# EXISTENCE AND UNIQUENESS THEOREMS FOR SEQUENTIAL LINEAR CONFORMABLE FRACTIONAL DIFFERENTIAL EQUATIONS

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*Abstract.* Recently, a new definition of fractional derivative called as the conformable fractional derivative which based on limits introduced by Khalil et al (2014). Later, Abdeljawad (2015) improved these definitions and gave the basic concepts in this new fractional calculus. In this paper, we generalize Abel's formula and Wronskian determinant definition and establish existence and uniqueness theorems for sequential linear conformable fractional differential equations.

## 1. INTRODUCTION

Although the idea of fractional derivative has been suggested by L'Hospital at the end 17th century, intensive studies about fractional derivative were carried out in the previous and current centuries. Integral form for fractional derivative are used by many researchers. It is known that the most popular definitions of fractional derivative are Riemann-Liouville and Caputo definitions. For Riemann-Liouville, Caputo and other definitions and the characteristics of these definitions, we refer to reader to [12],[15],[16].

(I) Riemann-Liouville definition:

$$D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_0^x (x-t)^{(n-\alpha-1)} f(t)\, dt, \ n-1 < \alpha \le n$$

(II) Caputo definition:

$$D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{(n-\alpha-1)} \left(\frac{d}{dt}\right)^n f(t)\, dt, \ n-1 < \alpha \le n$$

Recently, Khalil et al. introduced a new definition of fractional derivative and fractional integral [10]. This new definition is based on a limit form similar as in

usual definition of derivative. Later, this new theory is improved by Abdeljawad [1]. For instance, Taylor power series representation and Laplace transform of few certain functions, fractional integration by parts formulas, chain rule and Gronwall inequality are provided by him. In addition these studies, many studies with regard to conformable fractional calculus were done [2],[3],[9].

Several studies on theory and application of the sequential linear fractional differential equations which based on Hadamard, Riemann-Liouville and Caputo derivatives were carried out [13],[5],[14],[11],[6],[4]. In this study, we generalize Abel's formula and Wronskian determinant definition [8]. Further, we investigate existence and uniqueness theorems for sequential linear conformable fractional differential equations.

## 2. CONFORMABLE FRACTIONAL DERIVATIVE

Here, we give some basic definitions and properties of the conformable fractional calculus theory which can be found in [10],[1].

**Definition 1.** $f : [0, \infty) \to \mathbb{R}$ let a function. Then, for all $t > 0$, the conformable fractional derivative of $f$ of order $\alpha$ is defined as

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon} \qquad (2.1)$$

where $\alpha \in (0,1)$. If $f$ is $\alpha-$differentiable in some $(0,a)$, $a > 0$ and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then,

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t) \qquad (2.2)$$

is true.

**Theorem 1.** *let be $\alpha \in (0,1]$ and $f, g$ be $\alpha-$differentiable at a point $t > 0$. Then,*

(1) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, *for all $a, b \in \mathbb{R}$,*
(2) $T_\alpha(t^p) = pt^{p-\alpha}$ *for all $p \in \mathbb{R}$,*
(3) $T_\alpha(\lambda) = 0$ *for all constant functions $f(t) = \lambda$,*
(4) $T_\alpha(fg) = T_\alpha(f)g + fT_\alpha(g)$,
(5) $T_\alpha(f/g) = \frac{T_\alpha(f)g - fT_\alpha(g)}{g^2}$,
(6) *In addition, if $f$ is differentiable, then $T_\alpha(f(t)) = t^{1-\alpha}\frac{df}{dt}$.*
    *Additionally, conformable fractional derivatives of certain functions as follow:*
(i) $T_\alpha\left(\sin\frac{1}{\alpha}t^\alpha\right) = \cos\frac{1}{\alpha}t^\alpha$,
(ii) $T_\alpha\left(\cos\frac{1}{\alpha}t^\alpha\right) = -\sin\frac{1}{\alpha}t^\alpha$,
(iii) $T_\alpha\left(e^{\frac{1}{\alpha}t^\alpha}\right) = e^{\frac{1}{\alpha}t^\alpha}$.

