6 May 1996

PHYSICS LETTERS A

ELSEVIER

Physics Letters A 214 (1996) 21–26

Degenerate Svinolupov KdV systems

Metin Gürses [a], Atalay Karasu [b]

[a] *Department of Mathematics, Faculty of Sciences, Bilkent University, 06533 Ankara, Turkey*

[b] *Department of Physics, Faculty of Arts and Sciences, Middle East Technical University, 06531 Ankara, Turkey*

Received 8 January 1996; accepted for publication 14 February 1996
Communicated by A.P. Fordy

Abstract

We find infinitely many coupled systems of KdV type equations which are integrable. We give also their recursion operators.

Recently, Svinolupov [1] has introduced a class of integrable multicomponent KdV equations associated with Jordan algebras (JKdV). He has found a one-to-one correspondence between Jordan algebras and multicomponent KdV equations that possess infinitely many higher symmetries. In this work we extend his work on KdV systems to a more general form. In addition to the Jordan algebra related KdV systems found by Svinolupov [1,2] we find new integrable systems of equations.

We consider a system of N nonlinear equations of the form

$$q^i_t = b^i_j q^j_{xxx} + s^i_{jk} q^j q^k_x, \tag{1}$$

where $i, j, k = 1, 2, \ldots, N$, q^i are real and depend on the variables x and t, s^i_{jk} and b^i_j are constants. We assume that the recursion operator of this system is given by

$$R^i_j = b^i_j D^2 + a^i_{jk} q^k + c^i_{jk} q^k_x D^{-1} + F_{lk}{}^i{}_j q^l D^{-1} q^k D^{-1}, \tag{2}$$

where a^i_{jk}, c^i_{jk} and $F_{lk}{}^i{}_j$ are constants with

$$s^i_{j\,k} = a^i_{k\,j} + c^i_{j\,k}, \qquad F_{lk}{}^i{}_j = -F_{lj}{}^i{}_k. \tag{3}$$

The main purpose of this work is to find integrable subclasses of (1). In these classes the major problem is to determine a^i_{jk} , $c^i_{j\,k}$ and $F_{lk}{}^i{}_j$ in terms of b^i_j and s^i_{jk} and to find the conditions satisfied by b^i_j and s^i_{jk} (integrability conditions).

The recursion operator R^i_j satisfies the compatibility condition

$$R^i_{j,t} = F'^i_k R^k_j - R^i_k F'^k_j, \tag{4}$$

PII S0375-9601(96)00171-5

where $F_k^{\prime i}$ comes from the Fréchet derivative of system (1), which is given by

$$\sigma^i_t = F_j^{\prime i} \sigma^j, \tag{5}$$

where the σ^i are called the symmetries of system (1). Eq. (5) is called the symmetry equation of (1), with

$$F_j^{\prime i} = b^i_j D^3 + s^i_{jk} q^k_x + s^i_{kj} q^k D. \tag{6}$$

Recursion operators are defined as operators mapping symmetries to symmetries, i.e.

$$R^i_j \sigma^j = \lambda \sigma^i, \tag{7}$$

where λ is an arbitrary constant. Eqs. (5) and (7) imply (4). It is Eq. (4) which determines the constants a^i_{jk}, $c^i_{j\,k}$ and $F_{lk}{}^i{}_j$ in terms of the b^i_j and s^i_{jk}. The same equation (4) brings severe constraints on the b^i_j and s^i_{jk}.

We have two exclusive cases depending upon the matrix b^i_j. These are the nondegenerate Svinolupov system where $\det(b^i_j) \neq 0$ and the degenerate Svinolupov system where $\det(b^i_j) = 0$. Our major result in this work is the degenerate Svinolupov KdV system.

