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Chaotification of Impulsive Systems by Perturbations

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In this paper, we present a new method for chaos generation in nonautonomous impulsive systems. We prove the presence of chaos in the sense of Li–Yorke by implementing chaotic perturbations. An impulsive Duffing oscillator is used to show the effectiveness of our technique, and simulations that support the theoretical results are depicted. Moreover, a procedure to stabilize the unstable periodic solutions is proposed.

Keywords: Chaotic impulsive systems; Li–Yorke chaos; chaotic set of piecewise continuous functions; Duffing oscillator; discontinuous chaos control.

1. Introduction

It is well known how discrete dynamics is important for chaos theory [Devaney, 1989; Li & Yorke, 1975; Lorenz, 1963; Wiggins, 1988]. Very interesting examples of applications of discrete dynamics to continuous chaos analysis were provided in papers [Brown & Chua, 1993, 1996, 1997; Brown *et al.*, 2001]. In these studies, the general technique of dynamical synthesis [Brown & Chua, 1993] was developed. Besides that, it is of big interest to consider chaotic processes where continuous dynamics is intermingled with discontinuity [Akhmet, 2009a, 2009b; Battelli & Fečkan, 1997; Jiang *et al.*, 2007; Lin, 2005].

Impulsive differential equations describe the dynamics of real world processes in which abrupt changes occur. Such equations play an increasingly important role in various fields such as mechanics, electronics, neural networks, communication systems and population dynamics [Akhmet, 2010; Akhmet & Yılmaz, 2010; Khadra *et al.*, 2003; Liu, 1994; Ruiz-Herrera, 2012; Yang & Chua, 1997; Yang & Cao, 2007]. In this study, we present a rigorous method for chaotification of arbitrary high dimensional impulsive systems.

Throughout the paper $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ will denote the sets of real numbers, integers and natural numbers, respectively.

The main purpose of our investigation is as follows. Consider the collection of functions

$$\mathscr{A} = \left\{ \varphi(t) : \mathbb{R} \to \mathbb{R}^n \,\middle|\, \sup_{t \in \mathbb{R}} \|\varphi(t)\| \le H_0 \right\},$$

where H_0 is a positive number, and suppose that $\mathscr{A}$ is an equicontinuous family on $\mathbb{R}$. We perturb the impulsive system

$$\begin{aligned} x'(t) &= Ax(t) + f(t, x(t)), \quad t \neq \theta_k, \\ \Delta x|_{t=\theta_k} &= Bx(\theta_k) + W(x(\theta_k)), \end{aligned} \tag{1}$$

by the functions $\varphi(t) \in \mathscr{A}$ to obtain the system

$$\begin{aligned} y'(t) &= Ay(t) + f(t, y(t)) + \varphi(t), \quad t \neq \theta_k, \\ \Delta y|_{t=\theta_k} &= By(\theta_k) + W(y(\theta_k)), \end{aligned} \tag{2}$$

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where the functions $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $W : \mathbb{R}^n \to \mathbb{R}^n$ are continuous in all their arguments, A and B are $n \times n$ constant real valued matrices, the sequence $\{\theta_k\}$, $k \in \mathbb{Z}$, of impulsive moments is strictly increasing, $\Delta y|_{t=\theta_k} = y(\theta_k+) - y(\theta_k)$ and $y(\theta_k+) = \lim_{t\to\theta_k+} y(t)$. The main objective of the present article is the verification of chaos in the dynamics of system (2), provided that the collection $\mathscr{A}$ is chaotic. The description of chaotic collection of functions will be presented in the next section.

The term chaos, as a mathematical notion, has first been used in [Li & Yorke, 1975] for one-dimensional difference equations. According to Li and Yorke [1975], a continuous map $F : J \to J$, where $J \subset \mathbb{R}$ is an interval, exhibits chaos if: (i) For every natural number p, there exists a p-periodic point of F in J; (ii) there is an uncountable set $S \subset J$ containing no periodic points such that for every $s_1, s_2 \in S$ with $s_1 \neq s_2$ we have $\limsup_{j\to\infty} |F^j(s_1) - F^j(s_2)| > 0$ and $\liminf_{j\to\infty} |F^j(s_1) - F^j(s_2)| = 0$; (iii) for every $s \in S$ and periodic point $\sigma \in J$ we have $\limsup_{j\to\infty} |F^j(s) - F^j(\sigma)| > 0$.

The concept of snap-back repellers for high-dimensional maps was introduced in [Marotto, 1978]. According to Marotto [1978], if a multidimensional continuously differentiable map has a snap-back repeller, then it is Li–Yorke chaotic. Marotto's theorem was used in [Li *et al.*, 2007] to prove the existence of Li–Yorke chaos in a spatiotemporal chaotic system. Li–Yorke sensitivity, which links the Li–Yorke chaos with the notion of sensitivity, was studied in [Akin & Kolyada, 2003]. Moreover, generalizations of Li–Yorke chaos to mappings in Banach spaces and complete metric spaces were provided in [Kloeden & Li, 2006; Shi & Chen, 2004, 2005]. In the present article, we develop the concept of Li–Yorke chaos to piecewise continuous functions, and prove its presence rigorously in impulsive systems of the form (2) without any restriction on the dimension.

Taking advantage of chaotically changing impulsive moments, which are functionally dependent on the initial moment, the presence of Li–Yorke chaos in a nonautonomous impulsive differential equation was rigorously proved in [Akhmet, 2009b]. On the other hand, the existence of Li–Yorke chaos and its control in an autonomous impulsive differential system were discussed both theoretically and numerically in the paper [Jiang *et al.*, 2007], where the presence of a snap-back repeller was proved based on the qualitative analysis using the Poincaré map and the Lambert W-function. A system of impulsive differential equations with moments of impulses generated by a sensitive map which depends on a parameter was taken into account in [Lin, 2005], and sensitivity was considered as a chaotic property. The existence of chaos in singular impulsive systems was shown in [Battelli & Fečkan, 1997] by means of transversal homoclinic points. Moreover, chaos in the sense of Devaney [1989] was studied in an impulsive model of the cardiovascular system by means of chaotically changing impulsive moments within the scope of the article [Akhmet, 2009a]. Distinctively from the papers [Akhmet, 2009a, 2009b; Battelli & Fečkan, 1997; Jiang *et al.*, 2007; Lin, 2005], we make use of chaotic perturbations to prove the existence of Li–Yorke chaos, and this is the main novelty of our study.

Small perturbations applied to control parameters can be used to stabilize chaos, keeping the parameters in the neighborhood of their nominal values [Gonzales-Miranda, 2004; Schuster, 1999], and this idea was first introduced by Ott *et al.* [1990]. Experimental applications of the OGY control method requires a permanent computer analysis of the system's state. Since the method deals with a Poincaré map, the parameter changes are discrete in time. By this method, it is possible to stabilize only those periodic orbits whose maximal Lyapunov exponent is small compared to the reciprocal of the time interval between parameter changes [Pyragas, 1992].

In the example presented in Sec. 4, to obtain a collection of chaotic functions, we will use a Duffing oscillator which is forced by a relay function. On the other hand, to support our new theoretical results, an impulsive Duffing oscillator will be utilized. The presented example shows the effectiveness of our technique. Moreover, making use of the OGY control method [Ott *et al.*, 1990], we will demonstrate that the chaos of system (2) is controllable. This method is useful for visually discerning the periodic solutions, which are otherwise indistinguishable in the set of irregular motions.

A concept in which impulsive differential equations are effectively used is the impulsive synchronization of chaotic systems [Haeri & Dehghani, 2008; Li *et al.*, 2004; Li *et al.*, 2005; Liu *et al.*, 2005; Ren & Zhao, 2006; Sun *et al.*, 2009; Wan & Sun, 2011]. This technique is appropriate for the synchronization of Lorenz systems [Sun *et al.*, 2002;

Wu *et al.*, 2007; Yang *et al.*, 1997], Chua oscillators [Yang & Chua, 1997; Yang, 2001] and Rössler systems [Zhang *et al.*, 2007]. In the framework of impulsive synchronization, one can set up an impulsive error system which admits the synchronization error $e = y - x$ as a solution, where x and y denote the states of the drive and response systems, respectively, and require this system to possess a stable equilibrium point. According to the last equation, the synchronized systems must have the same dimensions. However, this is not requested in our results. Therefore, we significantly extend the chaos generation possibilities. Generally speaking, we consider the chaotification procedure in the most definitive and general form. On the other hand, in our theory, it is not necessary to use a drive-response couple. Instead, one can take into account an impulsive system and perturb it by a previously obtained chaotic data. Moreover, in the theory of impulsive synchronization the chaos type of the drive and response systems is not considered. Contrarily, in our results, we rigorously prove that the impulsive system exhibits the same type of chaos as the chaotic perturbations.

The rest of the paper is organized as follows. In Sec. 2, we introduce the ingredients of Li–Yorke chaos for collections of piecewise continuous functions and give sufficient conditions for the presence of chaotic dynamics in system (2). Moreover, we verify the attractiveness property of the bounded on $\mathbb{R}$ solutions of system (2). Section 3 is devoted to theoretical results such that the ingredients of Li–Yorke chaos for system (2) are rigorously proved. Our method of chaotification is applied to an impulsive Duffing oscillator in Sec. 4, and a procedure to stabilize the existing unstable periodic solutions is presented. Some concluding remarks are indicated in Sec. 5.

2. Preliminaries

We say that a function $\psi(t) : \mathbb{R} \to \mathbb{R}^l$, $l \in \mathbb{N}$, belongs to the set $\mathcal{PC}(\mathbb{R})$ if it is left-continuous and continuous except, possibly, at the points where it has discontinuities of the first kind. The definition of a Li–Yorke chaotic set of piecewise continuous functions is as follows.