**Theorem 2.** *Let $h(t) = f(g(t))$, $\alpha \in (0,1]$ and $f, g : (0, \infty) \to \mathbb{R}$ be $\alpha-$differentiable functions. Then, $h(t)$ is $\alpha-$differentiable and for all $t$ ($t \neq 0$ and $g(t) \neq 0$), we have*

$$T_\alpha(h)(t) = (T_\alpha f)(g(t))(T_\alpha g)(t) g(t)^{\alpha - 1}.$$

*If $t = 0$, we have*

$$T_\alpha(h)(t) = \lim_{t \to 0^+} (T_\alpha f)(g(t))(T_\alpha g)(t) g(t)^{\alpha - 1}. \tag{2.3}$$

**Theorem 3.** *Let $f, g : [a,b] \to \mathbb{R}$ be two functions such that $f, g$ are differentiable. Then,*

$$\int_a^b f(t) T_\alpha(g)(t) d_\alpha(t) = fg|_a^b - \int_a^b g(t) T_\alpha(f)(t) d_\alpha(t). \tag{2.4}$$

## 3. Conformable fractional integral

**Definition 2.** Let be $f : [0, \infty) \to \mathbb{R}$. Then, for all $t > 0$, $\alpha \in (0,1)$, the conformable fractional integral of $f$ of order $\alpha$ is defined by

$$(I_\alpha f)(t) = \int_0^t f(x) d_\alpha(x) = \int_0^t x^{\alpha - 1} f(x) dx \tag{3.1}$$

where the integral means as the usual Riemann improper integral.

**Theorem 4.** *Let $f : [0, \infty) \to \mathbb{R}$ be any continuous function and $0 < \alpha \leq 1$. Then, $\forall t > a$,*

$$T_\alpha I_\alpha f(t) = f(t). \tag{3.2}$$

**Theorem 5.** *Let $f : (a,b) \to \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > a$,*

$$I_\alpha T_\alpha f(t) = f(t) - f(a). \tag{3.3}$$

## 4. Existence and uniqueness theorems

Consider with consider linear sequential conformable fractional differential equations of order $n\alpha$

$$^nT_\alpha y + p_{n-1}(t) {}^{n-1}T_\alpha y + ... + p_2(t) {}^2T_\alpha y + p_1(t) T_\alpha y + p_0(t) y = 0, \tag{4.1}$$

$$^nT_\alpha y + p_{n-1}(t) {}^{n-1}T_\alpha y + ... + p_2(t) {}^2T_\alpha y + p_1(t) T_\alpha y + p_0(t) y = f(t) \tag{4.2}$$

where $^nT_\alpha y = T_\alpha T_\alpha ... T_\alpha$. We define $n^{th}$ order differential operator as following:

$$L_\alpha[y] = {}^nT_\alpha y + p_{n-1}(t) {}^{n-1}T_\alpha y + ... + p_2(t) {}^2T_\alpha y + p_1(t) T_\alpha y + p_0(t) y. \tag{4.3}$$

**Theorem 6.** *Let $t^{\alpha-1} p(t), t^{\alpha-1} q(t) \in C(a,b)$ and $y$ be $\alpha-$differentiable for $0 < \alpha \leq 1$. Then, the following initial value problem*

$$T_\alpha y + p(t) y = q(t), \tag{4.4}$$

$$y(t_0) = y_0 \tag{4.5}$$

*has a unique solution on the interval $(a,b)$ for any $t_0 \in (a,b)$.*

*Proof.* we can write

$$T_\alpha y + p(t) y = q(t), \tag{4.6}$$

$$t^{1-\alpha} y' + p(t) y = q(t), \tag{4.7}$$

$$y' + t^{\alpha-1} p(t) y = t^{\alpha-1} q(t). \tag{4.8}$$

So, existence and uniqueness follows from classical theorem for linear equation. □

