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Many closed K-magnetic geodesics on $\mathbb{S}^2$

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Abstract. In this paper we adopt an alternative, analytical approach to Arnol'd problem [4] about the existence of closed and embedded K-magnetic geodesics in the round 2-sphere $\mathbb{S}^2$, where $K : \mathbb{S}^2 \to \mathbb{R}$ is a smooth scalar function. In particular, we use Lyapunov-Schmidt finite-dimensional reduction coupled with a local variational formulation in order to get some existence and multiplicity results bypassing the use of symplectic geometric tools such as the celebrated Viterbo's theorem [21] and Bottkoll results [7].

1. Introduction

We deal with the motion $\gamma = \gamma(t)$ of a particle of unit mass and charge in $\mathbb{R}^3$, that experiences the Lorentz force $\mathbf{F}$ produced by a magnetostatic field $\mathbf{B}$. If the particle is constrained to the standard round sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, the motion law reads

$$\gamma'' + |\gamma'|^2\gamma = K(\gamma)\gamma \wedge \gamma', \tag{1.1}$$

where

$$K(p) := -\mathbf{B}(p) \cdot p\,, \quad p \in \mathbb{S}^2.$$

A trajectory $\gamma(t)$ satisfying (1.1) is called *K-magnetic geodesic*.

Let us recall the elementary derivation of (1.1). We have $\mathbf{F}(\gamma) = \gamma' \wedge \mathbf{B}(\gamma)$; due to the constraint $|\gamma| \equiv 1$, the vectors γ and γ' are orthogonal along the motion. It follows that the projection of $\mathbf{F}$ on $T_\gamma\mathbb{S}^2 = \langle\gamma\rangle^\perp$ is proportional to $\gamma \wedge \gamma'$, and in fact $\mathbf{F}^T(\gamma) = -(\mathbf{B}(\gamma) \cdot \gamma)\gamma \wedge \gamma' = K(\gamma)\gamma \wedge \gamma'$. Finally, by differentiating the identity $\gamma \cdot \gamma' \equiv 0$, we see that the tangent component of the acceleration vector

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is $\gamma'' - (\gamma'' \cdot \gamma)\gamma = \gamma'' + |\gamma'|^2\gamma$, and thus Newton's law gives (1.1). Notice that $\gamma'' - (\gamma'' \cdot \gamma)\gamma = \nabla^{\mathbb{S}^2}_{\gamma'}\gamma'$, where $\nabla^{\mathbb{S}^2}$ is the Levi-Civita connection of $\mathbb{S}^2$.

Two remarkable facts immediately follow from (1.1). First, we have $2\gamma'' \cdot \gamma' = (|\gamma'|^2)' = 0$. Thus the particle moves with constant scalar speed, say

$$|\gamma'| \equiv c,$$

for some $c > 0$. In particular, γ is a regular curve. Secondly, we learn from differential geometry that γ has geodesic curvature

$$\kappa(\gamma) = \frac{\gamma'' \cdot \gamma \wedge \gamma'}{|\gamma'|^3} = \frac{K(\gamma)}{c}.$$

Next, let $c > 0$ and $K : \mathbb{S}^2 \to \mathbb{R}$ be given. In [4], see also [5, Problems 1988/30, 1994/14, 1996/18], Arnol'd proposed the following question (actually in a more general setting, where $\mathbb{S}^2$ is replaced by an oriented Riemannian surface (Σ, g)):

Find closed and embedded K-magnetic geodesics $\gamma \subset \mathbb{S}^2$ with $|\gamma'| \equiv c$. $(\mathcal{P}_{K,c})$

Problem $(\mathcal{P}_{K,c})$, together with its generalizations, attracted the attention of many authors and has been studied via different mathematical tools, such as symplectic geometric [4,10,11,13,17] and variational arguments for multivalued functionals [6,15,19,20].