Suppose that $\mathcal{D}$ is a set of uniformly bounded functions $\psi(t) : \mathbb{R} \to \mathbb{R}^l$ which belong to $\mathcal{PC}(\mathbb{R})$ and have common points of discontinuity.

We say that a couple $(\psi(t), \tilde{\psi}(t)) \in \mathcal{D} \times \mathcal{D}$ is proximal if for arbitrary small $\epsilon > 0$ and arbitrary large $E > 0$, there exists an interval J with a length no less than E such that $\|\psi(t) - \tilde{\psi}(t)\| < \epsilon$ for $t \in J$. On the other hand, a couple $(\psi(t), \tilde{\psi}(t)) \in \mathcal{D} \times \mathcal{D}$ is called frequently (ϵ_0, Δ)-separated if there exist positive numbers ϵ_0, Δ and infinitely many disjoint intervals, each with a length no less than Δ, such that $\|\psi(t) - \tilde{\psi}(t)\| > \epsilon_0$ for each t from these intervals, and each of these intervals contains at most one discontinuity point of both $\psi(t)$ and $\tilde{\psi}(t)$. It is worth noting that the numbers ϵ_0 and Δ depend on the functions $\psi(t)$ and $\tilde{\psi}(t)$.

A couple $(\psi(t), \tilde{\psi}(t)) \in \mathcal{D} \times \mathcal{D}$ is a Li–Yorke pair if it is proximal and frequently (ϵ_0, Δ)-separated for some positive numbers ϵ_0 and Δ. Moreover, an uncountable set $\mathcal{C} \subset \mathcal{D}$ is called a scrambled set if $\mathcal{C}$ does not contain any periodic functions and each couple of different functions inside $\mathcal{C} \times \mathcal{C}$ is a Li–Yorke pair.

We say that the collection $\mathcal{D}$ is a Li–Yorke chaotic set if: (i) There exists a positive number T_0 such that $\mathcal{D}$ possesses a periodic function of period mT_0, for any $m \in \mathbb{N}$; (ii) $\mathcal{D}$ possesses a scrambled set $\mathcal{C}$; (iii) for any function $\psi(t) \in \mathcal{C}$ and any periodic function $\tilde{\psi}(t) \in \mathcal{D}$, the couple $(\psi(t), \tilde{\psi}(t))$ is frequently (ϵ_0, Δ)-separated for some positive numbers ϵ_0 and Δ.

One can obtain a new Li–Yorke chaotic set of functions from a given one as follows. Suppose that $h : \mathbb{R}^l \to \mathbb{R}^{\bar{l}}$ is a function which satisfies for all $u_1, u_2 \in \mathbb{R}^l$ that

$$\begin{aligned} L_1\|u_1 - u_2\| &\le \|h(u_1) - h(u_2)\| \\ &\le L_2\|u_1 - u_2\|, \end{aligned} \tag{3}$$

where L_1 and L_2 are positive numbers. One can verify that if $\mathcal{D}$ is a Li–Yorke chaotic set, then the collection $\mathcal{D}_h = \{h(\psi(t)) \,|\, \psi(t) \in \mathcal{D}\}$ is also Li–Yorke chaotic.

The following conditions are needed in the paper:

(A1) The matrices A and B commute and $\det(I + B) \neq 0$, where I is the $n \times n$ identity matrix.
(A2) There exists a positive number T and a natural number p such that $f(t + T, y) = f(t, y)$ for all $t \in \mathbb{R}$, $y \in \mathbb{R}^n$ and $\theta_{k+p} = \theta_k + T$ for all $k \in \mathbb{Z}$.
(A3) The eigenvalues of the matrix $A + \frac{p}{T} \times \ln(I + B)$ have negative real parts.

(A4) There exist positive numbers M_f and M_W such that
$$\sup_{t\in\mathbb{R},y\in\mathbb{R}^n}\|f(t,y)\|\leq M_f \quad \text{and}$$
$$\sup_{y\in\mathbb{R}^n}\|W(y)\|\leq M_W.$$

(A5) There exists a positive number L_f such that
$$\|f(t,y_1)-f(t,y_2)\|\leq L_f\|y_1-y_2\|,$$
for all $t\in\mathbb{R}$ and $y_1,y_2\in\mathbb{R}^n$.

(A6) There exists a positive number L_W such that
$$\|W(y_1)-W(y_2)\|\leq L_W\|y_1-y_2\|,$$
for all $y_1,y_2\in\mathbb{R}^n$.

Throughout the paper, the uniform norm $\|\Gamma\|=\sup_{\|v\|=1}\|\Gamma v\|$ for square matrices Γ will be used.

Let us denote by $U(t,s)$ the transition matrix of the linear homogeneous system
$$u'(t)=\mathrm{A}u(t),\quad t\neq\theta_k,\quad \Delta u|_{t=\theta_k}=Bu(\theta_k).$$
Under the conditions (A1)–(A3), there exist positive numbers N and ω such that $\|U(t,s)\|\leq Ne^{-\omega(t-s)}$ for $t\geq s$ [Akhmet, 2010; Samoilenko & Perestyuk, 1995].

The following conditions are also required:

(A7) $N\left(\dfrac{L_f}{\omega}+\dfrac{pL_W}{1-e^{-\omega T}}\right)<1;$

(A8) $-\omega+NL_f+\dfrac{p}{T}\ln(1+NL_W)<0;$

(A9) $L_W\|(I+B)^{-1}\|<1.$

We say that a left-continuous function $y(t):\mathbb{R}\to\mathbb{R}^n$ is a solution of (2) if: (i) It has discontinuities only at the points θ_k, $k\in\mathbb{Z}$, and these discontinuities are of the first kind; (ii) the derivative $y'(t)$ exists at each point $t\in\mathbb{R}\backslash\{\theta_k\}$, and the left-sided derivative exists at the points θ_k, $k\in\mathbb{Z}$; (iii) the differential equation is satisfied by $y(t)$ on $\mathbb{R}\backslash\{\theta_k\}$, and it holds for the left derivative of $y(t)$ at every point θ_k, $k\in\mathbb{Z}$; (iv) the jump equation is satisfied by $y(t)$ for every $k\in\mathbb{Z}$.

According to the results of [Akhmet, 2010; Samoilenko & Perestyuk, 1995], for any function $\varphi(t)\in\mathscr{A}$, one can confirm under the conditions (A1)–(A7) that there exists a unique bounded on $\mathbb{R}$ solution $\phi_\varphi(t)$ of system (2) which satisfies the relation
$$\phi_\varphi(t)=\int_{-\infty}^t U(t,s)[f(s,\phi_\varphi(s))+\varphi(s)]ds + \sum_{\theta_k<t}U(t,\theta_k)W(\phi_\varphi(\theta_k)).$$
It can be verified for each $\varphi(t)\in\mathscr{A}$ that the inequality $\sup_{t\in\mathbb{R}}\|\phi_\varphi(t)\|\leq K_0$ holds, where
$$K_0=\frac{N(M_f+H_0)}{\omega}+\frac{pNM_W}{1-e^{-\omega T}}.$$

By means of the collection $\mathscr{A}$, let us construct the set
$$\mathscr{B}=\{\phi_\varphi(t)\,|\,\varphi(t)\in\mathscr{A}\}.$$

For a given function $\varphi(t)\in\mathscr{A}$, let us denote by $y_\varphi(t,y_0)$ the solution of (2) with $y_\varphi(0,y_0)=y_0$. We say that the collection $\mathscr{B}$ is an attractor if for any $\varphi(t)\in\mathscr{A}$ and $y_0\in\mathbb{R}^n$, we have $\|y_\varphi(t,y_0)-\phi_\varphi(t)\|\to 0$ as $t\to\infty$. The attractiveness feature of the collection $\mathscr{B}$ is mentioned in the next assertion.

Lemma 1. *If the conditions (A1)–(A8) are valid, then the collection $\mathscr{B}$ is an attractor.*

Proof. Fix an arbitrary function $\varphi(t)\in\mathscr{A}$ and $y_0\in\mathbb{R}^n$. Taking advantage of the relations
$$y_\varphi(t,y_0)=U(t,0)y_0 + \int_0^t U(t,s)[f(s,y_\varphi(s,y_0))+\varphi(s)]ds + \sum_{0\leq\theta_k<t}U(t,\theta_k)W(y_\varphi(\theta_k,y_0))$$
and
$$\phi_\varphi(t)=U(t,0)\phi_\varphi(0) + \int_0^t U(t,s)[f(s,\phi_\varphi(s))+\varphi(s)]ds + \sum_{0\leq\theta_k<t}U(t,\theta_k)W(\phi_\varphi(\theta_k)),$$
for $t\geq 0$ we obtain the inequality
$$e^{\omega t}\|y_\varphi(t,y_0)-\phi_\varphi(t)\| \leq N\|y_0-\phi_\varphi(0)\|+\int_0^t NL_f e^{\omega s}\|y_\varphi(s,y_0)-\phi_\varphi(s)\| + \sum_{0\leq\theta_k<t}NL_W e^{\omega\theta_k}\|y_\varphi(\theta_k,y_0)-\phi_\varphi(\theta_k)\|.$$

Applying the Gronwall–Bellman lemma for piecewise continuous functions, one can verify that

$$\begin{aligned}&\|y_\varphi(t,y_0)-\phi_\varphi(t)\|\\&\quad\le N(1+NL_W)^p\|y_0-\phi_\varphi(0)\|\\&\qquad\times e^{[-\omega+NL_f+(p/T)\ln(1+NL_W)]},\quad t\ge 0.\end{aligned}$$

Consequently, in accordance with condition (A8), we have that $\|y_\varphi(t,y_0)-\phi_\varphi(t)\|\to 0$ as $t\to\infty$. ■

In the next section, we will prove that if the collection $\mathscr{A}$ is chaotic in the sense of Li–Yorke, then the same is true for the collection $\mathscr{B}$.