(I) *Nondegenerate Svinolupov KdV system.* $\det(b^i_j) \neq 0$. In this case the constant parameters $a^i_{j\,k}, c^i_{j\,k}, s^i_{j\,k}$ are symmetric with respect to the subindices and $a^i_{j\,k}, c^i_{j\,k}$ are given by

$$a^i_{j\,k} = \tfrac{2}{3} s^i_{j\,k}, \qquad c^i_{j\,k} = \tfrac{1}{3} s^i_{j\,k}, \tag{8}$$

where b^i_j and $s^i_{j\,k}$ have to satisfy the following constraints,

$$b^k_l s^i_{j\,k} = b^k_j s^i_{k\,l}, \tag{9}$$

$$s^k_{p\,r} F_{lj}{}^i{}_k + s^k_{j\,r} F_{lp}{}^i{}_k + s^k_{j\,p} F_{lr}{}^i{}_k = 0, \tag{10}$$

with

$$F_{pl}{}^r{}_j = \tfrac{1}{9} C^r_i (s^i_{j\,k} s^k_{l\,p} - s^i_{l\,k} s^k_{j\,p}), \tag{11}$$

and C^r_i is the inverse of b^i_r. Eq. (10) is the equation satisfied by the structure constants of the Jordan algebra [1]. We now consider some particular cases.

(i) If $F = 0$, we have the following equations,

$$a^i_{j\,k} = \tfrac{2}{3} s^i_{j\,k}, \qquad c^i_{j\,k} = \tfrac{1}{3} s^i_{j\,k}, \tag{12}$$

where b^i_j and s^i_{jk} satisfy

$$s^i_{j\,k} s^k_{l\,p} - s^i_{l\,k} s^k_{j\,p} = 0, \tag{13}$$

$$b^k_l s^i_{j\,k} - b^i_k s^k_{j\,l} = 0. \tag{14}$$

The recursion operator of this class is given by

$$R^i_j = b^i_j D^2 + \tfrac{2}{3} s^i_{jk} q^k + \tfrac{1}{3} s^i_{jk} q^k_x D^{-1}. \tag{15}$$

At this point we assume that the q^i are real and hence divide this class into two subcases. For the complex case such a division is irrelevant.

(a) If b^i_j is diagonalizable then the system in (1) decouples because the Jordan algebra becomes associative as well as commutative [5].

(b) If b^i_j is nondiagonalizable then we obtain distinct coupled systems. Let us consider the case where $N = 2$. Solving the constraint equations (13) and (14) we obtain the following integrable system,

$$u_t = v_{xxx} + ru_x - sv_x, \tag{16}$$

$$v_t = -u_{xxx} + rv_x + su_x, \tag{17}$$

where $r = c_0 u + c_1 v$ and $s = -c_1 u + c_0 v$. The recursion operator for the above system is given by

$$R = \begin{pmatrix} \frac{2}{3}r + \frac{1}{3}r_x D^{-1} & D^2 - \frac{2}{3}s - \frac{1}{3}s_x D^{-1} \\ -D^2 + \frac{2}{3}s + \frac{1}{3}s_x D^{-1} & \frac{2}{3}r + \frac{1}{3}r_x D^{-1} \end{pmatrix}. \tag{18}$$

In terms of the variables r and s the system of equations (16), (17) become

$$r_t = s_{xxx} + rr_x - ss_x, \tag{19}$$

$$s_t = -r_{xxx} + (rs)_x. \tag{20}$$

This system is nothing but the complex KdV equation $\mathrm{i}\rho_t = \rho_{xxx} - \rho\rho_x$ with $\rho = \mathrm{i}r - s$.

(ii) If $F \neq 0$, we obtain the Jordan KdV systems introduced by Svinolupov [1] with the following equations,

$$a^i_{jk} = \tfrac{2}{3}s^i_{jk}, \qquad c^i_{jk} = \tfrac{1}{3}s^i_{jk}, \tag{21}$$

where b^i_j and s^i_{jk} satisfy the following constraints,

$$b^k_l s^i_{jk} = b^i_k s^k_{jl}, \tag{22}$$

$$F_{pl}{}^r{}_j = \tfrac{1}{9}C^r_i(s^i_{jk}s^k_{lp} - s^i_{lk}s^k_{jp}), \tag{23}$$

$$s^k_{pr}F_{lj}{}^i{}_k + s^k_{jr}F_{lp}{}^i{}_k + s^k_{jp}F_{lr}{}^i{}_k = 0. \tag{24}$$