The relation between Problem $(\mathcal{P}_{K,c})$ and symplectic geometry can be explained as follows. Let us consider on $\mathbb{S}^2$ the (restriction of the) two-form $\beta := i_{\mathbf{B}}(dx \wedge dy \wedge dz)$ and let us define on the cotangent bundle $T^*\mathbb{S}^2$ endowed with coordinates (q, p) the symplectic form

$$\Omega = c\, dq \wedge dp - \pi^*\beta$$

where $dq \wedge dp = \sum_{i=1}^2 dq_i \wedge dp_i$ denotes the *standard symplectic form* on $T^*\mathbb{S}^2$ and $\pi : T^*\mathbb{S}^2 \to \mathbb{S}^2$ is the canonical projection.

It is not hard to show, via a straight calculation, that K-magnetic geodesics on $\mathbb{S}^2$ having constant speed c are exactly the projections $\pi(\gamma)$ of the integral curves of the vector field on $T^*\mathbb{S}^2$ defined by

$$dH = i_X\Omega, \tag{1.2}$$

where $H = \frac{1}{2}|p|^2$. In the language of symplectic geometry, X is the *Hamiltonian vector field* given by the Hamiltonian function H. Notice also that since γ' as observed above has constant speed, then $H(\gamma)$ is constant and then by (1.2) we have $i_{\gamma'}\Omega = 0$, which by definition means that γ is a *characteristic* of Ω.

Now, for any smooth K and every $c > 0$ large enough the existence of a solution to $(\mathcal{P}_{K,c})$ can be deduced via this symplectic geometric approach by applying the celebrated Viterbo result [21] on the existence of closed characteristics on compact hypersurfaces of contact type. It is worth to notice that this result can be generalized to any closed oriented surface Σ, yielding the existence of a solution for high energies c in every free homotopy class that can be represented by a non-degenerate geodesic [11, Theorem 2.1 (ii)].

For the case of low energy levels we cite [11, Theorem 2.1 (i)] and [17], where the author proves the existence of contractible periodic solutions for almost all sufficiently small energy levels and for arbitrary smooth magnetic fields.

The existence of at least two distinct solutions to $(\mathcal{P}_{K,c})$ in the case of the round two-sphere follows, always for $c > 0$ large enough, from a general result of Bottkoll [7] (see also [1]) about the number of periodic orbits of the flow of a Hamiltonian vector field which is close to a flow generating a free circle action (in our case, the geodesic flow on the round two-sphere), which implies that such periodic orbits are at least as many as one plus the cup-length of $\mathbb{S}^2$, i.e. two.

For other available results for $(\mathcal{P}_{K,c})$ showing the existence of at least two distinct solutions for arbitrary metrics on $\mathbb{S}^2$ let us mention [11, Theorem 2.1 (i) and Theorem 2.7], [18], [16]. Notice that all these results require that K has constant sign: indeed, in [11] the assumption $K > 0$ guarantees that $\Omega = K d\sigma$ is a symplectic form on $\mathbb{S}^2$; in [18], [16] an index-based topological argument is used to prove the existence of two distinct solutions for any $c > 0$, and the assumption $K > 0$ is needed to prove some crucial *a-priori* bound on the length of simple and closed K-magnetic geodesics. Schneider's multiplicity result is indeed sharp, that is, Problem $(\mathcal{P}_{K,c})$ might have exactly two distinct solutions, see [18, Theorem 1.3].

Let us however notice that from the physical point of view it is important to include sign-changing functions K, unless the existence of magnetic monopoles is admitted. In fact, the Gauss law for magnetism in absence of magnetic monopoles implies that

$$\int_{\mathbb{S}^2} K(p)\, d\sigma_p = 0,$$

see also [4, Problem 1996-17].

The aim of this paper is twofold. Firstly, we provide a more direct, self-contained and analytical approach to Viterbo's and Bottkoll's results, in the special case of the round sphere. Secondly, we provide sufficient conditions on K to obtain as many solutions as we wish, provided that c is large enough.

Our main results are stated in Sects. 4 and 5, see Theorems 4.1 and 5.2, respectively.

For the proofs we took inspiration from the breakthrough paper [2], where Ambrosetti and Badiale showed how merging the Lyapunov-Schmidt finite-dimensional reduction with variational arguments allows to obtain extremely powerful tools to get existence and multiplicity results. This idea has been applied to tackle quite a large number of variational problems arising from mathematical physics and differential geometry, see the exhaustive list of references in the monograph [3].

