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Certain fundamental properties of generalized natural transform in generalized spaces

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Abstract

This paper considers the definition and the properties of the generalized natural transform on sets of generalized functions. Convolution products, convolution theorems, and spaces of Boehmians are described in a form of auxiliary results. The constructed spaces of Boehmians are achieved and fulfilled by pursuing a deep analysis on a set of delta sequences and axioms which have mitigated the construction of the generalized spaces. Such results are exploited in emphasizing the virtual definition of the generalized natural transform on the addressed sets of Boehmians. The constructed spaces, inspired from their general nature, generalize the space of integrable functions of Srivastava et al. (Acta Math. Sci. 35B:1386–1400, 2015) and, subsequently, the extended operator with its good qualitative behavior generalizes the classical natural transform. Various continuous embeddings of potential interests are introduced and discussed between the space of integrable functions and the space of integrable Boehmians. On another aspect as well, several characteristics of the extended operator and its inversion formula are discussed.

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Keywords: Natural transform; Generalized natural transform; Boehmian; UltraBoehmian; Sumudu transform; Laplace transform

1 Introduction and preliminaries

The integral transform operators have attained their popularity due to their wide range of applications in various fields of science and engineering as, in most of cases, the physical phenomenon is converted into ordinary and partial differential equations. Along with interesting groups of integral transforms arising in literature, the natural transform NT was introduced by Khan and Khan [2] and renamed recently as the N-transform [3–6]. In addition to the shift and change of scale properties of the NT, the authors of [5] solved the unsteady fluid flow problem over a plane wall and highlighted that the transform converges to the Laplace and Sumudu transforms. Later, Belgacem et al. [3] defined the inverse natural transform formula and studied some properties and applications on Maxwell's equation. When the real sectionwise continuous function $\varphi(t) > 0$, $\varphi(t) = 0$ for $t < 0$ is of exponential order defined on A, where

$$A = \left\{\varphi(t) | \exists M, \epsilon_1, \epsilon_2 > 0, \left|\varphi(t)\right| < Me^{\frac{t}{\epsilon_j}}, \text{if } t \in (-1)^j \times [0, \infty)\right\},$$

the natural transform NT is given by [2]

$$N(\varphi)(u, v) = \int_0^\infty e^{-ut} \varphi(vt)\, dt, \quad u > 0, v > 0.$$

For $\alpha \in \mathbb{C}$; $\operatorname{Re}(\alpha) \geq 0$ and $k \in \mathbb{Z}_+$, the generalized natural transform GNT or the $M_{\alpha,k}$ transform of a function φ was proclaimed as [1, (1.1)]

$$M_{\alpha,k}(\varphi)(u, v) = \int_0^\infty \varphi(vt) \frac{e^{-ut}}{(t^k + v^k)^\alpha}\, dt,$$

provided the integral part exists. The integral part of the preceding equation can be indeed motivated to yield

$$M_{\alpha,k}(\varphi)(u, v) = v^{-k\alpha-1} \int_0^\infty \varphi(t) \frac{e^{-\frac{u}{v}t}}{(t^k + v^k)^\alpha}\, dt, \tag{1}$$

where u and v are the transform variables. The GNT transform corresponds to the NT for $\alpha = 0$ [8] and to the Stieltjes transform for $u = 0$ [9]. On top of that, it corresponds to the Laplace transform [10]

$$L(\varphi)(v) = \int_0^\infty e^{-vt} \varphi(t)\, dt$$

for $\alpha = 0$ and $u = 1$ and to the Sumudu transform [7]

$$S(\varphi)(u) = \int_0^\infty e^{-t} \varphi(ut)\, dt, \quad u \in (-\epsilon_1, \epsilon_2)$$

for $\alpha = 0$ and $v = 1$, see, e.g., [7, 11–17]. The Parseval type theorem of the GNT transform is given by [1, Theorem 3.1]

$$\int_0^\infty \frac{\varphi(vu)}{(u^k + v^k)^{\alpha_1}} M_{\alpha_2,k}(g)(t, v)\, dt = \int_0^\infty \frac{g(vt)}{(t^k + v^k)^{\alpha_2}} M_{\alpha_1,k}(\varphi)(t, v)\, dt. \tag{2}$$

The scaling property of the GNT transform for $\beta > 0$ is given by [1, (2.4)]

$$M_{\alpha,k}\big(\varphi\big(\beta^2 t\big)\big)(u, v) = \beta^{k\alpha-1} M_{\alpha,k}\big(\varphi(t)\big)\left(\frac{u}{\beta}, \beta v\right).$$

