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MEASURE AND INTEGRATION

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Measure and Integration

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirement for the Degree

Master of Arts

in

Mathematics

---

by

JeongHwan Lee

December 2021

Measure and Integration

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Abstract

Measure and Integral are important when dealing with abstract spaces such as function spaces and probability spaces. This thesis will cover Lebesgue Measure and Lebesgue integral. The Lebesgue integral is a generalized theory of Riemann integral learned in mathematics. The Riemann integral is centered on the Definition domain of the function, but the Lebesgue integral is different in that it is centered on the range of the function, and uses the basic concept of analysis. Measure and integral have widely applied not only to mathematics but also to other fields.

Acknowledgements

I would like to thank Professor Yuichiro Kakihara as I write my thesis. I would like to inform you that I was able to finish my thesis safely with the help of some people. I had a lot of problems writing the paper for the first time, but I was able to finish it with the help of Professor Yuichiro Kakihara and other professors. I would like to express my gratitude through this writing.

Introduction

General measures and integrals are used in many fields of mathematics. In this paper, I would like to introduce Lebesgue measure and integral. A Lebesgue measure provides mathematical abstraction of mass, distance, area, volume, probability, and general concepts to a subset of Euclidean space. The Lebesgue integral is an integral that can be defined on a general measure space. The Lebesgue measure and integral is mainly used in mathematics fields such as analysis and probability theory and other fields.

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Chapter 1

Preliminaries

1.1 Related Theorems and Definitions

Definition 1.1.1. *Measurable space and measurable set.*

Given a set X and a σ-algebra $\mathfrak{M} \subset \mathcal{P}(X)$, $(X, \mathfrak{M})$ is called a measurable space, and any set in $\mathfrak{M}$ is called a measurable set.

Definition 1.1.2. *σ-algebra.*

Given a set $\mathcal{A}(\neq \emptyset) \subset \mathcal{P}(X)$, the σ-algebra $\sigma(\mathcal{A}) \subset \mathcal{P}(X)$ obtained from *Theorem* 2.2.9 is called the σ-algebra generated by $\mathcal{A}$.

Definition 1.1.3. *Borel set and Bore measurable space.*

Given a topological space $(X, \mathcal{T})$, according to *Theorem* 2.2.9, there is a σ-algebra $\mathcal{G}(\mathcal{T})$ produced by $\mathcal{T}$. A topology $\mathcal{T}$ is collection of subsets of X. Let the σ-algebra $\mathfrak{B}_\sigma(X) = \mathcal{G}(\mathcal{T})$ be the *Borel* σ-algebra in the topological space $(X,\mathcal{T})$ or simply the *Borel* algebra. Any set in $\mathfrak{B}_\sigma$(X) is called a *Borel* Set. In particular, $(X, \mathfrak{B}_\sigma(\mathrm{X}))$ is called a *Borel* measurable space.

Definition 1.1.4. *F_{σ}- set and G_{δ}- set.*

The set represented by a union of countable closed sets is called an F_{σ}-set and the set represented by an intersection of countable open sets is called a G_{δ}-set.

Definition 1.1.5. *The extended Borel σ-algebra.*

The σ-algebra $\mathfrak{B}_\sigma(\bar{\mathbb{R}}) = \mathcal{G}\left(\mathcal{T}\left(\bar{\mathbb{R}}\right)\right) \subset \mathcal{P}(\bar{\mathbb{R}})$ generated by $\mathcal{T}(\bar{\mathbb{R}})$ is called the extended *Borel* σ-algebra.

Definition 1.1.6. *$\mathfrak{M}$-measurable.*

A function $f\colon X \to \bar{\mathbb{R}}$ defined on a measurable space $(X, \mathfrak{M})$ is called $\mathfrak{M}$-measurable.

Definition 1.1.7. *The positive part of the function $f\colon X \to \bar{\mathbb{R}}$ and the negative part of function $f\colon X \to \bar{\mathbb{R}}$.*

The two functions $f^{\pm}\colon X \to \bar{\mathbb{R}}$ from the function $f\colon X \to \bar{\mathbb{R}}$ defined on the set X are defined as

$$f^{+}(x) = \max\{f(x), 0\} \text{ and } f^{-}(x) = \max\{-f(x), 0\}.$$

Definition 1.1.8. *Measurable.*

Given the measurable space $(X,\, \mathfrak{M}_1)$ and $(Y,\, \mathfrak{M}_2)$, if the function $F\colon\ X \to Y$ satisfies the condition

$$F^{-1}(E) \in \mathfrak{M}_1\ (E \in \mathfrak{M}_2),$$

the function $F\colon\ X \to Y$ is called measurable.

Definition 1.1.9. *A measure and a measure space.*

Given a measurable space $(X,\, \mathfrak{M})$, if the function $\mu\colon\ \mathfrak{M} \to \bar{\mathbb{R}}$ with an extended real-valued and defined on σ-algebra $\mathfrak{M}$, satisfies the condition

(1) $\mu(\emptyset) = 0$,

(2) $\mu(E) \geq 0\ (E \in \mathfrak{M})$,

(3) $(E_n)_{n\geq 1} \subset \mathfrak{M}$ and $E_n \cap E_m = \phi(n \neq m) \Longrightarrow \mu\Big(\bigcup\limits_{k=1}^{\infty} E_k\Big) = \sum\limits_{k=1}^{\infty} \mu(E_k)$,

$\mu\colon\ \mathfrak{M} \to \bar{\mathbb{R}}$ is called a measure and $(X,\, \mathfrak{M},\, \mu)$ is called a measure space.

Definition 1.1.10. *A finite measure, finite measure space and σ-finite measure space.*

Given a measure space $(X,\, \mathfrak{M},\, \mu)$, it is defined as follows.

(a) If the condition $\mu(X) < +\infty$ is satisfied, μ is called a finite measure and $(X,\, \mathfrak{M},\, \mu)$ is called a finite measure space.

(b) If there exists a measurable set sequence $(E_n)_{n\geq 1} \subset \mathfrak{M}$ that satisfies the condition $X = \bigcup\limits_{n=1}^{\infty} E_n$ and $\mu(E_n) < +\infty\ (n \in \mathbb{N})$, μ is called a σ-finite measure and $(X,\, \mathfrak{M},\, \mu)$ is called a σ-finite measure space.

Definition 1.1.11. *The Borel measure space.*

For the Topological space $(X,\, \mathcal{T})$, if the measure $\mu\colon\ \mathfrak{B} \to \bar{\mathbb{R}}$ is given and the *Borel* measure space $\big(X,\, \mathfrak{B}_\sigma(X)\big)$ is considered, $\big(X,\, \mathfrak{B}_\sigma(X),\, \mu\big)$ is called the *Borel* measure space.

Definition 1.1.12. *A translation invariant measure space.*

If the measure space $(X,\, \mathfrak{M},\, \mu)$ satisfies the condition

$$E \in \mathfrak{M} \text{ and } a \in k \Longrightarrow E + a \in \mathfrak{M} \text{ and } \mu(E + a) = \mu(E)$$

for a closed set X by addition, the measure space $(X,\, \mathfrak{M},\, \mu)$ is called a translation invariant measure space.

Definition 1.1.13. *A complete measure space and complete measure.*

If a measure space $(X,\, \mathfrak{M},\, \mu)$ is given and the conditions

$$N \in \mathfrak{M} \text{ and } \mu(N) = 0 \Longrightarrow \mathcal{P}(N) \subset \mathfrak{M}$$

are satisfied, $(X,\, \mathfrak{M},\, \mu)$ is called a complete measure space and the function μ: $\mathfrak{M} \to [0,+\infty]$ is called a complete measure.

Definition 1.1.14. *The completion of the measure space, completion of $\mathfrak{M}$ and completion of μ.*

Given a measure space $(X,\, \mathfrak{M},\, \mu)$, a complete measure space $(X,\, \widehat{\mathfrak{M}},\, \hat{\mu})$ that satisfies the condition $[\mathfrak{M} \subset \widehat{\mathfrak{M}}$ and $\hat{\mu}|\mathfrak{M} = \mu]$ is called the completion of the measure space $(X,\, \mathfrak{M},\, \mu)$. The σ-algebra $\widehat{\mathfrak{M}}$ is called the completion of $\mathfrak{M}$ for a measure μ, and a measure $\hat{\mu}$ is called the completion of μ.

Definition 1.1.15. *The standard representation of a simple measurable function.*
Given a simple measurable function $\phi\colon X \to \mathbb{R}$, an expression such as (4.1) is called the standard representation of a simple measurable function $\phi\colon X \to \mathbb{R}$.

Definition 1.1.16. *The Lebesgue integral.*
Given a measure space $(X, \mathfrak{M}, \mu)$, we define

$$\int f\ d\mu = \sup\left\{ \int \phi\ d\mu\colon\ \phi \in \mathcal{S}_f^+(X, \mathfrak{M}) \right\} \tag{4.3}$$

for the measurable function $f \in \mathcal{M}^+(X, \mathfrak{M})$. (4.3) is called the Lebesgue integral [Wei73] of the measure function $f \in \mathcal{M}^+(X, \mathfrak{M})$ for measure $\mu\colon \mathfrak{M} \to [0, +\infty]$.

Theorem 1.1.17. *The standard expression.*
For the measurable function $f \in \mathcal{M}^+(X, \mathfrak{M})$ and measurable set $E \in \mathfrak{M}$, the following holds.

(a) $f(x) = 0\ (x \in E) \Longrightarrow \int_E f\ d\mu = 0$.
(b) $\mu(E) = 0 \Longrightarrow \int_E f\ d\mu = 0$.

$$\phi = \sum_{k=1}^{n} a_k \chi_{A_k}$$

is called the standard expression.

Chapter 2

Measurable Functions

2.1 Related Theorems and Definitions

Definition 2.1.3. *measurable space and measurable set*

Given a set X and a σ-algebra $\mathfrak{M} \subset \mathcal{P}(X)$, $(X, \mathfrak{M})$ is called a measurable space, and any set in $\mathfrak{M}$ is called a measurable set.

Definition 2.1.10. *σ-algebra.*

Given a set $\mathcal{A}(\neq \emptyset) \subset \mathcal{P}(X)$, the σ-algebra $\sigma(\mathcal{A}) \subset \mathcal{P}(X)$ obtained from *Theorem* 2.2.9 is called the σ-algebra generated by $\mathcal{A}$.

Definition 2.1.11. *Borel set and Borel measurable space.*

Given a topological space $(X, \mathcal{T})$, according to *Theorem* 2.2.9, there is a σ-algebra $\mathcal{G}(\mathcal{T})$ produced by $\mathcal{T}$. A topology $\mathcal{T}$ is collection of subsets of X. Let the σ-algebra $\mathfrak{B}_\sigma(X) = \mathcal{G}(\mathcal{T})$ be the Borel σ-algebra in the topological space $(X,\mathcal{T})$ or simply the Borel algebra. Any set in $\mathfrak{B}_\sigma$(X) is called a Borel Set. In particular, $(X$, $\mathfrak{B}_\sigma$(X)) is called a Borel measurable space.

Definition 2.1.12. *F_{σ^-} set and G_{δ^-} set.*

The set represented by a union of countable closed sets is called an F_{σ^-} set and the set represented by an intersection of countable open sets is called a G_{δ^-} set.

Definition 2.1.21. *The extended Borel σ-algebra.*

The σ-algebra $\mathfrak{B}_\sigma(\bar{\mathbb{R}}) = \mathcal{G}\left(\mathcal{T}\left(\bar{\mathbb{R}}\right)\right) \subset \mathcal{P}(\bar{\mathbb{R}})$ generated by $\mathcal{T}(\bar{\mathbb{R}})$ is called the extended Borel σ-algebra.

Definition 2.3.1. *$\mathfrak{M}$-measurable.*

A function $f\colon X \to \bar{\mathbb{R}}$ defined on a measurable space $(X, \mathfrak{M})$ is called $\mathfrak{M}$-measurable.

Definition 2.3.12. *The positive part of the function $f\colon X \to \bar{\mathbb{R}}$ and the negative part of function $f\colon X \to \bar{\mathbb{R}}$.*

The two functions $f^{\pm}\colon X \to \bar{\mathbb{R}}$ from the function $f\colon X \to \bar{\mathbb{R}}$ defined on the set X are defined as

$$f^{+}(x) = \max\{f(x),\, 0\} \text{ and } f^{-}(x) = \max\{-f(x),\, 0\}.$$

Definition 2.3.23. *measurable.*

Given the measurable space $(X, \mathfrak{M}_1)$ and $(Y, \mathfrak{M}_2)$, if the function $F\colon X \to Y$ satisfies the condition

$$F^{-1}(E) \in \mathfrak{M}_1 \ (E \in \mathfrak{M}_2),$$

the function $F\colon X \to Y$ is called measurable.

2.2 σ-Algebras

Given a set X, the set of all subsets of X is denoted by $\mathcal{P}(X)$. Therefore, $\mathcal{P}(X) = \{A\colon A \subset X\}$.

Definition 2.2.1. *If the set $\mathfrak{M} \subset \mathcal{P}(X)$ consisting of subsets of X satisfies*

(1) $\phi, X \in \mathfrak{M}$;

(2) $A \in \mathfrak{M} \Longrightarrow A^c = X - A \in \mathfrak{M}$;

(3) $A_k \in \mathfrak{M}\ (k \in \mathbb{N}) \Longrightarrow \bigcup_{k=1}^{\infty} A_k \in \mathfrak{M}$;

the set $\mathfrak{M}$ is called a σ-algebra. Instead of (3), if $\mathfrak{M}$ satisfies

$(3') A_1, A_2, \cdots, A_n \in \mathfrak{M}(n \in \mathbb{N}) \Longrightarrow \bigcup_{k=1}^{n} A_k \in \mathbb{N}$,

the set $\mathfrak{M}$ is called an algebra of sets.

Example 2.2.2. *For a set X, $\mathfrak{M}_0 = \{A \in \mathcal{P}(X)$: either A is a finite set or A^c is a finite set$\}$.*

If X is a finite set, then $\mathfrak{M}_0 = \mathcal{P}(X)$, so $\mathfrak{M}_0$ is both an algebra of set and a σ-algebra. If X is an infinite set, $\mathfrak{M}_0$ is an algebra of set, but not σ-algebra.

Definition 2.2.3. *Given a set X and a σ-algebra $\mathfrak{M} \subset \mathcal{P}(X)$, $(X, \mathfrak{M})$ is called a measurable space, and any set in $\mathfrak{M}$ is called a measurable set.*

Proposition 2.2.4. *For a measurable space $(X,\mathfrak{M})$,*

$$A_k \in \mathfrak{M}(k \in \mathbb{N}) \Longrightarrow \bigcap_{k=1}^{\infty} A_k \in \mathfrak{M}.$$

Proof.