**Theorem 7.** *Let $t^{\alpha-1} p_{n-1}(t), ..., t^{\alpha-1} p_1(t), t^{\alpha-1} p_0(t), t^{\alpha-1} q(t) \in C(a,b)$ and let $y$ be $n$ times $\alpha-$differentiable function. Then, a solution $y(t)$ of the next initial value problem*

$$^n T_\alpha \, y + p_{n-1}(t) \, ^{n-1} T_\alpha \, y + ... + p_2(t) \, ^2 T_\alpha \, y + p_1(t) T_\alpha y + p_0(t) y = q(t), \tag{4.9}$$

$$y(t_0) = y_0, T_\alpha y(t_0) = y_1, ..., \, ^{n-1} T_\alpha \, y(t_0) = y_{n-1}, \quad a < t_0 < b \tag{4.10}$$

*exists on the interval $[a,b]$ and it is unique.*

*Proof.* To show existence of a local solution, we reduce our problem to the first order system of differential equations. So, let's make the following change of variables

$$x_1 = y, \; x_2 = T_\alpha y, \; x_3 = \, ^2 T_\alpha \, y, \cdots, \; x_n = \, ^{n-1} T_\alpha \, y(t_0) = y_{n-1}. \tag{4.11}$$

Hence, we have

$$
\begin{aligned}
T_\alpha x_1 &= x_2, \\
T_\alpha x_2 &= x_3, \\
&\vdots \\
T_\alpha x_{n-1} &= x_n, \\
T_\alpha x_n &= -p_{n-1} x_n - \cdots - p_2 x_3 - p_1 x_2 - p_0 x_1 + q(t).
\end{aligned}
\tag{4.12}
$$

Now, we can rewrite the problem in the following form

$$T_\alpha \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}}_{X(t)} + \underbrace{\begin{bmatrix} 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \\ p_0 & p_1 & p_2 & p_3 & \cdots & p_{n-1} \end{bmatrix}}_{P(t)} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ q(t) \end{bmatrix}}_{Q(t)},$$

$$(4.13)$$

$$T_\alpha X(t) + P(t) X(t) = Q(t), \tag{4.14}$$

$$X'(t) + t^{\alpha-1} P(t) X(t) = t^{\alpha-1} Q(t). \tag{4.15}$$

Finally, the existence and uniqueness of solution (4.9)-(4.10) follows from classical theorem on existence and uniqueness for system equation [7]. □

**Theorem 8.** $L_\alpha$ *is linear, i.e.*

$$L_\alpha [c_1 y_1 + c_2 y_2] = c_1 L_\alpha [y_1] + c_2 L_\alpha [y_2] \tag{4.16}$$

*where* $y_1$, $y_2$ *are n times* $\alpha-$*differentiable functions and* $c_1$, $c_2$ *are arbitrary numbers.*

*Proof.* From the definition conformable fractional derivative it follows that

$$\begin{aligned} L_\alpha [c_1 y_1 + c_2 y_2] &= {}^n T_\alpha (c_1 y_1 + c_2 y_2) + p_{n-1}(t) \, {}^{n-1} T_\alpha (c_1 y_1 + c_2 y_2) + \cdots \\ &\quad + p_1(t) T_\alpha (c_1 y_1 + c_2 y_2) + p_0(t)(c_1 y_1 + c_2 y_2) \\ &= c_1 \left( {}^n T_\alpha \, y_1 + p_{n-1}(t) \, {}^{n-1} T_\alpha \, y_1 + \ldots + p_0(t) y_1 \right) \\ &\quad + c_2 \left( {}^n T_\alpha \, y_2 + p_{n-1}(t) \, {}^{n-1} T_\alpha \, y_2 + \ldots + p_0(t) y_2 \right). \end{aligned} \tag{4.17}$$

So, we have

$$L_\alpha [c_1 y_1 + c_2 y_2] = c_1 L_\alpha [y_1] + c_2 L_\alpha [y_2]. $$

□

**Theorem 9.** *let* $y_1, y_2, \ldots, y_n$ *be the solutions of the equation* $L_\alpha [y] = 0$. *In this case, the linear combination*

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \tag{4.18}$$

*is also its solution for the arbitrary constants* $c_k$, $k = 1, \ldots, n$.