Due to [1, Theorem 2.1], the GNT transform of φ exists for all $0 < v < \mu$ and $\operatorname{Re}(u) > \frac{\mu}{\beta}$, where the function φ is either continuous or piecewise continuous on $[0, \infty)$ and for certain given $K, T, \beta > 0$,

$$\left|\varphi(t)\right| \leq K t^{kRe(\alpha)} e^{t/\beta} \quad \text{for all } t > T. \tag{3}$$

Further, it converges uniformly with respect to the transform variable u provided $\mathrm{Re}(u) \geq \alpha > \frac{\mu}{\beta}$. The inversion of the GNT transform is defined by

$$\varphi(t) = \left(\frac{t^m}{v^m} + v^m\right)^{\alpha} L^{-1}\left(M_{\alpha,k}\left(\varphi(t)\right)(u,v)\right)\left(\frac{t}{v}\right), \quad v \in (0,\mu)$$

provided that the involved integral converges absolutely, where L^{-1} is the inverse Laplace transform operator. The Mellin-type convolution product of two integrable functions φ and g is defined by [18]

$$(\varphi \ominus g)(y) = \int_0^{\infty} \varphi(x) g\left(yx^{-1}\right)\frac{dx}{x} \tag{4}$$

when the integral part exists. Consequently, the natural properties of this convolution product are due to [19] given by

$$\begin{aligned} &\varphi \ominus g = g \ominus \varphi, \qquad (\varphi \ominus g_1) \ominus g_2 = \varphi \ominus (g_1 \ominus g_2) \quad \text{and} \\ &\varphi \ominus (g_1 + g_2) = \varphi \ominus g_1 + \varphi \ominus g_2. \end{aligned} \tag{5}$$

Evaluations of the GNT transform of various special functions, polynomials, and derivatives are computed in the above citations. In this article we derive a convolution theorem and establish sets of generalized functions for the considered GNT transform. In Sect. 1, we have already reviewed certain definitions and preliminaries from literature. In Sect. 2, we derive the convolution theorem and provide auxiliary results to facilitate our next investigations. In Sect. 3, we prove axioms and determine spaces of Boehmians and give the extension of the generalized natural transform to the Boehmian spaces. In Sect. 4, we derive some characteristics of the transform in a sense of generalized functions.

2 Convolution theorem and auxiliary results

To proceed in this study, we denote by $L^1(\mathbb{R}_+^2)$ the Lebesgue space of integrable functions over $\mathbb{R}_+^2 = \mathbb{R}_+ \times \mathbb{R}_+$, $\mathbb{R}_+ = (0,\infty)$ and by $C^{\infty}(\mathbb{R}_+)$ the Schwartz space of smooth functions of compact supports over $\mathbb{R}_+$. On the basis of the convolution product $\ominus$, we present a convolution formula that is very useful in the sequel.

Definition 1 Let $k \in \mathbb{Z}_+$ and $\alpha \in \mathbb{C}$ such that $\mathrm{Re}(\alpha) \geq 0$. We denote by $\oplus$ the integral equation

$$(\psi \oplus \varphi)(u,v) = \int_0^{\infty} \varphi(x)\psi(\sqrt{x}u, \sqrt{x}v)\frac{d_x}{x^{k\alpha}} \tag{6}$$

provided the right-hand side of the above equation exists for every $u > 0$ and $v > 0$.

By taking into account Eq. (4) and Eq. (6), we derive a convolution theorem as follows.

Theorem 2 *Let* $\varphi \in L^1(\mathbb{R}_+)$, $k \in \mathbb{Z}_+$, *and* $\alpha \in \mathbb{C}$ *such that* $\mathrm{Re}(\alpha) \geq 0$. *Then we have*

$$M_{\alpha,k}(\varphi \ominus g) = \varphi \oplus M_{\alpha,k} g$$

for every $g \in C^{\infty}(\mathbb{R}_+)$.

Proof Let $\varphi \in L^1(\mathbb{R}_+)$, $k \in \mathbb{Z}_+$, and $\alpha \in \mathbb{C}$ such that $\operatorname{Re}(\alpha) \geq 0$. Then invoking Eq. (4), Eq. (1) gives

$$M_{\alpha,k}(\varphi \ominus g)(u,v) = v^{-k\alpha-1} \int_0^\infty \frac{e^{-\frac{u}{v}y}}{(y^k + v^k)^\alpha} \int_0^\infty \varphi(x) g\big(yx^{-1}\big) \frac{dx}{x}\, dy.$$

