Dual Metrics for a Class of Radiative Spacetimes

D. Băleanu [1]

Department of Mathematics and Computer Sciences, Faculty of Arts and Sciences, Çankaya University, 06531 Ankara, Turkey

S. Başkal [2]

Physics Department, Middle East Technical University, Ankara, 06531, Turkey

Abstract

Second rank non-degenerate Killing tensors for some subclasses of spacetimes admitting parallel null one-planes are investigated. Lichnérowicz radiation conditions are imposed to provide a physical meaning to spacetimes whose metrics are described through their associated second rank Killing tensors. Conditions under which the dual spacetimes retain the same physical properties are presented.

1 Introduction

Killing tensors are known to be mathematical generalizations of Killing vector fields, although their defining symmetries are essentially different [1] [2]. Recently, the reciprocal relations between spaces admitting non-degenerate Killing tensors and the spaces whose metrics are specified through those Killing tensors have been investigated, together with their generalizations to Grassman variables [3] [4]. Despite the fact that the geometrical interpretation of the Killing metrics leads to the notion of geometrical duality, the physical signification of those metrics have not been fully understood. The non-degenerate Killing tensors corresponding to some very well known spacetimes, including the Kerr-Newman [5] and the Taub-NUT metrics [3], have non-vanishing Einstein tensors, the sources of which have neither been identified nor received any interpretation, and is still one of the main issues to be clarified. Mostly, the non-degenerate Killing tensors were constructed from Killing-Yano[6] tensors, but they are not the only solution of the Killing tensor equations on those manifolds [1]. Investigation of the non-degenerate Killing tensors corresponding to the Euclidean flat space suggests that there are in fact a class of tensors, that are not derived from Killing-Yano tensors [4]. In that context, the Stäckel systems of three-dimensional separable coordinates were recently investigated [7], but a four-dimensional and non-diagonal example is as yet missing. Furthermore,

[1]E-mail: dumitru@cankaya.edu.tr, Institute of Space Science, P.O. Box, MG-23, R 76900, Magurele-Bucharest, Romania

[2]E-mail: baskal@newton.physics.metu.edu.tr

finding a class of metrics admitting non-degenerate Killing tensors is not an easy task, because the condition of non-degeneracy is imposed by hand and has no connection with the symmetries of the equations.

The simplest way to attack these problems is to find an appropriate class of metrics such that their Killing tensors have the same form as the metric itself. One of those classes can be chosen from spacetimes admitting parallel fields of null 1-planes (PN1P), which are not only interesting in their own right, but also are appropriate candidates to represent radiation with their null vector field indicating the direction of propagation. The most general form of the metric for fields of parallel m-planes has been given by Walker [8]. In four dimensions, for each field of non-null and null 1-plane, it is possible to associate a massive and a massless particle, respectively, with their four momentum vector to be the basis of the 1-planes [9]. A very well known subclass of Walker's metric is due to Kundt [10] which has served as a background metric for diverse purposes [11] [12].

In this paper, we will classify metrics with PN1P admitting non-degenerate Killing tensors. We will investigate under what conditions, both the initial and its dual spacetimes have radiative properties, in an attempt to contribute to a deeper understanding of dual spacetimes.

2 Killing Spaces and Geometric Duality

A second rank Killing tensor is defined through the equation

$$\nabla_\lambda K_{\mu\nu} + \nabla_\mu K_{\nu\lambda} + \nabla_\nu K_{\lambda\mu} \,=\, 0. \tag{1}$$

If this tensor is non-degenerate, then it can be considered as a metric itself, defining a Killing space [7] or sometimes more specifically a Killing spacetime (KS). It has been shown in detail in reference [3] that $K^{\mu\nu}$ and $g^{\mu\nu}$ are reciprocally the contravariant components of the Killing tensors with respect to each other. Then, the second rank non-degenerate tensor $k_{\mu\nu}$, defined through $K^{\mu\alpha}k_{\alpha\nu} = \delta^\mu{}_\nu$, can be viewed as the metric on the "dual" space.