*Proof.* Let $y_1(t), y_2(t), \ldots, y_n(t)$ be solutions of the equation $L_\alpha [y] = 0$, for arbitrary constants $c_k$, $k = 1, \ldots, n$. Denote by

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n. \tag{4.19}$$

By using linear property of $L_\alpha$, we have

$$L_\alpha (y) = L_\alpha (c_1 y_1 + c_2 y_2 + \cdots + c_n y_n)$$

$$= c_1 L_\alpha(y_1) + c_2 L_\alpha(y_2) + \cdots + c_n L_\alpha(y_n)$$
$$= 0.$$

$\square$

**Definition 3.** We assume that $y_1(t)$, $y_2(t)$,..., $y_n(t)$ are at least $(n-1)$ times $\alpha-$differentiable functions. For any $0 < \alpha \leq 1$, the determinant

$$W_\alpha(y_1,\ y_2,\ldots,y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ T_\alpha y_1 & T_\alpha y_2 & \cdots & T_\alpha y_n \\ \vdots & \vdots & \cdots & \vdots \\ {}^{n-1}T_\alpha\ y_1 & {}^{n-1}T_\alpha\ y_2 & \cdots & {}^{n-1}T_\alpha\ y_n \end{vmatrix} \tag{4.20}$$

is called $\alpha-$Wronskian of the functions.

**Definition 4.** Any set $\{y_1,\ y_2,\ldots,y_n\}$ of n solutions of $L_\alpha[y] = 0$ is said to be a fundamental set of solutions if every solution of this equation can be written as a linear combination of these functions.

**Theorem 10.** *Let $y_1,\ y_2,\ldots,y_n$ be n solutions of $L_\alpha[y] = 0$. If there is a $t_0 \in (a,b)$ such that $W_\alpha(y_1,\ y_2,\ldots,y_n)(t_0) \neq 0$, then, $\{y_1,\ y_2,\ldots,y_n\}$ is a fundamental set of solutions.*

*Proof.* If $y(t)$ is a solution of $L_\alpha[y] = 0$, then we can write $y(t)$ as a linear combination of $y_1,\ y_2,\ldots,y_n$ which we need to show. That is

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n. \tag{4.21}$$

So, the problem is reduced to finding the constants $c_k$, $1 \leq k \leq n$. We can write the following linear system of equations

$$c_1 y_1(t_0) + c_2 y_2(t_0) + \cdots + c_n y_n(t_0) = y(t_0)$$
$$c_1 T_\alpha y_1(t_0) + c_2 T_\alpha y_2(t_0) + \cdots + c_n T_\alpha y_n(t_0) = T_\alpha y(t_0)$$

$$\vdots$$

$$c_1\ {}^{n-1}T_\alpha\ y_1(t_0) + c_2\ {}^{n-1}T_\alpha\ y_2(t_0) + \cdots + c_n\ {}^{n-1}T_\alpha\ y_n(t_0) = {}^{n-1}T_\alpha\ y(t_0). \tag{4.22}$$

Applying Cramer's rule, we can get

$$c_k = \frac{W_\alpha^k(t_0)}{W_\alpha(t_0)}, \quad 1 \leq k \leq n. \tag{4.23}$$

Since $W_\alpha(t_0) \neq 0$, it follows that $c_1,\ c_2,\ldots,c_n$ exist. $\square$

**Theorem 11.** *Suppose that $t^{\alpha-1}p_{n-1}(t),\ldots,t^{\alpha-1}p_1(t), t^{\alpha-1}p_0(t) \in C(a,b)$. Then, the equation $L_\alpha[y] = 0$ has a fundamental set of solutions $\{y_1,\ y_2,\ldots,y_n\}$.*

