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1. Introduction: the need for near-exact distributions and what are these distributions

We are all quite familiar with the concept of asymptotic distribution, at least those of us who usually work or do research in Statistics. Since there are random variables or statistics whose exact distribution is known to be quite elaborate and sometimes even non-manageable, the concept of asymptotic distribution was developed since quite early in the history of Statistics as a much useful one.

In simple terms an asymptotic distribution for some random variable or statistic is a distribution (of probability) which adequately approximates the exact distribution of that random variable or statistic, in such a way that when some relevant parameter of that distribution, usually related with the sample size, grows large, this approximate distribution improves its closeness to the exact distribution. These asymptotic distributions usually have much simpler expressions than the exact distribution and this is usually seen as their great advantage, allowing this way for a much easier computation of approximate quantiles and p-values for the statistic or random variable being studied. Such asymptotic distributions may

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also commonly arise from some standard results in Probability and Statistics, related with the convergence of sequences of random variables which verify some set of criteria.

For some sets of statistics, as it is for example the case with the so-called likelihood ratio test statistics, mainly those used in Multivariate Analysis, some authors developed what are nowadays seen as “standard” methods of building such asymptotic distributions, as it is the case of the seminal paper by Box (1949).

However, such asymptotic distributions may quite commonly yield approximations which may fall a bit short of the precision we need and/or may also exhibit some problems when some parameters in the exact distributions grow large, as it is indeed the case with many asymptotic distributions commonly used in Multivariate Analysis when the number of variables involved grows quite large (Coelho and Marques, 2011).

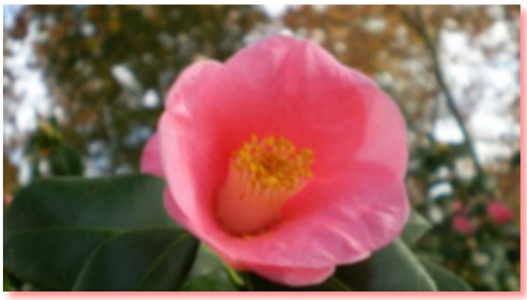
The pertinent question is thus the following one: are we willing to pay a bit more in terms of a more elaborate structure for the approximating distribution, anyway keeping it much manageable in terms of allowing for a quite easy computation of p-values and quantiles, if we will be able to keep untouched a good part of the original structure of the exact distribution of the random variable or statistic being studied, this way obtaining a much better approximation, which not only does not exhibit anymore the problems referred above and which on top of this exhibits extremely good performances even for very small sample sizes and large numbers of variables involved (which usually is not the case with the common asymptotic distributions), at the same time that these new approximations are asymptotic not only for increasing sample sizes but also (opposite to what happens with the common asymptotic distributions) for increasing values of the number of variables and any other parameters involved in the exact distribution of the random variable or statistic being studied?

If our answer to the above question is affirmative, then we are ready to enter the amazing world of the near-exact distributions.

Near-exact distributions are asymptotic distributions developed under a new concept of approximating distributions. Based on a decomposition (i.e., a factorization or a split in two or more terms) of the characteristic function of the statistic being studied, or of the characteristic function of its logarithm, they are asymptotic distributions which lie much closer to the exact distribution than common asymptotic distributions.

Exact

Near-exact



Asymptotic

Figure 1 – Illustrating the differences between exact, near-exact and asymptotic distributions.
(photos by the author)

In a figurative way we would say that the difference between near-exact and asymptotic distributions is that while in the first ones, as it happens with a good oil-on-canvas painting, the most of the original or real structure is kept, with some details even enhanced, in the asymptotic distributions it is as if the whole picture comes out blurred (see Figure 1 – and if differences are not evident please try a larger magnification).

Indeed the implementation of the process of developing near-exact distributions has usually a much useful by-product which is the study and understanding of the fine structure of the distribution of the random variable or statistic being considered, as a direct consequence of the study one has to carry on the characteristic function of this random variable or of its logarithm. This understanding of the fine structure of the distribution of the statistics may even enable us to devise ways to develop a family of near-exact distributions for sets of statistics, by better understanding the common traits among the exact distributions of these statistics, as it was done in Coelho, Arnold and Marques (2010) and Marques, Coelho and Arnold (2011).

The whole process of developing or obtaining near-exact distributions may be seen as parceled in two main steps: i) first of all one has to obtain a convenient decomposition of the characteristic function and then ii) identify the component part or parts which yield a manageable distribution and are to be left unchanged and the part or parts which not yielding a manageable distribution, have to be replaced by an asymptotic approximation which corresponds to the characteristic function of a manageable distribution. All this has to be done in such a way that the resulting characteristic function yields a manageable distribution, from which p-values and quantiles are easy to compute.

By using this procedure we are able to obtain very well-fitting near-exact distributions or approximations even in situations where common asymptotic distributions are not easy to be developed or they do not perform well. Actually, it is even possible to obtain near-exact distributions for statistics for which there are no asymptotic distributions developed. This is so, because we only need to obtain good asymptotic approximations for a part of the original characteristic function.

If the asymptotic replacement is adequately chosen, the resulting near-exact distributions, in case they refer to test statistics used in Multivariate Analysis, will be asymptotic not only for increasing sample sizes but also for increasing number of variables used, or even yet for increasing number of matrices or vectors being tested.

Although the whole process may seem a bit complicated, the correct choice of the part of the original c.f. to be left unchanged, together with an adequate choice for the asymptotic replacement(s) for the part of the original c.f. to be replaced, may quite easily lead to an overall quite simple process and to very well-fitting near-exact distributions.

So far, near-exact distributions have been developed for a wide range of statistics, namely likelihood ratio test statistics used in Multivariate Analysis, whose exact distributions have quite complicated structures (Coelho, 2004, 2006; Alberto and Coelho, 2007; Grilo and Coelho, 2007, 2010a, 2010b, 2011; Marques and Coelho, 2008, 2010, 2011a, 2011b; Coelho and Marques, 2009, 2010, 2011a, 2011b; Coelho and Mexia 2010; Coelho, Arnold and Marques, 2010, 2011; Marques, Coelho and Arnold, 2011).

Although dealing with these near-exact distributions, in terms of using them to compute the associated near-exact p-values and quantiles, will almost for sure require the use of a computer and some programming (preferably using one of the nowadays commonly available high level languages or symbolic softwares), the present wide availability of such machinery and software poses indeed no problem, on the other hand, by using these near-exact distributions we may easily obtain astonishing gains in precision, being almost simple to build approximations which cumulative distribution function lies away from the exact by less than a hundredth of a millionth part.