Therefore, Fubini's theorem leads to

$$M_{\alpha,k}(\varphi \ominus g)(u,v) = v^{-k\alpha-1} \int_0^\infty \varphi(x) \int_0^\infty \frac{e^{-\frac{u}{v}y}}{(y^k + v^k)^\alpha} g\big(yx^{-1}\big)\, dy \frac{dx}{x}. \tag{7}$$

By using the change of variables $y = zx$, along with simple computations, we get

$$\begin{aligned}
M_{\alpha,k}(\varphi \ominus g)(u,v) &= v^{-k\alpha-1} \int_0^\infty \varphi(x) \int_0^\infty \frac{e^{-\frac{u}{v}xz}}{((zx)^k + v^k)^\alpha} g(z)\, dz\, dx \\
\text{i.e.} \quad &= v^{-k\alpha-1} \int_0^\infty \varphi(x) \int_0^\infty \frac{e^{-\frac{u}{v}xz}}{x^{k\alpha}(z^k + (\frac{v}{\sqrt{x}})^k)^\alpha} g(z)\, dz\, dx \\
\text{i.e.} \quad &= v^{-k\alpha-1} \int_0^\infty \varphi(x) \int_0^\infty \frac{e^{-\frac{u\sqrt{x}}{\frac{v}{\sqrt{x}}}z}}{(z^k + (\frac{v}{\sqrt{x}})^k)^\alpha} g(z)\, dz \frac{dx}{x^{k\alpha}} \\
\text{i.e.} \quad &= \int_0^\infty \varphi(x) (M_{\alpha,k} g)(\sqrt{x}u, \sqrt{x}v) \frac{dx}{x^{k\alpha}}.
\end{aligned}$$

This completes the proof of the theorem. □

By Δ we denote the subset of the Schwartz space $C^\infty(\mathbb{R}_+)$ of delta sequences $\{\delta_0, \delta_1, \ldots, \delta_n, \ldots\}$ such that Eq. (8) to Eq. (10) hold

$$\int_0^\infty \delta_n(x)\, dx = 1, \quad \forall n \in \mathbb{N}, \tag{8}$$

$$\int_0^\infty \big\| \delta_n(x) \big\|_{L^1(\mathbb{R}_+^2)}\, dx \leq c, \quad \text{for some constant } c \text{ and all } n \in \mathbb{N}, \tag{9}$$

$$\lim_{n\to\infty} \int_{|x|>\varepsilon} |x|^k \big\| \delta_n(x) \big\|_{L^1(\mathbb{R}_+^2)}\, dx = 0, \quad \forall k \in \mathbb{N}, \varepsilon > 0. \tag{10}$$

Theorem 3 *Let $U \in L^1(\mathbb{R}_+^2)$ and $\phi_1, \phi_2 \in C^\infty(\mathbb{R}_+)$. Then we have*

$$U \oplus (\phi_1 \ominus \phi_2) = (U \oplus \phi_1) \oplus \phi_2 \quad \text{in } L^1\big(\mathbb{R}_+^2\big).$$

Proof Let $U \in L^1(\mathbb{R}_+^2)$ and ϕ_1, $\phi_2 \in C^\infty(\mathbb{R}_+)$ be given arbitrary. Then, by applying Eq. (6) and Eq. (4), we assert

$$\begin{aligned}
\big(U \oplus (\phi_1 \ominus \phi_2)\big)(u,v) &= \int_0^\infty U(\sqrt{x}u, \sqrt{x}v)(\phi_1 \ominus \phi_2)(x) \frac{dx}{x^{k\alpha}} \\
&= \int_0^\infty \phi_2(t) \int_0^\infty U(\sqrt{x}u, \sqrt{x}v) \phi_1\big(xt^{-1}\big) \frac{dx}{x^{k\alpha}} \frac{dt}{t}
\end{aligned}$$

$$= \int_0^\infty \phi_2(t) \int_0^\infty U(\sqrt{tz}u, \sqrt{tz}v)\phi_1(z) \frac{dz}{z^{k\alpha}} \frac{dt}{t^{k\alpha}}$$
$$= \int_0^\infty \phi_2(t)(U \oplus \phi_1)(\sqrt{t}u, \sqrt{t}v) \frac{dt}{t^{k\alpha}}.$$

Hence, we have obtained

$$\bigl(U \oplus (\phi_1 \ominus \phi_2)\bigr)(u,v) = \bigl((U \oplus \phi_1) \oplus \phi_2\bigr)(u,v).$$

This proves the first part. To complete the inclusion part of the theorem, we show that

$$U \oplus \phi \in L^1(\mathbb{R}_+^2) \quad \text{for } U \in L^1(\mathbb{R}_+^2) \text{ and } \phi \in C^\infty(\mathbb{R}_+). \tag{11}$$