We agree that the curves $\gamma_1(t), \gamma_2(t)$ are distinct if $\gamma_1 \neq \gamma_2 \circ g$, for any diffeomorphism g.

Notice however that Arnol'd problem on K-magnetic geodesics in $\mathbb{S}^2$ does not admit a (standard) variational formulation through a (non-multivalued) energy functional, due to obvious topological obstructions. To overcome this difficulty, we take advantage of a "local" variational approach which is developed in Sect. 2.

Notation.

The Euclidean space $\mathbb{R}^3$ is endowed with Euclidean norm $|p|$, scalar product $p \cdot q$, and exterior product $p \wedge q$. The canonical basis of $\mathbb{R}^3$ is $\{e_h\ ,\ h = 1, 2, 3\}$.

We isometrically embed the unit sphere $\mathbb{S}^2$ into $\mathbb{R}^3$, so that the tangent space $T_z\mathbb{S}^2$ at $z \in \mathbb{S}^2$ is identified with $\langle z\rangle^\perp = \{p \in \mathbb{R}^3 \mid p \cdot z = 0\}$. We denote by $\mathcal{D}_\rho(z) \subset \mathbb{S}^2$ the geodesic disk of radius $\rho \in (0, \frac{\pi}{2}]$ about $z \in \mathbb{S}^2$.

It is convenient to regard at $\mathbb{S}^1$ as the unit circle in the complex plane.

Function spaces. Let $m \geq 0$, $n \geq 1$ be integer numbers. We endow $C^m(\mathbb{S}^1, \mathbb{R}^n)$ with the standard Banach space structure. If $f \in C^1(\mathbb{S}^1, \mathbb{R}^n)$, we identify $f'(x) \equiv f'(x)(ix)$, so that $f' : \mathbb{S}^1 \to \mathbb{R}^n$.

We write $C^m(\mathbb{S}^1)$ instead of $C^m(\mathbb{S}^1, \mathbb{R})$ and C^m instead of $C^m(\mathbb{S}^1, \mathbb{R}^3)$. For $U \subseteq \mathbb{S}^2$ we put

$$C^m_U := C^m(\mathbb{S}^1, U) = \{u \in C^m \mid u(x) \in U \quad \text{for any} \ \ x \in \mathbb{S}^1\}.$$

We identify U with the set of constant functions in C^2_U, so that $C^2_U \setminus U = C^2_U \setminus \mathbb{S}^2$ contains only nonconstant curves.

The Hilbertian norm in $L^2 = L^2(\mathbb{S}^1, \mathbb{R}^3)$ is

$$\|u\|^2_{L^2} = \fint_{\mathbb{S}^1} |u(x)|^2 dx = \frac{1}{2\pi} \int_{\mathbb{S}^1} |u(x)|^2 dx\,,$$

and the orthogonal to $T \subseteq C^0$ with respect to the L^2 scalar product is given by

$$T^\perp = \{\varphi \in C^0 \mid \fint_{\mathbb{S}^1} u \cdot \varphi \, dx = 0 \quad \text{for any}\ u \in T\ \}.$$

We regard at $C^2_{\mathbb{S}^2}$ as a smooth complete submanifold of C^2. If $u \in C^2_{\mathbb{S}^2}$, the tangent space to $C^2_{\mathbb{S}^2}$ at u is

$$T_u C^2_{\mathbb{S}^2} = \{\varphi \in C^2 \mid u \cdot \varphi \equiv 0 \ \text{ on}\ \mathbb{S}^1\ \}.$$

If u is regular, that means $u'(x) \neq 0$ for any $x \in \mathbb{S}^1$, then

$$T_u C^2_{\mathbb{S}^2} = \{g_1 u' + g_2 u \wedge u' \mid g = (g_1, g_2) \in C^2(\mathbb{S}^1, \mathbb{R}^2)\ \}.$$

Rotations. Any complex number $\mathbb{S}^1$ is identified with the rotation $x \mapsto \xi x$. Recall that $\det(R) = +1$ and $R^{-1} =^t R$ for any $R \in SO(3)$, where $SO(3)$ is the group of rotations of $\mathbb{R}^3$ and ${}^tR$ is the transpose of R.