2. A very simple example

Let us suppose we have a statistic, let us call it Λ, whose exact distribution is known to be the same as that of

$$\prod_{j=1}^{3} Y_j \tag{2.1}$$

where $Y_j \quad (j=1,\dots,3)$ are three independent r.v.'s (random variables), with $Y_j \sim Beta\left(a-\frac{j}{2},\frac{5}{2}\right)$, for some $a>\frac{3}{2}$.

We know that the *h*-th moment of Y_j is

$$E\left(Y_j^h\right)=\frac{\Gamma\left(a-\frac{j}{2}+\frac{5}{2}\right)}{\Gamma\left(a-\frac{j}{2}\right)}\frac{\Gamma\left(a-\frac{j}{2}+h\right)}{\Gamma\left(a-\frac{j}{2}+\frac{5}{2}+h\right)} \quad \left(h>\frac{j}{2}-a\right), \tag{2.2}$$

and that as such, given the independence of the r.v.'s Y_j we have the *h*-th moment of Λ given by

$$\begin{aligned} E\left(\Lambda^h\right) &= E\left(\left(\prod_{j=1}^{3} Y_j\right)^h\right)=E\left(\prod_{j=1}^{3} Y_j^h\right)=\prod_{j=1}^{3} E\left(Y_j^h\right) \\ &= \prod_{j=1}^{3}\frac{\Gamma\left(a-\frac{j}{2}+\frac{5}{2}\right)}{\Gamma\left(a-\frac{j}{2}\right)}\frac{\Gamma\left(a-\frac{j}{2}+h\right)}{\Gamma\left(a-\frac{j}{2}+\frac{5}{2}+h\right)} \quad \left(h>\frac{3}{2}-a\right). \end{aligned} \tag{2.3}$$

Let us then consider the r.v. $W=-\log\Lambda$. Since the expression in (2.3) for the *h*-th moment of Λ is valid for *h* in a neighborhood of zero, we have the c.f. (characteristic function) of W given by

$$\begin{aligned} \Phi_W(t) &= E\left(e^{itW}\right)=E\left(e^{-it\log\Lambda}\right)=E\left(\Lambda^{-it}\right) \\ &= \prod_{j=1}^{3}\frac{\Gamma\left(a-\frac{j}{2}+\frac{5}{2}\right)}{\Gamma\left(a-\frac{j}{2}\right)}\frac{\Gamma\left(a-\frac{j}{2}-it\right)}{\Gamma\left(a-\frac{j}{2}+\frac{5}{2}-it\right)} \quad \left(t\in\mathbb{R}\right). \end{aligned} \tag{2.4}$$

But then using recursively the well-known relation

$$\Gamma(r+1)=r\,\Gamma(r) \tag{2.5}$$

for the Gamma function, we may write for $n \in \square$ and $a \in \square$,

$$\frac{\Gamma(a+n)}{\Gamma(a)} = \prod_{l=0}^{n-1}(a+l) \ ,$$

and then, using this relation, in (2.4) we may write

$$\begin{aligned}
\Phi_W(t) &= \prod_{j=1}^{3} \frac{\Gamma\left(a-\frac{j}{2}+2\right)}{\Gamma\left(a-\frac{j}{2}\right)} \frac{\Gamma\left(a-\frac{j}{2}-it\right)}{\Gamma\left(a-\frac{j}{2}+2-it\right)} \frac{\Gamma\left(a-\frac{j}{2}+\frac{5}{2}\right)}{\Gamma\left(a-\frac{j}{2}+2\right)} \frac{\Gamma\left(a-\frac{j}{2}+2-it\right)}{\Gamma\left(a-\frac{j}{2}+\frac{5}{2}-it\right)} \\
&= \left\{\prod_{j=1}^{3}\prod_{l=0}^{1}\left(a-\tfrac{j}{2}+l\right)\left(a-\tfrac{j}{2}+l-it\right)^{-1}\right\}\left\{\prod_{j=1}^{3}\frac{\Gamma\left(a-\frac{j}{2}+\frac{5}{2}\right)}{\Gamma\left(a-\frac{j}{2}+2\right)}\frac{\Gamma\left(a-\frac{j}{2}+2-it\right)}{\Gamma\left(a-\frac{j}{2}+\frac{5}{2}-it\right)}\right\} \\
&= \underbrace{\left\{\prod_{j=1}^{5}\left(a-2+\tfrac{j}{2}\right)^{r_j}\left(a-2+\tfrac{j}{2}-it\right)^{-r_j}\right\}}_{\Phi_{1,W}(t)}\underbrace{\left\{\prod_{j=1}^{3}\frac{\Gamma\left(a-\frac{j}{2}+\frac{5}{2}\right)}{\Gamma\left(a-\frac{j}{2}+2\right)}\frac{\Gamma\left(a-\frac{j}{2}+2-it\right)}{\Gamma\left(a-\frac{j}{2}+\frac{5}{2}-it\right)}\right\}}_{\Phi_{2,W}(t)}
\end{aligned} \qquad (2.6)$$

where

$$r_j = \{1,1,2,1,1\} \quad (j=1,\ldots,5)$$

or, using what we will see further ahead to be a more general notation,

$$r_j = \begin{cases} h_j & j=1,2 \\ h_j + r_{j-2} & j=3,\ldots,5 \end{cases} \qquad \text{with} \qquad h_j = \begin{cases} 1 & j=1,\ldots,3 \\ 0 & j=4 \\ -1 & j=5 \ . \end{cases} \qquad (2.7)$$

This form of the c.f. of W in (2.6) is most adequate for the development or construction of a near-exact distribution for W and then for Λ itself. In (2.6) $\Phi_{1,W}(t)$ is the c.f. of a Generalized Integer Gamma (GIG) distribution and $\Phi_{2,W}(t)$ the c.f. of a sum of independent Logbeta r.v.'s. The c.f. of W in (2.6) is thus the c.f. of the sum of a GIG distributed r.v. with and independent sum of independent Logbeta r.v.'s.