According to Definition 2.2.1 (2), $A_k^c \in \mathfrak{M}(k \in \mathbb{N})$ holds. Applying the Definition 2.2.1 (3), we get $\bigcup_k A_k^c \in \mathfrak{M}$ and apply the Definition 2.2.1 (2) and De Morgan's Law [Wei74], we get

$$\bigcap_{k=1}^{\infty} A_k = \left(\bigcup_{k=1}^{\infty} A_k^c\right)^c \in \mathfrak{M}.$$

Example 2.2.5. *Let $\mathfrak{M}_1 = \{\phi, X\}$ and $\mathfrak{M}_2 = \mathcal{P}(X)$ for a set X. Then, $(X, \mathfrak{M}_k)$ is a measurable space (k= 1, 2).*

Example 2.2.6. *Let $X = \mathbb{N}$, $A = \{1, 3, 5, \dots\}$ and B=$\{2, 4, 6, \dots\}$. Then, $\mathfrak{M} = \{\phi, A, B, X\} \subset \mathcal{P}(X)$ and $(X, \mathfrak{M})$ is a measurable space.*

Proposition 2.2.7. *Given an uncountable set X, let*

$$\mathfrak{M} = \{A \subset \mathcal{P}(X)\text{: } A \text{ is a countable set or } A^c \text{ is a countable set}\}.$$

$(X, \mathfrak{M})$ is a measurable space.

Proposition 2.2.8. *Let $(X, \mathfrak{M}_j)$ $(j = 1, 2)$ be a measurable space for a set X and*

$$\mathfrak{M} = \mathfrak{M}_1 \bigcap \mathfrak{M}_2.$$

Then, $(X, \mathfrak{M})$ is measurable space.

Theorem 2.2.9. *Given a set $\mathcal{A}(\neq \emptyset) \subset \mathcal{P}(X)$, there is one smallest σ-algebra $\sigma(\mathcal{A}) \subset \mathcal{P}(X)$ containing $\mathcal{A} \subset \mathcal{P}(X)$.*
Proof.
Consider the set $\mathcal{S} = \{\mathfrak{M} \subset \mathcal{P}(\mathrm{X})\text{: } \mathfrak{M} \text{ is a } \sigma\text{-algebra and } \mathcal{A} \subset \mathfrak{M}\}$. Now,

$$\sigma(\mathcal{A}) = \bigcap_{\mathfrak{M} \in \mathcal{S}} \mathfrak{M}.$$

It can be easily seen that $\sigma(\mathcal{A})$ is a σ-algebra. First, for all $\mathfrak{M} \in \mathcal{S}$, since $\mathcal{A} \subset \mathfrak{M}$, $\mathcal{A} \subset \sigma(\mathcal{A})$ clearly holds, and if $\mathfrak{M}' \subset \mathcal{P}(\mathrm{X})$ is a σ-algebra that satisfies the condition $\mathcal{A} \subset \mathfrak{M}'$, then by the *Definition* of $\mathcal{S}$, $\mathfrak{M}' \in \mathcal{S}$, so $\sigma(\mathcal{A}) \subset \mathfrak{M}'$ is satisfied. Therefore, $\sigma(\mathcal{A})$ is the smallest σ-algebra containing $\mathcal{A}$.

Definition 2.2.10. *Given a set $\mathcal{A}(\neq \emptyset) \subset \mathcal{P}(X)$, the σ-algebra $\sigma(\mathcal{A}) \subset \mathcal{P}(X)$ obtained from Theorem 2.2.9 is called the σ-algebra generated by $\mathcal{A}$.*

Definition 2.2.11. *Given a topological space $(X, \mathcal{T})$, according to Theorem 2.2.9, there is a σ-algebra $\mathcal{G}(\mathcal{T})$ produced by $\mathcal{T}$.*

A topology $\mathcal{T}$ is collection of subsets of X. Let the σ-algebra $\mathfrak{B}_\sigma(X) = \mathcal{G}(\mathcal{T})$ be the Borel σ-algebra in the topological space $(X,\mathcal{T})$ or simply the Borel algebra. Any set in $\mathfrak{B}_\sigma(\mathrm{X})$ is called a Borel Set. In particular, $(X, \mathfrak{B}_\sigma(\mathrm{X}))$ is called a Borel measurable space.

Definition 2.2.12. *The set represented by a union of countable closed sets is called an $F_\sigma-$set and the set represented by an intersection of countable open sets is called a $G_\delta-$set.*

Example 2.2.13. *Given a topological space $(X, \mathcal{T})$, any closed set is an $F_\sigma-$ set and any an open set is a $G_\delta-$ set.*

Also, the union of countable $F_\sigma-$ sets is an $F_\sigma-$ set and the intersection of countable $G_\delta-$ sets is a $G_\delta-$ set. In particular, a prerequisite for a set $A \in \mathcal{P}(\mathrm{X})$ to be an $F_\sigma-$ set is that the set $A^c \in \mathcal{P}(X)$ is a $G_\delta-$ set.

Note 2.2.14. *If the σ-algebra generated by the usual topology $\mathcal{U}_n$ has given on the set $\mathbb{R}^n = \overbrace{(\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R})}^{n}$ is expressed as $\mathfrak{B}_\sigma(\mathbb{R}^n) = \mathcal{G}(\mathcal{U}_n)$, then the elements of $\mathfrak{B}_\sigma(\mathbb{R}^n)$ are Borel sets in the topological space $\mathbb{R}^n$* [Wei74].

Note 2.2.15. *For the set $\mathbb{R}$, if $\mathcal{A} = \{(a, b)\text{: } -\infty < a < b < +\infty\} \subset \mathcal{P}(\mathbb{R})$, then $\mathcal{G}(\mathcal{A}) = \mathfrak{B}_\sigma(\mathbb{R})$ holds.*

Because we know that $\mathcal{U}_1 \subset \mathcal{G}(\mathcal{A})$, we get $\mathcal{G}(\mathcal{U}_1) \subset \mathcal{G}(\mathcal{A})$. Since $\mathcal{A} \subset \mathcal{U}_1$, $\mathcal{G}(\mathcal{A}) \subset \mathcal{G}(\mathcal{U}_1)$ holds. Therefore, $\mathfrak{B}_\sigma(\mathbb{R}) = \mathcal{G}(\mathcal{U}_1) \subset \mathcal{G}(\mathcal{A})$.

Proposition 2.2.16. *For the Borel σ-algebra $\mathfrak{B}_\sigma(\mathbb{R}) \subset \mathcal{P}(\mathbb{R})$ on the set $\mathbb{R}$, the following holds.*

(a) $\mathcal{A}_1 = \{(a, b]\colon -\infty < a < b < +\infty\} \subset \mathcal{P}(\mathbb{R}) \Longrightarrow \mathcal{G}(\mathcal{A}_1) = \mathfrak{B}_\sigma(\mathbb{R})$.

(b) $\mathcal{A}_2 = \{[a, b]\colon -\infty < a < b < +\infty\} \subset \mathcal{P}(\mathbb{R}) \Longrightarrow \mathcal{G}(\mathcal{A}_2) = \mathfrak{B}_\sigma(\mathbb{R})$.

(c) $\mathcal{A}_3 = \{(a, +\infty)\colon a \in \mathbb{R}\} \subset \mathcal{P}(\mathbb{R}) \Longrightarrow \mathcal{G}(\mathcal{A}_3) = \mathfrak{B}_\sigma(\mathbb{R})$.

Example 2.2.17. *Considering the usual topology $(\mathbb{R}, \mathcal{U}_1)$, for $a, b \in \mathbb{R}$ satisfying the condition $a < b$,* we get

$$(a, b) = \bigcup_{n=1}^{\infty} \left[a + \tfrac{1}{n}, b - \tfrac{1}{n}\right], \ \{a\} = \bigcap_{n=1}^{\infty} \left(a - \tfrac{1}{n}, a + \tfrac{1}{n}\right).$$

Therefore, an open interval is a G_δ–set and an F_σ–set. All open sets are both G_δ–sets and F_σ–sets. For any real number $p \in \mathbb{R}$, $\{p\}$ is a closed set, so $\{p\}$ is both an F_σ–set and a G_δ–set. Moreover, $\mathbb{Q} = \bigcup\limits_{q \in \mathbb{Q}} \{q\}$ is an F_σ–set.

Note 2.2.18. F_σ–sets and G_δ–sets are relatively simple Borel sets.

Moreover, the $F_{\sigma\delta}$–set is the intersection of countable F_σ–sets and the $G_{\delta\sigma}$–set is the union of countable G_δ–sets, which are also Borel sets,

$$\begin{cases} F_\sigma\text{–} \ set, F_{\sigma\delta}\text{–} \ set, F_{\sigma\delta\sigma}\text{–} \ set, \\ G_\delta\text{–} \ set, G_{\delta\sigma}\text{–} \ set, G_{\delta\sigma\delta}\text{–} \ set. \end{cases} \tag{2.1}$$

If the $F_{\sigma\delta}$–set is the countable union of the set, then the set is F_σ–. If the $G_{\delta\sigma}$–set is the countable intersection of the set, then the set is G_δ–. Thus, we define $F_{\sigma\delta}$– and $G_{\delta\sigma}$–. By repeating this process, a sequence consisting of Borel sets can be obtained.

Proposition 2.2.19. *Given a topological space $(X, \mathcal{T})$, let $[F]$ and $[G]$ be symbols representing all closed sets and all open sets, respectively.*

And all F_σ–sets are represented by $[F_\sigma]$, also $F_{\sigma\delta}$–set is represented by $[F_{\sigma\delta}]$. Define $[F_{\sigma\delta\sigma}]$ in the same way. All G_δ–sets are represented by $[G_\delta]$, also $G_{\delta\sigma}$– set is represented by $[G_{\delta\sigma}]$. Define $[G_{\delta\sigma\delta}]$ in the same way.

Given a topological space $(X, \mathcal{T})$, the following holds.

$$\begin{cases} [F] \subset [F_\sigma] \subset [F_{\sigma\delta}] \subset [F_{\sigma\delta\sigma}] \subset \cdots, \\ [G] \subset [G_\delta] \subset [G_{\delta\sigma}] \subset [G_{\delta\sigma\delta}] \subset \cdots. \end{cases}$$

First, the order and operation are determined as follows to deal with the set of all extended real numbers.

Note 2.2.20. *Assume the set of extended $\bar{\mathbb{R}} = [-\infty, +\infty]$, the following order and algebraic operations are given for $x \in \mathbb{R}$:*

- $-\infty < x < +\infty$,
- $(\pm\infty) + (\pm\infty) = \pm\infty$,
- $x + (\pm\infty) = (\pm\infty) + x = \pm\infty$,
- $\frac{x}{\infty} = 0 = \frac{x}{-\infty}$,
- $(\pm\infty)(\pm\infty) = +\infty$,
- $(\pm\infty)(\mp\infty) = -\infty$,

$$\circ\ x(\pm\infty) = (\pm\infty)x = \begin{cases} \pm\infty(x > 0), \\ 0(x = 0), \\ \mp\infty(x < 0), \end{cases}$$

$(+\infty) + (-\infty)$ and $(-\infty) + (+\infty)$ are not defined.
According to Note 2.2.20, we get:

(a) The set $\bar{\mathbb{R}}$ of all extended real numbers is an ordered set.

(b) Given a set $A \subset \bar{\mathbb{R}}$ arbitrarily, $\sup A$ and $\inf A$ exist within $\bar{\mathbb{R}}$.

(c) Given an arbitrarily point $a \in \mathbb{R}$, $(a,\, +\infty]$ is a neighborhood of $+\infty$.

Define

$$\mathcal{S}(\bar{\mathbb{R}}) = \{(a,\, +\infty],\, [-\infty,\, b)\colon\ a,\, b \in \mathbb{R}\} \subset \mathcal{P}(\bar{\mathbb{R}}).$$

The topology on $\bar{\mathbb{R}}$ generated by $\mathcal{S}(\bar{\mathbb{R}})$ is denoted by $\mathcal{T}(\bar{\mathbb{R}})$ [Kub07].

Definition 2.2.21. *The σ-algebra $\mathfrak{B}_\sigma(\bar{\mathbb{R}}) = \mathcal{G}\left(\mathcal{T}\left(\bar{\mathbb{R}}\right)\right) \subset \mathcal{P}(\bar{\mathbb{R}})$ generated by $\mathcal{T}(\bar{\mathbb{R}})$ is called the extended Borel σ-algebra.*

Proposition 2.2.22. *Let $\mathcal{A}_0 = \{(a,\, +\infty]\colon\ a \in \mathbb{R}\} \subset \mathcal{P}(\bar{\mathbb{R}})$. Then, $\mathfrak{B}_\sigma(\bar{\mathbb{R}}) = \mathcal{G}(\mathcal{A}_0)$, that is, $\mathcal{A}_0$ generates the extended Borel σ-algebra.*

Proof.

Let $b \in \mathbb{R}$ and a choose a sequence $(b_n)_{n\geq 1} \subset \mathbb{R}$ such that

$$\begin{cases} b_n \leq b_{n+1} < b \ \ (n \in \mathbb{N}), \\ \lim\limits_{n\to+\infty} b_n = b. \end{cases}$$

Then,

$$[-\infty,\, b) = \bigcup_{n=1}^{\infty} [-\infty,\, b_n] = \bar{\mathbb{R}} - \bigcap_{n=1}^{\infty} (b_n,\, +\infty]$$

holds, so that $[-\infty,\, b) \in \mathcal{G}(\mathcal{A}_0)$, and in particular, $\mathcal{S}(\bar{\mathbb{R}}) \subset \mathcal{G}(\mathcal{A}_0)$ is obtained. Therefore, $\mathcal{A}_0 \subset \mathcal{T}(\bar{\mathbb{R}}) \subset \mathcal{G}(\mathcal{A}_0)$ hold.

2.3 Measurable Functions

Definition 2.3.1. *A function $f\colon X \to \bar{\mathbb{R}}$ defined on a measurable space $(X,\, \mathfrak{M})$ is called $\mathfrak{M}$-measurable if*

$$\{x \in X\colon\ f(x) > \alpha\} \in \mathfrak{M}\ (\alpha \in \mathbb{R})$$

is satisfied. We denoted it $f \in \mathcal{M}(X,\, \mathfrak{M})$.

Notation 2.3.2. *The full set of measurable functions with extended real values on the measurable space $(X,\, \mathfrak{M})$ will be expressed as follows.*

$$\mathcal{M}(X, \mathfrak{M}) = \{f : X \to \bar{\mathbb{R}} \mid f \text{ is a measurable function}\}.$$

Lemma 2.3.3. *Given an extended real-valued function $f\colon X \to \bar{\mathbb{R}}$ for the measurable space $(X, \mathfrak{M})$, the following are equivalent to each other.*

(*a*) $f \in \mathcal{M}(X, \mathfrak{M})$.
(*b*) $A_\alpha = \{x \in X\colon f(x) > \alpha\} \in \mathfrak{M}(\alpha \in \mathbb{R})$.
(*c*) $B_\alpha = \{x \in X\colon f(x) \leq \alpha\} \in \mathfrak{M}(\alpha \in \mathbb{R})$.
(*d*) $C_\alpha = \{x \in X\colon f(x) \geq \alpha\} \in \mathfrak{M}(\alpha \in \mathbb{R})$.
(*e*) $D_\alpha = \{x \in X\colon f(x) < \alpha\} \in \mathfrak{M}(\alpha \in \mathbb{R})$.