*Proof.* Let $t_0 \in (a,b)$. Consider the following n initial value problems

$$L_\alpha [y] = 0, \ y(t_0) = 1, \ T_\alpha y(t_0) = 0, \ldots, \ {}^{n-1}T_\alpha \ y(t_0) = 0$$

$$L_\alpha [y] = 0, \ y(t_0) = 0, \ T_\alpha y(t_0) = 1, \ldots, \ {}^{n-1}T_\alpha \ y(t_0) = 0$$

$$\vdots$$

$$L_\alpha [y] = 0, \ y(t_0) = 0, \ T_\alpha y(t_0) = 0, \ldots, \ {}^{n-1}T_\alpha \ y(t_0) = 1 \qquad (4.24)$$

From 7, it follows that there is the solution $y_i$ of $i^{th}$ problem for each index $i$. By the Theorem 10, the set of $\{y_1, \ y_2, \ldots, y_n\}$ is a fundamental set of solutions since

$$W_\alpha(t) = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = 1 \neq 0. \qquad (4.25)$$

$\square$

**Theorem 12.** *Let $y_1, \ y_2, \ldots, y_n$ be n solutions of equation $L_\alpha [y] = 0$. Then*
(1) *$W_\alpha(t)$ satisfies the differential equation $T_\alpha W_\alpha + p_{n-1}(t) W_\alpha = 0$,*
(2) *For any $t_0 \in (a,b)$*

$$W_\alpha(t) = W_\alpha(t_0) e^{-\int_{t_0}^{t} x^{\alpha-1} p_{n-1}(x)dx}. \qquad (4.26)$$

*Moreover, if $W_\alpha(t_0) \neq 0$ then, $W_\alpha(t) \neq 0$ for all $t \in (a,b)$.*

*Proof.* Introduce new variables

$$x_1 = y, \ x_2 = T_\alpha y, \ x_3 = {}^2T_\alpha \ y, \cdots, \ x_n = {}^{n-1}T_\alpha \ y(t_0) = y_{n-1}. \qquad (4.27)$$

We can rewrite the differential in the matrix form

$$T_\alpha \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}}_{X(t)} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -p_0 & -p_1 & -p_2 & -p_3 & \cdots & -p_{n-1} \end{bmatrix}}_{P(t)} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}}_{X(t)}, \qquad (4.28)$$

$$T_\alpha X(t) = P(t) X(t). \qquad (4.29)$$

Therefore, we obtain

$$T_\alpha W_\alpha(t) = (p_{11} + p_{22} + \cdots + p_{nn}) W_\alpha(t). \qquad (4.30)$$

Really, if $\alpha-$conformable derivative of $W_\alpha(y_1, \ y_2, \ldots, y_n)$ is calculated, then, we can get

$$T_\alpha W_\alpha(t) = -p_{n-1}(t) W_\alpha(t). \qquad (4.31)$$

Hence,

$$\frac{T_\alpha W_\alpha(t)}{W_\alpha(t)} = -p_{n-1}(t), \tag{4.32}$$

$$\ln(W_\alpha(t)) - \ln(W_\alpha(t_0)) = -\int_{t_0}^t x^{\alpha-1} p_{n-1}(x)\,dx, \tag{4.33}$$

$$W_\alpha(t) = W_\alpha(t_0) e^{-\int_{t_0}^t x^{\alpha-1} p_{n-1}(x)dx}. \tag{4.34}$$

So, proof of theorem is finished. $\qquad\square$

**Theorem 13.** *Let* $t^{\alpha-1} p_{n-1}(t), ..., t^{\alpha-1} p_1(t), t^{\alpha-1} p_0(t) \in C(a,b)$. *If* $\{y_1,\ y_2,\dots,y_n\}$ *is a fundamental set of solutions of* $L_\alpha[y] = 0$, *then,* $W_\alpha(t) \neq 0$ *for all* $t \in (a,b)$.