The Generalized Integer Gamma (GIG) distribution (Coelho, 1998, 1999) is the distribution of the sum of independent Gamma r.v.'s, all with integer shape parameters and different rate parameters (see for example Marques, Coelho and Arnold (2011) or Coelho and Marques (2011b) for a simple introduction to the expressions of the probability density and cumulative distribution functions of this distribution). The Logbeta distribution is the distribution of the negative logarithm of a Beta distributed r.v.. We say that a r.v. X has a Gamma distribution with shape parameter *r* and rate parameter λ if the p.d.f. (probability density function) of X is

$$f_X(x)=\frac{\lambda^r}{\Gamma(r)}\,e^{-\lambda x}\,x^{r-1}\quad (x>0;\,r,\lambda>0)\,.$$

We will denote the fact that the r.v. X has this distribution by writing $X\sim\Gamma(r,\lambda)$. Then the c.f. of X is

$$\Phi_X(t)=E\left(e^{itX}\right)=\lambda^r(\lambda-it)^{-r}\ ,$$

so that if $X_j\sim\Gamma(r_j,\lambda_j)$ are independent r.v.'s for $j=1,\dots,p$, and if we consider the r.v.

$$V=\sum_{j=1}^{p}X_j$$

then the c.f. of V is

$$\Phi_V(t)=E\left(e^{itV}\right)=E\left(e^{it\sum_{j=1}^{p}X_j}\right)=E\left(\prod_{j=1}^{p}e^{itX_j}\right)=\prod_{j=1}^{p}E\left(e^{itX_j}\right)$$
$$=\prod_{j=1}^{p}\Phi_{X_j}(t)=\prod_{j=1}^{p}\lambda_j^{r_j}(\lambda_j-it)^{-r_j}$$

so that $\Phi_{1,W}(t)$ in (2.6) is easily recognized as the c.f. of a GIG distribution with shape parameters r_j and rate parameters $a-2+\frac{j}{2}$ $(j=1,\dots,5)$. On the other hand, if the r.v.'s Y_j $(j=1,\dots,q)$ are a set of q independent r.v.'s with $Beta(a_j,b_j)$ distributions, then the r.v.'s $Z_j=-\log Y_j$ are a set of q independent $Logbeta(a_j,b_j)$ r.v.'s, with (see expression (2.2) and the general expression for the *h*-th moment of a Beta distributed r.v.)

$$\Phi_{Z_j}(t)=E\left(e^{itZ_j}\right)=E\left(e^{-it\log Y_j}\right)=E\left(Y_j^{-it}\right)=\frac{\Gamma(a_j+b_j)}{\Gamma(a_j)}\,\frac{\Gamma(a_j-it)}{\Gamma(a_j+b_j-it)}$$

And as such, if we take

$$S=\sum_{j=1}^{q}Z_j$$

we will have

$$\Phi_S(t)=E\left(e^{itS}\right)=E\left(e^{it\sum_{j=1}^{p}Z_j}\right)=E\left(\prod_{j=1}^{p}e^{itZ_j}\right)=\prod_{j=1}^{p}E\left(e^{itZ_j}\right)$$
$$=\prod_{j=1}^{p}\Phi_{Z_j}(t)=\prod_{j=1}^{p}\frac{\Gamma(a_j+b_j)}{\Gamma(a_j)}\,\frac{\Gamma(a_j-it)}{\Gamma(a_j+b_j-it)}$$

so that $\Phi_{2,W}(t)$ in (2.6) is easily recognized as the c.f. of the sum of three independent $Logbeta\left(a-\frac{j}{2}+2,\frac{1}{2}\right)$ $(j=1,\dots,3)$ r.v.'s.

Then, since the sum of independent Logbeta r.v.'s does not have a manageable expression, while, on the other hand, the GIG distribution is a much manageable distribution, that is, one that has a much manageable expression for its c.d.f. (cumulative distribution function) and as such, one for which it is easy to compute exact p-values and quantiles, we will leave $\Phi_{1,W}(t)$ in (2.6) unchanged, while we will approximate $\Phi_{2,W}(t)$ asymptotically. The approximation we will use for $\Phi_{2,W}(t)$ is based on the fact that (see Tricomi and Erdélyi (1951)) we may asymptotically approximate, for large values of a, the distribution of any $Logbeta(a,b)$ r.v. by an infinite mixture of $\Gamma(b+l,a)$ distributions $(l=0,1,\dots)$. As such, it would be possible to replace $\Phi_{2,W}(t)$ asymptotically by the sum of three infinite mixtures of $\Gamma\left(\frac{1}{2}+l,a-\frac{j}{2}+2\right)$ $(l=0,1,\dots)$, which itself is an infinite mixture of sums of three $\Gamma\left(\frac{1}{2}+l,a-\frac{j}{2}+2\right)$ $(j=1,\dots,3)$. However, it happens that each of these sums is itself an infinite mixture. Indeed although the sum of independent Gamma r.v.'s, all with the same rate parameter is another Gamma distributed r.v., still with that same rate parameter and a shape parameter which is the sum of the original shape parameters, the distribution of the sum of independent Gamma distributed r.v.'s, with different rate parameters is indeed an infinite mixture of Gamma distributions (see Moschopoulos (1985)). This way, in order to simplify things, yet keeping a good accuracy, we will replace $\Phi_{2,W}(t)$ by the c.f. of a finite mixture of $m+1$ Gamma distributions, all with the same rate parameter, which will be the average of the three rate parameters involved and shape parameters $r+l$ $(l=0,\dots,m)$, where r is the sum of the three shape parameters, all indeed equal to 1/2. More precisely, we will replace $\Phi_{2,W}(t)$ in (2.6) by

$$\Phi^*_{2,W}(t)=\sum_{l=0}^{m}\pi_l\,\lambda^{r+l}(\lambda-it)^{-(r+l)}$$

where

$$\lambda = \frac{1}{3}\sum_{j=1}^{3}\left(a - \tfrac{j}{2} + 2\right) = a + 1 \qquad \text{and} \qquad r = 3 \times \frac{1}{2} = \frac{3}{2} \,, \tag{2.8}$$

and where the weights π_l are determined in such a way that

$$\left.\frac{\partial^h}{\partial t^h}\Phi^*_{2,W}(t)\right|_{t=0} = \left.\frac{\partial^h}{\partial t^h}\Phi_{2,W}(t)\right|_{t=0} \qquad \text{for} \qquad h = 1,\dots,m \tag{2.9}$$

and $\pi_m = 1 - \sum_{l=0}^{m-1}\pi_l$. This way we have

$$\Phi^*_W(t) = \Phi_{1,W}(t)\Phi^*_{2,W}(t) \tag{2.10}$$

as a near-exact c.f. for W . We should note that this near-exact distribution matches, by construction, the first m exact moments of W .