3. Chaotic Dynamics

The ingredients of Li–Yorke chaos for system (2) will be considered in Lemmas 2 and 3. The main conclusion of the present study will be stated in Theorem 1.

In the proof of the next assertion, we will denote by $i((a,b))$ the number of terms of the sequence $\{\theta_k\}$ which belong to the interval (a,b), where a and b are real numbers such that $a<b$. Clearly, $i((a,b))\le p+\frac{p}{T}(b-a)$.

Lemma 2. *Suppose that the conditions (A1)–(A8) hold. If a couple of functions $(\varphi(t),\overline{\varphi}(t))\in\mathscr{A}\times\mathscr{A}$ is proximal, then the same is true for the couple $(\phi_\varphi(t),\phi_{\overline{\varphi}}(t))\in\mathscr{B}\times\mathscr{B}$.*

Proof. Fix an arbitrary small positive number ϵ and an arbitrary large positive number E. Let us denote $\alpha=\omega-NL_f-\frac{p}{T}\ln(1+NL_W)$ and set

$$\begin{aligned}\gamma&=1+\frac{N}{\omega}+\frac{N^2L_f}{\omega\alpha}(1+NL_W)^p\\&\quad+\left(\frac{N^2L_W}{\omega}\right)\left(\frac{pe^{2\alpha T}}{e^{\alpha T}-1}\right)(1+NL_W)^p.\end{aligned}$$

Since the couple $(\varphi(t),\overline{\varphi}(t))\in\mathscr{A}\times\mathscr{A}$ is proximal, there exists an interval $J=[\sigma,\sigma+E_1]$, where $E_1\ge E$, such that $\|\varphi(t)-\overline{\varphi}(t)\|<\epsilon/\gamma$ for $t\in J$.

For the sake of clarity, we will denote $y(t)=\phi_\varphi(t)\in\mathscr{B}$ and $\overline{y}(t)=\phi_{\overline{\varphi}}(t)\in\mathscr{B}$. The functions $y(t)$ and $\overline{y}(t)$ satisfy the relations

$$\begin{aligned}y(t)&=\int_{-\infty}^t U(t,s)[f(s,y(s))+\varphi(s)]ds\\&\quad+\sum_{\theta_k<t}U(t,\theta_k)W(y(\theta_k))\end{aligned}$$

and

$$\begin{aligned}\overline{y}(t)&=\int_{-\infty}^t U(t,s)[f(s,\overline{y}(s))+\overline{\varphi}(s)]ds\\&\quad+\sum_{\theta_k<t}U(t,\theta_k)W(\overline{y}(\theta_k)),\end{aligned}$$

respectively. By means of these relations, we obtain that

$$\begin{aligned}y(t)-\overline{y}(t)&=\int_{-\infty}^\sigma U(t,s)[f(s,y(s))-f(s,\overline{y}(s))+\varphi(s)-\overline{\varphi}(s)]ds\\&\quad+\int_\sigma^t U(t,s)[f(s,y(s))-f(s,\overline{y}(s))+\varphi(s)-\overline{\varphi}(s)]ds\\&\quad+\sum_{\theta_k\le\sigma}U(t,\theta_k)[W(y(\theta_k))-W(\overline{y}(\theta_k))]+\sum_{\sigma<\theta_k<t}U(t,\theta_k)[W(y(\theta_k))-W(\overline{y}(\theta_k))].\end{aligned}$$

Using the inequalities

$$\begin{aligned}\left\|\sum_{\theta_k\le\sigma}U(t,\theta_k)[W(y(\theta_k))-W(\overline{y}(\theta_k))]\right\|&\le\sum_{\theta_k\le\sigma}2NM_We^{-\omega(t-\theta_k)}=2NM_We^{-\omega t}\sum_{l=0}^\infty\sum_{\sigma-(l+1)T<\theta_k\le\sigma-lT}e^{\omega\theta_k}\\&\le 2NM_We^{-\omega t}\sum_{l=0}^\infty pe^{\omega(\sigma-lT)}=\frac{2pNM_W}{1-e^{-\omega T}}e^{-\omega(t-\sigma)}\end{aligned}$$

and

$$\left\|\int_{-\infty}^\sigma U(t,s)[f(s,y(s))-f(s,\overline{y}(s))+\varphi(s)-\overline{\varphi}(s)]ds\right\|\le\frac{2N(M_f+H_0)}{\omega}e^{-\omega(t-\sigma)},$$

it can be verified for $t \in J$ that

$$\|y(t) - \overline{y}(t)\| \leq \left(\frac{2N(M_f + H_0)}{\omega} + \frac{2pNM_W}{1 - e^{-\omega T}} \right) e^{-\omega(t-\sigma)} + \frac{N\epsilon}{\gamma\omega}(1 - e^{-\omega(t-\sigma)})$$

$$+ NL_f \int_\sigma^t e^{-\omega(t-s)} \|y(s) - \overline{y}(s)\| ds + NL_W \sum_{\sigma < \theta_k < t} e^{-\omega(t-\theta_k)} \|y(\theta_k) - \overline{y}(\theta_k)\|.$$

Now, let $u(t) = e^{\omega t}\|y(t) - \overline{y}(t)\|$. Under the circumstances we have that

$$u(t) \leq c + \frac{N\epsilon}{\gamma\omega} e^{\omega t} + NL_f \int_\sigma^t u(s) ds + NL_W \sum_{\sigma < \theta_k < t} u(\theta_k), \quad t \in J,$$

where

$$c = e^{\omega\sigma} \left(\frac{2N(M_f + H_0)}{\omega} + \frac{2pNM_W}{1 - e^{-\omega T}} - \frac{N\epsilon}{\gamma\omega} \right).$$

Implication of the analogue of Gronwall's lemma for piecewise continuous functions leads to the inequality

$$u(t) \leq \frac{N\epsilon}{\gamma\omega} e^{\omega t} + c + \int_\sigma^t NL_f \left(\frac{N\epsilon}{\gamma\omega} e^{\omega s} + c \right) (1 + NL_W)^{i((s,t))} e^{NL_f(t-s)} ds$$

$$+ \sum_{\sigma < \theta_k < t} NL_W \left(\frac{N\epsilon}{\gamma\omega} e^{\omega\theta_k} + c \right) (1 + NL_W)^{i((\theta_k,t))} e^{NL_f(t-\theta_k)}.$$

With the aid of the equation

$$(1 + NL_W)^{i((\sigma,t))} e^{NL_f(t-\sigma)} = 1 + \int_\sigma^t NL_f (1 + NL_W)^{i((s,t))} e^{NL_f(t-s)} ds$$

$$+ \sum_{\sigma < \theta_k < t} NL_W (1 + NL_W)^{i((\theta_k,t))} e^{NL_f(t-\theta_k)}$$

one can attain that

$$u(t) \leq \frac{N\epsilon}{\gamma\omega} e^{\omega t} + c(1 + NL_W)^{i((\sigma,t))} e^{NL_f(t-\sigma)} + \frac{N^2 L_f \epsilon}{\gamma\omega} \int_\sigma^t e^{\omega s} (1 + NL_W)^{i((s,t))} e^{NL_f(t-s)} ds$$

$$+ \frac{N^2 L_W \epsilon}{\gamma\omega} \sum_{\sigma < \theta_k < t} e^{\theta_k} (1 + NL_W)^{i((\theta_k,t))} e^{NL_f(t-\theta_k)}$$

$$\leq \frac{N\epsilon}{\gamma\omega} e^{\omega t} + c(1 + NL_W)^p e^{(\omega-\alpha)(t-\sigma)} + \frac{N^2 L_f \epsilon}{\gamma\omega\alpha} (1 + NL_W)^p e^{\omega t} (1 - e^{-\alpha(t-\sigma)})$$

$$+ \frac{N^2 L_W \epsilon}{\gamma\omega} (1 + NL_W)^p e^{(\omega-\alpha)t} \sum_{\sigma < \theta_k < t} e^{\alpha\theta_k}.$$

Let $q = q(t) = \lfloor \frac{t-\sigma}{T} \rfloor$, that is, q is the greatest integer which is not larger than $\frac{t-\sigma}{T}$. Under the circumstances we have that

$$\sum_{\sigma < \theta_k < t} e^{\alpha\theta_k} \leq \sum_{\sigma < \theta_k < \sigma + (q+1)T} e^{\alpha\theta_k} \leq \sum_{l=0}^{q} \sum_{\sigma + lT \leq \theta_k < \sigma + (l+1)T} e^{\alpha\theta_k} < \sum_{l=0}^{q} p e^{\alpha[\sigma + (l+1)T]}$$

$$= pe^{(\sigma+T)\alpha} \frac{e^{(q+1)\alpha T} - 1}{e^{\alpha T} - 1} \leq \frac{pe^{(\sigma+T)\alpha}}{e^{\alpha T} - 1} (e^{\alpha(t-\sigma+T)} - 1) < \frac{pe^{2\alpha T}}{e^{\alpha T} - 1} e^{\alpha t}.$$

