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Shifted $1/N$ expansion for the Klein-Gordon equation with vector and scalar potentials

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The shifted $1/N$ expansion method has been extended to solve the Klein-Gordon equation with both scalar and vector potentials. The calculations are carried out to the third-order correction in the energy series. The analytical results are applied to a linear scalar potential to obtain the relativistic energy eigenvalues. Our numerical results are compared with those obtained by Gunion and Li [Phys. Rev. D **12**, 3583 (1975)].

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I. INTRODUCTION

Recently the shifted $1/N$ expansion technique [1] has received much attention in solving the Schrödinger equation. It has been applied to a large number of physically interesting potentials yielding highly accurate results [1–7]. Very recently the Klein-Gordon (KG) and Dirac equations have been studied by some authors [8–11] to determine the energy eigenvalues of these equations.

In the present paper we have extended the method to deal with the KG equation for any radially symmetric vector and scalar potentials. While the formalism applies for both vector- and scalar-type potentials, numerical results are obtained only for a scalar potential of the form $S(r)=Ar$, which is a Lorentz scalar used in the study of quarkonium systems [8,12–14]. Accurate numerical results are used to compare with our results.

In Sec. II we extend the formalism of Ref. [1] and apply a different approach [11] to deal with both vector and scalar potentials in the KG equation. In Sec. III we present our numerical results and compare with those given in Ref. [13]. Section IV is left for concluding remarks.

II. THE METHOD

The radial part of the N-dimensional KG equation (in units $\hbar=c=1$) for radially symmetric vector and scalar potentials [15] can be written as

$$\left[-\frac{d^2}{dr^2}+\frac{(k-1)(k-3)}{4r^2}+\{[m+S(r)]^2-[E-V(r)]^2\}\right]\phi(r)=0 \quad (1)$$

where $S(r)$ is a scalar potential and $V(r)$ the fourth component of a vector potential, $k=N+2l$, and $\phi(r)$ is the radial wave function.

Following Ref. [1], we use $\bar{k}$, which is defined as

$$\bar{k}=k-a \quad (2)$$

and shift the origin of coordinate by

$$x=\bar{k}^{1/2}(r-r_0)/r_0 , \quad (3)$$

and also accordingly expand $V(r)$, $S(r)$, and E as

$$V(r)=(\bar{k}^2/Q)[V(r_0)+V'(r_0)r_0x/\bar{k}^{1/2}+V''(r_0)r_0^2x^2/(2\bar{k}+\cdots] , \quad (4a)$$

$$S(r)=(\bar{k}^2/Q)[S(r_0)+S'(r_0)r_0x/\bar{k}^{1/2}+S''(r_0)r_0^2x^2/(2\bar{k})+\cdots] , \quad (4b)$$

$$E=E_0+E_1/\bar{k}+E_2/\bar{k}^2+E_3/\bar{k}^3+\cdots , \quad (4c)$$

where Q is a scale to be determined later. After substituting Eqs. (4a)–(4c) in Eq. (1), we obtain a Schrödinger-like equation which has been solved by Imbo, Pagnamenta, and Sukhatme [1]. We therefore just quote the results and give the final expression for the energy eigenvalue.

$$E=E_0+[\beta(1)+\beta(2)/\bar{k}]/2E_0r_0^2 \quad (5)$$

where $\beta(1)$ and $\beta(2)$ are defined in Ref. [8] and

$$E_0=V(r_0)+\{[S(r_0)+m]^2+Q/4r_0^2\}^{1/2} , \quad (6)$$

where r_0 is chosen to be the minimum of E_0. Hence r_0 satisfies the relation

$$Q=b(r_0)+[b^2(r_0)+c(r_0)] , \quad (7)$$

in which

$$b(r_0)=4r_0^3S'(r_0)[S(r_0)+m]+2r_0^4V'(r_0)^2 , \quad (8)$$

$$c(r_0)=16r_0^6[S(r_0)+m]^2[V'(r_0)^2-S'(r_0)^2] . \quad (9)$$

The shifting parameter a is chosen so as to make the first-order correction $E_1/\bar{k}$ vanish. Consequently

$$a=2-(1+2n_r)w , \quad (10)$$

where

$$w=\{3+(4r_0^4/Q)[[S(r_0)+m](S''(r_0)+\{1+(Q/4r_0^2)[S(r_0)+m]^2\}^{1/2}V''(r_0))+S'(r_0)^2-V'(r_0)^2]\} . \quad (11)$$

TABLE I. Klein-Gordon results for part of the energy levels (in GeV) of ψ,ψ' system with $A=0.137$ GeV^2 and $m=1.12$ GeV. The values in parentheses are those given by Gunion and Li [13].