We should also note that $\Phi^*_W(t)$ in (2.10) is the c.f. of a mixture of $m+1$ Generalized Near-Integer Gamma (GNIG) distributions of depth 6.

The GNIG distribution (Coelho 2004) is the distribution of the sum of a GIG distributed r.v. with an independent r.v. with a $\Gamma(r,\lambda)$ distribution, where r is not integer and λ is different from any of the rate parameters in the GIG distribution. The expressions for the p.d.f. and c.d.f. of the GNIG distribution are quite manageable, although they involve the Kummer confluent hypergeometric function, which is both highly convergent as well as handled by a number of high level languages easily available. To see the expressions for the p.d.f. and/or c.d.f. of the GNIG distribution, please refer to Coelho (2004) or Marques and Coelho (2011b).

Using the notation in Appendix A of the above last reference, we may write the p.d.f. and the c.d.f. of these near-exact distributions for W as

$$f^*_W(w) = \sum_{l=0}^{m}\pi_l \; f^{GNIG}\left(w \,|\, r_1,\dots,r_5, r+l; a-\tfrac{3}{2},\dots,a+\tfrac{1}{2},\lambda;6\right) \quad (w>0)$$

and

$$F^*_W(w) = \sum_{l=0}^{m}\pi_l \; F^{GNIG}\left(w \,|\, r_1,\dots,r_5, r+l; a-\tfrac{3}{2},\dots,a+\tfrac{1}{2},\lambda;6\right) \quad (w>0)$$

and the p.d.f. and c.d.f. of the corresponding near-exact distributions for Λ as

$$f_\Lambda^*(z)=\sum_{l=0}^{m}\pi_l\ f^{GNIG}\left(-\log z\,|\,r_1,\dots,r_5,r+l;a-\tfrac{3}{2},\dots,a+\tfrac{1}{2},\lambda;6\right)\frac{1}{z}\quad(0<z<1)$$

and

$$F_\Lambda^*(z)=1-\sum_{l=0}^{m}\pi_l\ F^{GNIG}\left(-\log z\,|\,r_1,\dots,r_5,r+l;a-\tfrac{3}{2},\dots,a+\tfrac{1}{2},\lambda;6\right)\quad(0<z<1)\,,$$

where w represents the running value of the r.v. W and z the running value of the r.v. Λ and where $r_1,\dots,r_5$ are given by (2.7) and r and λ by (2.8).

A pertinent question now is: how can we evaluate the closeness of these near-exact distributions to the exact distribution, moreover since we do not have a closed form expression for either the exact p.d.f. or c.d.f.?

The answer is to use the measure

$$\Delta=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\frac{\Phi_W(t)-\Phi_W^*(t)}{t}\right|dt \tag{2.11}$$

with

$$\max_{w\in S_W}\left|F_W(w)-F_W^*(w)\right|\le\Delta \qquad \text{and} \qquad \max_{z\in S_\Lambda}\left|F_\Lambda(z)-F_\Lambda^*(z)\right|\le\Delta$$

where $F_W(w)$ and $F_\Lambda(z)$ represent respectively the exact c.d.f. of W and Λ and $F_W^*(w)$ and $F_\Lambda^*(z)$ respectively the near-exact c.d.f. of W and Λ, with S_W and S_Λ representing respectively the supports of W and Λ. For more details on this measure, which may be seen as based on the Berry-Esseen bound (Berry, 1941; Esseen, 1945; Hsu, 1945; Hwang, 1998), see Coelho and Mexia (2010).

In Table 1 we may analyze the values of the measure Δ for the near-exact distributions whose c.f. is in (2.10), for different values of m, and for increasing values of a. We may see how the near-exact distributions exhibit a marked asymptotic behavior for increasing values of a, which is a parameter whose value is usually directly related with the sample size in the distribution of many likelihood ratio test statistics. We must keep in mind that (see the definition of the measure Δ in (2.11)) the lower the value of Δ, the better is the performance of the associated asymptotic distribution.

Table 1. – Values of Δ for the near-exact distribution with c.f. given by (2.10), for increasing values of a and different values of m

	Value of m (number of exact moments matched)		
a	4	6	10
2.6	7.26×10^{-8}	5.38×10^{-10}	5.46×10^{-14}
5.6	7.12×10^{-9}	1.77×10^{-11}	8.67×10^{-17}
10.6	5.03×10^{-10}	4.32×10^{-13}	5.89×10^{-19}
25.6	8.62×10^{-12}	1.46×10^{-15}	3.09×10^{-22}

Also, as expected, the more exact moments that the near-exact distribution matches, that is, the larger the value of m, the lower the value of the measure Δ, showing a closer near-exact distribution to the exact distribution. We may see as the near-exact distribution that matches 4 exact moments shows a difference between its c.d.f. and the exact c.d.f. of at most 7.26 hundredths of a millionth part.

Right now some questions may be building in our minds, as for example:

1 – Why did we use a r.v. Λ with the structure in (2.1) and why did we use for the r.v.'s Y_j the distributional structure we did use ?

2 – Can we still do better than what we have done so far ?

3 – Why did we not use the same approach we used to approximate $\Phi_{2,W}(t)$ in order to approximate the whole c.f. $\Phi_W(t)$?

4 – Why did we do not use the results available for the distribution of the product of independent Beta r.v.'s ?

The answers to these questions are given in each of the subsections below.

2.1 – *Why did we use a r.v. Λ with the structure in (2.1) and why did we use for the r.v.'s Y_j the distributional structure we did use ?*

We have done so because this way the r.v. Λ would have a distributional structure similar to the one that many likelihood ratio test statistics, namely many of the ones used in Multivariate Analysis, have.

2.2 – *Can we still do better than what we have done so far ?*

The answer to this question is indeed affirmative! Actually, it happens that we did not completely explore the fine structure of the exact distribution of the r.v. Λ in order to keep as much as possible of it unchanged when building the near-exact distribution.