Indeed, from Eq. (6) and Fubini's theorem, we obtain

$$\|U \oplus \phi\|_{L^1(\mathbb{R}_+^2)} = \int_{\mathbb{R}_+^2} \bigl|(U \oplus \phi)(u,v)\bigr|\, d(u,v)$$
$$\leq \int_{\mathbb{R}_+^2} \int_{\mathbb{R}_+} \bigl|U(\sqrt{x}u, \sqrt{x}v)\bigr|\bigl|\phi(x)\bigr|\, dx\, d(u,v).$$

The definition of the norm $\|\cdot\|_{L^1(\mathbb{R}_+^2)}$ of $L^1(\mathbb{R}_+^2)$ indeed implies that

$$\|U \oplus \phi\|_{L^1(\mathbb{R}_+^2)} \leq \|U\|_{L^1(\mathbb{R}_+^2)} \int_{\mathbb{R}_+} \bigl|\phi(x)\bigr|\, dx. \tag{12}$$

Therefore, the right-hand side of Eq. (12) is bounded by the compactness of the support of ϕ. Hence, our theorem is completely proved. □

3 The spaces B_1 and B_2

To proceed in the construction of the abstract Boehmian space, we demand two sets, say G and S, and two operations, say $\star$ and $*$, where G is a topological vector space, S is a subspace of G and, for $\alpha, \beta \in G$ and $x, y \in S$, the operation $\star : G \times S \to G$ and $*$ satisfy the axioms: $x * y = y * x \in S$, $(\alpha \star x) \star y = \alpha \star (x * y)$, $(\alpha + \beta) \star x = \alpha \star x + \beta \star x$; and as $\alpha_n \to \alpha$ in G, we have $\alpha_n \star y \to \alpha \star y$ for sufficiently large values of n. Besides, there should be a collection Δ of sequences in S such that:

(i) If $\{y_1, y_2, \ldots, y_n, \ldots\}, \{x_1, x_2, \ldots, x_n, \ldots\} \in \Delta$, then $\{x_1 * y_1, x_2 * y_2, \ldots, x_n * y_n, \ldots\} \in \Delta$.

(ii) If $\alpha \in G$ and $\{y_1, y_2, \ldots, y_n, \ldots\} \in \Delta$, then $\alpha \star y_n \to \alpha$ in G as $n \to \infty$.

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\} \in G$ and $\{y_1, y_2, \ldots, y_n, \ldots\} \in \Delta$, then by A we denote the collection of all pairs of sequences (α_n, y_n) such that $\alpha_n \star y_m = \alpha_m \star y_n$, $m, n \in \mathbb{N}$. Each element of A is said to be a quotient of sequences and is denoted by $\frac{\alpha_n}{y_n}$. We define a relation $\sim$ on A by $\alpha_n / y_n \sim \beta_n / x_n$ if

$$\alpha_n \star x_m = \beta_m \star y_n, \quad \forall m, n \in \mathbb{N}.$$

The relation $\sim$ is an equivalence relation on A and decomposes it into disjoint equivalence classes. Each equivalence class is said to be a Boehmian. Every Boehmian is denoted by $X_{\alpha_n} = (\frac{\alpha_n}{y_n})$. The collection of all Boehmians is, for more convenience, denoted by $B(G, S, \star, \Delta)$ or B. Every element α of G is identified uniquely as a member of B by $(\frac{\alpha\star y_n}{y_n})$ where $(y_n) \in \Delta$. The space B is a vector space with

$$\left(\frac{\alpha_n}{y_n}\right) + \left(\frac{\beta_n}{x_n}\right) = \left(\frac{\alpha_n \star x_n + \beta_n \star x_n}{y_n * x_n}\right) \quad \text{and} \quad \mu\left(\frac{\alpha_n}{y_n}\right) = \left(\frac{\mu\alpha_n}{y_n}\right), \quad \mu \in \mathbb{C}.$$

In what follows, we construct the Boehmian spaces $B_1 \approx B(L^1(\mathbb{R}_+), (C^\infty(\mathbb{R}_+), \Delta, \ominus), \ominus)$ and $B_2 \approx B(L^1(\mathbb{R}_+^2), (C^\infty(\mathbb{R}_+), \Delta, \ominus), \oplus)$ with the products $\ominus$ (to act as $*$) and $\oplus$ (to act as $\star$) that seem to comply with the delta sequences and the operator $M_{\alpha,k}$. We refer to [7, 14, 15, 20–34] for an outright description and full details of abstract constructions of various Boehmian spaces and transform operators.

