in the table.

Another point that must be mentioned is, in contrast with the bipartite case, the insufficiency of the geometric measures alone for deciding the possibility of a given entanglement transformation. As an example consider the following state in GHZ class,

$$|\Phi(z; c_1, c_2, c_3)\rangle = \frac{1}{\sqrt{N}}(|000\rangle + z|\beta_1\beta_2\beta_3\rangle) , \quad (47)$$

where $|\beta_i\rangle = c_i|0\rangle + \sqrt{1-c_i^2}|1\rangle$, c_i are real with $0 \leq c_i < 1$ and z is a complex number. It is obvious that the states $|\Phi(z; c_1, c_2, c_3)\rangle$ and $|\Phi(z^*; c_1, c_2, c_3)\rangle$, where z^* represents the complex conjugate, have the same values for the three-tangle and all geometric monotones. In

a recent study[23], the rules for deterministic entanglement transformations between multipartite states with tensor rank 2 have been established. According to these rules, when none of c_i are zero and z is neither real nor on the unit circle, then these two states cannot be converted into each other. Moreover, if it is possible to transform $|\Phi(z; c_1, c_2, c_3)\rangle$ to some GHZ state $|\psi\rangle$, then it is not possible to convert $|\Phi(z^*; c_1, c_2, c_3)\rangle$ into $|\psi\rangle$. This example clearly shows that the geometric monotones and the three tangle are not sufficient for deciding on the possibility of transformations. Hence, there must be another monotone that is not derivable from all of these, which changes value under the complex conjugation of the z parameter.

V. CONCLUSION

In this article, a more general approach is taken to the geometric measure of entanglement in pure states by replacing the set of product states with a set which is invariant under stochastic reducibility relation. In this way, a number of new entanglement monotones can be obtained. Moreover, it is argued that these measures exhaust all pure-state monotones whose definition are based on the geometric distance to a set, since the closure of such sets must be SI. Consequently, these monotones contain previous generalizations[7, 16] of the geometric measure.

These measures quantify not only the amount, but also the flavor of entanglement where by flavor we mean the type of entanglement associated with each SLOCC class. The original geometric measure $E(\psi, S_P)$, where S_P is the set of product states, quantifies the property of being entangled, meaning that the state $|\psi\rangle$ is unentangled if and only if this measure vanishes. In contrast to this, $E(\psi, S)$ for non-product SI sets S essentially quantifies the difference of the flavor of entanglement in $|\psi\rangle$ from the flavor associated with S . In other words, an S based characterization of entanglement is obtained. We have $E(\psi, S) = 0$ if and only if either $|\psi\rangle$ has an identical flavor with the states in S or otherwise it can be well approximated with those states with desirably high fidelity.

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[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

[2] M. B. Plenio and S. Virmani, Quant. Inf. Comp. **7**, 1

- (2007); arXiv:quant-ph/0504163.
- [3] C. H. Bennett, S. Popescu, D. Rohrlich, J. A. Smolin, and A. V. Thapliyal, Phys. Rev. A **63**, 012307 (2000).
 - [4] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
 - [5] T.-C. Wei and P. M. Goldbart, Phys. Rev. A **68**, 042307 (2003).
 - [6] A. Shimony, Fundamental Problems in Quantum Theory, edited by D. M. Greenberger and A. Zeilinger (1995), vol. 755 of New York Academy Sciences Annals, pp. 675-679.
 - [7] H. Barnum and N. Linden, J. Phys.: Math. and Gen. **34**, 6787 (2001).
 - [8] G. Vidal, J. Mod. Opt. **47**, 355 (2000).
 - [9] O. Biham, M. A. Nielsen, T. A. Osborne Phys. Rev. A **65**, 062312 (2001).
 - [10] L. K. Grover, Phys. Rev. Lett. **79**, 325 (1997).
 - [11] M. Hayashi, D. Markham, M. Muraio, M. Owari and S. Virmani, Phys. Rev. Lett. **96**, 040501 (2006).
 - [12] D. Gross, S. T. Flammia and J. Eisert, Phys. Rev. A **102**, 190501 (2009).
 - [13] T.-C. Wei, D. Das, S. Mukhopadhyay, S. Vishveshwara, P. M. Goldbart, Paul M. Phys. Rev. A **71**, 060305 (2005).
 - [14] R. Orús, Phys. Rev. Lett. **100**, 130502 (2008).
 - [15] P. Walther, K. J. Resch, and A. Zeilinger, Phys. Rev. Lett. **94** 240501 (2005).
 - [16] M. Blasone, F. Dell'Anno, S. De Siena and F. Illuminati, Phys. Rev. A, **77**, 062304 (2008).
 - [17] L. P. Hughston, R. Jozsa, and W. K. Wootters, Phys. Lett. A **183**, 14 (1993).
 - [18] R. Bhatia, *Matrix Analysis* (Springer-Verlag, New York, 1997).
 - [19] M. A. Nielsen, Phys. Rev. Lett. **83**, 436 (1999).
 - [20] D. Jonathan and M. B. Plenio, Phys. Rev. Lett. **83**, 1455 (1999).
 - [21] V. Coffman, J. Kundu and W. K. Wootters, Phys. Rev. A **61**, 052306 (2000).
 - [22] R. M. Gingrich, Phys. Rev. A **65**, 052302 (2002).
 - [23] S. Turgut, Y. Gül, and N. K. Pak, arXiv:0907.3960 (2009).