	L			
n_r	0	1	2	3
0	3.12	3.46	3.74	3.99
	(3.1)	(3.47)	(3.73)	(3.98)
1	3.70	3.96	4.18	4.387
	(3.7)	(3.95)	(4.17)	(4.39)
2	4.16	4.36	4.556	4.738
	(4.17)	(4.38)	(4.56)	(4.73)
3	4.537	4.719	4.89	5.06
	(4.54)	(4.72)	(4.90)	(5.05)
4	4.9	5.039	5.196	5.346
	(4.8)	(5.04)	(5.2)	(5.35)

Equations (7)–(9) along with $Q=\bar{k}^2$ and Eqs. (10) and (11) read

$$\{b(r_0)+[b(r_0)^2+c(r_0)]^{1/2}\}^{1/2}=1+2l+(1-2n_r)w\ . \tag{12}$$

III. APPLICATION TO A PURE SCALAR POTENTIAL [$V(r)=0, S(r)=Ar$]

By this case we are precisely referring to the quark-confining linear potential regarded as a Lorentz scalar. From Eq. (5) the eigenvalue E is calculated and accordingly the Klein-Gordon results for the energy levels are listed in Tables I–III. Our results are compared with the numerical results obtained by Gunion and Li [13].

IV. CONCLUDING REMARKS

We have developed a general formalism for the shifted $1/N$ expansion of the Klein-Gordon equation with both vector and scalar potentials. The case where $V(r)=0$ and $S(r)=Ar$ only has been treated in this paper, leaving the other cases for later investigations. The comparison of our results with those of Gunion and Li [13] gives no doubt about the good agreement between them. In Table I the accuracy ranges from 97.96% to 100.00%. In Table II the accuracy is noted to range between 99.21% and 99.88%. In Table III the accuracy is between 98.87% and 99.62%. It has also been noted that the term contributing most to the energy levels is the leading term E_0 of Eq. (5) in the sense that the ratio of the leading term contribution to the contribution of E, Eq. (5), ranges between 0.9962 and 0.9998 for Table I, 0.9987 and 0.9997 for Table II, and 0.9992 and 0.9998 for Table III. All in all we can say that the shifted $1/N$ expansion works well for the KG equation with a scalar potential of the form $S(r)=Ar$.

APPENDIX

We list below the definitions of ϵ_j and δ_j:

$$\begin{aligned}
&\epsilon_1=(2-a),\quad \epsilon_2=-3(2-a)/2\ ,\\
&\epsilon_3=-1+(r_0^5/3Q)[mS'''(r_0)+E_0V'''(r_0)+S(r_0)S'''(r_0)-V(r_0)V'''(r_0)+3S'(r_0)S''(r_0)-3V'(r_0)V''(r_0)]\ ,\\
&\epsilon_4=\tfrac{5}{4}+(r_0^6/12Q)[mS''''(r_0)S''''(r_0)+E_0V''''(r_0)+S(r_0)S''''(r_0)+4S'(r_0)S'''(r_0)+3S''(r_0)^2\\
&\qquad -V(r_0)V''''(r_0)-4V'(r_0)V'''(r_0)-3V''(r_0)^2]\ ,
\end{aligned} \tag{A1}$$

and

TABLE II. Klein-Gordon results for part of the energy levels (in GeV) of the ρ system, using ρ' (1.25) as a first excitation, with $A=0.07$ GeV^2 and $m=0.15$ GeV. The values in parentheses are those given by Gunion and Li [13].

	L	
n_r	0	1
0	1.127	1.396
	(1.13)	(1.4)
1	1.60	1.792
	(1.61)	(1.79)
2	1.955	2.11
	(1.96)	(2.12)
3	2.247	2.38
	(2.25)	(2.39)
4	2.50	2.62
	(2.52)	(2.63)

TABLE III. Klein-Gordon results for part of the energy levels (in GeV) of the ρ system, using ρ' (1.6) as a first excitation, with $A=0.21$ GeV^2 and $m=0.15$ GeV. The values in parentheses are those given by Gunion and Li [13].

	L	
n_r	0	1
0	1.79	2.26
	(1.8)	(2.28)
1	2.63	2.95
	(2.64)	(2.98)
2	3.24	3.50
	(3.27)	(3.54)
3	3.74	3.98
	(3.78)	(4.0)

$$\delta_1=-(1-a)(3-a)/2,\quad \delta_2=3(1-a)(3-a)/4\ ,$$

$$\delta_3=2(2-a),\quad \delta_4=-5(2-a)/2\ ,$$

$$\begin{aligned}\delta_5=-\tfrac{3}{2}+(r_0^7/60Q)[&mS'''''(r_0)+E_0V'''''(r_0)+S(r_0)S'''''(r_0)+5S'(r_0)S''''(r_0)+15S''(r_0)S'''(r_0)\\&-V(r_0)V'''''(r_0)-5V'(r_0)V''''(r_0)-15V''(r_0)V'''(r_0)]\ ,\end{aligned}\tag{A2}$$

$$\begin{aligned}\delta_6=\tfrac{7}{4}+(r_0^8/360Q)[&mS''''''(r_0)+E_0V''''''(r_0)+S(r_0)S''''''(r_0)+6S'(r_0)S'''''(r_0)+15S''(r_0)S''''(r_0)\\&+10S'''(r_0)^2-V(r_0)V''''''(r_0)-6V'(r_0)V'''''(r_0)-15V''(r_0)V''''(r_0)-10V'''(r_0)^2]\ .\end{aligned}$$

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