*Proof.* Suppose that $t_0$ be any point in $(a,b)$. By Theorem 7, there is a unique solution $y(t)$ of the initial value problem

$$L_\alpha[y] = 0,\ y(t_0) = 1,\ T_\alpha y(t_0) = 0,\dots, {}^{n-1}T_\alpha\, y(t_0) = 0. \tag{4.35}$$

There exist unique constants $c_1,\ c_2,\dots,c_n$ such that

$$c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) = y(t),$$
$$c_1 T_\alpha y_1(t) + c_2 T_\alpha y_2(t) + \cdots + c_n T_\alpha y_n(t) = T_\alpha y(t),$$
$$\vdots$$
$$c_1\, {}^{n-1}T_\alpha\, y_1(t) + c_2\, {}^{n-1}T_\alpha\, y_2(t) + \cdots + c_n\, {}^{n-1}T_\alpha\, y_n(t) = {}^{n-1}T_\alpha\, y(t) \tag{4.36}$$

for all $t \in (a,b)$ since $\{y_1,\ y_2,\dots,y_n\}$ is a fundamental set of solutions. In particular, for $t = t_0$, we obtain the system

$$c_1 y_1(t_0) + c_2 y_2(t_0) + \cdots + c_n y_n(t_0) = 1,$$
$$c_1 T_\alpha y_1(t_0) + c_2 T_\alpha y_2(t_0) + \cdots + c_n T_\alpha y_n(t_0) = 0,$$
$$\vdots$$
$$c_1\, {}^{n-1}T_\alpha\, y_1(t_0) + c_2\, {}^{n-1}T_\alpha\, y_2(t_0) + \cdots + c_n\, {}^{n-1}T_\alpha\, y_n(t_0) = 0. \tag{4.37}$$

This system has a unique solution

$$c_k = \frac{W_\alpha^k}{W_\alpha(t_0)}, \quad 1 \le k \le n. \tag{4.38}$$

Here, for every $k$

$$W_\alpha^k = \tag{4.39}$$

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_{k-1}(t_0) & 1 & y_k(t_0) & \cdots & y_n(t_0) \\ T_\alpha y_1(t_0) & T_\alpha y_2(t_0) & \cdots & T_\alpha y_{k-1}(t_0) & 0 & T_\alpha y_k(t_0) & \cdots & T_\alpha y_n(t_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ {}^{n-1}T_\alpha\, y_1(t_0) & {}^{n-1}T_\alpha\, y_2(t_0) & \cdots & {}^{n-1}T_\alpha\, y_{k-1}(t_0) & 0 & {}^{n-1}T_\alpha\, y_k(t_0) & \cdots & {}^{n-1}T_\alpha\, y_n(t_0) \end{bmatrix}.$$

For existence $c_1$, $c_2, \ldots, c_n$ it should be $W_\alpha(t_0) \neq 0$. By Theorem 12, we conclude that $W_\alpha(t_0) \neq 0$ for all $t_0 \in (a, b)$. $\qquad \square$

**Theorem 14.** *Let $t^{\alpha-1} p_{n-1}(t), \ldots, t^{\alpha-1} p_1(t), t^{\alpha-1} p_0(t) \in C(a, b)$. The solution set $\{y_1, y_2, \ldots, y_n\}$ is a fundamental set of solutions of equation $L_\alpha[y] = 0$ if and only if the functions $y_1, y_2, \ldots, y_n$ are linearly independent.*

*Proof.* Suppose that $\{y_1, y_2, \ldots, y_n\}$ is a fundamental set of solutions, then by Theorem 13 it follows that there is $t_0 \in (a, b)$ such that $W_\alpha(t_0) \neq 0$. Assume that

$$c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) = 0 \tag{4.40}$$

for all $t \in (a, b)$. By repeated $\alpha$−differentiation of the equation (4.40), we find

$$c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) = 0,$$
$$c_1 T_\alpha y_1(t) + c_2 T_\alpha y_2(t) + \cdots + c_n T_\alpha y_n(t) = 0,$$
$$\vdots$$
$$c_1{}^{n-1}T_\alpha\, y_1(t) + c_2{}^{n-1}T_\alpha\, y_2(t) + \cdots + c_n{}^{n-1}T_\alpha\, y_n(t) = 0. \tag{4.41}$$