Indeed, by using (2.5) we may write the c.f. $\Phi_{2,W}(t)$ in (2.6) as

$$\begin{aligned}\Phi_{2,W}(t) &= \frac{\Gamma(a+2)}{\Gamma(a+\frac{3}{2})}\frac{\Gamma(a+\frac{3}{2}-it)}{\Gamma(a+2-it)}\left\{\frac{\Gamma(a+\frac{3}{2})\Gamma(a+1)}{\Gamma(a+1)\Gamma(a+\frac{1}{2})}\frac{\Gamma(a+1-it)\Gamma(a+\frac{1}{2}-it)}{\Gamma(a+\frac{3}{2}-it)\Gamma(a+1-it)}\right\}\\
&= \frac{\Gamma(a+2)}{\Gamma(a+\frac{3}{2})}\frac{\Gamma(a+\frac{3}{2}-it)}{\Gamma(a+2-it)}\left\{\frac{\Gamma(a+\frac{3}{2})}{\Gamma(a+\frac{1}{2})}\frac{\Gamma(a+\frac{1}{2}-it)}{\Gamma(a+\frac{3}{2}-it)}\right\}\\
&= \underbrace{\frac{\Gamma(a+2)}{\Gamma(a+\frac{3}{2})}\frac{\Gamma(a+\frac{3}{2}-it)}{\Gamma(a+2-it)}}_{\Phi_{2a,W}(t)}\underbrace{(a+\tfrac{1}{2})(a+\tfrac{1}{2}-it)^{-1}}_{\Phi_{2b,W}(t)}\end{aligned}$$

where $\Phi_{2b,W}(t)$ is the c.f. of an Exponential distribution with rate parameter equal to $a+\frac{1}{2}$ and as such may be combined with $\Phi_{1,W}(t)$ in (2.6) in order to enable us to write $\Phi_W(t)$ as

$$\Phi_W(t)=\underbrace{\prod_{j=1}^{5}\left(a-2+\tfrac{j}{2}\right)^{r_j^+}\left(a-2+\tfrac{j}{2}-it\right)^{-r_j^+}}_{\Phi_{1,W}^*(t)}\underbrace{\frac{\Gamma(a+2)}{\Gamma(a+\frac{3}{2})}\frac{\Gamma(a+\frac{3}{2}-it)}{\Gamma(a+2-it)}}_{\Phi_{2a,W}(t)} \qquad (2.12)$$

where

$$r_j^+=\{1,1,2,1,2\}$$

or, using what we will see further ahead to be a more general notation,

$$r_j^+=\begin{cases}h_j^+ & j=1,2\\ h_j^+ + r_{j-2}^+ & j=3,\dots,5\end{cases} \qquad \text{with} \qquad h_j^+=\begin{cases}1 & j=1,\dots,3\\ 0 & j=4,5\ .\end{cases}$$

In (2.12) the c.f. $\Phi_{1,W}^*(t)$ is the c.f. of a GIG distribution of depth 5, with shape parameters r_j^+ and rate parameters $a-2+\frac{j}{2}$ $(j=1,\dots,5)$, while $\Phi_{2a,W}(t)$ is the c.f. of a $Logbeta\left(a+\frac{3}{2},\frac{1}{2}\right)$ distribution.

As such, in order to obtain a near-exact distribution we will leave $\Phi_{1,W}^*(t)$ unchanged, while we replace $\Phi_{2a,W}(t)$ asymptotically by the c.f. of a finite mixture of $\Gamma\left(\frac{1}{2}+l,a+\frac{3}{2}\right)$ distributions $(l=0,\dots,m)$, more precisely, by

$$\Phi_{2a,W}^*(t)=\sum_{l=0}^{m}\pi_l^*\left(a+\tfrac{3}{2}\right)^{\frac{1}{2}+l}\left(a+\tfrac{3}{2}-it\right)^{-\left(\frac{1}{2}+l\right)}$$

where the weights π_l^* $(l=0,\dots,m-1)$ are determined in such a way that

$$\left.\frac{\partial^h}{\partial t^h}\Phi^*_{2a,W}(t)\right|_{t=0} = \left.\frac{\partial^h}{\partial t^h}\Phi_{2a,W}(t)\right|_{t=0} \qquad \text{for} \qquad h=1,\dots,m$$

and $\pi^*_m = 1-\sum_{l=0}^{m-1}\pi^*_l$.

Now, in this case, we will then use

$$\Phi^{**}_W(t) = \Phi^*_{1,W}(t)\,\Phi^*_{2a,W}(t) \tag{2.13}$$

as near-exact c.f. for W .

As before, these near-exact distributions match the first m exact moments of W , and they are mixtures of $m+1$ GNIG distributions of depth 6. The near-exact p.d.f. and c.d.f. for Λ are now respectively given by

$$f^*_\Lambda(z) = \sum_{l=0}^{m}\pi^*_l\; f^{GNIG}\left(-\log z\,|\,r_1,\dots,r_5,\tfrac{1}{2}+l;a-\tfrac{3}{2},\dots,a+\tfrac{1}{2},a+\tfrac{3}{2};6\right)\frac{1}{z} \quad (0<z<1)$$

and

$$F^*_\Lambda(z) = 1-\sum_{l=0}^{m}\pi^*_l\; F^{GNIG}\left(-\log z\,|\,r_1,\dots,r_5,\tfrac{1}{2}+l;a-\tfrac{3}{2},\dots,a+\tfrac{1}{2},a+\tfrac{3}{2};6\right) \quad (0<z<1)\ .$$

Clearly, the question now is: what gains did we get with the implementation of this near-exact distribution?

Well, the values for the measure Δ in (2.11) for the same values of a that were used in Table 1 are now the ones in Table 2. We may see they are generally lower than the values in Table 1, what shows that indeed the near-exact distributions developed in this subsection give a better fit. That is, the small extra work we had in building this new version of near-exact distributions ended up paying off.