The notion of geometric duality extends to that of phase space. The constant of motion $K = \frac{1}{2}K^{\mu\nu}p_\mu p_\nu$, generates symmetry transformations on the phase space linear in momentum: $\{x^\mu, K\} = K^{\mu\nu}p_\nu$, and in view of (1) the Poisson brackets satisfy $\{H, K\} = 0$, where $H = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu$. Thus, in the phase space there is a reciprocal model with constant of motion H and the Hamiltonian K.

The relation between the Christoffel symbols, $\hat{\Gamma}^\mu{}_{\alpha\beta}$ of the KS and of the initial manifold has been expressed earlier [4]. By writing $\hat{\Gamma}^\mu{}_{\alpha\beta}$ in terms of the Killing tensor and taking (1) into account we have:

$$\hat{\Gamma}^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\alpha\beta} - \mathcal{K}^{\mu\delta}\nabla_\delta K_{\alpha\beta}, \tag{2}$$

where $\mathcal{K}^{\mu\alpha}K_{\alpha\nu} = \delta^\mu{}_\nu$.

3 Parallel Null One-Planes and Lichnérowicz Radiation Conditions

A parallel field of null 1-plane in spacetime consists in a recurrent field of null vectors. If l_μ is a basis for the plane we have:

$$\nabla_\nu l_\mu = \kappa_\nu l_\mu, \qquad l_\mu l^\mu = 0, \quad l_\mu \neq 0 \tag{3}$$

where κ_μ is the recurrence vector of the plane. It can be seen that such a vector field is geodesic and non-rotating. If the PN1P is strictly parallel then the recurrence vector vanishes identically.

One criteria for the existence of radiation in spacetime has been proposed by Lichnérowicz, relies on an analogy with electromagnetism, and is based on the solution of Cauchy's problem for Einstein-Maxwell equations [13]. In brief, the spacetime metric is subject to the conditions:

$$l^\mu R_{\mu\nu\,\alpha\beta} = 0, \tag{4}$$

$$l_{[\mu} R_{\nu\sigma]\,\alpha\beta} = 0 \tag{5}$$

with $l_\mu \neq 0$ and $R_{\mu\nu\,\alpha\beta} \neq 0$. He also proved that the trajectories of l^μ are null geodesics if $R_{\mu\nu\alpha\beta} \neq 0$ [14]. The applicability of Lichnérowicz radiation conditions (LRC) to some alternative approaches to gravity has been another subject of interest [12].

From (3) and the Ricci identity we have

$$l^\nu R_{\mu\nu\,\alpha\beta} = l_\mu f_{\alpha\beta}, \tag{6}$$

where

$$f_{\alpha\beta} = \partial_\alpha \kappa_\beta - \partial_\beta \kappa_\alpha. \tag{7}$$

From above, it is seen that when $f_{\alpha\beta} = 0$, one of the radiation conditions due to Lichnérowicz is satisfied.

The canonical form for PN1P has been given by Walker as: [8]

$$ds^2 = 2\,dv\,du + A'\,dx^2 + 2D'\,dx\,dy + 2E'\,dx\,du + B'\,dy^2 + 2F'\,dy\,du + H'\,du^2\,, \tag{8}$$

where H' depends on v, x, y, u and A', B', D', E', F' depend only on x, y, u, with $(A'B' - D'^2) > 0$. The plane is strictly parallel when H' is also independent of v. It is apparent that the existence of a geodesic null vector field is crucial for the description of a radiative spacetime. Therefore, Walker's metric is an appropriate candidate if we are to seek spacetimes having radiative properties.