It is well-known that $SO(3)$ is a connected three-dimensional manifold. More precisely, it is a Lie group whose Lie algebra is given by the skew-symmetric matrices, and the tangent space $T_{\mathrm{Id}_3} SO(3)$ at the identity matrix is spanned by

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \ T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \ T_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A simple explanation of this elementary fact follows by introducing the matrices

$$R_1^\xi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \xi_1 & -\xi_2 \\ 0 & \xi_2 & \xi_1 \end{pmatrix}, \quad R_2^\xi = \begin{pmatrix} \xi_1 & 0 & -\xi_2 \\ 0 & 1 & 0 \\ \xi_2 & 0 & \xi_1 \end{pmatrix}, \quad R_3^\xi = \begin{pmatrix} \xi_1 & -\xi_2 & 0 \\ \xi_2 & \xi_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $\xi = \xi_1 + i\xi_2 \in \mathbb{S}^1$. Clearly R_h^ξ is a rotation about the $\langle e_h \rangle$ axis. By differentiating R_h^ξ with respect to $\xi \in \mathbb{S}^1$ at $\xi = 1$ one gets $T_h = dR_h^\xi\big|_{\xi=1}$, and thus infers that $\{T_h\}$ is a basis for $T_{\mathrm{Id}_3}SO(3)$. In accordance with the Lie group structure of $SO(3)$, the tangent space to $SO(3)$ at $R \in SO(3)$ is obtained by rotating $T_{\mathrm{Id}_3}SO(3)$. Hence

$$T_R SO(3) = \langle RT_1, RT_2, RT_3 \rangle.$$

Finally, for any $q \in \mathbb{S}^2$ we denote by d_R the differential of the function $SO(3) \to \mathbb{S}^2$, $R \mapsto Rq$, so that $d_R(Rq)\tau \in T_{Rq}\mathbb{S}^2$ for any $\tau \in T_R SO(3)$. We have the formula

$$d_R(Rq)(RT_h) = R(e_h \wedge q) = Re_h \wedge Rq. \tag{1.3}$$

2. A "local" variational approach

We put $\varepsilon = c^{-1}$ and study Problem $(\mathcal{P}_{K,\varepsilon^{-1}})$ for ε close to 0. We take advantage of its geometrical interpretation to rewrite it in an equivalent way. Let γ be a solution to $(\mathcal{P}_{K,\varepsilon^{-1}})$, and let $\mathcal{L}_\gamma$ be its length. Extend γ to an $\varepsilon\mathcal{L}_\gamma$-periodic function on $\mathbb{R}$ and consider the curve $u \in C^2_{\mathbb{S}^2}$, $u(e^{i\theta}) = \gamma\big(\frac{\varepsilon\mathcal{L}_\gamma}{2\pi}\theta\big)$. Evidently u and γ have the same length $\mathcal{L}_\gamma$ and curvature εK. Moreover $|u'| \equiv \mathcal{L}_\gamma/2\pi$ and u solves the system

$$u'' + |u'|^2 u = |u'|\varepsilon K(u)\, u \wedge u' \qquad \text{on } \mathbb{S}^1, \tag{2.1}$$

because γ solves (1.1). Conversely, any solution $u \in C^2_{\mathbb{S}^2} \setminus \mathbb{S}^2$ to (2.1) has constant speed $|u'|$, curvature $\varepsilon K(u)$ and gives rise to a solution to $(\mathcal{P}_{K,\varepsilon^{-1}})$.