The recursion operator becomes

$$R^i_j = b^i_j D^2 + \tfrac{2}{3}s^i_{jk}q^k + \tfrac{1}{3}s^i_{jk}q^k_x D^{-1} + \tfrac{1}{9}C^i_r(s^r_{jm}s^m_{kl} - s^r_{km}s^m_{jl})q^l D^{-1} q^k D^{-1}. \tag{25}$$

Here there is only one choice $b^i_j = b_0\,\delta^i_j$ where b_0 can be taken as unity without losing any generality. The special cases $N = 2$ and $N = 3$ are given in Refs. [1,2].

(II) *Degenerate Svinolupov KdV system.* $\det(b^i_j) = 0$ or b^i_j is singular. Here we consider only the case where $F_{pl}{}^k{}_j = 0$ and in addition we assume that the rank of the matrix b^i_j is $N - 1$. In this case we may take $b^i_j = \delta^i_j - k^i k_j$, where k_i is a unit vector, $k^i k_i = 1$. In this work we use the Einstein convention, i.e., repeated indices are summed up from 1 to N. We then have the following solution for all N,

$$a^i_{kj} = \tfrac{2}{3}s^i_{jk} + \tfrac{1}{3}[k^i(k_j n_k - 2k_k n_j) + k_k k_j b^i], \tag{26}$$

$$c^i_{jk} = \tfrac{1}{3}s^i_{jk} - \tfrac{1}{3}[k^i(k_j n_k - 2k_k n_j) + k_k k_j b^i], \tag{27}$$

where

$$n_l = k_i k^j s^i_{lj}, \tag{28}$$

$$b^i = k^j k^l s^i_{lj}. \tag{29}$$

The vectors k^i and s^i_{jk} are not arbitrary, they satisfy the following constraints,

$$s^i_{jk}s^k_{lm} - s^i_{lk}s^k_{jm} = 2(k_j n_l - k_l n_j)(-k^i n_m + b^i k_m), \tag{30}$$

$$k^i n_i = 0, \tag{31}$$

$$s^i_{jk} = s^i_{kj}. \tag{32}$$

The recursion operator is given by

$$R^i_j = b^i_j D^2 + \{\tfrac{2}{3}s^i_{j\,k} + \tfrac{1}{3}[k^i(k_j n_k - 2k_k n_j) + k_k k_j b^i]\}q^k + \{\tfrac{1}{3}s^i_{j\,k} - \tfrac{1}{3}[k^i(k_j n_k - 2k_k n_j) + k_k k_j b^i]\}q^k_x D^{-1}. \tag{33}$$

The first generalised symmetry is found as

$$\frac{\partial q^i}{\partial \tau} = R^i_j q^j_t = b^i_m q^m_{xxxxx} + b^i_j s^j_{mn}(q^m q^n_{xxx} + 3q^m_x q^n_{xx}) + \tfrac{2}{3}b^j_m s^i_{kj} q^k q^m_{xxx} + \tfrac{1}{3}b^j_m s^i_{jk} q^k_x q^m_{xx} + \tfrac{2}{3}s^i_{kj}s^j_{mn}q^k q^m q^n_x + \tfrac{1}{6}s^i_{jk}s^j_{mn}q^k_x q^m q^n + \tfrac{1}{3}k^i k_k n_m q^k q^m_{xxx} + \tfrac{2}{3}k^i k_k n_m q^k_x q^m_{xx} + \tfrac{2}{3}k^i k_k k_m k_j (b.n) q^j q^m q^k_x + \tfrac{1}{3}k^i k_n n_k n_m q^k q^m q^n_x + \tfrac{1}{3}b^i k_k k_m n_n q^k q^m q^n_x. \tag{34}$$

We have some particular solutions of Eqs. (30)–(32).