The last inequality implies that

$$u(t) < \frac{N\epsilon}{\gamma\omega}e^{\omega t}(1-(1+NL_W)^p e^{-\alpha(t-\sigma)}) + \frac{N^2L_f\epsilon}{\gamma\omega\alpha}(1+NL_W)^p e^{\omega t}(1-e^{-\alpha(t-\sigma)})$$

$$+\left(\frac{N^2L_W\epsilon}{\gamma\omega}\right)\left(\frac{pe^{2\alpha T}}{e^{\alpha T}-1}\right)(1+NL_W)^p e^{\omega t} + (1+NL_W)^p\left(\frac{2N(M_f+H_0)}{\omega}+\frac{2pNM_W}{1-e^{-\omega T}}\right)e^{\omega t}e^{-\alpha(t-\sigma)}$$

and multiplying both sides by $e^{-\omega t}$ one can obtain the inequality

$$\begin{aligned}\|y(t)-\overline{y}(t)\| &< \frac{N\epsilon}{\gamma\omega}(1-(1+NL_W)^p e^{-\alpha(t-\sigma)}) + \frac{N^2L_f\epsilon}{\gamma\omega\alpha}(1+NL_W)^p(1-e^{-\alpha(t-\sigma)})\\ &\quad+\left(\frac{N^2L_W\epsilon}{\gamma\omega}\right)\left(\frac{pe^{2\alpha T}}{e^{\alpha T}-1}\right)(1+NL_W)^p\\ &\quad+(1+NL_W)^p\left(\frac{2N(M_f+H_0)}{\omega}+\frac{2pNM_W}{1-e^{-\omega T}}\right)e^{-\alpha(t-\sigma)}\\ &< \frac{N\epsilon}{\gamma\omega}+\frac{N^2L_f\epsilon}{\gamma\omega\alpha}(1+NL_W)^p+\left(\frac{N^2L_W\epsilon}{\gamma\omega}\right)\left(\frac{pe^{2\alpha T}}{e^{\alpha T}-1}\right)(1+NL_W)^p\\ &\quad+(1+NL_W)^p\left(\frac{2N(M_f+H_0)}{\omega}+\frac{2pNM_W}{1-e^{-\omega T}}\right)e^{-\alpha(t-\sigma)}.\end{aligned}$$

Set $\beta=(1+NL_W)^p(\frac{2N(M_f+H_0)}{\omega}+\frac{2pNM_W}{1-e^{-\omega T}})$, and suppose that the number E is sufficiently large such that $E\geq\frac{2}{\alpha}\ln(\frac{\gamma\beta}{\epsilon})$. In this case, $\beta e^{-\alpha(t-\sigma)}<\epsilon/\gamma$ for $t\in[\sigma+E/2,\sigma+E_1]$. Thus, the inequality

$$\|y(t)-\overline{y}(t)\| < \frac{\epsilon}{\gamma}\left[1+\frac{N}{\omega}+\frac{N^2L_f}{\omega\alpha}(1+NL_W)^p+\left(\frac{N^2L_W}{\omega}\right)\left(\frac{pe^{2\alpha T}}{e^{\alpha T}-1}\right)(1+NL_W)^p\right]=\epsilon$$

holds for $t\in[\sigma+E/2,\sigma+E_1]$. The interval $\tilde{J}=[\sigma+E/2,\sigma+E_1]$ has a length no less than $E/2$. Consequently, the couple $(\phi_\varphi(t),\phi_{\overline{\varphi}}(t))\in\mathscr{B}\times\mathscr{B}$ is proximal. ■

Next, we shall continue with the second ingredient of Li–Yorke chaos in the following lemma.

Lemma 3. *Suppose that the conditions (A1)–(A7), (A9) are fulfilled. If a couple of functions $(\varphi(t),\overline{\varphi}(t))\in\mathscr{A}\times\mathscr{A}$ are frequently (ϵ_0,Δ)-separated for some positive numbers ϵ_0 and Δ, then the couple $(\phi_\varphi(t),\phi_{\overline{\varphi}}(t))\in\mathscr{B}\times\mathscr{B}$ are frequently $(\epsilon_1,\overline{\Delta})$-separated for some positive numbers ϵ_1 and $\overline{\Delta}$.*

Proof. Since the couple of functions $(\varphi(t),\overline{\varphi}(t))\in\mathscr{A}\times\mathscr{A}$ is frequently (ϵ_0,Δ)-separated for some positive numbers ϵ_0 and Δ, there exist infinitely many disjoint intervals $J_i, i\in\mathbb{N}$, each with a length no less than Δ, such that $\|\varphi(t)-\overline{\varphi}(t)\|>\epsilon_0$ for each t from these intervals. Without loss of generality, suppose that the intervals $J_i, i\in\mathbb{N}$, are all open subsets of $\mathbb{R}$. In that case, one can find a sequence $\{\Delta_i\}$ satisfying $\Delta_i\geq\Delta$, $i\in\mathbb{N}$, and a sequence $\{\alpha_i\}$, $\alpha_i\to\infty$ as $i\to\infty$, such that $J_i=(\alpha_i,\alpha_i+\Delta_i)$.

In the proof, we will verify the existence of positive numbers ϵ_1, $\overline{\Delta}$ and infinitely many disjoint intervals J_i^1, $i\in\mathbb{N}$, each with length $\overline{\Delta}$, such that the inequality $\|\phi_\varphi(t)-\phi_{\overline{\varphi}}(t)\|>\epsilon_1$ holds for each t from these intervals.

Suppose that $\varphi(t)=(\varphi_1(t),\varphi_2(t),\ldots,\varphi_n(t))$ and $\overline{\varphi}(t)=(\overline{\varphi}_1(t),\overline{\varphi}_2(t),\ldots,\overline{\varphi}_n(t))$, where φ_j and $\overline{\varphi}_j$, $1\leq j\leq n$, are real valued functions. According to the equicontinuity of the collection $\mathscr{A}$, one can find a positive number $\tau<\Delta$, such that for any $t_1,t_2\in\mathbb{R}$ with $|t_1-t_2|<\tau$, the inequality

$$|(\varphi_j(t_1)-\overline{\varphi}_j(t_1))-(\varphi_j(t_2)-\overline{\varphi}_j(t_2))|<\frac{\epsilon_0}{2n} \tag{4}$$

holds for all $1\leq j\leq n$.

For each i, let $\eta_i=\alpha_i+\frac{\Delta_i}{2}$. That is, η_i is the midpoint of the interval J_i. Moreover, define a sequence $\{\zeta_i\}$, $i\in\mathbb{N}$, through the equation $\zeta_i=\eta_i-\frac{\tau}{2}$.

Fix a natural number i. For each $t \in J_i$, there exists an integer $j_i = j_i(t)$, $1 \le j_i \le n$, such that

$$|\varphi_{j_i}(t) - \overline{\varphi}_{j_i}(t)| \ge \frac{1}{n}\|\varphi(t) - \overline{\varphi}(t)\|.$$

Otherwise, if there exists $t_0 \in J_i$ such that for all $1 \le j \le n$ the inequality

$$|\varphi_j(t_0) - \overline{\varphi}_j(t_0)| < \frac{1}{n}\|\varphi(t_0) - \overline{\varphi}(t_0)\|$$

holds, then we encounter with a contradiction since

$$\|\varphi(t_0) - \overline{\varphi}(t_0)\| \le \sum_{j=1}^{n} |\varphi_j(t_0) - \overline{\varphi}_j(t_0)| < \|\varphi(t_0) - \overline{\varphi}(t_0)\|.$$

For this reason, there exists an integer $j_i = j_i(\eta_i)$, $1 \le j_i \le n$, such that

$$|\varphi_{j_i}(\eta_i) - \overline{\varphi}_{j_i}(\eta_i)| \ge \frac{1}{n}\|\varphi(\eta_i) - \overline{\varphi}(\eta_i)\| > \frac{\epsilon_0}{n}. \qquad (5)$$

On the other hand, making use of the inequality (4), it is easy to verify for all $t \in [\zeta_i, \zeta_i + \tau]$ that

$$|\varphi_{j_i}(\eta_i) - \overline{\varphi}_{j_i}(\eta_i)| - |\varphi_{j_i}(t) - \overline{\varphi}_{j_i}(t)| \le |(\varphi_{j_i}(t) - \overline{\varphi}_{j_i}(t)) - (\varphi_{j_i}(\eta_i) - \overline{\varphi}_{j_i}(\eta_i))| < \frac{\epsilon_0}{2n}.$$

Therefore, by virtue of (5), we obtain the inequality

$$|\varphi_{j_i}(t) - \overline{\varphi}_{j_i}(t)| > |\varphi_{j_i}(\eta_i) - \overline{\varphi}_{j_i}(\eta_i)| - \frac{\epsilon_0}{2n} > \frac{\epsilon_0}{2n}, \quad t \in [\zeta_i, \zeta_i + \tau]. \qquad (6)$$

It is possible to find numbers $s_1^i, s_2^i, \ldots, s_n^i \in [\zeta_i, \zeta_i + \tau]$ such that

$$\int_{\zeta_i}^{\zeta_i+\tau} (\varphi(s) - \overline{\varphi}(s))ds = \tau(\varphi_1(s_1^i) - \overline{\varphi}_1(s_1^i), \varphi_2(s_2^i) - \overline{\varphi}_2(s_2^i), \ldots, \varphi_n(s_n^i) - \overline{\varphi}_n(s_n^i)).$$

Hence, the inequality (6) implies that

$$\left\|\int_{\zeta_i}^{\zeta_i+\tau} (\varphi(s) - \overline{\varphi}(s))ds\right\| \ge \tau|\varphi_{j_i}(s_{j_i}^i) - \overline{\varphi}_{j_i}(s_{j_i}^i)| > \frac{\tau\epsilon_0}{2n}. \qquad (7)$$