From (3) one can observe that the principal null vector $l_\mu = \delta_\mu{}^4 = \partial_\mu u$, is hypersurface-orthogonal and the recurrence vector for the PN1P is $\kappa_\mu = -\Gamma^4{}_{\mu 4}$. If both the initial and its KS admit PN1P with the same principal null vector then the relation between their recurrence vectors can be expressed as

$$\hat{\kappa}_\mu = \kappa_\mu + \mathcal{K}^{4\delta} \nabla_\delta K_{\mu 4}. \tag{9}$$

4 Subclasses of Parallel Null 1-Planes

It has been shown that the canonical form of Walker's metric can be brought into a simpler form, by an appropriate choice of the coordinate system where $E = F = 0$ [15]. Then the resulting form can always be diagonalized within the metric functions A', B' and D'. As such the simplified form becomes:

$$ds^2 = 2\,dv\,du + A(x,y,u)\,dx^2 + B(x,y,u)\,dy^2 + H(v,x,y,u)\,du^2 \tag{10}$$

where $A(x,y,u), B(x,y,u)$ and $H(v,x,y,u)$ are functions of their arguments.

4.1 PN1P satisfying LRC

We are looking for a subclass of the above metric satisfying LRC. Condition (4) suggests that all $R_{1\nu\,\alpha\beta}$ vanish for all ν, α, β. This in return yields the metric function H in (10) to be linear in v. With this there are left only six non-vanishing components of the Riemann tensor which are: $R_{23\,23}$, $R_{23\,24}$, $R_{23\,34}$, $R_{24\,24}$, $R_{24\,34}$, $R_{34\,34}$. Furthermore, condition (5) imposes the following restriction on the Riemann tensor:

$$R_{23\,23} = R_{23\,24} = R_{23\,34} = 0, \tag{11}$$

which are second order non-linear coupled equations with respect to the metric functions $A(x,y,u)$ and $B(x,y,u)$. Analyzing (11), two possible solutions can be distinguished:
i. $A(x,u)$ is independent of y and $B(y,u)$ is independent of x.
ii. $A(x,y,u) = B(x,y,u)$.
The above spacetimes satisfy Lichnérowicz radiation conditions.

4.2 Subclasses of PN1P admitting non-degenerate Killing tensors

The second step is to classify the above metrics whose Killing tensors are of the same form. Thus their KS will retain the same properties as that of the initial ones. Corresponding to the cases which we have found in the preceding section, we have the following classifications:
Case i. The metrics satisfying the first case in above are of the following form:

$$\begin{aligned} &A(x,u) = a(x)\,(s_2(u) - 1), \qquad B(y,u) = b(y)\,(s_3(u) - 1), \\ &H(v,x,y,u) = v\,h(u) + (r_1(x)\,q_1(u) + r_2(y)\,q_2(u))(s_2(u) - s_3(u))^{-1}\,, \end{aligned} \tag{12}$$

with $s_2(u) \neq s_3(u) \neq 1$. Here, once and for all, the functions specifying the metric and the Killing tensor are arbitrary functions of their arguments, unless restrictions are explicitly stated. Solving equation (1) the non-vanishing components of the corresponding Killing tensor are found to be

$$\begin{aligned} &K_{14} = 1, \qquad K_{22} = A(x,u)\, s_2(u), \qquad K_{33} = B(y,u)\, s_3(u), \\ &K_{44} = H(v,x,y,u) + [r_1(x)q_1(u)\,(1 - s_2(u)) + r_2(y)q_2(u)\,(1 - s_3(u))] \\ &\quad (s_2(u) - s_3(u))^{-1} . \end{aligned} \tag{13}$$

Here, the functions $q_1(u)$ and $q_2(u)$ are subject to the following equations:

$$\begin{aligned} f_1(u)\, q_1(u) - g_1(u)\, q_1(u)_{,u} = 0, \\ f_2(u)\, q_2(u) + g_2(u)\, q_2(u)_{,u} = 0 \end{aligned} \tag{14}$$