Table 2. – Values of Δ for the near-exact distribution with c.f. given by (2.13), for increasing values of a and different values of m

	Value of m (number of exact moments matched)		
a	4	6	10
2.6	3.56×10^{-10}	7.99×10^{-13}	2.32×10^{-15}
5.6	1.36×10^{-10}	3.61×10^{-13}	9.13×10^{-17}
10.6	1.73×10^{-11}	2.71×10^{-14}	8.09×10^{-20}

25.6	4.32×10^{-13}	1.68×10^{-16}	1.96×10^{-22}

2.3 – *Why did we not use the same approach we used to approximate $\Phi_{2,W}(t)$ in order to approximate the whole c.f. $\Phi_W(t)$?*

We may indeed use a similar technique to the one used to approximate $\Phi_{2,W}(t)$ in order to approximate the whole c.f. $\Phi_W(t)$, but there is a big drawback in doing this. The whole c.f. $\Phi_W(t)$ is the c.f. of the sum of three independent Logbeta r.v.'s with different second parameters. Thus, in trying to apply to the whole c.f. $\Phi_W(t)$ a similar technique of approximation to the one used to approximate $\Phi_{2,W}(t)$ this would lead us to use as an approximation a finite mixture of $m+1$ Gamma distributions with shape parameters $\frac{15}{2}+l$ $(l=0,\ldots,m)$ and a rate parameter equal to $a-1$. This would correspond to use as approximation for the whole $\Phi_W(t)$ a c.f.

$$\Phi_W^{***}(t)=\sum_{l=0}^{m}\pi_l^{***}\left(a-1\right)^{\frac{15}{2}+l}\left(a-1-it\right)^{-\left(\frac{15}{2}+l\right)} \tag{2.14}$$

where the weights π_l^{***} $(l=0,\ldots,m-1)$ will be determined in such a way that

$$\left.\frac{\partial^h}{\partial t^h}\Phi_W^{***}(t)\right|_{t=0}=\left.\frac{\partial^h}{\partial t^h}\Phi_W(t)\right|_{t=0} \qquad \text{for} \qquad h=1,\ldots,m$$

with $\pi_m^{***}=1-\sum_{l=0}^{m-1}\pi_l^{***}$.

But, this approximation would correspond to a common asymptotic approximation (actually with a further improvement which is not common to the usual asymptotic approximations which is the fact that this asymptotic approximation matches, by construction, the first m exact moments of W) since there is no part of the exact c.f. of W which is left unchanged. This is the big drawback in doing things this way. This approximation, although working quite well, as it may be seen from the values of the measure Δ in Table 3, has a much worse performance than any of the two near-exact distributions developed.

Table 3. – Values of Δ for the asymptotic distribution with c.f. in (2.14), for increasing values of a and different values of m

	Value of m (number of exact moments matched)		
a	4	6	10
2.6	7.36×10^{-3}	1.46×10^{-3}	6.39×10^{-5}
5.6	2.36×10^{-4}	1.21×10^{-5}	3.77×10^{-8}
10.6	1.13×10^{-5}	1.78×10^{-7}	5.37×10^{-11}
25.6	1.54×10^{-7}	4.43×10^{-10}	4.60×10^{-15}

2.4 – *Why did we do not use the results available for the distribution of the product of independent Beta r.v.'s ?*

In simple terms the answer is: because if we had done so the results would be even worse than the ones in Table 3 above, and as such much worse than the ones obtained using the near-exact approach.

There are indeed many results available concerning the distribution of the product of independent Beta r.v.'s as for example the ones in Tukey and Wilks (1946), Springer and Thompson (1966, 1970), Tretter and Walster (1975), Carter and Springer (1977), Springer (1979), Walster and Tretter (1980), Bhargava and Khatri (1981) and Pederzoli (1985) and also the ones in Nagarsenker and Das (1975), Nandi (1980), Nagarsenker and Suniaga (1983), Tang and Gupta (1984, 1986), Mathai (1984) and Nagar, Jain and Gupta (1985). However, any of these results would yield a less good approximation than the one in subsection 2.3 above. Actually, only those in the second group of references would yield distributions close to the ones in section 2.3 above. Anyway, since such representations of the distribution of the product of independent Beta r.v.'s are not designed to match any of the exact moments of the distribution, for the same number of terms used in subsection 2.3, the results obtained from those distributions would always be worse than the ones that subsection.

3. A more elaborate example

The example we are going to consider in this section, although based on the one addressed in the previous section, has a much wider scope. Anyway it will take much less page space than the previous simple example since all the notation and base methodology used to build a near-exact distribution were settled in the previous section.

Let us consider the r.v.

$$\Lambda=\prod_{j=1}^{p} Y_j \qquad \text{with} \qquad Y_j \sim Beta\left(a-\tfrac{j}{2},\tfrac{b}{2}\right) \tag{3.1}$$

where p and b are two odd integers (in case any of p or b is even, we actually do not need to resort to the use of near-exact distributions since in that case it may be shown that the exact distribution of Λ is indeed a GIG distribution – see Coelho (1998, 1999) for a proof of this result).

Let $W=-\log\Lambda$. Then, using an argument in all similar to the one used in Section 2, the c.f. of W is

$$\Phi_W(t)=\prod_{j=1}^{p}\frac{\Gamma\left(a-\frac{j}{2}+\frac{b}{2}\right)}{\Gamma\left(a-\frac{j}{2}\right)}\,\frac{\Gamma\left(a-\frac{j}{2}-it\right)}{\Gamma\left(a-\frac{j}{2}+\frac{b}{2}-it\right)}$$

which, on using once again techniques in all similar to the ones used in the first part of Section 2 and skipping all the algebraic details, may be written as

$$\Phi_W(t)=\underbrace{\left\{\prod_{j=1}^{p+b-3}\left(a-\tfrac{p+1}{2}+\tfrac{j}{2}\right)^{r_j}\left(a-\tfrac{p+1}{2}+\tfrac{j}{2}-it\right)^{-r_j}\right\}}_{\Phi_{1,W}(t)}\underbrace{\left\{\prod_{j=1}^{p}\frac{\Gamma\left(a-\frac{j}{2}+\frac{b}{2}\right)}{\Gamma\left(a-\frac{j}{2}+\frac{b-1}{2}\right)}\,\frac{\Gamma\left(a-\frac{j}{2}+\frac{b-1}{2}-it\right)}{\Gamma\left(a-\frac{j}{2}+\frac{b}{2}-it\right)}\right\}}_{\Phi_{2,W}(t)}, \tag{3.2}$$

where

$$r_j=\begin{cases}h_j & j=1,2\\ h_j+r_{j-2} & j=3,\dots,p+b-3\,,\end{cases} \qquad \text{with} \qquad h_j=\begin{cases}1 & j=1,\dots,\min(p,b-1)\\ 0 & j=1+\min(p,b-1),\dots,\max(p,b-1)\\ -1 & j=1+\max(p,b-1),\dots,p+b-3\,.\end{cases}$$

Then, since $\Phi_{2,W}(t)$ is the c.f. of a sum of p independent $Logbeta\left(a-\frac{j}{2}+\frac{b-1}{2},\frac{1}{2}\right)$ r.v.'s, we will replace this c.f., asymptotically, by

$$\Phi_{2,W}^{*}(t)=\sum_{l=0}^{m}\pi_l\,\lambda^{r+l}(\lambda-it)^{-(r+l)}\,,$$

where

$$r = p \times \tfrac{1}{2} \qquad \text{and} \qquad \lambda = \tfrac{1}{p}\sum_{j=1}^{p}\left(a - \tfrac{j}{2} + \tfrac{b-1}{2}\right) = a + \tfrac{b}{2} - \tfrac{p+3}{4}, \tag{3.3}$$

and where the weights π_l $(l = 0,\ldots,m-1)$ are determined in such a way that a relation similar to the one in (2.9) is verified.