Thus, one finds $c_1$, $c_2, \ldots, c_n$ by solving the system

$$c_1 y_1(t_0) + c_2 y_2(t_0) + \cdots + c_n y_n(t_0) = 0,$$
$$c_1 T_\alpha y_1(t_0) + c_2 T_\alpha y_2(t_0) + \cdots + c_n T_\alpha y_n(t_0) = 0,$$
$$\vdots$$
$$c_1{}^{n-1}T_\alpha\, y_1(t_0) + c_2{}^{n-1}T_\alpha\, y_2(t_0) + \cdots + c_n{}^{n-1}T_\alpha\, y_n(t_0) = 0. \tag{4.42}$$

Namely, by using Cramer's rule one finds

$$c_1 = c_2 = \ldots = c_n = \frac{0}{W_\alpha(t_0)} = 0 \tag{4.43}$$

Thus, set of functions $\{y_1, y_2, \ldots, y_n\}$ are linearly independent. Contrarily, suppose $\{y_1, y_2, \ldots, y_n\}$ is a linearly independent set and suppose $\{y_1, y_2, \ldots, y_n\}$ is not a fundamental set of solutions. Then, by Theorem 10, we get $W_\alpha(t) = 0$, for all $t \in (a, b)$. We choose any $t_0 \in (a, b)$. Then $W_\alpha(t_0) = 0$. But, this says $W_\alpha(t_0) \neq 0$ that the matrix

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ T_\alpha y_1(t_0) & T_\alpha y_2(t_0) & \cdots & T_\alpha y_n(t_0) \\ \vdots & \vdots & \cdots & \vdots \\ {}^{n-1}T_\alpha\, y_1(t_0) & {}^{n-1}T_\alpha\, y_2(t_0) & \cdots & {}^{n-1}T_\alpha\, y_n(t_0) \end{bmatrix} \tag{4.44}$$

is not invertible which means that there exist $c_1$, $c_2, \ldots, c_n$, $c_1^2 + c_2^2 + \ldots + c_n^2 \neq 0$ such that

$$c_1 y_1(t_0) + c_2 y_2(t_0) + \cdots + c_n y_n(t_0) = 0,$$
$$c_1 T_\alpha y_1(t_0) + c_2 T_\alpha y_2(t_0) + \cdots + c_n T_\alpha y_n(t_0) = 0,$$
$$\vdots$$
$$c_1 {}^{n-1}T_\alpha \, y_1(t_0) + c_2 {}^{n-1}T_\alpha \, y_2(t_0) + \cdots + c_n {}^{n-1}T_\alpha \, y_n(t_0) = 0. \qquad (4.45)$$

Now, let

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) \qquad (4.46)$$

for all $t \in (a, b)$. Then, $y(t)$ is the solution of the differential equation and

$$y(t_0) = T_\alpha y(t_0) = \cdots = {}^{n-1}T_\alpha \, y(t_0) = 0. \qquad (4.47)$$

But, the zero function is also the solution of the initial value problem. By Theorem 7, it should be

$$c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) = 0 \qquad (4.48)$$

for all $t \in (a, b)$ with $c_1$, $c_2, \ldots, c_n$ not all equal to zero which means that $y_1, y_2, \ldots, y_n$ are linearly dependent which is not in agreement with our earlier assumption that $y_1, y_2, \ldots, y_n$ are linearly independent. $\qquad \square$

**Theorem 15.** *Let $y_1, y_2, \ldots, y_n$ be linearly independent solutions of the equation $L_\alpha[y] = 0$. Then, the general solution of the equation is*