Proof.

(a)$\Longleftrightarrow$(b) is clear by *Definition* 2.3.1.
(b)$\Longleftrightarrow$(c) is obtained immediately by applying $B_\alpha = A_\alpha^c$ and $A_\alpha = B_\alpha^c$.
(b)$\Longrightarrow$(d). Assuming that (b) is established, $A_{\alpha-\frac{1}{n}} \in \mathfrak{M}$ for any natural number $n \in \mathbb{N}$. However, if you apply the Archimedes property [Hal74], we get

$$\bigcap_{n=1}^{\infty} A_{\alpha-\frac{1}{n}} \subset C_\alpha.$$

Since the opposite inclusion is obvious,

$$C_\alpha = \bigcap_{n=1}^{\infty} A_{\alpha-\frac{1}{n}}$$

is established and thus $C_\alpha \in \mathfrak{M}$ is known.
(d)$\Longrightarrow$(b). Assuming that (d) is established, $C_{\alpha+\frac{1}{n}} \in \mathfrak{M}$ for any $n \in \mathbb{N}$. However, if you apply the Archimedes property [Hal74], we get

$$A_\alpha \subset \bigcup_{n=1}^{\infty} C_{\alpha+\frac{1}{n}}.$$

Since the opposite inclusion is clear,

$$A_\alpha = \bigcup_{n=1}^{\infty} C_{\alpha+\frac{1}{n}}.$$

is established and thus $A_\alpha \in \mathfrak{M}$ is known.
(d)$\Longleftrightarrow$(e), If $C_\alpha = D_\alpha^c$ and $D_\alpha = C_\alpha^c$ are applied, the equivalence can be obtained immediately.

Example 2.3.4. *All constant functions are measurable. Let $f\colon X \to \bar{\mathbb{R}}$ be a constant function defined by $f(x) = c\,(x \in X)$ and let $\alpha \in \mathbb{R}$.*
If $c \in \mathbb{R}$:

$$\{x \in X\colon f(x) > \alpha\} = \begin{cases} \emptyset & (\alpha \leq c), \\ X & (\alpha < c). \end{cases}$$

If $c \in \{-\infty, +\infty\}$: since $-\infty < \alpha < +\infty$, we get

$$\{x \in X\colon f(x) > \alpha\} = \begin{cases} \emptyset & (c = -\infty), \\ X & (c = +\infty). \end{cases}$$

Since $\{x \in X\colon f(x) > \alpha\} \in \mathfrak{M}$ is established in either case, the constant function $f\colon X \to \bar{\mathbb{R}}$ is measurable.

Example 2.3.5. *A function* χ_E*:* $X \to \mathbb{R}$ *defined as*

$$\chi_E(x) = \begin{cases} 1 & (x \in E) \\ 0 & (x \notin E) \end{cases}$$

for $E \in \mathcal{P}(X)$ is called a characteristic function of the set $E \in \mathcal{P}(X)$. If $E \in \mathfrak{M}$, the function χ_E: $X \to \mathbb{R}$ is measurable. Because we can see

$$\{x \in X\colon\ \chi_E(x) > \alpha\} = \begin{cases} \emptyset & (\alpha\ \leq 1) \\ E & (0 \leq \alpha < 1) \\ X & (\alpha\ < 0) \end{cases}$$

for any $\alpha \in \mathbb{R}$. In either case, $\{x \in X\colon\ \chi_E > \alpha\} \in \mathfrak{M}$ is established and the function χ_E: $X \to \mathbb{R}$ is measurable.

Proposition 2.3.6. *When considering the Borel measurable space* $(X,\ \mathfrak{B}_\sigma(X))$ *for the Topological space* $(X,\ \mathcal{T})$*, all continuous functions* $f\colon X \to \bar{\mathbb{R}}$ *are* $\mathfrak{B}_\sigma(X)$*-measurable.*
Proof.

We see that $\{x \in X\colon f(x) > \alpha\} = f^{-1}((\alpha,\ +\infty])$ for any $\alpha \in \mathbb{R}$. $(\alpha,\ +\infty]$ is an open set on $\bar{\mathbb{R}}$ and the function f is continuous, so $f^{-1}((\alpha, +\infty]) \in \mathcal{T}$. Therefore, $\{x \in X\colon f(x) > \alpha\} \in \mathfrak{B}_\sigma(X)$ is established, and thus the continuous function f: $X \to \mathbb{R}$ is $\mathfrak{B}_\sigma(X)$-measurable.

Lemma 2.3.7. *When considering the Borel measurable space* $(\mathbb{R},\ \mathfrak{B}_\sigma(\bar{\mathbb{R}}))$ *for the Topological space* $(X,\ \mathcal{T})$*, all continuous functions* $f\colon \mathbb{R} \to \bar{\mathbb{R}}$ *is Borel measurable.*

Example 2.3.8. *Considering the Borel measurable space* $(\mathbb{R},\ \mathfrak{B}_\sigma(\bar{\mathbb{R}}))$*, all monotone functions* $f\colon \mathbb{R} \to \bar{\mathbb{R}}$ *are measurable.* Let's only look at the case where the function f: $\mathbb{R} \to \bar{\mathbb{R}}$ is a monotone increase. That is, for any $\alpha \in \mathbb{R}$, there is a real number $\beta \in \mathbb{R}$ that satisfies

$$\{x \in \mathbb{R}\colon f(x) > \alpha\} \in \{(\beta,\ +\infty),\ [\beta,\ +\infty),\ \phi,\ \mathbb{R}\}.$$

In any case, $\{x \in \mathbb{R}\colon f(x) > \alpha\} \in \mathfrak{B}_\sigma(\mathbb{R})$ holds, and the monotone increasing function f: $\mathbb{R} \to \bar{\mathbb{R}}$ is measurable.

Note 2.3.9. *For the measurable space* $(X, \mathfrak{M})$*, the following holds.*

- $\chi_A \in \mathcal{M}(X,\ \mathfrak{M}) \Longleftrightarrow A \in \mathfrak{M}$.
- If $\mathfrak{M} = \{\phi,\ X\}$, $f \in \mathcal{M}(X,\ \mathfrak{M}) \Longleftrightarrow f$ is a constant function.
- If $\mathfrak{M} = \mathcal{P}(X)$, any function f: $X \to \bar{\mathbb{R}}$ is measurable.

Example 2.3.10. *Given a measurable function* $f \in \mathcal{M}(X,\ \mathfrak{M})$ *and a positive number* $r > 0$*, if the function* f_r*:* $X \to \mathbb{R}$ *is defined as*

$$f_r(x) = \begin{cases} r & (f(x) > r), \\ f(x) & (|f(x)| \leq r), \\ -r & (f(x) < -r)), \end{cases}$$

then $f_r \in \mathcal{M}(X,\ \mathfrak{M})$. If any real number $\alpha \in \mathbb{R}$,

$$\{x \in X\colon f_r > \alpha\} = \begin{cases} X & (\alpha < -r), \\ \{x \in X\colon f(x) > \alpha\} & (-r \leq \alpha < r), \\ \emptyset & (r \leq \alpha) \end{cases}$$

is established, so $f_r \in \mathcal{M}(X, \mathfrak{M})$.

Lemma 2.3.11. *Given measurable functions f, $g\colon X \to \mathbb{R}$ on the measurable space $(X, \mathfrak{M})$ and a real number $c \in \mathbb{R}$, we have cf, f^2, $f+g$, fg, and $|f| \in \mathcal{M}(X, \mathfrak{M})$.*

Proof.

(1) Consider the case of cf.

- If $c = 0$, cf is a constant function, so cf is a measurable function.
- If $c \neq 0$,

$$\{x \in X\colon (cf)(x) > \alpha\} = \begin{cases} \{x \in X\colon f(x) > \frac{\alpha}{c}\} & (c > 0), \\ \{x \in X\colon f(x) < \frac{\alpha}{c}\} & (c < 0) \end{cases}$$

is established for any $\alpha \in \mathbb{R}$. In any case, $\{x \in X\colon (cf)(x) > \alpha\} \in \mathfrak{M}$ is established, so cf is a measurable function.

(2) In the case of f^2,

$$\{x \in X\colon f^2(x) > \alpha\} =$$

$$\begin{cases} X & (\alpha < 0), \\ \{\text{x} \in X\colon f(x) > \sqrt{\alpha}\} \cup \{x \in X\colon f(x) < -\sqrt{\alpha}\} & (\alpha \geq 0) \end{cases}$$

is established for any $\alpha \in \mathbb{R}$. In any case, $\{x \in X\colon f^2(x) > \alpha\} \in \mathfrak{M}$ is established, so f^2 is a measurable function.

(3) In the case of $f+g$, if any real number $\alpha \in \mathbb{R}$ is fixed and the set $\mathcal{S}_r$ is given by

$$\mathcal{S}_r = \{x \in X\colon f(x) > r\} \cap \{x \in X\colon f(x) > \alpha - r\}$$

for a rational number $r \in \mathbb{Q}$, then $\mathcal{S}_r \in \mathfrak{M}$ is established. Also, $f+g$ is a measurable function because

$$\{x \in X\colon (f+g)(x) > \alpha\} = \bigcup_{r \in \mathbb{Q}} \mathcal{S}_r$$

is established by the density of the rational numbers in $\mathbb{R}$.

(4) In case of fg, if equation

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$$

is used and (1) – (3) are applied, then fg is a measurable function.

(5) In case of $|f|$,

$$\{x \in X\colon |f|(x) > \alpha\} =$$

$$\begin{cases} X & (\alpha < 0), \\ \{x \in X\colon f(x) > \alpha\} \cup \{x \in X\colon f(x) < -\alpha\} & (\alpha \geq 0) \end{cases}$$

is established for any real number $\alpha \in \mathbb{R}$. In either case,
$\{x \in X\colon |f|(x) > \alpha\} \in \mathfrak{M}$ is established, so $|f|$ is a measurable function.

Definition 2.3.12. *The two functions* $f^{\pm}\colon X \to \bar{\mathbb{R}}$ *from the function* $f\colon X \to \bar{\mathbb{R}}$ *defined on the set* X *are defined as*

$$f^{+}(x) = max\{f(x),\, 0\} \text{ and } f^{-}(x) = max\{-f(x),\, 0\}.$$

The function $f^{+}\colon X \to \bar{\mathbb{R}}$ is called the positive part of the function $f\colon X \to \bar{\mathbb{R}}$ and $f^{-}\colon X \to \bar{\mathbb{R}}$ is called the negative part of function $f\colon X \to \bar{\mathbb{R}}$.

Note 2.3.13. *For the function* $f\colon X \to \bar{\mathbb{R}}$ *defined on the set* X*, the following holds.*

$$\begin{cases} f = f^{+} - f^{-} \text{ and } |f| = f^{+} + f^{-}, \\ f^{+} = \frac{1}{2}\big(|f| + f\big) \text{ and } f^{-} = \frac{1}{2}\big(|f| - f\big). \end{cases}$$

Proposition 2.3.14. *If two functions* f*,* g *and* $h\colon X \to \bar{\mathbb{R}}$ *defined on the set* X *satisfy*

$$\begin{cases} f(x) = g(x) - h(x) \quad (x \in X), \\ min\{g(x),\, h(x)\} \geq 0 \; (x \in X), \end{cases}$$

$f^{+}(x) \leq g(x)$ and $f^{-}(x) \leq h(x)$ $(x \in X)$ is established. In other words, the positive part f^{+} and the negative part f^{-} of the function f are the smallest among the functions where the function f is expressed as the difference of the non-negative functions.

Note 2.3.15. *If* $f \in \mathcal{M}(X,\, \mathfrak{M})$*, the following is established by Archimedes Property* [Hal74].

$$\{x \in X\colon f(x) = +\infty\} = \bigcap_{n=1}^{\infty} \{x \in X\colon f(x) > n\} \in \mathfrak{M},$$
$$\{x \in X\colon f(x) = -\infty\} = \bigcup_{n=1}^{\infty} \{x \in X\colon f(x) > -n\}^{c} \in \mathfrak{M}.$$

Theorem 2.3.16. *The following is established for* $f\colon X \to \bar{\mathbb{R}}$ *having the measurable space* $(X,\, \mathfrak{M})$ *and extended real number value.*

$$f \in \mathcal{M}(X,\, \mathfrak{M}) \iff \begin{cases} \{x \in X\colon f(x) = -\infty\} \in \mathfrak{M}, \\ \{x \in X\colon \alpha < f(x) < +\infty\} \in \mathfrak{M} \; (\alpha \in \mathbb{R}), \\ \{x \in X\colon f(x) = \infty\} \in \mathfrak{M}. \end{cases}$$

Note 2.3.17. *Given* $f \in \mathcal{M}(X,\, \mathfrak{M})$*,*

$$\{x \in X\colon -\infty < f(x) < \alpha\} = [\{x \in X\colon \alpha < f(x) < +\infty\} \cup \{x \in X\colon f(x) = \alpha\}]^{c}$$

is established for any $\alpha \in \mathbb{R}$, so it can be seen that $\{x \in X\colon -\infty < f(x) < \alpha\} \in \mathfrak{M}$ is established.

Theorem 2.3.18. *Given a measurable function* $f \in \mathcal{M}(X,\, \mathfrak{M})$ *and any real number* $c \in \mathbb{R}$*,* cf*,* f^2*,* $|f|$ *and* $f^{\pm} \in \mathcal{M}(X,\, \mathfrak{M})$*.*

Theorem 2.3.19. *Given a measurable function sequence* $(f_n)_{n\geq 1} \subset \mathcal{M}(X,\, \mathfrak{M})$*, we define the function,* f*,* F*,* f^{*}*,* $F^{*}\colon X \to \bar{\mathbb{R}}$ *as follows.*

$$f(x) = \inf_{n} f_n(x),\; F(x) = \sup_{n} f_n(x),$$
$$f^{*}(x) = \liminf_{n\to+\infty} f_n(x),\; F^{*}(x) = \limsup_{n\to+\infty} f_n(x).$$

Then, $f,\ F,\ f^*,\ F^* \in \mathcal{M}(X, \mathfrak{M})$.