For r and λ given by (3.3), the resulting near-exact distribution for W is a mixture of $m+1$ GNIG distributions of depth $p+b-2$, with p.d.f.

$$f_W^*(w) = \sum_{l=0}^{m} \pi_l \; f^{GNIG}\left(w \,|\, r_1,\ldots,r_{p+b-3}, r+l; a-\tfrac{p}{2},\ldots,a-2+\tfrac{b}{2},\lambda; p+b-2\right) \quad (w>0)$$

and c.d.f.

$$F_W^*(w) = \sum_{l=0}^{m} \pi_l \; F^{GNIG}\left(w \,|\, r_1,\ldots,r_{p+b-3}, r+l; a-\tfrac{p}{2},\ldots,a-2+\tfrac{b}{2},\lambda; p+b-2\right) \quad (w>0).$$

Such near-exact distribution will be asymptotic not only for increasing values of a but also for increasing values of b, as it may be checked by analyzing the values of the measure Δ in Table 4. However, as it may be seen from the values of Δ in that same table, this near-exact distribution is not asymptotic for increasing values of p, which is a much annoying drawback.

Table 4. – Values of Δ for the near-exact distribution with c.f. given by $\Phi_{1,W}(t)\Phi^*_{2,W}(t)$ in this section, for increasing values of a and different values of m

			Value of m (number of exact moments matched)		
a	b	p	4	6	10
2.6	5	5	2.57×10^{-7}	7.23×10^{-9}	1.93×10^{-11}
5.6	"	"	1.33×10^{-7}	1.13×10^{-9}	1.71×10^{-13}
10.6	"	"	7.48×10^{-9}	1.86×10^{-11}	2.24×10^{-16}
25.6	"	"	1.04×10^{-10}	4.52×10^{-14}	1.59×10^{-20}
25.6	5	5	1.04×10^{-10}	4.52×10^{-14}	1.59×10^{-20}
"	7	"	3.63×10^{-11}	1.10×10^{-14}	2.03×10^{-21}
"	9	"	1.53×10^{-11}	3.41×10^{-15}	3.57×10^{-22}
"	15	"	2.08×10^{-12}	2.20×10^{-16}	5.66×10^{-24}
25.6	15	5	2.08×10^{-12}	2.20×10^{-16}	5.66×10^{-24}
"	"	7	8.67×10^{-12}	1.62×10^{-15}	1.30×10^{-22}
"	"	9	2.64×10^{-11}	7.41×10^{-15}	1.35×10^{-21}
"	"	15	3.40×10^{-10}	2.28×10^{-13}	2.25×10^{-19}
"	"	25	8.29×10^{-9}	1.58×10^{-11}	1.15×10^{-16}

The question now is of course: 'Is there something we can do in order to obtain a near-exact distribution which is also asymptotic for increasing values of p ?

The answer is in fact affirmative. All we have to do is indeed to follow a procedure in all similar to the one used in subsection 2.2. As a matter of fact, the importance of such a procedure is not only to obtain a better version of a near-exact distribution but rather to obtain a true near-exact distribution, asymptotic for every parameter in the exact distribution of the statistic it refers to.

Using that procedure we will be able to write the c.f. of W as

$$\Phi_W(t) = \underbrace{\left\{ \prod_{j=1}^{p+b-3} \left(a - \tfrac{p+1}{2} + \tfrac{j}{2}\right)^{r_j^+} \left(a - \tfrac{p+1}{2} + \tfrac{j}{2} - it\right)^{-r_j^+} \right\}}_{\Phi_{1,W}(t)} \underbrace{\frac{\Gamma\left(a + \frac{b-1}{2}\right)}{\Gamma\left(a + \frac{b-2}{2}\right)} \frac{\Gamma\left(a + \frac{b-2}{2} - it\right)}{\Gamma\left(a + \frac{b-1}{2} - it\right)}}_{\Phi_{2a,W}(t)}, \qquad (3.4)$$

where

$$r_j^+ = \begin{cases} h_j^+ & j = 1, 2 \\ h_j^+ + r_{j-2}^+ & j = 3, \ldots, p+b-3, \end{cases} \qquad \text{with} \qquad h_j^+ = \begin{cases} 1 & j = 1, \ldots, \min(p,b) \\ 0 & j = 1 + \min(p,b), \ldots, \max(p,b) \\ -1 & j = 1 + \max(p,b), \ldots, p+b-3, \end{cases}$$

which actually seems to be a more natural way to compute the shape parameters in $\Phi_{1,W}(t)$.

Then, in order to build a near-exact distribution we would approximate $\Phi_{2,W}(t)$ in (3.4) by a mixture of $\Gamma\left(\frac{1}{2} + l, a + \frac{b-1}{2}\right)$ distributions $(l = 0, \ldots, m)$, whose weights π_l^* $(l = 0, \ldots, m-1)$ will be determined in such a way that this mixture of Gamma distributions and $\Phi_{2,W}(t)$ in (3.4) have the same first m derivatives at $t = 0$.

This near-exact distribution yields, for W, a distribution with p.d.f.

$$f_W^*(w) = \sum_{l=0}^{m} \pi_l^* f^{GNIG}\left(w \mid r_1^+, \ldots, r_{p+b-3}^+, \tfrac{1}{2} + l; a - \tfrac{p}{2}, \ldots, a - 2 + \tfrac{b}{2}, a + \tfrac{b-1}{2}; p+b-2\right) \quad (w > 0)$$

and c.d.f.

$$F_W^*(w) = \sum_{l=0}^{m} \pi_l^* F^{GNIG}\left(w \mid r_1^+, \ldots, r_{p+b-3}^+, \tfrac{1}{2} + l; a - \tfrac{p}{2}, \ldots, a - 2 + \tfrac{b}{2}, a + \tfrac{b-1}{2}; p+b-2\right) \quad (w > 0)$$

and, for the same values of a, b and p used in Table 4 the values of Δ which are in Table 5. Not only these values are smaller than the ones in Table 4 for all cases considered, but also they show that this new near-exact distribution is indeed asymptotic also for increasing values of p, which is indeed a much desirable feature.