For the sake of clarity, let us denote $y(t) = \phi_\varphi(t)$ and $\overline{y}(t) = \phi_{\overline{\varphi}}(t)$. For $t \in [\zeta_i, \zeta_i + \tau]$, using the couple of relations

$$y(t) = y(\zeta_i) + \int_{\zeta_i}^{t} [Ay(s) + f(s, y(s)) + \varphi(s)]ds + \sum_{\zeta_i \le \theta_k < t} [By(\theta_k) + W(y(\theta_k))]$$

and

$$\overline{y}(t) = \overline{y}(\zeta_i) + \int_{\zeta_i}^{t} [A\overline{y}(s) + f(s, \overline{y}(s)) + \overline{\varphi}(s)]ds + \sum_{\zeta_i \le \theta_k < t} [B\overline{y}(\theta_k) + W(\overline{y}(\theta_k))],$$

one can verify that

$$\begin{aligned}
\|y(\zeta_i + \tau) - \overline{y}(\zeta_i + \tau)\| &\ge \left\|\int_{\zeta_i}^{\zeta_i+\tau} (\varphi(s) - \overline{\varphi}(s))ds\right\| - \|y(\zeta_i) - \overline{y}(\zeta_i)\| \\
&\quad - \int_{\zeta_i}^{\zeta_i+\tau} (\|A\| + L_f)\|y(s) - \overline{y}(s)\|ds \\
&\quad - \sum_{\zeta_i \le \theta_k < \zeta_i+\tau} (\|B\| + L_W)\|y(\theta_k) - \overline{y}(\theta_k)\|.
\end{aligned}$$

Making use of the last inequality together with (7), we obtain that

$$\sup_{t\in[\zeta_i,\zeta_i+\tau]} \|y(t) - \overline{y}(t)\| > \frac{\tau\epsilon_0}{2n} - [1 + \tau(\|A\| + L_f)] \sup_{t\in[\zeta_i,\zeta_i+\tau]} \|y(t) - \overline{y}(t)\| - \frac{p}{T}(T + \tau)(\|B\| + L_W) \sup_{t\in[\zeta_i,\zeta_i+\tau]} \|y(t) - \overline{y}(t)\|$$

and therefore $\sup_{t\in[\zeta_i,\zeta_i+\tau]} \|y(t) - \overline{y}(t)\| > M_0$, where

$$M_0 = \frac{\tau\epsilon_0}{2n\left[2 + \tau(\|A\| + L_f) + \frac{p}{T}(T + \tau)(\|B\| + L_W)\right]}.$$

Set $\underline{\theta} = \min_{1\le k\le p}(\theta_{k+1} - \theta_k)$, and define the numbers

$$\epsilon_1 = \frac{M_0}{2}\min\left\{\frac{1 - L_W\|(I+B)^{-1}\|}{\|(I+B)^{-1}\|}, \frac{1}{1+\|B\|+L_W}\right\}$$

and

$$\overline{\Delta} = \min\left\{\underline{\theta}, \frac{M_0}{4[(\|A\|+L_f)K_0+H_0](2+\|B\|+L_W)}, \frac{M_0(1-L_W\|(I+B)^{-1}\|)}{4[(\|A\|+L_f)K_0+H_0][1+(1-L_W)\|(I+B)^{-1}\|]}\right\}.$$

First, suppose that there exists $\xi_i \in [\zeta_i, \zeta_i+\tau]$ such that $\sup_{t\in[\zeta_i,\zeta_i+\tau]}\|y(t)-\overline{y}(t)\| = \|y(\xi_i) - \overline{y}(\xi_i)\|$.

Let

$$\zeta_i^1 = \begin{cases} \xi_i, & \text{if } \xi_i \le \zeta_i + \dfrac{\tau}{2} \\ \xi_i - \overline{\Delta}, & \text{if } \xi_i > \zeta_i + \dfrac{\tau}{2} \end{cases}.$$

Since $\overline{\Delta} \le \underline{\theta}$, there exists at most one impulsive moment on the interval $(\zeta_i^1, \zeta_i^1+\overline{\Delta})$.

We shall start by considering the case $\xi_i > \zeta_i + \frac{\tau}{2}$. Assume that there exists an impulsive moment $\theta_j \in (\zeta_i^1, \zeta_i^1+\overline{\Delta})$. Under the circumstances, one can verify for $t \in (\theta_j, \zeta_i^1+\overline{\Delta})$ that

$$\begin{aligned}
&\|y(t)-\overline{y}(t)\| \\
&\quad \ge \|y(\xi_i)-\overline{y}(\xi_i)\| - \left\|\int_{\xi_i}^t A(y(s)-\overline{y}(s))ds\right\| \\
&\qquad - \left\|\int_{\xi_i}^t [f(s,y(s)) - f(s,\overline{y}(s))]ds\right\| \\
&\qquad - \left\|\int_{\xi_i}^t [\varphi(s) - \overline{\varphi}(s)]ds\right\| \\
&\quad > M_0 - 2\overline{\Delta}[(\|A\|+L_f)K_0+H_0] \\
&\quad > \frac{M_0}{2} > \epsilon_1.
\end{aligned}$$

In particular, the inequality $\|y(\theta_j+) - \overline{y}(\theta_j+)\| > M_0 - 2\overline{\Delta}[(\|A\|+L_f)K_0+H_0]$ is valid. Taking advantage of the relations $y(\theta_j+) = (I+B)y(\theta_j) + W(y(\theta_j))$ and $\overline{y}(\theta_j+) = (I+B)\overline{y}(\theta_j) + W(\overline{y}(\theta_j))$, we obtain that

$$\begin{aligned}
\|y(\theta_j) - \overline{y}(\theta_j)\| &\ge \frac{\|y(\theta_j+) - \overline{y}(\theta_j+)\|}{1+\|B\|+L_W} \\
&> \frac{M_0 - 2\overline{\Delta}[(\|A\|+L_f)K_0+H_0]}{1+\|B\|+L_W}.
\end{aligned}$$

The last inequality implies for $t \in (\zeta_i^1, \theta_j]$ that

$$\begin{aligned}
&\|y(t)-\overline{y}(t)\| \\
&\quad \ge \|y(\theta_j)-\overline{y}(\theta_j)\| - \left\|\int_{\theta_j}^t A(y(s)-\overline{y}(s))ds\right\| \\
&\qquad - \left\|\int_{\theta_j}^t [f(s,y(s)) - f(s,\overline{y}(s))]ds\right\| \\
&\qquad - \left\|\int_{\theta_j}^t [\varphi(s) - \overline{\varphi}(s)]ds\right\| \\
&\quad > \frac{M_0 - 2\overline{\Delta}[(\|A\|+L_f)K_0+H_0]}{1+\|B\|+L_W} \\
&\qquad - 2\overline{\Delta}[(\|A\|+L_f)K_0+H_0] \\
&\quad = \frac{1}{1+\|B\|+L_W}[M_0 - 2\overline{\Delta}(2+\|B\|+L_W) \\
&\qquad \times ((\|A\|+L_f)K_0+H_0)] \\
&\quad \ge \frac{M_0}{2(1+\|B\|+L_W)} \ge \epsilon_1.
\end{aligned}$$

On the other hand, if none of the impulsive moments belong to $(\zeta_i^1, \zeta_i^1+\overline{\Delta})$, then for each t from this interval we have that $\|y(t)-\overline{y}(t)\| > \frac{M_0}{2}$. Therefore, in the case of $\xi_i > \zeta_i + \frac{\tau}{2}$, the inequality $\|y(t)-\overline{y}(t)\| > \epsilon_1$ holds for all $t \in (\zeta_i^1, \zeta_i^1+\overline{\Delta})$, regardless of the existence of an impulsive moment in this interval.

Next, we continue with the case $\xi_i \le \zeta_i + \frac{\tau}{2}$. If there exists an impulsive moment $\theta_j \in (\zeta_i^1, \zeta_i^1+\overline{\Delta})$, then it is easy to show for $t \in (\zeta_i^1, \theta_j]$ that

$$\|y(t)-\overline{y}(t)\| > M_0 - 2\overline{\Delta}[(\|A\|+L_f)K_0+H_0] > \epsilon_1.$$

Since $\|y(\theta_j) - \overline{y}(\theta_j)\| > M_0 - 2\overline{\Delta}[(\|A\|+L_f)K_0+H_0]$, the condition (A9) implies that

$$\begin{aligned}
&\|y(\theta_j+) - \overline{y}(\theta_j+)\| \\
&\quad \ge \|(I+B)(y(\theta_j) - \overline{y}(\theta_j))\| - L_W\|y(\theta_j) - \overline{y}(\theta_j)\|
\end{aligned}$$

$$\geq \frac{\|y(\theta_j) - \overline{y}(\theta_j)\|}{\|(I+B)^{-1}\|} - L_W \|y(\theta_j) - \overline{y}(\theta_j)\|$$
$$> \left(\frac{1 - L_W\|(I+B)^{-1}\|}{\|(I+B)^{-1}\|}\right)$$
$$\times [M_0 - 2\overline{\Delta}((\|A\| + L_f)K_0 + H_0)].$$

Making use of the last inequality, we attain for all $t \in (\theta_j, \zeta_i^1 + \overline{\Delta})$ that

$$\begin{aligned}\|y(t) - \overline{y}(t)\| &\geq \|y(\theta_j+) - \overline{y}(\theta_j+)\| \\ &\quad - 2\overline{\Delta}[(\|A\| + L_f)K_0 + H_0] \\ &> \left(\frac{1 - L_W\|(I+B)^{-1}\|}{\|(I+B)^{-1}\|}\right) \\ &\quad \times [M_0 - 2\overline{\Delta}((\|A\| + L_f)K_0 + H_0)] \\ &\quad - 2\overline{\Delta}[(\|A\| + L_f)K_0 + H_0] \\ &\geq \frac{M_0}{2}\left(\frac{1 - L_W\|(I+B)^{-1}\|}{\|(I+B)^{-1}\|}\right).\end{aligned}$$

Therefore, for all $t \in (\zeta_i^1, \zeta_i^1 + \overline{\Delta})$ it is clear that $\|y(t) - \overline{y}(t)\| > \epsilon_1$. Besides, the same inequality holds even if the interval $(\zeta_i^1, \zeta_i^1 + \overline{\Delta})$ does not contain an impulsive moment.