Corollary 2.3.20. *If the measurable function sequence* $(f_n)_{n\geq 1} \subset \mathcal{M}(X, \mathfrak{M})$ *satisfies the condition*

$$\lim_{n\to\infty} f_n(x) = f(x)\,(x \in X),\text{ then } f \in \mathcal{M}(X, \mathfrak{M}).$$

Corollary 2.3.21. *Given two measurable functions* $f, g \in \mathcal{M}(X, \mathfrak{M})$*, then* $fg \in \mathcal{M}(X, \mathfrak{M})$. That is,

$$f, g \in \mathcal{M}(X, \mathfrak{M}) \Longrightarrow fg \in \mathcal{M}(X, \mathfrak{M}).$$

Theorem 2.3.22. *Given a measurable space* $(X,\ \mathfrak{M})$ *and the function* $f\colon\ X \to \bar{\mathbb{R}}$*, there is a simple function sequence* $(\phi_n)_{n\geq 1}$ *that satisfies the condition*

$$f(x) = \lim_{n\to+\infty} \phi_n(x)\ (x \in X).$$

Moreover, the following holds.

- If $f \in \mathcal{M}(X,\ \mathfrak{M})$, then $(\phi_n)_{n\geq 1} \subset \mathcal{S}(X,\ \mathfrak{M})$.
- If $f^* \in \mathcal{M}^+(X,\ \mathfrak{M})$, the simple measurable function sequence $(\phi_n)_{n\geq 1} \subset \mathcal{S}^*(X,\ \mathfrak{M})$ increases monotonically. That is, it holds that $0 \leq \phi_n(x) \leq \phi_{n+1}(x) \leq f(x)$ ($n \in \mathbb{N}$ and $x \in X$).
- If $f \in \mathcal{M}(X,\ \mathfrak{M})$ is bounded, the simple measurable function sequence $(\phi_n)_{n\geq 1} \subset \mathcal{S}(X,\ \mathfrak{M})$ uniformly converges to the measurable function $f \in \mathcal{M}(X,\ \mathfrak{M})$ on the measurable space $(X,\ \mathfrak{M})$.

Definition 2.3.23. *Given two measurable spaces* $(X,\ \mathfrak{M}_1)$ *and* $(Y,\ \mathfrak{M}_2)$*, if the function* $F\colon\ X \to Y$ *satisfies the condition*

$$F^{-1}(E) \in \mathfrak{M}_1\ (E \in \mathfrak{M}_2),$$

the function $F\colon\ X \to Y$ is called measurable.

Theorem 2.3.24. *Given two measurable spaces* $(X,\ \mathfrak{M}_1)$ *and* $(Y,\ \mathfrak{M}_2)$ *and the measurable function* $F\colon\ X \to Y$*, the following holds.*

$$f \in \mathcal{M}(Y,\ \mathfrak{M}_2) \Longrightarrow f \circ F \in \mathcal{M}(X,\ \mathfrak{M}_1).$$

Chapter 3

Measure Spaces

3.1 Related Theorems and Definitions

Definition 3.1.1. *A measure and a measure space.*

Given a measurable space $(X,\, \mathfrak{M})$, if the function $\mu\colon\, \mathfrak{M} \to \bar{\mathbb{R}}$ with an extended real-valued and defined on σ-algebra $\mathfrak{M}$, satisfies the condition

(1) $\mu(\emptyset) = 0$,

(2) $\mu(E) \geq 0$ $(E \in \mathfrak{M})$,

(3) $(E_n)_{n\geq 1} \subset \mathfrak{M}$ and $E_n \cap E_m = \phi(n \neq m) \Longrightarrow \mu\Big(\bigcup\limits_{k=1}^{\infty} E_k\Big) = \sum\limits_{k=1}^{\infty} \mu(E_k)$,

$\mu\colon\, \mathfrak{M} \to \bar{\mathbb{R}}$ is called a measure and $(X,\, \mathfrak{M},\, \mu)$ is called a measure space.

Definition 3.1.2. *A finite measure, finite measure space and σ-finite measure space.*

Given a measure space $(X,\, \mathfrak{M},\, \mu)$, it is defined as follows.

(a) If the condition $\mu(X) < +\infty$ is satisfied, μ is called a finite measure and $(X,\, \mathfrak{M},\, \mu)$ is called a finite measure space.

(b) If there exists a measurable set sequence $(E_n)_{n\geq 1} \subset \mathfrak{M}$ that satisfies the condition $X = \bigcup\limits_{n=1}^{\infty} E_n$ and $\mu(E_n) < +\infty$ $(n \in \mathbb{N})$, μ is called a σ-finite measure and $(X,\, \mathfrak{M},\, \mu)$ is called a σ-finite measure space.

Definition 3.1.3. *The Borel measure space.*

For the Topological space $(X,\, \mathcal{T})$, if the measure $\mu\colon\, \mathfrak{B} \to \bar{\mathbb{R}}$ is given and the *Borel* measure space $\big(X,\, \mathfrak{B}_\sigma(X)\big)$ is considered, $\big(X,\, \mathfrak{B}_\sigma(X),\, \mu\big)$ is called the *Borel* measure space.

Definition 3.1.4. *A translation invariant measure space.*

If the measure space $(X,\, \mathfrak{M},\, \mu)$ satisfies the condition

$$E \in \mathfrak{M} \text{ and } a \in k \Longrightarrow E + a \in \mathfrak{M} \text{ and } \mu(E + a) = \mu(E)$$

for a closed set X by addition, the measure space $(X,\, \mathfrak{M},\, \mu)$ is called a translation invariant measure space.

Definition 3.1.5. *A complete measure space and complete measure.*

If a measure space $(X,\, \mathfrak{M},\, \mu)$ is given and the conditions

$$N \in \mathfrak{M} \text{ and } \mu(N) = 0 \Longrightarrow \mathcal{P}(N) \subset \mathfrak{M}$$

are satisfied, $(X, \mathfrak{M}, \mu)$ is called a complete measure space and the function μ: $\mathfrak{M} \to [0, +\infty]$ is called a complete measure.

Definition 3.1.6. *The completion of the measure space, completion of* $\mathfrak{M}$ *and completion of* μ.

Given a measure space $(X, \mathfrak{M}, \mu)$, a complete measure space $(X, \widehat{\mathfrak{M}}, \hat{\mu})$ that satisfies the condition $[\mathfrak{M} \subset \widehat{\mathfrak{M}}$ and $\hat{\mu}|\mathfrak{M} = \mu]$ is called the completion of the measure space $(X, \mathfrak{M}, \mu)$. The σ-algebra $\widehat{\mathfrak{M}}$ is called the completion of $\mathfrak{M}$ for a measure μ, and a measure $\hat{\mu}$ is called the completion of μ.

3.2 Measures

Definition 3.2.1. *Given a measurable space* $(X, \mathfrak{M})$*, if the function* μ*:* $\mathfrak{M} \to \bar{\mathbb{R}}$ *with an extended real-valued and defined on* σ*-algebra* $\mathfrak{M}$*, satisfies the condition*

(1) $\mu(\emptyset) = 0$,

(2) $\mu(E) \geq 0$ $(E \in \mathfrak{M})$,

(3) $(E_n)_{n\geq 1} \subset \mathfrak{M}$ and $E_n \cap E_m = \phi(n \neq m) \Longrightarrow \mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$,

μ: $\mathfrak{M} \to \bar{\mathbb{R}}$ is called a measure and $(X, \mathfrak{M}, \mu)$ is called a measure space.

Note 3.2.2. *A measurable set sequence* $(E_n)_{n\geq 1} \subset \mathfrak{M}$ *satisfies the condition* $E_n \cap E_m = \phi(n \neq m)$ *is called a disjoint measurable set sequence.* The property

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

is called countably additivity of μ for a disjoint measurable set sequence $(E_n)_{n\geq 1} \subset \mathfrak{M}$.

Definition 3.2.3. *Given a measure space* $(X, \mathfrak{M}, \mu)$*, it is defined as follows.*

(a) If the condition $\mu(X) < +\infty$ is satisfied, μ is called a finite measure and $(X, \mathfrak{M}, \mu)$ is called a finite measure space.

(b) If there exists a measurable set sequence $(E_n)_{n\geq 1} \subset \mathfrak{M}$ that satisfies the condition $X = \bigcup_{n=1}^{\infty} E_n$ and $\mu(E_n) < +\infty$ $(n \in \mathbb{N})$, μ is called a σ-finite measure and $(X, \mathfrak{M}, \mu)$ is called a σ-finite measure space.

Example 3.2.4. *If function* μ_1*:* $\mathcal{P}(X) \to \bar{\mathbb{R}}$ *is defined as* $\mu_1(E) = 0$ $(E \in \mathcal{P}(X))$ *for the measurable space* $(X, \mathcal{P}(X))$*, then the function* μ_1*:* $\mathcal{P}(X) \to \bar{\mathbb{R}}$ *is a finite measure.* Also, if the function μ_2: $\mathcal{P}(X) \to \bar{\mathbb{R}}$ is defined as

$$\mu_2(E) = \begin{cases} 0 & (E = \emptyset), \\ +\infty & (E \neq \emptyset), \end{cases}$$

then the function μ_2: $\mathcal{P}(X) \to \bar{\mathbb{R}}$ is neither a finite measure nor a σ-finite measure.

Example 3.2.5. *If a point* $p \in X$ *is fixed and the function* μ*:* $\mathcal{P}(X) \to \bar{\mathbb{R}}$ *is defined as*

$$\mu(E) = \begin{cases} 0 & (p = E), \\ 1 & (p \neq E) \end{cases}$$

for the measurable space $(X, \mathcal{P}(X))$, then the function μ: $\mathcal{P}(X) \to \bar{\mathbb{R}}$ is a finite measure. The measure μ: $\mathcal{P}(X) \to \bar{\mathbb{R}}$ is called a Dirac measure or a point mass measure at point $p \in X$.

Example 3.2.6. (Counting measure [Kub07]) *If the function μ: $\mathcal{P}(\mathbb{N}) \to \bar{\mathbb{R}}$ is defined as*

$$\mu(E) = \begin{cases} \#(\mathrm{E}) & (\text{finite set}), \\ +\infty & (\text{infinite set}) \end{cases}$$

for the measurable space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, then the function μ: $\mathcal{P}(X) \to \bar{\mathbb{R}}$ is a measure. In particular, the function μ: $\mathcal{P}(X) \to \bar{\mathbb{R}}$ is a σ-finite measure, but not a finite measure.

Definition 3.2.7. *For the Topological space $(X, \mathcal{T})$, if the measure μ: $\mathfrak{B} \to \bar{\mathbb{R}}$ is given and the Borel measure space $\big(X, \mathfrak{B}_\sigma(X)\big)$ is considered, $\big(X, \mathfrak{B}_\sigma(X), \mu\big)$ is called the Borel measure space.*

Example 3.2.8. *Given the Borel measurable space $(\mathbb{R}, \mathfrak{B}_\sigma(\mathbb{R}))$, there is a unique measure λ: $\mathfrak{B}_\sigma(\mathbb{R}) \to \bar{\mathbb{R}}$ that satisfies the condition*

$$\lambda\big((a, b]\big) = b - a \ (-\infty \leq a < b < +\infty).$$

The measure λ: $\mathfrak{B}_\sigma(\mathbb{R}) \to \bar{\mathbb{R}}$ is called the *Borel* measure, and $(\mathbb{R}, \mathfrak{B}_\sigma(\mathbb{R}), \lambda)$ is called the *Borel* measure space. The *Borel* measure λ: $\mathfrak{B}_\sigma(\mathbb{R}) \to \bar{\mathbb{R}}$ is a σ-finite measure.

Example 3.2.9. *Given a Borel measurable space $(\mathbb{R}, \mathfrak{B}_\sigma(\mathbb{R}))$ and a monotone increasing continuous function f: $\mathbb{R} \to \mathbb{R}$, there is a unique measure λ_f: $\mathfrak{B}_\sigma(\mathbb{R}) \to \bar{\mathbb{R}}$ that satisfies the condition*

$$\lambda_f((a, b]) = f(b) - f(a) \ (-\infty \leq a < b < +\infty).$$

The measure λ_f: $\mathfrak{B}_\sigma(\mathbb{R}) \to \bar{\mathbb{R}}$ is a *Borel* measure derived by the function f: $\mathbb{R} \to \mathbb{R}$, and $(\mathbb{R}, \mathfrak{B}_\sigma(\mathbb{R}), \lambda_f)$ is called the Borel measure space.

Definition 3.2.10. *If the measure space $(X, \mathfrak{M}, \mu)$ satisfies the condition*

$$E \in \mathfrak{M} \text{ and } a \in k \Longrightarrow E + a \in \mathfrak{M} \text{ and } \mu(E + a) = \mu(E)$$

for a closed set X by addition, the measure space $(X, \mathfrak{M}, \mu)$ is called a translation invariant measure space.

Lemma 3.2.11. *Given a measurable set sequence $(A_n)_{n\geq 1} \subset \mathfrak{M}$ for a measurable space $(X, \mathfrak{M})$, there exists a monotone increasing measurable set sequence $(E_n)_{n\geq 1} \subset \mathfrak{M}$ and a disjoint measurable set sequence $(F_n)_{n\geq 1} \subset \mathfrak{M}$ that satisfies condition*

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} F_n.$$

Proposition 3.2.12. *Given a measure space $(X, \mathfrak{M}, \mu)$ and $E, F \in \mathfrak{M}$, the following holds.*

(a) $E \subset F \Longrightarrow \mu(E) \leq \mu(F)$.

(b) $E \subset F$ and $\mu(E) < +\infty \Longrightarrow \mu(F - E) = \mu(F) - \mu(E)$.

Proposition 3.2.13. *Given a measure space $(X, \mathfrak{M}, \mu)$, the following holds.*

(a) $(E_n)_{n\geq 1} \subset \mathfrak{M}$: monotone increasing set sequence

$\Longrightarrow \mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \lim_{n\to+\infty} \mu(E_n).$

(b) $(E_n)_{n\geq 1} \subset \mathfrak{M}$: monotone decreasing set sequence and $\mu(E_1) < +\infty$

$\Longrightarrow \mu\Big(\bigcap_{n=1}^{\infty} E_n\Big) = \lim_{n\to+\infty} \mu(E_n).$

3.3 Completion of Measures

3.3.1 Complete Measures

Definition 3.3.1.1. *If a measure space $(X, \mathfrak{M}, \mu)$ is given and the conditions*

$$N \in \mathfrak{M} \text{ and } \mu(N) = 0 \Longrightarrow \mathcal{P}(N) \subset \mathfrak{M}$$

are satisfied, $(X, \mathfrak{M}, \mu)$ is called a complete measure space and the function μ: $\mathfrak{M} \to [0, +\infty]$ is called a completion.