Table 5. – Values of Δ for the near-exact distribution with c.f. given by $\Phi_{1,W}(t)\Phi^*_{2,W}(t)$ in this section, for increasing values of a and different values of m

			Value of m (number of exact moments matched)		
a	b	p	4	6	10
2.6	5	5	2.40×10^{-12}	2.09×10^{-15}	9.99×10^{-19}
5.6	"	"	2.23×10^{-11}	3.25×10^{-14}	3.46×10^{-18}
10.6	"	"	4.15×10^{-12}	4.14×10^{-15}	5.14×10^{-21}
25.6	"	"	1.25×10^{-13}	3.34×10^{-17}	2.07×10^{-23}
25.6	5	5	1.25×10^{-13}	3.34×10^{-17}	2.07×10^{-23}
"	7	"	4.40×10^{-14}	8.19×10^{-18}	2.69×10^{-24}
"	9	"	1.88×10^{-14}	2.57×10^{-18}	4.87×10^{-25}
"	15	"	2.65×10^{-15}	1.76×10^{-19}	8.62×10^{-27}
25.6	15	5	2.65×10^{-15}	1.76×10^{-19}	8.62×10^{-27}
"	"	7	1.05×10^{-15}	4.84×10^{-20}	1.20×10^{-27}
"	"	9	5.05×10^{-16}	1.77×10^{-20}	2.52×10^{-28}
"	"	15	1.01×10^{-16}	1.87×10^{-21}	7.73×10^{-30}
"	"	25	1.41×10^{-17}	1.20×10^{-22}	1.05×10^{-31}

3.1 – *On the interchangeability of p and b and the truly near-exact distribution*

We should note that in (3.1) p and b are indeed interchangeable, that is, the distribution of Λ in (3.1) is also the distribution of

$$\prod_{j=1}^{b} Y_j^* \qquad \text{with} \qquad Y_j^* \sim Beta\left(a^* - \tfrac{j}{2}, \tfrac{p}{2}\right)$$

where $a^* = a + \frac{b}{2} - \frac{p}{2}$, since indeed

$$\prod_{j=1}^{p} \frac{\Gamma\left(a-\frac{j}{2}+\frac{b}{2}\right)}{\Gamma\left(a-\frac{j}{2}\right)} \frac{\Gamma\left(a-\frac{j}{2}-it\right)}{\Gamma\left(a-\frac{j}{2}+\frac{b}{2}-it\right)} = \prod_{j=1}^{b} \frac{\Gamma\left(a^*-\frac{j}{2}+\frac{p}{2}\right)}{\Gamma\left(a^*-\frac{j}{2}\right)} \frac{\Gamma\left(a^*-\frac{j}{2}-it\right)}{\Gamma\left(a^*-\frac{j}{2}+\frac{p}{2}-it\right)}$$
$$= \prod_{j=1}^{b} \frac{\Gamma\left(a-\frac{j}{2}+\frac{b}{2}\right)}{\Gamma\left(a+\frac{b}{2}-\frac{p}{2}-\frac{j}{2}\right)} \frac{\Gamma\left(a+\frac{b}{2}-\frac{p}{2}-\frac{j}{2}-it\right)}{\Gamma\left(a-\frac{j}{2}+\frac{b}{2}-it\right)}$$

although this relation may be not completely evident at first sight. But, if we look well at (3.4), the interchangeability of p and b is evident in $\Phi_{1,W}(t)$, while from the relation between a and a^* we may easily see that we may write $\Phi_{2,W}(t)$ as

$$\frac{\Gamma\left(a^*+\frac{p-1}{2}\right)}{\Gamma\left(a^*+\frac{p-2}{2}\right)}\frac{\Gamma\left(a^*+\frac{p-2}{2}-it\right)}{\Gamma\left(a^*+\frac{p-1}{2}-it\right)}$$

since from the relation between a and a^* we may also write $a^*+\frac{p}{2}=a+\frac{b}{2}$. But this interchangeability of p and b is not clear in (3.2), where the roles of p and b in the definition of the r_j are not clearly interchangeable.

Indeed, the near-exact distribution expressed by the decomposition of $\Phi_W(t)$ in (3.4) is the most proper one.

4. As a conclusion

In this paper the author tried to show the need for the near-exact distributions as providers of much better approximations then the usual asymptotic distributions, at the same time that he tries to introduce these distributions in a simple way. Then he gave a couple of examples to illustrate how these distributions may be built, all the steps involved and some details to which one has to pay attention in case a really good and well-developed near-exact approximation is sought.

The author hopes to have been able to have sparked the attention towards this method of approximation for the exact distribution of statistics with complex distributions and also to have been able to have shed the seeds of desire in some of the readers to explore and to learn more about the amazing world of probability distributions, namely the one of the near-exact distributions.

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RESUMO

As distribuições quase-exactas são distribuições assimptóticas construídas sobre uma abordagem diferente no que diz respeito ao princípio e à técnica da aproximação da distribuição de estatísticas cuja distribuição exacta tem uma estrutura e expressão complexas.

No presente artigo o autor apresenta, em termos simples e através de dois exemplos, as distribuições quase-exactas como alternativa vantajosa às usuais distribuições assimptóticas. Em primeiro lugar são apresentadas as características e forma de construção destas distribuições e daí intuídas as suas vantagens em relação às usuais distribuições assimptóticas. Depois, através de dois exemplos, um primeiro muito simples e um segundo mais elaborado, tenta-se ilustrar como se constroem na prática estas distribuições e mostrar o seu excelente desempenho.

ABSTRACT

Near-exact distributions are asymptotic distributions built using a different concept and technique in what concerns the approximation of the distribution of given statistics whose exact distribution has a complex structure and expression.

In the present paper the author introduces, in simple terms and through two examples, the near-exact distributions as an alternative to common asymptotic distributions. First are introduced the characteristics and the general building process of these distributions. Then, through a couple of examples, a very simple first one and a more elaborate second one, we try to illustrate how in practice these distributions may be developed and to show their very good performance.