with

$$\begin{aligned} f_1(u) &= s_2(u)_{,u}\,(s_3(u) - 1) - s_3(u)_{,u}(s_2(u) - 1) - h(u)\,(s_2(u) - 1)(s_2(u) - s_3(u)), \\ f_2(u) &= s_2(u)_{,u}(s_3(u) - 1) - s_3(u)_{,u}(s_2(u) - 1) - h(u)\,(s_3(u) - 1)(s_2(u) - s_3(u)), \\ g_1(u) &= (s_2(u) - 1)(s_3(u) - s_2(u)), \\ g_2(u) &= (s_3(u) - 1)(s_3(u) - s_2(u)), \end{aligned} \tag{15}$$

where the comma denotes partial differentiation. We found the class of dual metrics corresponding to (12) as:

$$\begin{aligned} &k_{14} = 1, \qquad k_{22} = A(x,u)s_2(u)^{-1}, \qquad k_{33} = B(y,u)s_3(u)^{-1}, \\ &k_{44} = v\, h_1(u) + [r_1(x)q_1(u)s_2(u) + r_2(y)q_2(u)s_3(u)]\,(s_2(u) - s_3(u))^{-1} , \end{aligned} \tag{16}$$

where $s_2 \neq 0$, $s_3 \neq 0$, with further restrictions as in (12).

Case ii. For the second case we discussed in the previous section, we have the following metrics:

$$A(x,y,u) = e^{x+y}a(u), \qquad H(v,x,y,u) = vh_1(u) + h_2(x,y,u). \tag{17}$$

The non-vanishing components of the Killing tensors are now found as:

$$\begin{aligned} &K_{14} = 1, \quad K_{22} = K_{33} = A(x,y.u)(1 + a(u)), \\ &K_{44} = H(v,x,y,u) - a(u)h_2(x,y,u) + k_1(u), \end{aligned} \tag{18}$$

where

$$h_2(x,y,u) = \frac{e^{P(u)}}{a(u)} \left[\int e^{P(u)}(h_1(u)k_1(u) + k_1(u)_{,u})du + k_2(x,y) \right], \tag{19}$$

with $P(u) = \int h_1(u)du$. The corresponding dual metrics become:

$$\begin{aligned} &k_{14} = 1, \qquad k_{22} = k_{33} = A(x,y,u), \\ &k_{44} = H(v,x,y,u) + a(u)h_2(x,y,u) - k_1(u). \end{aligned} \tag{20}$$

In the following we investigate a very familiar subclass of Walker's metric, recognized as Kundt's metric [10], to provide an example whose Killing tensor is not of the same form of the initial metric but still falls into the class of PN1P. They describe plane fronted waves with parallel rays, admitting a non-expanding shear-free and twist-free null geodesic congruence. This metric is expressed as:

$$ds^2 = 2\,dv\,du + dx^2 + dy^2 + H(x, y, u)\,du^2. \tag{21}$$

The metric function $H(x, y, u)$ has either of the following forms:

$$H^{(1)}(x, y, u) = h_1(u) - h_2(x-y) + x, \quad H^{(2)}(x, y, u) = h_1(u) - h_2(x-y) + y \tag{22}$$

and admits a Killing tensor in a more general form, whose non-vanishing components are

$$\begin{aligned} &K_{14} = K_{22} = K_{33} = 1, \qquad K_{24} = K_{34} = u, \\ &K^{(1)}_{44} = H^{(1)}(x, y, u) - 2(x + y) - u^2/2, \\ &K^{(2)}_{44} = H^{(2)}(x, y, u) - 2(x + y) - u^2/2. \end{aligned} \tag{23}$$

The associated dual metrics are found as:

$$\begin{aligned} &k_{14} = k_{22} = k_{33} = 1, \qquad k_{24} = k_{34} = -u, \\ &k^{(1)}_{44} = H^{(1)}(x, y, u) + 2(x + y) + \tfrac{5}{2}u^2, \\ &k^{(2)}_{44} = H^{(2)}(x, y, u) + 2(x + y) + \tfrac{5}{2}u^2. \end{aligned} \tag{24}$$

Since the Killing and the dual metrics both satisfy the conditions we have presented in Sec.4.1, they are also radiative in the sense of Lichnérowicz.