Now, suppose that there exists an impulsive moment $\theta_l \in [\zeta_i, \zeta_i + \tau]$ such that

$$\sup_{t \in [\zeta_i, \zeta_i + \tau]} \|y(t) - \overline{y}(t)\| = \|y(\theta_l+) - \overline{y}(\theta_l+)\|.$$

Let us define

$$\zeta_i^1 = \begin{cases} \theta_l, & \text{if } \theta_l \leq \zeta_i + \dfrac{\tau}{2} \\ \theta_l - \overline{\Delta}, & \text{if } \theta_l > \zeta_i + \dfrac{\tau}{2} \end{cases}.$$

In the case that $\theta_l > \zeta_i + \frac{\tau}{2}$, taking advantage of the inequality

$$\begin{aligned}\|y(\theta_l) - \overline{y}(\theta_l)\| &\geq \frac{\|y(\theta_l+) - \overline{y}(\theta_l+)\|}{1 + \|B\| + L_W} \\ &> \frac{M_0}{1 + \|B\| + L_W},\end{aligned}$$

one can verify that

$$\begin{aligned}\|y(t) - \overline{y}(t)\| &\geq \|y(\theta_l) - \overline{y}(\theta_l)\| \\ &\quad - 2\overline{\Delta}[(\|A\| + L_f)K_0 + H_0] \\ &> \frac{M_0}{2(1 + \|B\| + L_W)} \geq \epsilon_1,\end{aligned}$$

for all $t \in (\zeta_i^1, \zeta_i^1 + \overline{\Delta})$. In a similar way, if $\theta_l \leq \zeta_i + \frac{\tau}{2}$, then we have for $t \in (\zeta_i^1, \zeta_i^1 + \overline{\Delta})$ that

$$\begin{aligned}\|y(t) - \overline{y}(t)\| &\geq \|y(\theta_l+) - \overline{y}(\theta_l+)\| \\ &\quad - 2\overline{\Delta}[(\|A\| + L_f)K_0 + H_0] \\ &> \frac{M_0}{2} > \epsilon_1.\end{aligned}$$

Consequently, on each of the intervals $J_i^1 = (\zeta_i^1, \zeta_i^1 + \overline{\Delta})$, $i \in \mathbb{N}$, the inequality $\|y(t) - \overline{y}(t)\| > \epsilon_1$ holds. Therefore, the couple of functions $(\phi_\varphi(t), \phi_{\overline{\varphi}}(t)) \in \mathscr{B} \times \mathscr{B}$ is frequently $(\epsilon_1, \overline{\Delta})$-separated. ■

The main theorem of the present study is as follows.

Theorem 1. *Suppose that the conditions (A1)–(A9) are valid. If $\mathscr{A}$ is a Li–Yorke chaotic set which possesses an mT-periodic function for each natural number m, then $\mathscr{B}$ is also a Li–Yorke chaotic set.*

Proof. By means of conditions (A1)$-$(A7), one can confirm that if $\varphi(t) \in \mathscr{A}$ is mT-periodic for some natural number m, then $\phi_\varphi(t) \in \mathscr{B}$ is a periodic function with the same period, and vice versa.

Suppose that the set $\mathscr{C}_{\mathscr{A}}$ is a scrambled set inside $\mathscr{A}$, and define the set

$$\mathscr{C}_{\mathscr{B}} = \{\phi_{\varphi(t)}(t) \,|\, \varphi(t) \in \mathscr{C}_{\mathscr{A}}\}.$$

It is easy to verify that there is a one-to-one correspondence between the sets $\mathscr{C}_{\mathscr{A}}$ and $\mathscr{C}_{\mathscr{B}}$. Since the set $\mathscr{C}_{\mathscr{A}}$ is uncountable, the same is true for $\mathscr{C}_{\mathscr{B}}$. Moreover, no periodic functions exist inside $\mathscr{C}_{\mathscr{B}}$, since no such functions take place inside the set $\mathscr{C}_{\mathscr{A}}$.

Because each pair of functions that belong to $\mathscr{C}_{\mathscr{A}} \times \mathscr{C}_{\mathscr{A}}$ is proximal, Lemma 2 implies the same feature for each pair inside $\mathscr{C}_{\mathscr{B}} \times \mathscr{C}_{\mathscr{B}}$. In connection with Lemma 3, there exist positive numbers ϵ_1 and $\overline{\Delta}$ such that each couple of functions from $\mathscr{C}_{\mathscr{B}} \times \mathscr{C}_{\mathscr{B}}$ are frequently $(\epsilon_1, \overline{\Delta})$-separated. Hence, $\mathscr{C}_{\mathscr{B}}$ is a scrambled set inside $\mathscr{B}$. If we denote by $\mathscr{P}_{\mathscr{A}}$ and $\mathscr{P}_{\mathscr{B}}$ the sets of periodic functions inside $\mathscr{A}$ and $\mathscr{B}$, respectively, then a similar discussion holds for each pair inside $\mathscr{C}_{\mathscr{B}} \times \mathscr{P}_{\mathscr{B}}$, since the same is true for any pair from the set $\mathscr{C}_{\mathscr{A}} \times \mathscr{P}_{\mathscr{A}}$. Consequently, the collection $\mathscr{B}$ is Li–Yorke chaotic. ■

In the next section, we will apply our results to an impulsive Duffing oscillator.

4. An Example

This part of the paper is devoted to an illustrative example. In Theorem 1 we have shown that if the perturbation term $\varphi(t)$ in system (2) belongs to a collection which is chaotic in the sense of Li–Yorke, then system (2) exhibits chaotic motions. Therefore, to actualize our results, we need a source of Li–Yorke chaotic functions. To construct such a collection, we will consider a Duffing oscillator which is forced with a relay function [Akhmet, 2009c, 2009d, 2009e; Akhmet & Fen, 2012, 2013]. The switching moments of the relay function are generated through the logistic map

$$F_\mu(s) = \mu s(1-s), \tag{8}$$

which is chaotic in the sense of Li–Yorke for the parameter μ between 3.84 and 4 [Li & Yorke, 1975]. We note that the interval $[0,1]$ is invariant under the iterations of the map $F_\mu(s)$ if $0 < \mu \leq 4$ [Hale & Koçak, 1991]. An impulsive Duffing oscillator will be used for the main illustration. Moreover, in the example, a procedure to control the chaos of the impulsive system will be presented. In our evaluations, we will make use of the usual Euclidean norm [Horn & Johnson, 1992].

Example. Consider the forced Duffing oscillator

$$\begin{aligned} z''(t) + 0.6z'(t) + 5z(t) - 0.02z^3(t) \\ = \nu(t, t_0, \mu), \end{aligned} \tag{9}$$

where $t \in \mathbb{R}$, t_0 belongs to the interval $[0,1]$ and the relay function $\nu(t, t_0)$ is defined as

$$\nu(t, t_0, \mu) = \begin{cases} 0.6, & \text{if } \zeta_{2j}(t_0,\mu) < t \leq \zeta_{2j+1}(t_0,\mu), \\ & \qquad j \in \mathbb{Z}, \\ 2.5, & \text{if } \zeta_{2j-1}(t_0,\mu) < t \leq \zeta_{2j}(t_0,\mu), \\ & \qquad j \in \mathbb{Z}. \end{cases} \tag{10}$$

In Eq. (10), the switching moments $\zeta_j(t_0,\mu)$, $j \in \mathbb{Z}$, are defined through the equation $\zeta_j(t_0,\mu) = j + \kappa_j(t_0,\mu)$, where the sequence $\{\kappa_j(t_0,\mu)\}$, $\kappa_0(t_0,\mu) = t_0$, is generated by the logistic map (8), that is, $\kappa_{j+1}(t_0,\mu) = F_\mu(\kappa_j(t_0,\mu))$. More information about the dynamics of relay systems can be found in [Akhmet, 2009c, 2009d, 2009e; Akhmet & Fen, 2012, 2013].

By means of the variables $z_1 = z$ and $z_2 = z'$, Eq. (9) can be reduced to the system

$$\begin{aligned} z_1'(t) &= z_2(t), \\ z_2'(t) &= -5z_1(t) - 0.6z_2(t) \\ &\quad + 0.02z_1^3(t) + \nu(t, t_0, \mu). \end{aligned} \tag{11}$$

According to the results of [Akhmet, 2009d], system (11) with the parameter value $\mu = 3.9$ is Li–Yorke chaotic. Moreover, for each natural number m, the system admits different unstable periodic solutions with periods $2m$.

In system (11) we set $\mu = 3.9$, and represent in Fig. 1 the z_1- and z_2-coordinates of the solution of the system with $z_1(t_0) = 0.492$ and $z_2(t_0) = -0.143$, where $t_0 = 0.385$. It is seen in Fig. 1 that system (11) possesses chaotic motions.