However, we provide several systematic hypotheses to generate the space B_2 of Boehmians. The following theorem includes a straightforward proof alluded to a simple integral calculus. Hence it has been detailed.

Theorem 4 *Let $U_n, U, U_1, U_2 \in L^1(\mathbb{R}_+^2)$ and $\phi \in C^\infty(\mathbb{R}_+)$. Then we have*

$$(U_1 + U_2) \oplus \phi = U_1 \oplus \phi + U_2 \oplus \phi, \qquad (\zeta U) \oplus \phi = \zeta(U \oplus \phi),$$

where $\zeta \in C$, and, if $U_n \to U$ in $L^1(\mathbb{R}_+^2)$, then

$$U_n \oplus \phi \to U \oplus \phi$$

as $n \to \infty$ in $L^1(\mathbb{R}_+^2)$.

Theorem 5 *If $U \in L^1(\mathbb{R}_+^2)$ and $(\delta_n) \in \Delta$, then $U \oplus \delta_n \to U$ as $n \to \infty$.*

Proof By the property $\int_{-\infty}^{\infty} \delta_n = 1$ of delta sequences and the definition of the norm $\|\cdot\|_{L^1(\mathbb{R}_+^2)}$ together with the facts that $\delta_n \in C^\infty(\mathbb{R}_+)$ and $\operatorname{supp}\delta_n \subset (\alpha_n, \beta_n)$, for all $n \in \mathbb{N}$, we write

$$\begin{aligned}
&\|U \oplus \delta_n - U\|_{L^1(\mathbb{R}_+^2)} \\
&\quad = \int_{\mathbb{R}_+^2} \left| \int_{\mathbb{R}_+} U(\sqrt{x}u, \sqrt{x}v)\delta_n(x)\frac{dx}{x^{k\alpha}} - U(u,v) \right| d(u,v) \\
&\quad = \int_{\mathbb{R}_+^2} \left| \int_{\mathbb{R}_+} U(\sqrt{x}u, \sqrt{x}v)\delta_n(x)\frac{d_x}{x^{k\alpha}} - U(u,v)\int_{\mathbb{R}_+} \delta_n(x)\,dx \right| d(u,v) \\
&\quad \le \int_{\alpha_n}^{\beta_n} \int_{\mathbb{R}_+^2} \left| \frac{U(\sqrt{x}u, \sqrt{x}v)}{x^{k\alpha}} - U(u,v) \right| \left|\delta_n(x)\right| dx\, d(u,v).
\end{aligned}$$

By applying certain favorable computations and considering the facts

$$\left| \frac{U(\sqrt{x}u, \sqrt{x}v)}{x^{k\alpha}} - U(u,v) \right| \le 2\|U\|_{L^1(\mathbb{R}_+^2)} \quad \text{and}$$

$$\left|\delta_n(x)\right| \le A \quad \text{for some real number } A,$$

we end up with

$$\|U \oplus \delta_n - U\|_{L^1(\mathbb{R}_+^2)} \leq 2A\|U\|_{L^1(\mathbb{R}_+^2)}(\beta_n - \alpha_n) \to 0 \quad \text{as } n \to \infty.$$

The proof is therefore ended. □

The trustworthy conclusion which can be drawn from Theorems 3, 4, and 5 is the presence of the space B_2 as a Boehmian space. A Boehmian in B_2 is defined as $X_{\varphi_n} = (\frac{\varphi_n}{\delta_n})$. In B_2, if $X_{\varphi_n} = (\frac{\varphi_n}{\delta_n})$ and $X_{g_n} = (\frac{g_n}{\varepsilon_n})$ are two Boehmians, then typically we define

$$X_{\varphi_n} + X_{g_n} = \left(\frac{\varphi_n \oplus \delta_n + g_n \oplus \delta_n}{\delta_n \ominus \varepsilon_n}\right), \qquad \beta X_{\varphi_n} = \left(\frac{\beta \varphi_n}{\delta_n}\right) \quad \text{for all } \beta \in \mathbb{C}.$$

Also, for $\alpha \in \mathbb{R}$ and $\omega \in L^1(\mathbb{R}_+^2)$, we resp. define $\oplus$, the differentiation D^α, and the extension of $\oplus$ to $B_2 \oplus L^1(\mathbb{R}_+^2)$ in B_2 as

$$X_{\varphi_n} \oplus X_{g_n} = \left(\frac{\varphi_n \oplus g_n}{\delta_n \oplus \varepsilon_n}\right), \qquad D^\alpha X_{\varphi_n} = \left(\frac{D^\alpha \varphi_n}{\delta_n}\right), \quad \text{and} \quad X_{\varphi_n} \oplus \omega = \frac{\varphi_n \oplus \omega}{\delta_n}.$$