Given a measure space $(X, \mathfrak{M}, \mu)$, let $\mathfrak{N}$ be the collection of all sets in $\mathfrak{M}$:

$$\mathfrak{N} = \{N \in \mathfrak{M}\colon\ \mu(N) = 0\}.$$

If μ is not complete, then $N \in \mathfrak{M}$ and $\mu(N) = 0$. There is $F \in \mathcal{P}(N)$ that satisfies $F \notin \mathfrak{M}$. Now, consider the completion of a given measure space $(X, \mathfrak{M}, \mu)$.

Proposition 3.3.1.2. *Given a measure space $(X, \mathfrak{M}, \mu)$, the σ-algebra $\mathcal{G}(\mathfrak{M} \cup \mathfrak{D})$ generated by $\mathfrak{M} \cup \mathfrak{D}$ is expressed as follows.*

$$\mathcal{G}(\mathfrak{M} \cup \mathfrak{D}) = \{E \cup F\colon\ E \in \mathfrak{M} \text{ and } F \in \mathfrak{D}\}. \tag{3.1}$$

Theorem 3.3.1.3. *Given a measure space $(X, \mathfrak{M}, \mu)$, there exists only one complete measure space $(X, \mathfrak{M}, \mu)$ that satisfies the condition*

$$\begin{cases} \mathfrak{M} \subset \widehat{\mathfrak{M}}, \\ \hat{\mu}|\mathfrak{M} = \mu. \end{cases}$$

Definition 3.3.1.4. *Given a measure space $(X, \mathfrak{M}, \mu)$, a complete measure space $(X, \widehat{\mathfrak{M}}, \hat{\mu})$ that satisfies the condition $[\mathfrak{M} \subset \widehat{\mathfrak{M}}$ and $\hat{\mu}|\mathfrak{M} = \mu]$ is called the completion of the measure space $(X, \mathfrak{M}, \mu)$.* The σ-algebra $\widehat{\mathfrak{M}}$ is called the completion of $\mathfrak{M}$ for a measure μ, and a measure $\hat{\mu}$ is called the completion of μ.

3.3.2 Almost Everywhere

Definition 3.3.2.5. *Given a measure space $(X, \mathfrak{M}, \mu)$ and $P(x)$ is a Proposition for $x \in X$, the condition*

$$\begin{cases} \mu(N) = 0, \\ \{x \in X\colon \sim P(x)\} \subset N, \end{cases}$$

is satisfied. If $N \in \mathfrak{M}$ exists, almost everywhere $P(x)$ is established. It is denoted as

$$P\ \mu\text{-a.e } [X] \text{ or } P(x)\ \mu\text{-a.e } x \in X.$$

Note 3.3.2.6. *If the measure space (X, $\mathfrak{M}$, μ) is complete, it can be seen that the following holds.*

$$P\mu\text{-a.e [X]} \Longleftrightarrow \mu\big(\{x \in X\colon \sim P(x)\}\big) = 0.$$

Example 3.3.2.7. *Considering the Lebesgue measure space* ($\mathbb{R}$, $\mathfrak{M}$, γ) *and the characteristic function* $\chi_{\mathbb{Q}}$ *of the set of all rational numbers* $\mathbb{Q} \subset \mathbb{R}$, $\{x \in \mathbb{R} : \chi_{\mathbb{Q}}(x) \neq 0\} = \mathbb{Q}$ *and* $\gamma(\mathbb{Q}) = 0$, *so* $\chi_{\mathbb{Q}} = 0$ γ*-a.e* [$\mathbb{R}$]. Almost all real numbers are irrational numbers.

Proposition 3.3.2.8. *Given a complete measure space* (X, $\mathfrak{M}$, μ) *and the functions* $f, g\colon X \to \bar{\mathbb{R}}$, *if* $f \in \mathcal{M}(X, \mathfrak{M})$ *and* $f = g$ μ*-a.e [X], the following holds.*

(a) $g \in \mathcal{M}(X, \mathfrak{M})$.

(b) $\mu(\{x \in X\colon g > \alpha\}) = \mu(\{x \in X\colon f(x) > \alpha\}$ $(\alpha \in \mathbb{R})$.

Proof.

If $N = \{x \in X\colon f(x) = g(x)\}$, since $f = g$ μ-a.e $[X]$ is established, $\mu(N^c) = 0$. All subsets of N^c are measurable and all measures are zero. For any real number $\alpha \in \mathbb{R}$,

$$\begin{aligned}\{x \in X\colon g(x) > \alpha\} &= \{x \in N\colon g(x) > \alpha\} \cup \{x \in N^c\colon g(x) > \alpha\} \\ &= \{x \in N\colon f(x) > \alpha\} \cup \{x \in N^c\colon g(x) > \alpha\}\end{aligned} \tag{3.2}$$

is established, especially $\{x \in N^c\colon g(x) > \alpha\} \subset N^c$. However, since ($X$, $\mathfrak{M}$, μ) is a complete measure space, $\{x \in N^c\colon g(x) > \alpha\} \in \mathfrak{M}$ is established. Meanwhile, since it is

$$\{x \in N\colon f(x) > \alpha\} = \{x \in X\colon f(x) > \alpha\} \cap N \in \mathfrak{M},$$

we get $g \in \mathcal{M}(X, \mathfrak{M})$ from (3.2). Also, since it is

$\{x \in X : f(x) > \alpha\} = \{x \in N\colon f(x) > \alpha\} \cup \{x \in N^c\colon f(x) > \alpha\}$,

$\{x \in N^c\colon f(x) > \alpha\} \subset N^c$,

we get $\mu(\{x \in N^c\colon f(x) > \alpha\}) = 0$. it can be seen that

$$\mu(\{x \in X\colon g(x) > \alpha\}) = \mu(\{x \in X\colon f(x) > \alpha\})$$

is established.

Example 3.3.2.9. *Two functions* $f, g\colon \mathbb{R} \to \mathbb{R}$ *on the Lebesgue measure space* ($\mathbb{R}$, $\mathfrak{M}$, γ) *are defined as* $f(x) = sin\, x$ $(x \in \mathbb{R})$ *and*

$$g(x) = \begin{cases} cos\ x\ (x \in \mathbb{Q}), \\ sin\ x\ (x \in \mathbb{R} - \mathbb{Q}). \end{cases}$$

Since f is continuous, $f \in \mathcal{M}\,(\mathbb{R}, \mathfrak{M})$. Meanwhile, since $\mathbb{Q}$ is a countable set, so it is $\gamma(\{x \in \mathbb{R}\colon f(x) \neq g(x)\}) = \gamma(\mathbb{Q}) = 0$. Thus, it can be seen that $f = g$ γ-a.e [$\mathbb{R}$]. Therefore, applying *Proposition* 3.3.2.8, we get $g \in \mathcal{M}(\mathbb{R}, \mathfrak{M})$.

Proposition 3.3.2.10. *Given a measure space* (X, $\mathfrak{M}$, μ), *if* $\lim_{n\to+\infty} f_n = f$ μ*-a.e* $[X]$ *and* $\lim_{n\to+\infty} f_n = g$ μ*-a.e* $[X]$ *for the measurable function sequence* $(f_n)_{n\geq 1} \subset \mathcal{M}(X,M)$ *and the measurable functions* $f, g \in \mathcal{M}(X, \mathfrak{M})$, *then* $f = g$ μ*-a.e* $[X]$ *is established.*

Proposition 3.3.2.11. *Given a measure space* $(X,\ \mathfrak{M},\ \mu)$*, if* $\lim_{n\to+\infty} f_n = f$ μ*-a.e* $[X]$ *and* $f = g$ μ*-a.e* $[X]$ *for the measurable function sequence* $(f_n)_{n\geq 1} \subset \mathcal{M}(X,\ \mathfrak{M})$ *and the measurable functions* $f, g \in \mathcal{M}(X,\ \mathfrak{M})$*, then* $\lim_{n\to+\infty} f_n = g$ μ*-a.e* $[X]$ *is established.*

Proposition 3.3.2.12. *Given a measure space* $(X,\ \mathfrak{M},\ \mu)$*, if* $\lim_{n\to+\infty} f_n = f$ μ*-a.e* $[X]$ *and* $f_n \leq g$ μ*-a.e* $[X]$ $(n \in \mathbb{N})$ *for the measurable function sequence* $(f_n)_{n\geq 1} \subset \mathcal{M}(X,$ $\mathfrak{M})$ *and the measurable functions* $f, g \in \mathcal{M}(X,\ \mathfrak{M})$*, then* $f \leq g$ μ*-a.e* $[X]$ *is established.*

3.3.3 Cantor Sets

Consider the Borel measure space $(\mathbb{R},\ \mathfrak{B}_\alpha(\mathbb{R}),\ \lambda)$ and the measurable set $E_0 = [0,\ 1] \in \mathfrak{B}_{(\mathbb{R})}$ on the set of all real numbers $\mathbb{R}$. If the open interval $\left(\frac{1}{3},\ \frac{2}{3}\right)$ is removed from the set $E_0 = [0,\ 1]$ and the remaining set is E_1, then

$$E_1 = \left[0,\ \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3},\ 1\right].$$

Also, if the closed interval $\left[0,\ \frac{1}{3}\right]$ and $\left[\frac{2}{3},\ 1\right]$ are divided into thirds, the middle open interval is removed, and the remaining set is E_2, then

$$E_2 = \left[0,\ \tfrac{1}{9}\right] \cup \left[\tfrac{2}{9},\ \tfrac{3}{9}\right] \cup \left[\tfrac{6}{9},\ \tfrac{7}{9}\right] \cup \left[\tfrac{8}{9},\ 1\right].$$

By repeating this process, we obtain a compact set sequence $(E_n)_{n\geq 1} \subset \mathfrak{B}_\alpha\ (\mathbb{R})$ that satisfies the condition

$$\begin{cases} E_n \in \mathfrak{B}_\alpha(\mathbb{R})\ (n \in \mathbb{N}), \\ E_1 \supset E_2 \supset E_3 \supset \cdots, \\ \lambda(E_n) = 3^{-n}\ \cdot\ 2^n = \left(\tfrac{2}{3}\right)^n\ (n \in \mathbb{N}). \end{cases}$$

In this case, $P = \bigcap_{n=1}^{\infty} E_n$ is called the Cantor set. In particular, P is a compact set and $P \neq \emptyset$. Also, applying Proposition 3.2.12,

$$\lambda(P) = \lim_{n\to+\infty} \lambda E_n = \lim_{n\to+\infty} \left(\tfrac{2}{3}\right)^n = 0$$

is established.

Proposition 3.3.3.13. *The Cantor set* P *has the following properties.*

(a) P is a compact set and $P \in \mathfrak{B}_\alpha(\mathbb{R})$.

(b) P does not include any open intervals.

(c) P is a non-countable set.

3.3.4 Cantor-Lebesgue Functions

Given a the set $D_n = [0,\ 1] - E_n$ $(n \in \mathbb{N})$, the set D_n means $2^n - 1$ open intervals removed when constructing the nth *Cantor* set. The removed intervals are sorted from the left and denoted by I_j^n $(j = 1,\ 2,\ \cdots,\ 2^n - 1)$. The function f_n: $[0,\ 1] \to \mathbb{R}$ is defined as

$$f_n(x) = \begin{cases} 0 & (x = 0), \\ j2^{-n} & (x \in I^n), \\ linear & (x \in E_n), \\ 1 & (x = 1) \end{cases}$$

for each $x \in [0, 1]$. However, the linear is selected the function f_n is continuous on E_n. Then the function f_n: $[0, 1] \to \mathbb{R}$ increases monotonically. Moreover,

$$\begin{cases} f_{n+1}(\text{x}) = f_n(\text{x}) \ (x \in I_j^n \text{ and } j \in \{1, 2, \cdots, 2^n - 1\}), \\ |f_n(x) - f_{n+1}(x)| < 2^{-n} \ (x \in [0, 1]) \end{cases}$$

is established. However, since

$$\begin{aligned} |f_m(x) - f_n(x)| &= \left| \sum_{j=k}^{m-1} \Big(f_j(x) - f_{j+1}(x) \Big) \right| \\ &\leq \sum_{j=k}^{m-1} \Big| f_j(x) - f_{j+1}(x) \Big| \\ &\leq \sum_{j=k}^{\infty} \Big| f_j(x) - f_{j+1}(x) \Big| \\ &\leq \sum_{j=k}^{\infty} 2^{-j} = 2^{-k+1} \ (x \in [0, 1]) \end{aligned}$$

is satisfied for any natural numbers $k, m \in \mathbb{N}$ that satisfies the condition $k \leq m$, the function sequence $(f_n)_{n \geq 1}$ uniformly converges on the interval [0, 1] by the Cauchy Test [Kub07]. In this case, the limit function f: $[0, 1] \to \mathbb{R}$ defined as $f(x) = \lim_{n \to +\infty} f_n(x)$ $(x \in [0, 1])$ is called the Cantor-Lebesgue [Wei74] functions.

Proposition 3.3.4.14. *The Cantor Lebesgue function* $f : [0, 1] \to \mathbb{R}$ *has the following properties.*

(a) $f(0) = 0$ and $f(1) = 1$.

(b) f is continuous and increases monotonically on the interval [0, 1].

(c) f is constant on each open interval removed when constructing the Cantor set P.

Chapter 4

Integral of Non-negative Measurable Functions

4.1 Related Theorems and Definitions

Definition 4.1.1. *The standard representation of a simple measurable function.*
Given a simple measurable function $\phi\colon X \to \mathbb{R}$, an expression such as (4.1) is called the standard representation of a simple measurable function $\phi\colon X \to \mathbb{R}$.

Definition 4.1.2. *The Lebesgue integral.*
Given a measure space $(X, \mathfrak{M}, \mu)$, we define

$$\int f \, d\mu = \sup \left\{ \int \phi \, d\mu\colon \phi \in \mathcal{S}_f^+(X, \mathfrak{M}) \right\} \tag{4.3}$$

for the measurable function $f \in \mathcal{M}^+(X, \mathfrak{M})$. (4.3) is called the Lebesgue integral [Wei73] of the measure function $f \in \mathcal{M}^+(X, \mathfrak{M})$ for measure $\mu\colon \mathfrak{M} \to [0, +\infty]$.

Theorem 4.1.3. *The standard expression.*
For the measurable function $f \in \mathcal{M}^+(X, \mathfrak{M})$ and measurable set $E \in \mathfrak{M}$, the following holds.

(a) $f(x) = 0 \ (x \in E) \Longrightarrow \int_E f \, d\mu = 0.$
(b) $\mu(E) = 0 \Longrightarrow \int_E f \, d\mu = 0.$

$\phi = \sum\limits_{k=1}^{n} a_k \chi_{A_k}$ is called the standard expression.