The main goal of the present section is to show that for any point $p \in \mathbb{S}^2$, the problem of finding solutions to (2.1) in $C^2_{\mathbb{S}^2\setminus\{p\}}$, that is an open subset of $C^2_{\mathbb{S}^2}$, can be faced by using variational methods. First, we need to introduce the functional

$$L(u) = \Big(\fint_{\mathbb{S}^1} |u'|^2 dx \Big)^{\frac{1}{2}}, \quad L : C^2_{\mathbb{S}^2} \setminus \mathbb{S}^2 \to \mathbb{R}. \tag{2.2}$$

Notice that the Cauchy-Schwarz inequality gives $\mathcal{L}_u \le 2\pi L(u)$, and equality holds if and only if $|u'|$ is constant. Moreover, it holds that

$$L(Ru \circ \xi) = L(u) \quad \text{for any } \xi \in \mathbb{S}^1,\ R \in SO(3). \tag{2.3}$$

Finally, we notice that L is Fréchet differentiable at any $u \in C^2_{\mathbb{S}^2} \setminus \mathbb{S}^2$, with differential

$$L'(u)\varphi = \frac{1}{L(u)} \fint_{\mathbb{S}^1} u' \cdot \varphi' \, dx = \frac{1}{L(u)} \fint_{\mathbb{S}^1} (-u'' - |u'|^2 u) \cdot \varphi \, dx \quad \text{for any } \varphi \in T_u C^2_{\mathbb{S}^2}. \tag{2.4}$$

In the next lemma we provide a variational reading of the right-hand side of (2.1), see also [15] and [11, Remark 2.2].

Lemma 2.1. *Let $K \in C^0(\mathbb{S}^2)$ and let U, V be open and contractible subsets of $\mathbb{S}^2$.*

$i)$ There exists a unique C^1 functional $\mathcal{A}^U_K : C^2_U \to \mathbb{R}$, such that $\mathcal{A}^U_K(u) = 0$ if u is constant, and

$$(\mathcal{A}^U_K)'(u)\phi = \fint_{S^1} K(u)\phi \cdot u \wedge u' \, dx \quad \textit{for any } u \in C^2_U,\ \phi \in T_u C^2_{\mathbb{S}^2}; \tag{2.5}$$

$ii)$ If $R \in SO(3)$, $\xi \in \mathbb{S}^1$ and $u \in C^2_U$, then $\mathcal{A}^{RU}_{K \circ {}^t R}(Ru \circ \xi) = \mathcal{A}^U_K(u)$;

$iii)$ If $U \cap V$ is nonempty and contractible, then $\mathcal{A}^U_K(u) = \mathcal{A}^V_K(u)$ for any $u \in C^2_{U \cap V}$;

$iv)$ Let $u \in C^2_{\mathbb{S}^2}$. The function $p \mapsto \mathcal{A}^{\mathbb{S}^2 \setminus \{p\}}_K(u)$ is constant on each connected component of $\mathbb{S}^2 \setminus u(\mathbb{S}^1)$;

$v)$ Let $u \in C^2_U$ be a positively oriented parametrization of the boundary of a regular open set $\Omega_u \subset U$. Then

$$\mathcal{A}^U_K(u) = -\frac{1}{2\pi} \int_{\Omega_u} K(q) \, d\sigma_q \, .$$

Proof. Take a 1-form β^U_K on U, such that

$$d\beta^U_K = -K(q) \, d\sigma_q \, , \tag{2.6}$$

where $d\sigma_q$ is the restriction of the volume form on the sphere. We put

$$\mathcal{A}^U_K(u) = \fint_{\mathbb{S}^1} u^* \beta^U_K = \fint_{\mathbb{S}^1} \beta^U_K(u) u' \, dx \, , \quad u \in C^2_U \, .$$

It is evident that $\mathcal{A}^U_K(u) = 0$ if u is constant. Formula (2.5) can be derived by using Lie differential calculus or local coordinates, like in the proof of [6, Lemma 3]. Elementary arguments and (2.5) give the C^1 differentiability of the functional $\mathcal{A}^U_K$. Uniqueness is trivial, because C^2_U is a connected manifold. In particular, for $u \in C^2_U$ the real number $\mathcal{A}^U_K(u)$ does not depend on the choice of β^U_K.