For all of the subclasses we have investigated above we have found that both the initial metric and its KS has only the G_{44} component of the Einstein tensor surviving. In a tensorial form this can be expressed as

$$G_{\mu\nu} \,=\, \rho\, l_\mu l_\nu, \tag{25}$$

where ρ is the energy density, and its expression can be found straightforwardly, for each subclass. Within the framework of Einstein's theory of relativity, this means that they describe pure radiative spacetimes [1].

Once the arbitrary functions are specified then $k_{\mu\nu}$ can be found explicitly. Finally, we would like to emphasize that, independent of the explicit forms of those arbitrary functions, the dual metrics are also in the form of PN1P, satisfy LRC and are pure radiative.

5 Conclusion

In this paper, we have classified spacetimes with a field of parallel null one-planes admitting non-degenerate Killing tensors. In general, for an arbitrary metric, one cannot predict in advance that the Killing tensor equations admit non-degenerate and non-trivial solutions, because there is not a well defined technique to solve this problem. For this purpose we have analyzed in detail equation (1) and looked for non-trivial Killing tensors that are of the same form as that of the initial metric.

The next step has been to evaluate the dual spacetimes associated with those Killing metrics. We have put some additional conditions on the metrics defining PN1P so that they describe radiative spacetimes; namely we have imposed Lichnérowicz radiation conditions. Furthermore, it can be seen by direct calculations that, to generate pure radiative spacetimes, it is sufficient to impose LRC.

We have found out that the dual spaces also satisfy the same properties as that of the initial ones, endowing a physical significance to dual spaces as being pure radiative spacetimes.

Spacetimes with PN1P are under further investigation from a supersymmetric point of view in connection with their Killing-Yano tensors [16].

Acknowledgments

We would like to thank to M. Cahen and F. Öktem for stimulating discussions.

References

[1] D. Kramer, H. Stephani, E. Herlt and M. MacCallum, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, 1980).

[2] K. Rosquist, *J. Math. Phys.* **30**, 2319 (1989).

[3] R. H. Rietdijk and J. W. van Holten, *Nucl. Phys.* **B472** , 427 (1996).

[4] D. Baleanu, gr-qc/9805053, to be published in *Int. J. Mod. Phys.* **D**.

[5] B. Carter, *Phys. Rev.* **174**, 1559 (1968);
G. W. Gibbons, R. H. Rietdijk and J. W. van Holten, *Nucl. Phys.* **B404**, 559 (1993)

[6] C. D. Collinson, *Int. J. Theor. Phys.* **15**, 311 (1976);
J. W. van Holten, *Phys. Lett.* **B342**, 47 (1995);
J. W. van Holten, A. Waldron and K. Peeters, *Class. Quant. Grav.* **16**, 2537 (1999).

[7] F. Hinterleitner, *Ann. Phys.* **271**, 23 (1999).

[8] A. G. Walker, *Quart. Journ. Math.* **1**, 69 (1950).

[9] F. Öktem, *Nuovo Cimento* **A15**, 189 (1973);
F. Öktem, *Nuovo Cimento* **B34**, 169 (1976).

[10] W. Kundt, *Zeit. Phys.* **163**, 77 (1961).

[11] J. Podolsky, *Class. Quant. Grav.* **15**, 3229 (1998);
J. Bicak and J. Podolsky *J. Math. Phys.* **40**, 4495 (1999).

[12] S. Başkal, *Prog. Theor. Phys.* **102**, 803 (1999).

[13] A. Lichnérowicz, *Compt. Rend.* **246**, 893 (1958);
A. Lichnérowicz, in *Colloques Internationaux du CNRS* p.93 (Paris, 1962);

[14] A. Lichnérowicz, *Ann. Mat. Pura ed Appl.* **50**, 1 (1960).

[15] Ş. Akçağ and F. Öktem, *METU J. Pure Appl. Sci.* **11**, 23 (1978).

[16] D. Baleanu and S. Başkal, in preparation.