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \qquad (4.49)$$

*where the arbitrary constants $c_k$, $k = 1, \ldots, n$.*

*Proof.* Particular solution at any $t = t_0$ is obtained by the help of initial conditions as following:

$$y(t_0) = \gamma_0, \ T_\alpha y(t_0) = \gamma_1, \cdots, \ {}^{n-1}T_\alpha \, y(t_0) = \gamma_{n-1} \qquad (4.50)$$

where $t_0 \in (a, b)$ and $\gamma_0, \gamma_1, \ldots \gamma_{n-1}$ are arbitrary constants. If we choose $c_1, c_2, \ldots, c_n$ constants to protect conditions (4.50), then, proof is completed. To accomplish this, we can write following system of equations:

$$c_1 y_1(t_0) + c_2 y_2(t_0) + \cdots + c_n y_n(t_0) = \gamma_0,$$
$$c_1 T_\alpha y_1(t_0) + c_2 T_\alpha y_2(t_0) + \cdots + c_n T_\alpha y_n(t_0) = \gamma_1,$$
$$\vdots$$
$$c_1 {}^{n-1}T_\alpha \, y_1(t_0) + c_2 {}^{n-1}T_\alpha \, y_2(t_0) + \cdots + c_n {}^{n-1}T_\alpha \, y_n(t_0) = \gamma_{n-1}. \qquad (4.51)$$

Writing the above system in matrix form, we get

$$
\begin{bmatrix}
y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\
T_\alpha y_1(t_0) & T_\alpha y_2(t_0) & \cdots & T_\alpha y_n(t_0) \\
\vdots & \vdots & \cdots & \vdots \\
{}^{n-1}T_\alpha\, y_1(t_0) & {}^{n-1}T_\alpha\, y_2(t_0) & \cdots & {}^{n-1}T_\alpha\, y_n(t_0)
\end{bmatrix}
\begin{bmatrix}
c_1 \\ c_2 \\ \vdots \\ c_n
\end{bmatrix}
=
\begin{bmatrix}
\gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{n-1}
\end{bmatrix}.
$$

(4.52)

$y_1,\ y_2,\ldots,y_n$ are linearly independent solutions of $L_\alpha[y] = 0$, so $W_\alpha(t_0) \neq 0$. In this case, according to the fundamental theorem of algebra, system (4.51) has a unique solution. Hence, proof is completed. $\square$

**Theorem 16.** *Let $y_p$ be any particular solution of the nonhomogeneous linear $n^{th}$ order sequential differential equation (4.2) and $\{y_1,\ y_2,\ldots,y_n\}$ be a fundamental set of solutions of the associated homogeneous differential equation (4.1). Then, the general solution of the equation is*

$$
y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n + y_p
$$

(4.53)

*where the arbitrary constants $c_k$, $k = 1,\ldots,n$.*

*Proof.* Let $L_\alpha$ be the differential operator and let $Y(t)$ and $y_p(t)$ be particular solutions of the nonhomogeneous equation $L_\alpha[y] = q(t)$. If we define $u(t) = Y(t) - y_p(t)$, then, by linearity of $L_\alpha$, we have

$$
L_\alpha[u] = L_\alpha\big[Y(t) - y_p(t)\big] = L_\alpha[Y(t)] - L_\alpha\big[y_p(t)\big] = q(t) - q(t) = 0. \quad (4.54)
$$

This shows that $u(t)$ is a solution of the homogeneous equation $L_\alpha[y] = 0$. Hence, by Theorem 9,

$$
u(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)
$$

(4.55)

and so

$$
Y(t) - y_p(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)
$$

(4.56)

or

$$
Y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + y_p(t)
$$

(4.57)

$\square$

## 5. CONCLUSION

In our work, we generalize Abel's formula and Wronskian determinant and present existence and uniqueness theorems for sequential linear conformable fractional differential equations. It has been found that results obtained from this work is analogous to the results obtained from the ordinary case.

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