The function $h(z_1, z_2) = (z_1 + 0.5z_1^3, z_2)$ satisfies the inequality (3) with $L_1 = 1/\sqrt{2}$ and

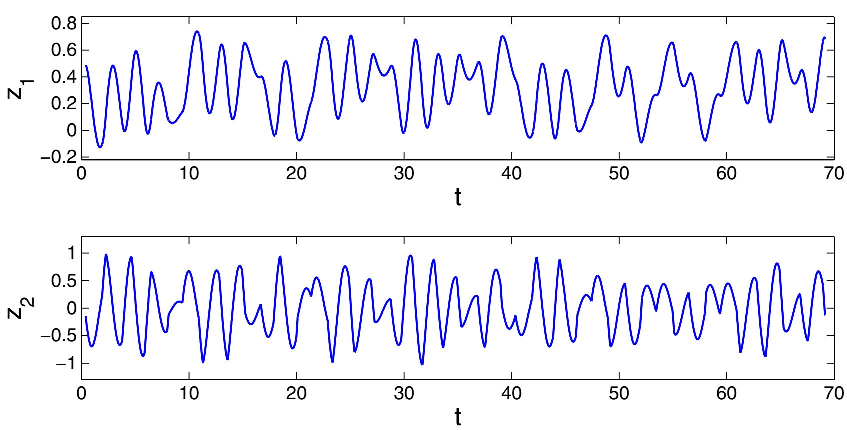

Fig. 1. The chaotic behavior of system (11) with $\mu = 3.9$. The initial data $z_1(t_0) = 0.492$, $z_2(t_0) = -0.143$, where $t_0 = 0.385$, is used in the simulation.

$L_2 = 1.96\sqrt{2}$ on the compact region in which the chaotic attractor of system (11) with $\mu = 3.9$ takes place. Therefore, the collection consisting of functions of the form $(z_1(t) + 0.5z_1^3(t), z_2(t))$, where $(z_1(t), z_2(t))$ is a bounded on $\mathbb{R}$ solution of system (11), is Li–Yorke chaotic.

Next, we take into account the impulsive Duffing oscillator

$$\begin{aligned}
&x''(t) + 2x'(t) + 3x(t) + 0.025x^3(t) \\
&\quad = 0.2\cos(\pi t), \quad t \neq \theta_k, \\
&\Delta x|_{t=\theta_k} = -\frac{3}{4}x(\theta_k), \\
&\Delta x'|_{t=\theta_k} = -\frac{3}{4}x'(\theta_k) + 0.05(x'(\theta_k))^2,
\end{aligned} \tag{12}$$

where $t \in \mathbb{R}$ and $\theta_k = 2k$, $k \in \mathbb{Z}$. Clearly, $\theta_{k+p} = \theta_k + T$, where $p = 1$ and $T = 2$.

Defining the new variables $x_1 = x$ and $x_2 = x'$, one can reduce (12) to the system

$$\begin{aligned}
x_1'(t) &= x_2(t), \\
x_2'(t) &= -3x_1(t) - 2x_2(t) - 0.025x_1^3(t) \\
&\quad + 0.2\cos(\pi t), \quad t \neq \theta_k, \\
\Delta x_1|_{t=\theta_k} &= -\frac{3}{4}x_1(\theta_k), \\
\Delta x_2|_{t=\theta_k} &= -\frac{3}{4}x_2(\theta_k) + 0.05(x_2(\theta_k))^2.
\end{aligned} \tag{13}$$

Let us demonstrate numerically that the system (13) possesses an asymptotically stable periodic solution. Figure 2 shows the graphs of the x_1- and x_2-coordinates of system (13). The initial data $x_1(1) = -0.019$, $x_2(1) = -0.056$ is used in the simulation. The existence of an asymptotically stable periodic solution is observable in the figure, and therefore, one can conclude that system (13) is not chaotic.

We perturb (13) by the solutions of (11) to set up the system

$$\begin{aligned}
y_1'(t) &= y_2(t) + z_1(t) + 0.5z_1^3(t), \\
y_2'(t) &= -3y_1(t) - 2y_2(t) - 0.025y_1^3(t) \\
&\quad + 0.2\cos(\pi t) + z_2(t), \quad t \neq \theta_k, \\
\Delta y_1|_{t=\theta_k} &= -\frac{3}{4}y_1(\theta_k), \\
\Delta y_2|_{t=\theta_k} &= -\frac{3}{4}y_2(\theta_k) + 0.05(y_2(\theta_k))^2.
\end{aligned} \tag{14}$$

System (14) is in the form of (2), where

$$A = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -\frac{3}{4} & 0 \\ 0 & -\frac{3}{4} \end{pmatrix},$$

$$f(t, y_1, y_2) = (0, -0.025y_1^3 + 0.2\cos(\pi t)) \quad \text{and}$$

$$W(y_1, y_2) = (0, 0.05y_2^2).$$

The matrices A and B commute, and the matrix

$$A + \frac{p}{T}\ln(I + B) = \begin{pmatrix} -\ln 2 & 1 \\ -3 & -2 - \ln 2 \end{pmatrix}$$

has eigenvalues $\lambda_{1,2} = -1 - \ln 2 \pm i\sqrt{2}$.

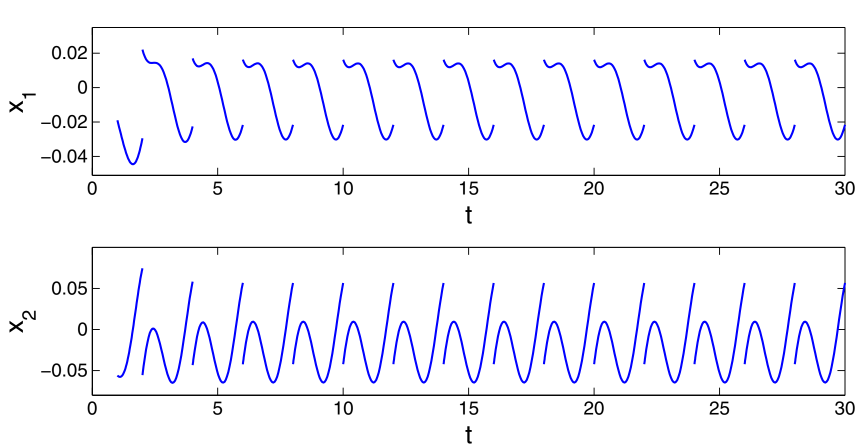

Fig. 2. The graphs of the x_1- and x_2-coordinates of system (13).

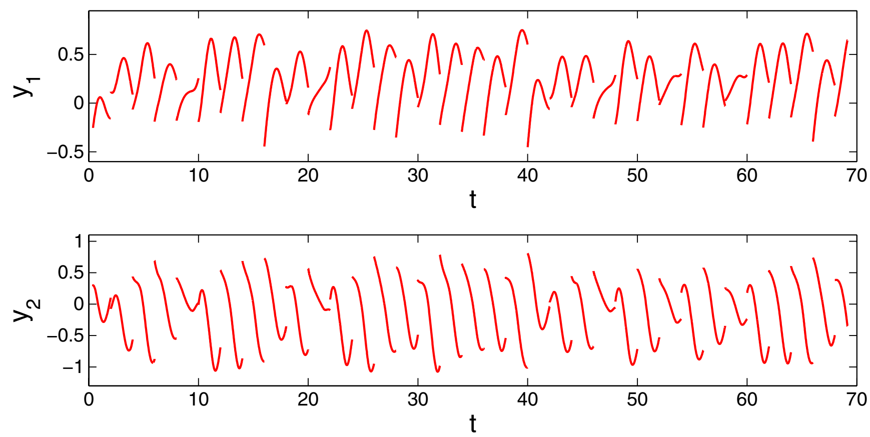

Fig. 3. The chaotic behavior of the perturbed impulsive Duffing oscillator (14). The initial data $y_1(t_0) = -0.254$, $y_2(t_0) = 0.297$ is used, where $t_0 = 0.385$. The irregular behavior observed in the graphs support our theoretical results.

Let us denote by $U(t,s)$ the transition matrix of the linear homogeneous system

$$\begin{aligned} &u_1'(t) = u_2(t), \quad u_2'(t) = -3u_1(t) - 2u_2(t), \quad t \neq \theta_k, \\ &\Delta u_1|_{t=\theta_k} = -\frac{3}{4}u_1(\theta_k), \quad \Delta u_2|_{t=\theta_k} = -\frac{3}{4}u_2(\theta_k). \end{aligned} \tag{15}$$

One can verify that

$$U(t,s) = e^{-(t-s)} P \begin{pmatrix} \cos(\sqrt{2}(t-s)) & -\sin(\sqrt{2}(t-s)) \\ \sin(\sqrt{2}(t-s)) & \cos(\sqrt{2}(t-s)) \end{pmatrix} P^{-1}(I+B)^{i([s,t))}, \quad t > s,$$

where $i([s,t))$ is the number of the terms of the sequence $\{\theta_k\}$ that belong to the interval $[s,t)$ and $P = \begin{pmatrix} 0 & 1 \\ \sqrt{2} & -1 \end{pmatrix}$. Making use of the matrix norm which is induced by the usual Euclidean norm in $\mathbb{R}^n$ [Horn & Johnson, 1992], it is easy to show that $\|U(t,s)\| \leq Ne^{-\omega(t-s)}$, $t \geq s$, where $\omega = 1$ and $N = 2.415$.

The conditions $(A4)$–$(A9)$ hold for system (14) with $M_f = 0.2183$, $M_W = 0.0845$, $L_f = 0.0608$ and $L_W = 0.13$. Thus, according to Theorem 1, system (14) is Li–Yorke chaotic.

Making use of the initial data $y_1(t_0) = -0.254$ and $y_2(t_0) = 0.297$, where $t_0 = 0.385$, we illustrate in Fig. 3 the y_1- and y_2-coordinates of the solution of system (14) with the solution $(z_1(t), z_2(t))$ of system (11) which is illustrated in Fig. 1. On the other hand, Fig. 4 depicts the trajectory of the same solution on the y_1–y_2 plane. Even if the system (13) is not chaotic, the simulation results shown in Figs. 3 and 4 support our theoretical results such that a chaotic attractor takes place in the dynamics of system (14).