(i) For $N = 2$ we have

$$b^i_j = \delta^i_j - y^i y_j = x^i x_j, \qquad s^i_{j\,k} = \tfrac{3}{2}\alpha_1 x^i x_j x_k + \alpha_2 x^i y_j y_k + \tfrac{1}{2}\alpha_1 y^i(y_j x_k + y_k x_j), \tag{35}$$

where $i, j = 1, 2$ and

$$x^i = \delta^i_1, \qquad y^i = \delta^i_2, \tag{36}$$

and

$$k_i = y_i, \qquad n_i = \tfrac{1}{2}\alpha_1 x_i, \qquad b_i = \alpha_2 x_i. \tag{37}$$

The constants a^i_{jk} and c^i_{jk} appearing in the recursion operator are given by

$$a^i_{j\,k} = \alpha_1 x^i x_j x_k + \alpha_2 x^i y_j y_k + \tfrac{1}{2}\alpha_1 y^i x_j y_k, \qquad c^i_{j\,k} = \tfrac{1}{2}\alpha_1 x^i x_j x_k + \tfrac{1}{2}\alpha_1 y^i x_j y_k. \tag{38}$$

Taking $\alpha_1 = 2$ and $\alpha_2 = 1$ (without loss of generality), we obtain the following coupled system,

$$u_t = u_{xxx} + 3uu_x + vv_x, \tag{39}$$

$$v_t = (uv)_x. \tag{40}$$

The above system was first introduced by Ito [3] and the bi-Hamiltonian structure has been studied by Olver and Rosenau [4]. The recursion operator of this system is given by

$$R = \begin{pmatrix} D^2 + 2u + u_x D^{-1} & v \\ v + v_x D^{-1} & 0 \end{pmatrix}. \tag{41}$$

The first generalised symmetry of the system is found as

$$\frac{\partial u}{\partial \tau} = u_{xxxxx} + 5uu_{xxx} + vv_{xxx} + 10u_x u_{xx} + 3v_x v_{xx} + 3uvv_x + \tfrac{15}{2}u^2 u_x + \tfrac{3}{2}v^2 u_x, \tag{42}$$

$$\frac{\partial v}{\partial \tau} = vu_{xxx} + 3uvu_x + \tfrac{3}{2}v^2 v_x + v_x u_{xx} + \tfrac{3}{2}u^2 v_x. \tag{43}$$

(ii) For $N = 3$, we have $b^i_j = \delta^i_j - z^i z_j = x^i x_j + y^i y_j$, where $i, j, k = 1, 2, 3$ and $s^i_{j\,k}$ are found as

$$s^i_{j\,k} = x^i[3\alpha_0 x_j x_k + 3\alpha_1(x_k y_j + x_j y_k) + 3\alpha_2 y_j y_k + \beta_1 z_j z_k] + y^i[3\alpha_3 x_j x_k + 3\alpha_4(x_k y_j + x_j y_k) + 3\alpha_5 y_j y_k + \beta_2 z_j z_k] + z^i[\alpha_6(x_j z_k + x_k z_j) + \alpha_7(y_j z_k + y_k z_j)], \tag{44}$$

where for simplicity we have taken

$$x^i = \delta^i_1, \qquad y^i = \delta^i_2, \qquad z^i = \delta^i_3, \tag{45}$$

and

$$k_i = z_i, \qquad n_i = \alpha_6 x_i + \alpha_7 y_i, \qquad b_i = \beta_1 x_i + \beta_2 y_i, \tag{46}$$

and the constants $(\alpha_0, \ldots, \alpha_7)$ and $(\beta_1, \beta_2, \beta_3)$ have the following relations,

$$\alpha_2 = \frac{\alpha_1(\alpha_0\alpha_6 + \alpha_1\alpha_3 - \alpha_6^2)}{\alpha_3(\alpha_0 - \alpha_6)}, \qquad \beta_3 = \alpha_6, \qquad \alpha_4 = \frac{\alpha_2\alpha_3}{\alpha_1},$$
$$\alpha_5 = \frac{-\alpha_0\alpha_1\alpha_2 + \alpha_1^3 + \alpha_2^2\alpha_3}{\alpha_1^2}, \qquad \alpha_7 = \frac{-\alpha_0\alpha_2 + \alpha_1^2 + \alpha_2\beta_3}{\alpha_1}, \qquad \beta_2 = \frac{\beta_1(\beta_3 - \alpha_0)}{\alpha_1}. \tag{47}$$