4.2 Integral of Non-negative Measurable Functions

4.2.1 Integral of Simple Measurable Functions

Given a function $\phi\colon X \to \mathbb{R}$, since the range $\phi(X)$ is a finite set, if we put

$$\phi(X) = \{a_1, a_2, \cdots, a_n\} \text{ and } a_j \neq a_k \ (j \neq k)$$

and let $A_k = \phi^{-1}(\{a_k\})$ for each $k \in \{1, 2, \cdots, n\}$, the following holds.

$$A_j \cap A_k = \emptyset \ (j \neq k), \ X = \bigcup_{k=1}^{n} A_k \text{ and } \phi = \sum_{k=1}^{n} a_k \chi_{A_k}. \tag{4.1}$$

Here, χ_{A_k}: $X \to \mathbb{R}$ is a characteristic function of $A_k \in \mathcal{P}(X)$.

Proposition 4.2.1.1. If the simple function ϕ: $X \to \mathbb{R}$ defined on the measurable space $(X, \mathfrak{M})$ is expressed as (4.1), the following holds.

$$\phi \in \mathcal{S}(X, \mathfrak{M}) \Longleftrightarrow A_k \in \mathfrak{M}(k \in 1, 2, \cdots, n).$$

Definition 4.2.1.2. *Given a simple measurable function ϕ: $X \to \mathbb{R}$, an expression such as* (4.1) *is called the standard representation of a simple measurable function ϕ: $X \to \mathbb{R}$.*

Definition 4.2.1.3. *If the non-negative simple measurable function $\phi \in \mathcal{S}^+ (X, \mathfrak{M})$ has a standard expression*

$$\phi = \sum_{k=1}^{n} a_k \chi_{A_k}$$

as shown in (4.1), the integration for ϕ is defined as follows.

$$\int \phi \; d\mu = \sum_{k=1}^{n} a_k \mu(A_k). \tag{4.2}$$

Note 4.2.1.4. *In* (4.2)*, the already defined rule* $0 \cdot (+\infty) = 0$ *is applied, and* $\mu(A_k) = +\infty$ *for any* $A_k \in \mathfrak{M}$.
If $a_k > 0$,

$$\int \phi \; d\mu = \sum_{k=1}^{n} a_k \mu(A_k) \in [0, +\infty]$$

is established because $a_k \mu(A_k) = +\infty$.
If $\phi = 0$, the standard expression of ϕ is $\phi = 0 \cdot \chi_X$, so

$$\int \phi \; d\mu = 0 \cdot \mu(X) = 0.$$

Lemma 4.2.1.5. *If the non-negative measurable simple function $\phi \in \mathcal{S}^+ (X, \mathfrak{M})$ is expressed as*

$$\phi = \sum_{i=1}^{m} b_i \chi_{B_i}, \; B_i \cap B_j = \emptyset \; (i \neq j) \text{ and } X = \bigcap_{i=1}^{m} B_i,$$

the following holds.

$$\int \phi \; d\mu = \sum_{i=1}^{m} b_i \mu(B_i).$$

Theorem 4.2.1.6. *Given a non-negative measurable simple functions ϕ, $\psi \in \mathcal{S}^+ (X, \mathfrak{M})$ and a non-negative real number $c \in \mathbb{R}_{\geq 0}$, the following holds.*

(a) $\int c\phi \; d\mu = c \int \phi \; d\mu$.

(b) $\int (\phi + \psi) \; d\mu = \int \phi \; d\mu + \int \psi \; d\mu$.

(c) $\phi(x) \leq \psi(x) \; (x \in X) \Longrightarrow \int \phi \; d\mu \leq \int \psi \; d\mu$.

Proof.

(a) If $c = 0$, since $c\phi = 0$,

$$\int c\phi \; d\mu = 0 = c \int \phi \; d\mu$$

is established.

If $c > 0$, it is $c\phi \in \mathcal{S}^+(X, \mathfrak{M})$. Moreover, if we put

$$\phi = \sum_{j=1}^{n} a_j \chi_{E_j}, \ \ E_i \cap E_j = \emptyset \ (i \neq j) \text{ and } X = \bigcup_{j=1}^{n} E_j,$$

we get

$$c\phi = \sum_{j=1}^{n} ca_j \chi_{E_j}, \ \ E_i \cap E_j = \emptyset \ (i \neq j) \text{ and } X = \bigcup_{j=1}^{n} E_j.$$

If Lemma 4.2.1.5 is applied,

$$\int c\phi \, d\mu = \sum_{j=1}^{n} ca_j \mu(E_j) = c \sum_{j=1}^{n} a_j \mu E_j = c \int \phi \, d\mu$$

is established.

(b) The standard expression of the simple measurable functions $\phi, \psi \in \mathcal{S}^+(X, \mathfrak{M})$ are set as

$$\phi = \sum_{j=1}^{n} a_j \chi_{E_j} \text{ and } \psi = \sum_{k=1}^{n} b_k \chi_{F_k}$$

respectively. If we consider the subset $\{E_j \cap F_k \colon E_j \cap F_k \neq \emptyset \ (1 \leq j \leq n \text{ and } 1 \leq k \leq n)\} = \{A_1, A_2, \cdots, A_N\}$ of X, we can rewrite them as

$$\phi = \sum_{i=1}^{N} a_i \chi_{A_i} \text{ and } \psi = \sum_{i=1}^{N} b_i \chi_{A_i}$$

and get

$$\phi + \psi = \sum_{i=1}^{n} (a_i + b_i) \chi_{A_i}.$$

Now, if Lemma 4.2.1.5 is applied,

$$\begin{aligned} \int (\phi + \psi) \, d\mu &= \sum_{i=1}^{n} (a_i + b_i) \mu(A_i) \\ &= \sum_{i=1}^{n} a_i \mu(A_i) + \sum_{i=1}^{n} b_i \mu(A_i) \\ &= \int \phi \, d\mu + \int \psi \, d\mu \end{aligned}$$

is established.

(c) The standard expression of the simple measurable functions $\phi, \psi \in \mathcal{S}^+(X, \mathfrak{M})$ are set as

$$\phi = \sum_{j=1}^{n} a_j \chi_{E_j} \text{ and } \psi = \sum_{k=1}^{n} b_k \chi_{F_k},$$

respectively. By applying the same method as in the proof (b) is applied, it can be rewritten as

$$\phi = \sum_{i=1}^{N} a_i \chi_{A_i} \text{ and } \psi = \sum_{i=1}^{N} b_i \chi_{A_i}.$$

However, since it is $\phi(x) \leq \psi(x)$ $(x \in X)$, $a_i \leq b_i$ is established for each $i \in \{1, 2, \cdots, N\}$. Moreover, if Lemma 4.2.1.5 is applied,

$$\int \phi \; d\mu = \sum_{i=1}^{n} a_i \mu(A_i) \leq \sum_{i=1}^{n} b_i \mu(A_i) = \int \psi \; d\mu$$

is obtained.

Theorem 4.2.1.7. *If the function λ_ϕ: $\mathfrak{M} \to \bar{\mathbb{R}}$ is defined as $\lambda_\phi(E) = \int \phi \chi_E \; d\mu = \int_E \phi \; d\mu$ for the non-negative simple measurable function $\phi \in \mathcal{S}^+(X, \mathfrak{M})$, the function $\lambda_\phi : \mathfrak{M} \to \bar{\mathbb{R}}$ is a measure.*

4.2.2 Integral of Non-negative Measurable Functions

Definition 4.2.2.8. *Given a measure space $(X, \mathfrak{M}, \mu)$, we define*

$$\int f \; d\mu = \sup \left\{ \int \phi \; d\mu\text{: } \phi \in \mathcal{S}_f^+(X, \mathfrak{M}) \right\} \tag{4.3}$$

for the measurable function $f \in \mathcal{M}^+(X, \mathfrak{M})$. (4.3) is called the Lebesgue integral [Wei73] of the measure function $f \in \mathcal{M}^+(X, \mathfrak{M})$ for measure μ: $\mathfrak{M} \to [0, +\infty]$. If $E \in \mathfrak{M}$ is given, it is defined as

$$\int_E f \; d\mu = \int f \chi_E \; d\mu \tag{4.4}$$

because it is $f\chi_E \in \mathcal{M}^+(X, \mathfrak{M})$. (4.4) is called the Lebesgue integral [Wei73] of the measurable function $f \in \mathcal{M}^+(X, \mathfrak{M})$ for the measure μ: $\mathfrak{M} \to [0, +\infty]$ on the measurable set $E \in \mathfrak{M}$.

Theorem 4.2.2.9. *For the measure space $(X, \mathfrak{M}, \mu)$, the following holds.*

(a) $f, g \in \mathcal{M}^+(X, \mathfrak{M})$ and $f(x) \leq g(x)$ $(x \in X) \Longrightarrow \int f \; d\mu \leq \int g \; d\mu$.

(b) $\begin{cases} f \in \mathcal{M}^+(X, \mathfrak{M}) \\ E, F \in \mathfrak{M} \text{ and } E \subset F \end{cases} \Longrightarrow \int_E f \; d\mu \leq \int_F f \; d\mu.$

Theorem 4.2.2.10. *For the measurable function $f \in \mathcal{M}^+(X, \mathfrak{M})$ and measurable set $E \in \mathfrak{M}$, the following holds.*

(a) $f(x) = 0$ $(x \in E) \Longrightarrow \int_E f \; d\mu = 0$.

(b) $\mu(E) = 0 \Longrightarrow \int_E f \; d\mu = 0$.

Proof.

(a) It is $f = 0$ $(x \in E)$ because it is $f\chi_E = 0 \in \mathcal{S}^+(X, \mathfrak{M})$. Therefore, according to Note 4.2.1.4,

$$\int_E f \; d\mu = \int f \chi_E \; d\mu = 0$$

is established.

(b) If positive number $\epsilon > 0$ is arbitrarily determined, there is a simple measurable function $\phi \in \mathcal{S}^+_{f\chi_E}(X, \mathfrak{M})$ that satisfies the inequality

$$\int_E f\chi_E \, d\mu < \int \phi \, d\mu + \epsilon$$

according to Definition 4.2.2.8. However, since it is

$$0 \leq \phi(x) \leq (f\chi_E)(x) \leq f(x)\chi_E(x)(x \in X),$$

we can be seen that it is $\phi(X) = 0 \ (x \in E^c)$.

Now, $\phi = \sum_{k=1}^{n} a_k \chi_{A_k}$ is called the standard expression. Applying Theorem 4.2.1.7, (a) and Proposition 3.3.2.11, we get

$$\begin{aligned}\int \phi \, d\mu =& \int_E \phi \, d\mu + \int_{E^c} \phi \, d\mu \text{ (by Theorem 4.2.1.7)}\\ =& \int \phi\chi_E \, d\mu \left(\phi(x) = 0 \ (x \in E^c) \text{ and } \int_E \phi \, d\mu = \int \phi\chi_E \, d\mu\right)\\ =& \sum_{k=1}^{n} a_k \mu(E \cap A_k) = 0 \left(\phi\chi_E = \sum_{k=1}^{n} a_k \chi_{E\cap A_k} \text{ and } \mu(E) = 0\right).\end{aligned}$$

Therefore, $\int_E f \, d\mu = \int f\chi_E \, d\mu < \epsilon$ is established. However, since $\epsilon > 0$ is an arbitrarily given positive number, the conclusion is established according to Proposition 4.2.2.10.1.

Proposition 4.2.2.10.1. (Infinitesimal Principle [Wei74]) *For real Numbers $a, b \in \mathbb{R}$, the following holds.*

$$a \leq b + \epsilon \, (\epsilon > 0) \Longrightarrow a \leq b. \tag{1}$$

Proof.

Assuming $a > b$ by denying the conclusion, it is

$$\epsilon = \tfrac{a-b}{2} > 0 \text{ and } b + \epsilon = b + \tfrac{a-b}{2} = \tfrac{a+b}{2} < \tfrac{a+a}{2} = a.$$

However, this contradicts the assumption of (1).

Corollary 4.2.2.11. *For the measurable functions $f, g \in \mathcal{M}^+(X, \mathfrak{M})$, the following holds.*

$$f \leq g \ \mu\text{-a.e } [X] \Longrightarrow \int f \, d\mu \leq \int g \, d\mu.$$

4.3 The Monotone Convergence *Theorem*

Theorem 4.3.1. (Monotone Convergence *Theorem* [MCT] [Kub07])
If the measurable function sequence $(f_n)_{n\geq 1} \subset \mathcal{M}^+(X, \mathfrak{M})$ satisfies the condition

$$\begin{cases} 0 \leq f_n(x) \leq f_{n+1}(x) \ (x \in X \text{ and } n \in \mathbb{N}),\\ \lim_{n\to+\infty} f_n(x) = f(x) \ (x \in X),\end{cases}$$

the following holds.

$$f \in \mathcal{M}^+(X, \mathfrak{M}) \text{ and } \lim_{n\to+\infty} \int f_n \, d\mu = \int f \, d\mu.$$

Proof.

According to Corollary 2.3.21, it can be seen that $f \in \mathcal{M}^+(X, \mathfrak{M})$. Since it is $f_n(x) \leq f_{n+1}(x) \leq f(x) \ (x \in X)$ for each natural number $n \in \mathbb{N}$,

$$\int f_n \, d\mu \leq \int f_{n+1} \, d\mu \leq \int f \, d\mu \ (n \in \mathbb{N})$$

is established if Theorem 4.2.2.9 is applied. So we can get

$$\lim_{n \to +\infty} \int f_n \, d\mu \leq \int f \, d\mu. \tag{4.5}$$

In order to obtain an inequality in the opposite direction, any $\alpha \in (0, 1)$ and any $\phi \in \mathcal{S}_f^+ \ (X, \mathfrak{M})$ are selected. For each natural number $n \in \mathbb{N}$, we set it as a set $A_n = \{x \in X\colon f_n(x) \geq \alpha\phi(x)\}$. Then

$$A_n \in \mathfrak{M}, \ A_n \subset A_{n+1} \text{ and } X = \bigcup_{n=1}^{\infty} A_n$$

are established. Therefore, we get

$$\int_{A_n} \alpha\phi \, d\mu \leq \int_{A_n} f_n \, d\mu = \int f_n \chi_{A_n} \, d\mu \leq \int f_n \, d\mu$$

by Theorem 4.2.2.9. Meanwhile, according to Theorem 4.2.1.7, function $\lambda_{\alpha\phi}\colon \mathfrak{M} \to [0, +\infty] \ (E \to \int_E \alpha\phi \, d\mu)$ is a measure, and

$$\int \alpha\phi \, d\mu = \lambda_{\alpha\phi}(X) = \lambda_{\alpha\phi}\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to +\infty} \lambda_{\alpha\phi}(A_n) = \lim_{n \to +\infty} \int_{A_n} \alpha\phi \, d\mu$$

is established using Proposition 3.2.13. Therefore, it is

$$\alpha \int \phi \, d\mu = \int \alpha\phi \, d\mu = \lim_{n \to +\infty} \int_{A_n} \alpha\phi \, d\mu \leq \lim_{n \to +\infty} \int f_n \, d\mu$$

according to Theorem 4.2.1.6. In particular, since $\alpha \in (0, 1)$ is arbitrary,

$$\int \phi \, d\mu \leq \lim_{n \to +\infty} \int f_n \, d\mu$$

is obtained by applying Proposition 4.3.1.1. However, since $\phi \in \mathcal{S}_f^+ \ (X, \mathfrak{M})$ was randomly selected, it can be seen that it is

$$\int f \, d\mu \leq \lim_{n \to +\infty} \int f_n \, d\mu \tag{4.6}$$

by the integral Definition. Combining (4.5) and (4.6) yields

$$\int f \, d\mu = \lim_{n \to +\infty} \int f_n \, d\mu.$$

Proposition 4.3.1.1. (Infinitesimal Principle [Wei74]) *Given the non-negative real number $a \geq 0$ and $b \geq 0$, the following holds.*

$$\epsilon b \leq a \, (\epsilon \in (0, 1)) \Longrightarrow b \leq a.$$

Theorem 4.3.2. *For non-negative measurable functions, the following holds.*

(a) $f \in \mathcal{M}^+(X, \mathfrak{M})$ and $c \geq 0 \Longrightarrow cf \in \mathcal{M}^+(X, \mathfrak{M})$ and $\int cf \, d\mu = c \int f \, d\mu$.