Definition 6 Let $\beta_n, \beta \in B_2$, $n = 1,2,3,\ldots$. Then the sequence $\{\beta_1, \beta_2, \ldots, \beta_n, \ldots\}$ is δ-convergent to β, denoted by $\delta - \lim_{n\to\infty} \beta_n = \beta (\beta_n \xrightarrow{\delta} \beta)$, provided there can be found a delta sequence (δ_n) such that

- $(\beta_n \oplus \delta_k)$ and $(\beta \oplus \delta_k) \in L^1(\mathbb{R}_+^2)$ for all $n, k \in \mathbb{N}$,
- $\lim_{n\to\infty} \beta_n \oplus \delta_k = \beta \oplus \delta_k$ in $L^1(\mathbb{R}_+^2)$ for all $k \in \mathbb{N}$.

Or, equivalently, $\delta - \lim_{n\to\infty} \beta_n = \beta$ if and only if there are $\varphi_{n,k}$, $\varphi_k \in L^1(\mathbb{R}_+^2)$ and $(\delta_k) \in \Delta$ such that

- $\beta_n = (\frac{\varphi_{n,k}}{\delta_k})$, $\beta = (\frac{\varphi_k}{\delta_k})$,
- to every $k \in \mathbb{N}$, we have$\lim_{n\to\infty} \varphi_{n,k} = \varphi_k$ in $L^1(\mathbb{R}_+^2)$.

Definition 7 Let $\beta_n, \beta \in B_2$ for $n = 1,2,3,\ldots$. Then the sequence $\{\beta_1, \beta_2, \ldots, \beta_n, \ldots\}$ is Δ-convergent to β, denoted by $\Delta\text{-}\lim_{n\to\infty} \beta_n = \beta (\beta_n \xrightarrow{\Delta} \beta)$, provided there can be found a delta sequence $\{\delta_1, \delta_2, \ldots, \delta_n, \ldots\}$ such that

- $(\beta_n - \beta) \oplus \delta_n \in L^1(\mathbb{R}_+^2)$ $(\forall n \in \mathbb{N})$
- $\lim_{n\to\infty} (\beta_n - \beta) \oplus \delta_n = 0$ in $L^1(\mathbb{R}_+^2)$.

Similarly, for $U_n, U, U_1, U_2 \in L^1(\mathbb{R}_+)$ and $\phi \in C^\infty(\mathbb{R}_+)$, we can easily check the construction of the space B_1 by using the familiar properties of the Mellin type convolution (see Eq. (5)), which are $U \ominus \phi = \phi \ominus U$ and $(U \ominus \phi_1) \ominus \phi_2 = U \ominus (\phi_1 \ominus \phi_2)$ for $U \in L^1(\mathbb{R}_+)$ and $\phi_1, \phi_2 \in C^\infty(\mathbb{R}_+)$, and applying analogous techniques in proving the axioms:

(i) $(U_1 + U_2) \ominus \phi = U_1 \ominus \phi + U_2 \ominus \phi$.
(ii) $(\zeta U) \ominus \phi = \zeta (U \ominus \phi)$, where $\zeta \in \mathbb{C}$.
(iii) If $U_n \to U$ in $L^1(\mathbb{R}_+)$, then $U_n \ominus \phi \to U \ominus \phi$ as $n \to \infty$ in $L^1(\mathbb{R}_+)$.
(iv) If $U \in L^1(\mathbb{R}_+)$ and $(\delta_n) \in \Delta$, then $U \ominus \delta_n \to U$ as $n \to \infty$.

Operations on B_1 can be stated as they have already been defined on the space B_2. Therefore, in B_1, if $X_{U_n} = (\frac{U_n}{\delta_n})$ and $X_{V_n} = (\frac{V_n}{\varepsilon_n})$ are two Boehmians, then we define

$$X_{U_n} + X_{V_n} = \left(\frac{U_n \ominus \delta_n + V_n \ominus \delta_n}{\delta_n \ominus \varepsilon_n}\right), \qquad \beta X_{U_n} = \left(\frac{\beta U_n}{\delta_n}\right) \quad \text{for all } \beta \in \mathbb{C}.$$

Also, for $\alpha \in \mathbb{R}$ and $U \in L^1(\mathbb{R}_+)$, we resp. define the application of $\ominus$ to Boehmians, the differentiation D^α, and the extension of $\ominus$ to $B_1 \ominus L^1(\mathbb{R}_+)$ in B_1 as

$$X_{U_n} \ominus X_{V_n} = \left(\frac{U_n \ominus V_n}{\delta_n \ominus \varepsilon_n}\right), \qquad D^\alpha X_{U_n} = \left(\frac{D^\alpha U_n}{\delta_n}\right), \quad \text{and} \quad X_{U_n} \ominus U = \frac{U_n \ominus U}{\delta_n}.$$

Hence we have the following definition.