Now, we shall present a method to control the chaos of system (14). This procedure depends on the idea that to control the chaos of system (14) it is sufficient to stabilize an unstable periodic solution of system (11). For this reason, we will apply the OGY control method for the logistic map [Ott *et al.*, 1990; Schuster, 1999], since the map gives

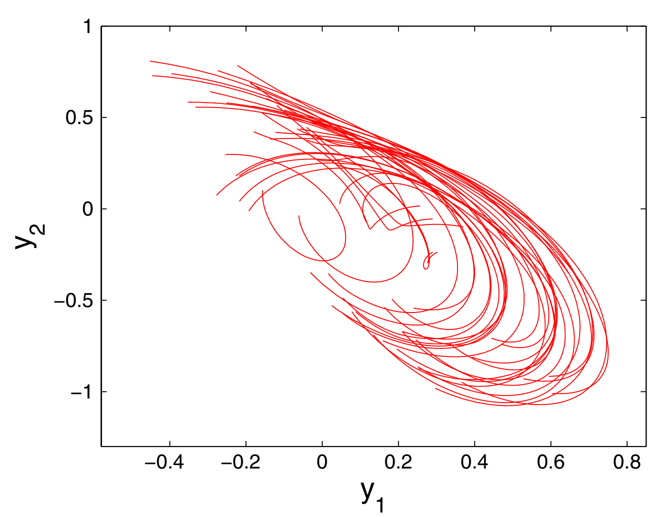

Fig. 4. The chaotic trajectory of system (14).

rise to the chaotic behavior in system (11). Let us explain the method briefly.

Suppose that the parameter μ in the logistic map (8) is allowed to vary in the range $[3.9-\varepsilon, 3.9+\varepsilon]$, where ε is a given small positive number. Consider an arbitrary solution $\{\kappa_j\}$, $\kappa_0 \in [0,1]$, of the map and denote by $\kappa^{(i)}$, $i = 1, 2, \ldots, p_0$, the target unstable p_0-periodic orbit to be stabilized. In the OGY control method [Schuster, 1999], at each iteration step j after the control mechanism is switched on, we consider the logistic map with the parameter value $\mu = \overline{\mu}_j$, where

$$\overline{\mu}_j = 3.9\left(1 + \frac{(2\kappa^{(i)} - 1)(\kappa_j - \kappa^{(i)})}{\kappa^{(i)}(1-\kappa^{(i)})}\right), \qquad (16)$$

provided that the number on the right-hand side of the formula (16) belongs to the interval $[3.9-\varepsilon, 3.9+\varepsilon]$. In other words, formula (16) is valid if the trajectory $\{\kappa_j\}$ is sufficiently close to the target periodic orbit. Otherwise, we take $\overline{\mu}_j = 3.9$, so that the system evolves at its original parameter value, and wait until the trajectory $\{\kappa_j\}$ enters in a sufficiently small neighborhood of the periodic orbit $\kappa^{(i)}$, $i = 1, 2, \ldots, p_0$, such that the inequality $-\varepsilon \leq 3.9\frac{(2\kappa^{(i)}-1)(\kappa_j-\kappa^{(i)})}{\kappa^{(i)}(1-\kappa^{(i)})} \leq \varepsilon$ holds. If this is the case, the control of chaos is not achieved immediately after switching on the control mechanism. Instead, there is a transition time before the desired periodic orbit is stabilized. The transition time increases if the number ε decreases [Gonzales-Miranda, 2004].

To apply the OGY method for controlling the chaos of system (11), we replace the parameter μ in system (11) with $\overline{\mu}_j$, which is introduced by formula (16), and set up the system

$$\begin{aligned} z_1'(t) &= z_2(t), \\ z_2'(t) &= -5z_1(t) - 0.6z_2(t) \\ &\quad + 0.02z_1^3(t) + \nu(t, t_0, \overline{\mu}_j). \end{aligned} \qquad (17)$$

System (17) is the control system conjugate to (11).

We consider the solution of system (17) with $z_1(t_0) = 0.492$ and $z_2(t_0) = -0.143$, where $t_0 = 0.385$, and apply the OGY control method around the period-1 orbit, that is the fixed point 2.9/3.9, of the logistic map $F_{3.9}(s)$. Figure 5 depicts the simulation results. One can observe in the figure that a 2-periodic solution of system (17) is stabilized. The value $\varepsilon = 0.05$ is used. The control mechanism is switched on at $t = \zeta_{20}$ and switched off at $t = \zeta_{40}$. The control becomes dominant approximately at $t = 37$ and its effect lasts approximately until $t = 93$, after which the instability becomes dominant and irregular behavior develops again.

In the next simulation, we demonstrate that the chaos of system (14) can be controlled by stabilizing an unstable periodic solution of system (11). We consider system (14) with the solution $(z_1(t), z_2(t))$ of system (17) which is illustrated in Fig. 5, and simulate in Fig. 6 the solution $(y_1(t), y_2(t))$ of system (14) with $y_1(t_0) = -0.254$ and $y_2(t_0) = 0.297$, where $t_0 = 0.385$. It is seen in Fig. 6 that a 2-periodic solution of the system is stabilized. The moments where the control is switched on and switched off and the period of time in which the stabilization becomes dominant are the same with the results presented in Fig. 5. The simulations seen

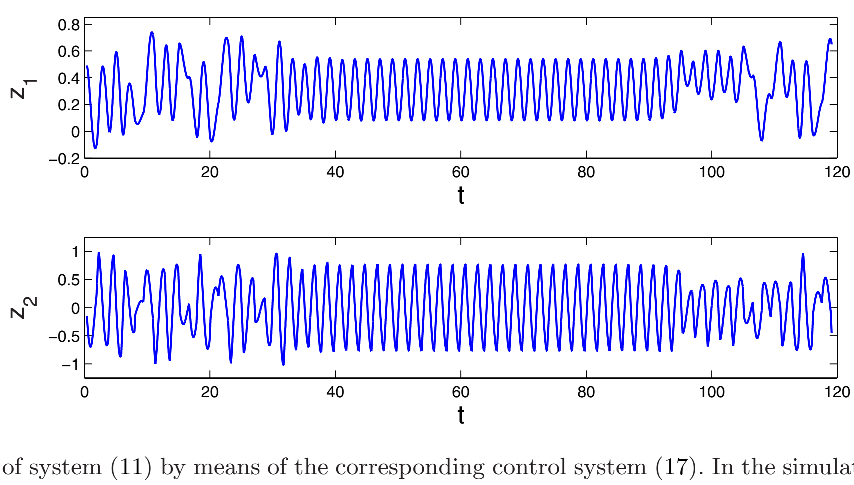

Fig. 5. Chaos control of system (11) by means of the corresponding control system (17). In the simulation, the value $\varepsilon = 0.05$ is used. The control is switched on at $t = \zeta_{20}$ and switched off at $t = \zeta_{40}$. It is seen in the figure that an unstable 2-periodic solution of system (11) is stabilized.

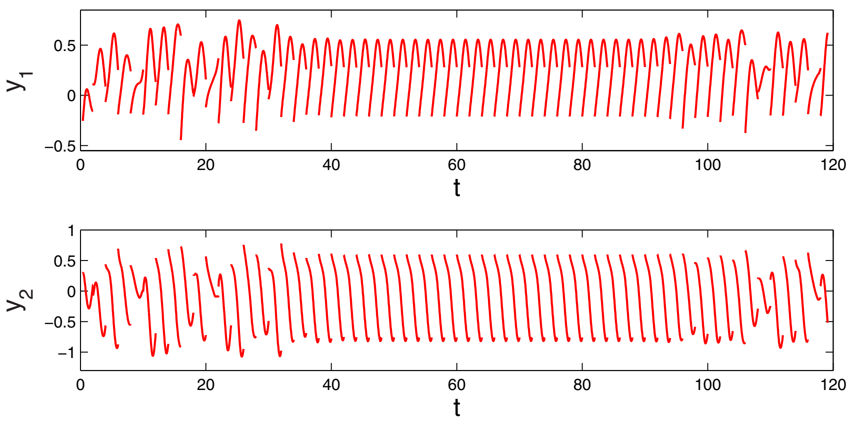

Fig. 6. Chaos control of the perturbed impulsive Duffing oscillator (14). To control the chaos of this system, the OGY control method is applied to system (11), which gives rise to the presence of chaos in (14).

in Fig. 6 confirm that to control the chaos of system (14) it is sufficient to stabilize an unstable periodic solution of system (11).

5. Conclusions

In this article, we present a technique to obtain chaotic impulsive systems with the aid of chaotic perturbations. Chaotic collections of piecewise continuous functions are introduced based on the Li–Yorke definition of chaos. Our results are useful for generating multidimensional discontinuous chaos, especially if one requires a rigorous proof for the phenomenon.

We applied our method to an impulsive Duffing oscillator to show the feasibility. According to their instability, the existing periodic solutions of system (2) are invisible in the simulations. A periodic solution of the perturbed impulsive Duffing oscillator is illustrated by means of the OGY control method [Ott *et al.*, 1990] applied to the logistic map. Other control procedures, such as the Pyragas method [Pyragas, 1992], can also be used for this purpose. The results of the present study are convenient for construction and stabilization of chaotic mechanical systems and electrical circuits with impulses. Moreover, our approach can be applied to other types of chaos such as the one analyzed through period-doubling cascade [Sander & Yorke, 2011].

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