(b) $f, \, g \in \mathcal{M}^+(X, \mathfrak{M}) \Longrightarrow f + g \in \mathcal{M}^+(X, \mathfrak{M})$ and $\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$.

Proof.

(a) The case of $c = 0$ is clear.
If $c > 0$, by applying *Theorem* 2.3.24, we can select a simple measurable increasing function sequence $(\phi_n)_{n \geq 1} \subset \mathcal{S}^+(X, \mathfrak{M})$ that satisfies

$$\lim_{n\to+\infty} \phi_n(x) = f(x)\ (x \in X).$$

Therefore, $(c\phi_n)_{n\geq 1} \subset \mathcal{S}^+ (X, \mathfrak{M})$ is also a simple measurable increasing function sequence and furthermore satisfies

$$\lim_{n\to+\infty} c\phi_n(x) = cf(x)\ (x \in X).$$

Therefore, it is $cf \in \mathcal{M}^+(X, \mathfrak{M})$, then

$$\int cf\ d\mu = \lim_{n\to+\infty} \int c\phi_n\ d\mu = c \lim_{n\to+\infty} \int \phi_n\ d\mu = c \int f\ d\mu$$

is established by *Theorem* 4.3.1 and *Theorem* 4.2.1.6.

(b) By applying *Theorem* 2.3.24, we can choose a simple measurable increasing function sequence $(\phi_n)_{n\geq 1}, (\psi_n)_{n\geq 1} \subset \mathcal{S}^+(X, \mathfrak{M})$ that satisfies

$$\lim_{n\to+\infty} \phi_n(x) = f(x) \text{ and } \lim_{n\to+\infty} \psi_n(x) = g(x)$$

for an arbitrarily given $x \in X$. Therefore, $(\phi_n + \psi_n)_{n\geq 1} \subset \mathcal{S}^+(X, \mathfrak{M})$ is also a simple measurable increasing function sequence and furthermore satisfies

$$\lim_{n\to+\infty} \Big(\phi_n(x) + \psi_n(x)\Big) = f(x) + g(x)\ (x \in X).$$

Therefore, it is $f + g \in \mathcal{M}^+(X, \mathfrak{M})$.

$$\begin{aligned}
\int \big(f+g\big)\ d\mu &= \lim_{n\to+\infty} \int \big(\phi_n + \psi_n\big)\ d\mu \\
&= \lim_{n\to+\infty} \left\{ \int \phi_n\ d\mu + \int \psi_n\ d\mu \right\} \\
&= \lim_{n\to+\infty} \int \phi_n\ d\mu + \lim_{n\to+\infty} \int \psi_n\ d\mu \\
&= \int f\ d\mu + \int g\ d\mu
\end{aligned}$$

is established by *Theorem* 4.3.1 and *Theorem* 4.2.1.6.

Proposition 4.3.3. *If the measurable functions* $f, g \in \mathcal{M}^+(X, \mathfrak{M})$ *satisfies the condition*

$$\begin{cases} f(x) \leq g(x)\ (x \in X), \\ \{x \in X\colon f(x) = +\infty\} = \emptyset, \\ \int f\ d\mu < +\infty, \end{cases}$$

the following equation is established for an arbitrarily given measurable set $E \in \mathfrak{M}$.

$$\int_E (g-f)\ d\mu = \int_E g\ d\mu - \int_E f\ d\mu.$$

Theorem 4.3.4. *If the measurable function* $f \in \mathcal{M}^+(X, \mathfrak{M})$ *satisfies the condition* $\int f\ d\mu < +\infty$*, there is a positive number* $\varrho > 0$ *that satisfies*

$$[E \in \mathfrak{M} \text{ and } \mu(E) < \varrho] \Longrightarrow \int_E f \, d\mu < \epsilon$$

for an arbitrarily given positive number $\epsilon > 0$.

Theorem 4.3.5. (Fatou *Lemma* [Bog07]) *The inequality*

$$\int \left(\liminf_{n\to+\infty} f_n \right) d\mu \leq \liminf_{n\to+\infty} \int f_n \, d\mu$$

is established for the measurable function sequence $(f_n)_{n\geq 1} \subset \mathcal{M}^+(X, \mathfrak{M})$.
Proof.

Let $g_m = \inf_{n\geq m} f_n$ for any natural number $m \in \mathbb{N}$, then

$$\begin{cases} g_m \in \mathcal{M}^+(X, \mathfrak{M}), \\ 0 \leq g_m(x) \leq g_{m+1}(x) \ (x \in X \text{ and } m \in \mathbb{N}), \\ g_m \leq f_n (n \in \mathbb{N} \text{ and } n \geq m), \\ \lim_{m\to+\infty} g_m(x) = \sup_{m\geq 1} \left(\inf_{n\geq m} f_n(x) \right) = \liminf_{n\to+\infty} f_n(x) \ (x \in X) \end{cases}$$

is established, we get

$$\int g_m \, d\mu \leq \int f_n \, d\mu \ (n \in \mathbb{N} \text{ and } n \geq m).$$

Therefore, since

$$\int g_m \, d\mu \leq \inf_{n\geq m} \left\{ \int f_n \, d\mu \right\} (m \in \mathbb{N})$$

is obtained,

$$\int g_k \, d\mu \leq \sup_{n\geq 1} \left\{ \int g_m \, d\mu \right\} \leq \sup_{n\geq 1} \left(\inf_{n\geq m} \left\{ \int f \, d\mu \right\} \right) = \liminf_{n\to+\infty} \int f_n \, d\mu \qquad (4.7)$$

is established for any $k \in \mathbb{N}$. Meanwhile,

$$\int \left(\liminf_{n\to+\infty} f_n \right) d\mu = \int \lim_{k\to+\infty} g_k \, d\mu = \lim_{k\to+\infty} \int g_k \, d\mu \qquad (4.8)$$

is established by *Theorem* 4.3.1.

$$\int \left(\liminf_{n\to+\infty} f_n \right) d\mu \leq \liminf_{n\to+\infty} \int f_n \, d\mu$$

is obtained by (4.7) and (4.8).

Theorem 4.3.6. *For the measurable function $f \in \mathcal{M}^+(X, \mathfrak{M})$, the following holds.*

$$f = 0 \ \mu\text{-a.e } [X] \Longleftrightarrow \int f \, d\mu = 0.$$

Theorem 4.3.7. *If the function $\lambda_f \colon \mathfrak{M} \to \bar{\mathbb{R}}$ is defined as*

$$\lambda_f(E) = \int_E f \, d\mu = \int f \chi_E \, d\mu$$

for the measurable function $\lambda_f \colon \mathfrak{M} \to \bar{\mathbb{R}}$, the following holds.

(a) The function $\lambda_f \colon \mathfrak{M} \to [0, +\infty]$ is a measure.

(b) $E \in \mathfrak{M}$ and $\mu(E) = 0 \Longrightarrow \lambda_f(E) = 0$.

Corollary 4.3.8. *If the monotone increasing function sequence* $(f_n)_{n\geq 1} \subset \mathcal{M}^+(X, \mathfrak{M})$ *and the function* $f \in \mathcal{M}^+(X, \mathfrak{M})$ *satisfy*

$$\lim_{n\to+\infty} f_n = f\,\mu\text{-a.e } [X],$$

the equation $\int f \; d\mu = \lim_{n\to+\infty} \int f_n \; d\mu$ is established.

Corollary 4.3.9. *Given a measurable function sequence* $(f_n)_{n\geq 1} \subset \mathcal{M}^+(X, \mathfrak{M})$*, the following holds.*

$$\int \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \int f_n \; d\mu \; (x \in X).$$

Proof.

If $g_n = \sum_{k=1}^{n} f_k$ is set for each natural number $n \in \mathbb{N}$, the measurable function sequence $(g_n)_{n\geq 1} \subset \mathcal{M}^+(X, \mathfrak{M})$ increases monotonically and furthermore satisfies

$$\lim_{n\to+\infty} g_n(x) = \sum_{k=1}^{\infty} f_k(x) \; (x \in X).$$

Therefore, applying *Theorem* 4.3.1 and *Theorem* 4.3.2, we get

$$\int \left(\sum_{k=1}^{\infty} f_k \right) d\mu = \lim_{n\to+\infty} \int g_n \; d\mu = \lim_{n\to+\infty} \sum_{k=1}^{n} \int f_k \; d\mu = \sum_{k=1}^{\infty} \int f_k \; d\mu.$$

Corollary 4.3.10. *For the double sequence* $a_{i,j} \geq 0 \; \big((i, j) \in \mathbb{N} \times \mathbb{N}\big)$*, the following holds.*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{i,j}.$$

Proof.

Consider the counting measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu). If the function f_n : \mathbb{N} \to \mathbb{R}$ for each natural number $n \in \mathbb{N}$ is defined as $f_n(j) = a_{n,j} (j \in \mathbb{N})$, it can be seen that $f_n \in \mathcal{M}^+(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. However, since

$$\int \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} f_n(j) = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} a_{n,j},$$
$$\sum_{n=1}^{\infty} \int f_n \; d\mu = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} f_n(j) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} a_{n,j},$$

applying *Corollary* 4.3.9 leads to the required conclusion.

Chapter 5

Integrable Functions

5.1 Integrable Function Spaces

5.1.1 Integrable Functions

Given a measurable function $f \in \mathcal{M}(X, \mathfrak{M})$, it can see that it is

$$0 \in \int f^{+} \, d\mu \leq +\infty \text{ and } 0 \leq \int f^{-} \, d\mu \leq +\infty \tag{5.1}$$

because it is $f^{\pm} \in \mathcal{M}^{+}(X, \mathfrak{M})$. If at least one of the two integrals in (5.1) is a finite, it is expressed as

$$\int f \, d\mu = \int f^{+} \, d\mu - \int f^{-} \, d\mu \in [-\infty, +\infty], \tag{5.2}$$

and the integral $\int f d\mu$ is defined.
If both integrals in (5.1) are finite , it is defined as

$$\int f \, d\mu = \int f^{+} \, d\mu - \int f^{-} \, d\mu \in \mathbb{R}. \tag{5.3}$$

The function $f \in \mathcal{M}(X, \mathfrak{M})$ is called the Lebesgue integrable [Wei73] for the measure $\mu : \mathfrak{M} \to [0, +\infty]$. The set of all integrable functions is denoted by

$$\mathcal{L}_1(X, \mathfrak{M}, \mu) = \{f \in \mathcal{M}(X, \mathfrak{M}) \colon \int f^{\pm} \, d\mu < +\infty\}.$$

Definition 5.1.1.1. *For the integrable function $f \in \mathcal{L}_1(X, \mathfrak{M}, \mu)$ and the measurable set $E \in \mathfrak{M}$, it is defined as*

$$\int_E f \, d\mu = \int_E f^{+} \, d\mu - \int_E f^{-} \, d\mu.$$

Proposition 5.1.1.2. *For the integrable function $f \in \mathcal{L}_1(X, \mathfrak{M}, \mu)$ and the disjoint measurable set sequence $(E_n)_{n \leq 1} \subset \mathfrak{M}$,*

$$\int_{\bigcup_{k=1}^{\infty} E_k} f \, d\mu = \sum_{k=1}^{\infty} f \, d\mu$$

is established.

Proposition 5.1.1.3. *If the integrals $\int f \, d\mu$ and $\int g \, d\mu$ are both defined for the measurable functions $f, \ g \in \mathcal{M}(X, \mathfrak{M})$,*

$$f \leq g \ \mu\text{-}a.e \ [X] \Longrightarrow \int f d\mu \leq \int g d\mu$$

is established.

Proposition 5.1.1.4. *If the measurable functions* $f,\ g \in \mathcal{M}(X,\ \mathfrak{M})$ *satisfies* $f = g$ *μ-a.e* $[X]$, *the following holds.*

(a) $f \in \mathcal{L}_1(X,\ \mathfrak{M},\ \mu) \Longleftrightarrow g \in \mathcal{L}_1(X,\ \mathfrak{M},\ \mu)$.

(b) $\int f\ d\mu = \int g\ d\mu$.

Proof.

Since $f = g$ μ-a.e $[X]$ by the assumption, there is a measurable set $N \in \mathfrak{M}$ that satisfies the condition

$$\begin{cases} \mu\,(N) = 0, \\ f\,(x) = g\,(x)\,(x \in N^c). \end{cases} \tag{5.4}$$

Therefore, $f^{\pm} = g^{\pm}$ μ-a.e$[X]$ is established, and thus

$$\int f^{\pm}\ d\mu = \int g^{\pm}\ d\mu$$

is obtained. In particular, $f \in \mathcal{L}_1(X,\ \mathfrak{M},\ \mu) \Longleftrightarrow g \in \mathcal{L}_1(X,\ \mathfrak{M},\ \mu)$ is established. Moreover, it is

$$\int g\ d\mu = \int g^+\ d\mu - \int g^-\ d\mu = \int f^+\ d\mu - f^-\ d\mu = \int f\ d\mu.$$

5.1.2 Properties of Integrable Functions

Theorem 5.1.2.5. $f \in \mathcal{L}_1(X,\ \mathfrak{M},\ \mu) \Longrightarrow |f| \in \mathcal{L}_1(X,\ \mathfrak{M},\ \mu)$.

Proof.