Definition 8 Let $(\frac{U_n}{\delta_n}) \in B_1$, then we define the generalized $M_{\alpha,k}$ of $X_{U_n} = (\frac{U_n}{\delta_n})$ as

$$\bar{M}_{\alpha,k}\left(\frac{U_n}{\delta_n}\right) = \bar{M}_{\alpha,k} X_{U_n} = \left(\frac{M_{\alpha,k} U_n}{\delta_n}\right).$$

By the fact that $M_{\alpha,k} U_n \in L^1(\mathbb{R}_+^2)$, the formula in the above equation is well-defined.

Remark 9 Let $\{\Omega_1, \Omega_2, \dots, \Omega_n, \dots\} \in \Delta$ and $U \in L^1(\mathbb{R}_+^2)$, $U = M_{\alpha,k}\psi$ for some fixed $\psi \in L^1(\mathbb{R}_+^2)$, then we have:

(i) If $X_{\tilde{\psi}} = (\frac{\psi \ominus \Omega_n}{\Omega_n})$, then the mapping $\psi \to X_{\tilde{\psi}}$ from $L^1(\mathbb{R}_+)$ into B_1 is an injective.
(ii) If $Y_{\tilde{U}} = (\frac{U \oplus \Omega_n}{\Omega_n})$, then the mapping $U \to Y_{\tilde{U}}$ from $L^1(\mathbb{R}_+^2)$ into B_2 is an injective.

From Remark 9, it may be said that $L^1(\mathbb{R}_+)$ (resp. $L^1(\mathbb{R}_+^2)$) can be identified as subspaces of B_1 (resp. B_2).

Remark 10

(i) Let $(\psi_n) \in \Delta$. Then, if $f_n \to f$ in $L^1(\mathbb{R}_+)$ as $n \to \infty$, then for all $k \in \mathbb{N}$,

$$f_n \ominus \psi_k \to f \ominus \psi_k \quad \text{as } n \to \infty.$$

(ii) Let $(\Omega_n) \in \Delta$. Then, if $U_n \to U$ in $L^1(\mathbb{R}_+^2)$ as $n \to \infty$, then for all $k \in \mathbb{N}$,

$$U_n \oplus \Omega_k \to U \oplus \Omega_k \quad \text{as } n \to \infty.$$

It follows from above that $X_{f_n} \to X_f$ in B_1 and $Y_{U_n} \to Y_U$ in B_2 as $n \to \infty$. Moreover, the following can also be inferred.

Theorem 11 *The mappings defined in Remark 10 are continuous embedding of $L^1(\mathbb{R}_+)$ (resp. $L^1(\mathbb{R}_+^2)$) into the space B_1 (resp. B_2).*

4 General properties

In this section, we provide certain properties of the generalized natural integral transform. In fact, the results here are brief and concise, and give the reader a general overview of the generalized operator as most of similar properties are enumerated in the previous work of the author.

Theorem 12 *Let $X_{U_n} = (\frac{U_n}{\delta_n})$. Then the mapping $X_{U_n} \to Y_{U_n}$, defined by*

$$U_A X_{U_n} = Y_{U_n},$$

is linear and coincides with the classical transform $M_{\alpha,k} : L^1(\mathbb{R}_+) \to L^1(\mathbb{R}_+^2)$.

Proof Linearity is obvious. To show consistency of the transform $M_{\alpha,k}$, let $U \in L^1(\mathbb{R}_+)$, then U can be identified in B_1 as X_U where $X_U = (\frac{U\ominus\delta_n}{\delta_n})$, which is the representation of U in B_1. Indeed, $\{\delta_1, \delta_2, \ldots, \delta_n, \ldots\}$ is independent of $(\frac{U\ominus\delta_n}{\delta_n})$. Now by the convolution theorem, we have

$$\bar{M}_{\alpha,k} X_U = \left(\frac{M_{\alpha,k}(U \ominus \delta_n)}{\delta_n}\right) = \left(\frac{M_{\alpha,k} U \oplus \delta_n}{\delta_n}\right) = Y_U.$$

Therefore, Y_U is the identification in B_2 of $M_{\alpha,k}U$ in $L^1(\mathbb{R}^2_+)$.