The constants c^i_{jk} are given by

$$c^i_{j\,k} = x^i[\alpha_0 x_j x_k + \alpha_1(x_k y_j + x_j y_k) + \alpha_2 y_j y_k] + y^i[\alpha_3 x_j x_k + \alpha_4(x_k y_j + x_j y_k) + \alpha_5 y_j y_k] + z^i(\alpha_6 x_j z_k + \alpha_7 y_j z_k), \tag{48}$$

and $a^i_{j\,k} = s^i_{k\,j} - c^i_{k\,j}$. Letting $q^i = (u, v, w)$, the system integrable equations for u, v and w are found as

$$u_t = u_{xxx} + 3\alpha_0 u u_x + 3\alpha_1(uv)_x + \beta_1 w w_x + 3\alpha_2 v v_x, \tag{49}$$

$$v_t = v_{xxx} + 3\alpha_3 u u_x + 3\alpha_4(uv)_x + 3\alpha_5 v v_x + \beta_2 w w_x, \tag{50}$$

$$w_t = \alpha_6(uw)_x + \alpha_7(vw)_x, \tag{51}$$

and the recursion operator is given by

$$R = \begin{pmatrix} D^2 + 2\alpha_0 u + 2\alpha_1 v + r_0 D^{-1} & 2\alpha_1 u + 2\alpha_2 v + r_1 D^{-1} & \beta_1 w \\ 2\alpha_3 u + 2\alpha_4 v + r_2 D^{-1} & D^2 + 2\alpha_4 u + 2\alpha_5 v + r_3 D^{-1} & \beta_2 w \\ \alpha_6(w + w_x D^{-1}) & \alpha_7(w + w_x D^{-1}) & 0 \end{pmatrix}, \tag{52}$$

where

$$r_0 = \alpha_0 u_x + \alpha_1 v_x, \qquad r_1 = \alpha_1 u_x + \alpha_2 v_x, \qquad r_2 = \alpha_3 u_x + \alpha_4 v_x, \qquad r_3 = \alpha_4 u_x + \alpha_5 v_x.$$

The first generalised symmetry of the system (49)–(51) as an expression is too long, hence we do not give it here.

As a summary, we have found infinitely many integrable systems of nonlinear partial differential equations corresponding to each value of N. We have also given the recursion operator of each system. In this work we took the rank of the matrix b^i_j as $N - 1$. It is also possible to have integrable systems of nonlinear partial differential equations with lower rank b^i_j's. In the general case with arbitrary rank we have

$$q_t^{\mu_1} = q_{xxx}^{\mu_1} + s_{ij}^{\mu_1} q^i q^j_x, \tag{53}$$

$$q_t^{\mu_2} = s_{ij}^{\mu_2} q^i q^j_x, \tag{54}$$

where $\mu_1 = 1, 2, \dots, r$, $\mu_2 = r, r+1, \dots, N$ and r is the rank of the matrix b^i_j. We have found the constants a^i_{jk}, c^i_{jk} for $r = N - 1$. For other values of r these constants will be different from those given in (24) and (25). These systems and a detailed discussion of this work will be published elsewhere.

We thank Maxim Pavlov for discussions on the Svinolupov systems. This work is partially supported by the Scientific and Technical Research Council of Turkey (TUBITAK). M.G is an associate member of the Turkish Academy of Sciences (TUBA).

References

[1] S.I. Svinolupov, Theor. Mat. Fiz. 87 (1991) 391.
[2] S.I. Svinolupov, Functional Anal. Appl. 27 (1994) 257.
[3] M. Ito, Phys. Lett. A 91 (1982) 335.
[4] P.J. Olver and P. Rosenau, Tri-Hamiltonian soliton–compacton duality, preprint (4 April 1995).
[5] Ö. Oğuz, A canonical Lagrangian formalism of multicomponent KdV equations, preprint (1995).