First, it should be noted that $f \in \mathcal{L}_1(X,\ \mathfrak{M},\ \mu) \Longleftrightarrow [f \in \mathcal{M}(X,\ \mathfrak{M})$ and $\int f^{\pm}\ d\mu < +\infty]$ by Definition. If $f \in \mathcal{L}_1(X,\ \mathfrak{M},\ \mu)$ is assumed, it is

$$|f|^+ = |f| = f^+ + f^- \in \mathcal{M}^+(X,\ \mathfrak{M}) \text{ and } |f|^- = 0 \in \mathcal{M}^+(X,\ \mathfrak{M}).$$

However, according to *Theorem* 4.3.2, since

$$\int |f|^-\ d\mu = 0 < +\infty \text{ and } \int |f|^+\ d\mu = \int |f|\ d\mu = \int f^+\ d\mu + \int f^-\ d\mu < +\infty,$$

it is $|f| \in \mathcal{L}_1(X,\ \mathfrak{M},\ \mu)$.

Example 5.1.2.6. *Let* $\mathfrak{M} = \{\phi,\ X\}$ *for a set* X, and if we define the function μ: $\mathfrak{M} \to [0,\ +\infty]$ as $\mu\,(\phi) = 0$ and $\mu(X) = 1$, the measure space $(X,\ \mathfrak{M},\ \mu)$ is obtained. For a set $E \in \mathcal{P}\,(X)$ that satisfies $E \notin \{\phi,\ X\}$, if the function $f\colon\ X \to \bar{\mathbb{R}}$ is defined as

$$f\,(x) = \begin{cases} 1 & (x \in E)\,, \\ -1 & (x \in E)\,, \end{cases}$$

it is $f \notin \mathcal{M}(X,\ \mathfrak{M})$ but $|f| \in \mathcal{M}(X,\ \mathfrak{M})$. Moreover, since

$$\int |f|\ d\mu = \int\ d\mu = \mu(X) = 1,$$

it is $|f| \in \mathcal{L}_1(X,\ \mathfrak{M},\ \mu)$.

Theorem 5.1.2.7. $[f \in \mathcal{M}(X,\ \mathfrak{M})$ and $|f| \in \mathcal{L}_1(X,\ \mathfrak{M},\ \mu)] \Longrightarrow f \in \mathcal{L}_1(X,\ \mathfrak{M},\ \mu)$.

Proof.

Assuming $f \in \mathcal{M}(X,\ \mathfrak{M})$ and $|f| \in \mathcal{L}_1\,(X\,,\ \mathfrak{M},\ \mu)$, since $f^+ \leq |f| = |f|^+$ and $f^- \leq |f| = |f|^+$, it is $f \in \mathcal{L}_1(X,\ \mathfrak{M},\ \mu)$ according to Theorem 4.2.2.9.

Corollary 5.1.2.8. *For the measurable function* $f \in \mathcal{M}(X, \mathfrak{M})$, $f \in \mathcal{L}_1(X, \mathfrak{M}, \mu) \Longleftrightarrow |f| \in \mathcal{L}_1(X, \mathfrak{M}, \mu)$ *is established.*

Theorem 5.1.2.9. *If an integral* $\int f \, d\mu$ *is defined for the measurable function* $f \in \mathfrak{M}(X, \mathfrak{M})$, *an inequality*

$$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu$$

is established.

Proof.

If $\int |f| \, d\mu = +\infty$, an inequality holds without proof, so we assume $\int |f| \, d\mu < +\infty$. Since $f^{\pm} \leq |f|$ on the set X,

$$0 \leq \int f^+ \, d\mu < +\infty \text{ and } 0 \leq \int f^- \, d\mu < +\infty$$

are established to obtain $f \in \mathcal{L}_1(X, \mathfrak{M}, \mu)$. Moreover, applying *Theorem* 4.3.2 establishes

$$\begin{aligned} \left| f \, \mathrm{d}\mu \right| &= \left| \int f^+ \, d\mu - \int f^- \, d\mu \right| \\ &\leq \int f^+ \, d\mu + \int f^- \, d\mu \\ &= \int \left(f^+ + f^- \right) \, d\mu \\ &= \int |f| \, d\mu. \end{aligned}$$

Corollary 5.1.2.10. *If the measurable function* $f \in \mathcal{M}(X, \mathfrak{M})$ *satisfies the condition*

$$g \in \mathcal{L}_1(X, \mathfrak{M}, \mu)\colon \ |f(x)| \leq |g(x)| \ (x \in X),$$

the following holds.

(a) $f \in \mathcal{L}_1(X, \mathfrak{M}, \mu)$.

(b) $\int |f| \, d\mu \leq \int |g| \, d\mu$.

Theorem 5.1.2.11. (Chebychev inequality [Kub07]) *An inequality*

$$\alpha\mu(\{x \in X\colon \ |f\,(x)\,| > \alpha\}) \geq \int |f| \, d\mu$$

is established for the measurable function $f \in \mathcal{M}(X, \mathfrak{M})$ and the positive number $\alpha > 0$.

Theorem 5.1.2.12. *Given a measurable function* $f \in \mathcal{M}(X, \mathfrak{M})$, *the following holds.*

$$f \in \mathcal{L}_1(X, \mathfrak{M}, \mu) \Longrightarrow f(x) \in \mathbb{R} \ \mu\text{-a.e } x \in X.$$

Theorem 5.1.2.13. *For the integrable functions* $f, g \in \mathcal{L}_1(X, \mathfrak{M}, \mu)$ *and* $\alpha \in \mathbb{R}$, *the following holds.*

(a) αf, $f + g \in \mathcal{L}_1(X, \mathfrak{M}, \mu)$. The set $\mathcal{L}_1(X, \mathfrak{M}, \mu)$ is the vector space on $\mathbb{R}$.

(b) $\int \alpha f \, d\mu = \alpha \int f \, d\mu$ and $\int (f + g) \, d\mu = \int f \, d\mu + \int g] \, d\mu$.

5.2 Lebesgue's Dominated Convergence Theorem

Theorem 5.2.1. (*Lebesgue's Dominated Convergence Theorem* [Bog07])
If the measurable function sequence $(f_n)_{n\geq 1} \subset \mathcal{M}(X, \mathfrak{M})$ holds the condition

$$\begin{cases} \lim\limits_{n\to+\infty} f_n(x) = f(x)(x \in X), \\ g \in \mathcal{L}_1(X, \mathfrak{M}, \mu)\colon\ |f_n(x)| \leq g(x)\ (x \in X \text{ and } n \in \mathbb{N}), \end{cases}$$

the following holds.

(a) $f \in \mathcal{L}_1(X, \mathfrak{M}, \mu)$.

(b) $\lim\limits_{n\to+\infty} \int |f_n - f|\ d\mu = 0$.

(c) $\lim\limits_{n\to+\infty} \int f_n\ d\mu = \int f\ d\mu$.

Theorem 5.2.2. (*Lebesgue's Dominated Convergence Theorem* 1 [Bog07])
Given a measurable function sequence $(f_n)_{n\geq 1} \subset \mathcal{M}(X, \mathfrak{M})$ and a measurable function $f\colon X \to \mathbb{R}$ having a real-valued, if the condition

$$\begin{cases} \lim\limits_{n\to+\infty} f_n = f\ \mu\text{-a.e[X]}, \\ g \in \mathcal{L}_1(X, \mathfrak{M}, \mu)\colon\ |f_n(x)| \leq g(x)\ (x \in X \text{ and } n \in \mathbb{N}) \end{cases}$$

is satisfies, the following holds.

(a) $f \in \mathcal{L}_1(X, \mathfrak{M}, \mu)$.

(b) $\lim\limits_{n\to+\infty} \int |f_n - f|\ d\mu = 0$.

(c) $\lim\limits_{n\to+\infty} \int f_n\ d\mu = \int f\ d\mu$.

Theorem 5.2.3. (*Lebesgue's Dominated Convergence Theorem* 2 [Bog07])
Given a measurable function sequence $(f_n)_{n\geq 1} \subset \mathcal{M}(X, \mathfrak{M})$ and a measurable function $f\colon X \to \mathbb{R}$ with real-valued, if the condition

$$\begin{cases} \lim\limits_{n\to+\infty} f_n = f\ \mu\text{-a.e [X]}, \\ g \in \mathcal{L}]_1(X, \mathfrak{M}, \mu)\colon\ |f_n| \leq g\mu\text{-a.e [X]}\ (n \in \mathbb{N}) \end{cases}$$

is satisfied, the following holds.

(a) $f \in \mathcal{L}_1(X, \mathfrak{M}, \mu)$.

(b) $\lim\limits_{n\to+\infty} \int |f_n - f|\ d\mu = 0$.

(c) $\lim\limits_{n\to+\infty} \int f_n\ d\mu = \int f\ d\mu$.

5.3 Absolute Convergence of Integrals

Given a measure space $(X, \mathfrak{M}, \mu)$ and a measurable set $A \in \mathfrak{M}$, if

$$\mathfrak{M}_A = \{A \cap E\colon\ E \in \mathfrak{M}\} \text{ and } \mu_A = \mu\big|_{\mathfrak{M}_A},$$

then $(X, \mathfrak{M}_A, \mu_A)$ becomes a measure space. The measure space $(X, \mathfrak{M}_A, \mu_A)$ obtained in this way is called a subspace of the measure space $(X, \mathfrak{M}, \mu)$.

Theorem 5.3.1. *For a measure space* $(X, \mathfrak{M}, \mu)$*, a monotone increasing measurable set* $(E_n)_{n\leq 1} \subset \mathfrak{M}$ with $X = \bigcup_{n=1}^{\infty} E_n$ and a measurable function $f \in \mathcal{M}(X, \mathfrak{M})$ are given, and if

$$\begin{cases} f \in \mathcal{L}_1(\mathrm{E}_n, \mathfrak{M}, \mu) \quad (\mathrm{n} \in \mathbb{N}), \\ \lim\limits_{n\to+\infty} \int_{E_n} |f|\, d\mu < +\infty \end{cases} \tag{5.5}$$

is satisfied, then $f \in \mathcal{L}_1(E_n, \mathfrak{M}, \mu)$. Moreover, the equation

$$\int f\, d\mu = \lim_{n\to+\infty} \int_{E_n} f\, d\mu$$

is established.

Corollary 5.3.2. *In Theorem 5.3.1, even if the monotone increasing measurable set* $(E_n)_{n\geq 1} \subset \mathfrak{M}$ satisfying the condition (5.5) is selected differently, the integrable value

$$\lim_{n\to+\infty} \int_{E_n} f\, d\mu$$

remains unchanged.

Proof.

If monotone increasing measurable set $(F_n)_{n\geq 1} \subset \mathfrak{M}$ satisfying the assumption of Theorem 5.2.1 is selected,

$$\lim_{n\to+\infty} \int_{F_n} f\, d\mu = \int f\, d\mu = \lim_{n\to+\infty} \int_{E_n} f\, d\mu.$$

is obtained.

5.4 Functions of Complex Numeric Values

Given a measurable function $f\colon X \to \mathbb{C}$ having a complex-valued for the measure space $(X, \mathfrak{M}, \mu)$, if $Re\ f,\ Im\ f \in \mathcal{L}_1(X, \mathfrak{M}, \mu)$, a measurable function $f\colon X \to \mathbb{C}$ having a complex-valued is integrable with respect to measure μ. Therefore, it is defined as

$$\int f\, d\mu = \int Re\ f\, d\mu + i \int Im\ f.$$

In order to distinguish $\mathcal{L}_1(X, \mathfrak{M}, \mu)$, it is denoted by $f \in \mathcal{L}_1(\mu)$.

Lemma 5.4.1. *Given a measure space* $(X, \mathfrak{M}, \mu)$ *and a measurable function* f: $X \to \mathbb{C}$ having a complex-valued, the following holds.

(a) $f \in \mathcal{L}_1(\mu) \Longleftrightarrow |f| \in \mathcal{L}_1(X, \mathfrak{M}, \mu)$.

(b) $\left|\int f\, d\mu\right| \leq \int |f|\, d\mu\, (f \in \mathcal{L}_1(\mu))$.

Theorem 5.4.2. *Given a measure space* $(X, \mathfrak{M}, \mu)$ *and a measurable function* $(f_n)_{n\leq 1}$ with a complex-valued, if

$$\begin{cases} \lim\limits_{n\to+\infty} f_n(x) = f(x)\ (x \in X), \\ g \in \mathcal{L}_1(X, \mathfrak{M}, \mu)\colon |f_n(x)| \leq g(x)\, (x \in X) \end{cases}$$

is satisfied, the following holds.

(a) $f \in \mathcal{L}_1(\mu)$.

(b) $\lim_{n\to+\infty} \int \left|f_n - f\right| \, d\mu = 0$.

(c) $\lim_{n\to+\infty} \int f_n \, d\mu = \int f d\mu$.

Proof.

Let $f_n = \phi_n + i\psi_n$ for each natural number $n \in \mathbb{N}$, and if $f_n = \phi_n + i\psi_n$, remember that

$$\lim_{n\to+\infty} f_n(x) = f(x) \Longleftrightarrow \begin{cases} \lim_{n\to+\infty} \phi_n(x) = \phi(x), \\ \lim_{n\to+\infty} \psi_n(x) = \psi(x) \end{cases}$$

for any point $x \in X$.

(a) Since $|f_n(x)| \leq g(x)\,(x \in X)$ for each natural number $n \in \mathbb{N}$, if $\max\{|\phi_n(x)|, |\psi_n(x)|\} \leq |f_n(x)|\,(x \in X)$ is considered, $\max\{|\phi_n(x)|, |\psi_n(x)|\} \leq |g_n(x)|\,(x \in X)$ is established. Now, applying Corollary 5.1.2.10, since $\phi, \psi \in \mathcal{L}_1(X, \mathfrak{M}, \mu)$, it is $f \in \mathcal{L}_1(\mu)$ by Definition.

(b) If Lebesgue's Dominated Convergence Theorem [LDCT] [Bog07] is applied,

$$\lim_{n\to+\infty} \int |\phi_n - \phi| \, d\mu = 0 = \lim_{n\to+\infty} \int |\psi_n - \psi| \, d\mu$$

is satisfied. However, since $|f_n - f| \leq |\phi_n - \phi| + |\psi_n - \psi|\,(x \in X)$ holds on set X, we get

$$\lim_{n\to+\infty} \int |f_n - f| \, d\mu = 0.$$

(c) Finally, if Lebesgue's Dominated Convergence Theorem [LDCT] [Bog07] is applied, since

$$\lim_{n\to+\infty} \int \phi_n \, d\mu = \int \phi \, d\mu \text{ and } \lim_{n\to+\infty} \int \psi_n \, d\mu = \int \psi \, d\mu,$$

we get

$$\begin{aligned} \lim_{n\to+\infty} \int f_n \, d\mu &= \lim_{n\to+\infty} \left[\int \phi_n \, d\mu + i \int \psi_n \, d\mu \right] \\ &= \lim_{n\to+\infty} \int \phi_n \, d\mu + i \lim_{n\to+\infty} \int \psi_n \, d\mu \\ &= \int \phi \, d\mu + i \int \psi \, d\mu \\ &= \int f \, d\mu. \end{aligned}$$

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