The proof is therefore finished. □

Theorem 13 *Let $X_{f_n} = (\frac{U_n}{\delta_n})$ and $\bar{M}_{\alpha,k} X_{U_n} = Y_{U_n}$. Then the mapping $X_{U_n} \to Y_{U_n}$ is one-to-one, onto, and continuous with respect to the convergence of the Boehmian spaces. A similar proof for this theorem can be performed by a similar way to that of* [27, 28]. *Hence it has been omitted.*

We introduce the inverse integral operator of U_A as follows.

Definition 14 Let $Y_{U_n} \in B_2$, $Y_{U_n} = \bar{M}_{\alpha,k} X_{U_n} = \frac{M_{\alpha,k}\chi_n}{\delta_n}$, $(\delta_n) \in \Delta$, $X_{U_n} = (\frac{U_n}{\delta_n})$. We define the inverse $\bar{M}_{\alpha,k}$ integral operator of Y_{U_n} as

$$\bar{M}^{-1}_{\alpha,k} Y_{U_n} = X_{U_n}. \tag{13}$$

Theorem 15 *The inverse mapping $Y_{U_n} \to X_{U_n}$ is linear.*

Proof Consider two Boehmians Y_{V_n} and Y_{U_n} in B_2, where $Y_{V_n} = (\frac{M_{\alpha,k}V_n}{\delta_n})$ and $Y_{U_n} = (\frac{M_{\alpha,k}U_n}{\epsilon_n})$. Then, for all $n \in \mathbb{N}$, the convolution theorem and the linearity of the integral reveal

$$Y_{V_n} + Y_{U_n} = \left(\frac{M_{\alpha,k}V_n \oplus \epsilon_n + M_{\alpha,k}U_n \oplus \epsilon_n}{\delta_n \ominus \epsilon_n}\right) = \left(\frac{M_{\alpha,k}(V_n \ominus \epsilon_n + U_n \ominus \delta_n)}{\delta_n \ominus \epsilon_n}\right).$$

Hence, Definition 14 yields

$$\bar{M}^{-1}_{\alpha,k}(Y_{V_n} + Y_{U_n}) = \left(\frac{V_n \ominus \epsilon_n + U_n \ominus \delta_n}{\delta_n \ominus \epsilon_n}\right).$$

Notion of addition in B_1 implies

$$\bar{M}^{-1}_{\alpha,k}(Y_{V_n} + Y_{U_n}) = X_{V_n} + X_{U_n},$$

where $X_{V_n} = (\frac{V_n}{\delta_n})$ and $X_{U_n} = (\frac{U_n}{\epsilon_n})$. To complete the proof of the theorem, we indeed, for some $\eta \in \mathbb{C}$ and all $n \in N$, have

$$\bar{M}^{-1}_{\alpha,k}(\eta Y_{V_n}) = \eta \bar{M}^{-1}_{\alpha,k} Y_{V_n}.$$

This finishes the proof of the theorem. □

The generalized convolution theorem can be drawn as follows.

Theorem 16 *Let $Y_{U_n} \in B_2$ and $U \in D$. Then we have*
(i) $\bar{M}^{-1}_{\alpha,k}(Y_{U_n} \oplus U) = X_{U_n} \ominus U$,
(ii) $\bar{M}_{\alpha,k}(X_{U_n} \ominus U) = Y_{U_n} \oplus U$.

Proof Assume $Y_{U_n} \in B_2$ and $U \in D$. Then we have

$$\bar{M}^{-1}_{\alpha,k}(Y_{U_n} \oplus U) = \bar{M}^{-1}_{\alpha,k}\left(\frac{M_{\alpha,k}U_n}{\epsilon_n} \oplus U\right) = \bar{M}^{-1}_{\alpha,k}\left(\frac{M_{\alpha,k}U_n \oplus U}{\epsilon_n}\right).$$

By using Theorem 2 and Eq. (13), the above equation reveals

$$\bar{M}^{-1}_{\alpha,k}(Y_{U_n} \ominus U) = \bar{M}^{-1}_{\alpha,k}\left(\frac{M_{\alpha,k}(U_n \ominus U)}{\epsilon_n}\right) = \left(\frac{U_n}{\epsilon_n} \ominus U\right) = X_{U_n} \ominus U.$$

Proof of the part $\bar{M}_{\alpha,k}(X_{U_n} \ominus U) = Y_{U_n} \oplus U$ is quite similar.

This finishes the proof of the theorem. □

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