To prove $ii)$ take a 1-form β in the domain RU such that $d\beta = -(K \circ {}^t R) d\sigma_q$. Clearly $R^*\beta$ is a 1-form in U, and $d(R^*\beta) = R^*(d\beta) = -K(q) d\sigma_q$. Thus we can take $\beta^U_K = R^*\beta$ in formula (2.6) and we obtain

$$\mathcal{A}^{RU}_{K \circ {}^t R}(Ru) = \fint_{\mathbb{S}^1} (Ru)^* \beta = \fint_{\mathbb{S}^1} u^*(R^*\beta) = \mathcal{A}^U_K(u)$$

for any $u \in C^2_U$. The invariance of the area functional with respect to composition with rotations of $\mathbb{S}^1$ is immediate.

Now we prove iii). If $V \subset U$ and $u \in C^2_V$, then the restriction of β^U_K to V can be used to compute $\mathcal{A}^V_K(u)$. Thus $\mathcal{A}^V_K(u) = \mathcal{A}^U_K(u)$. It follows that if two open, connected sets U, V have contractible intersection and $u \in C^2_{U\cap V}$, then $\mathcal{A}^{U\cap V}_K(u) = \mathcal{A}^U_K(u)$ and $\mathcal{A}^{U\cap V}_K(u) = \mathcal{A}^V_K(u)$.

Claim iv) readily follows from iii). In fact, take $p_0 \in \mathbb{S}^2 \setminus u(\mathbb{S}^1)$ and a small disk $\mathcal{D}_\delta(p_0) \subset \mathbb{S}^2 \setminus u(\mathbb{S}^1)$. For any $p \in \mathcal{D}_\delta(p_0)$ we have

$$\mathcal{A}^{\mathbb{S}^2\setminus\{p\}}(u) = \mathcal{A}^{\mathbb{S}^2\setminus\mathcal{D}_\delta(p_0)}(u) = \mathcal{A}^{\mathbb{S}^2\setminus\{p_0\}}(u).$$

We proved that the function $p \mapsto \mathcal{A}^{\mathbb{S}^2\setminus\{p\}}(u)$ is locally constant on $\mathbb{S}^2 \setminus u(\mathbb{S}^1)$, and hence is constant on each connected component of $\mathbb{S}^2 \setminus u(\mathbb{S}^1)$.

For the last claim we use Stokes' theorem to get

$$2\pi\,\mathcal{A}^U_K(u) = \int\limits_{\mathbb{S}^1} u^*\beta^U_K = \int\limits_{\partial\Omega_u} \beta^U_K = \int\limits_{\Omega_u} d\beta^U_K = -\int\limits_{\Omega_u} K(q)d\sigma_q$$

by (2.6). The lemma is completely proved. □

From now on we write

$$A_K(p;u) = \mathcal{A}^{\mathbb{S}^2\setminus\{p\}}_K(u)\,, \quad p \in \mathbb{S}^2\,, \;\; u \in C^2_{\mathbb{S}^2\setminus\{p\}}.$$

By Lemma 2.1, the functional A_K enjoys the following properties,

$A1)$ The functional $A_K(p;\cdot)$ is of class C^1 on $C^2_{\mathbb{S}^2\setminus\{p\}}$, and

$$A'_K(p;u)\phi = \fint_{\mathbb{S}^1} K(u)\phi\cdot u\wedge u'\,dx \;\text{ for any } u \in C^2_{\mathbb{S}^2\setminus\{p\}},\ \phi \in T_uC^2_{\mathbb{S}^2}.$$

$A2)$ If $R \in SO(3)$, $\xi \in \mathbb{S}^1$, and $u \in C^2_{\mathbb{S}^2\setminus\{p\}}$, then $A_{K\circ^tR}(Rp; Ru\circ\xi) = A_K(p;u)$.

$A3)$ Let $u \in C^2_{\mathbb{S}^2}$. The function $p \mapsto A_K(p;u)$ is locally constant on $\mathbb{S}^2 \setminus u(\mathbb{S}^1)$.

$A4)$ Let $u \in C^2_{\mathbb{S}^2\setminus\{p\}}$ be a positively oriented parametrization of the boundary of a regular open set $\Omega_u \subset \mathbb{S}^2 \setminus \{p\}$. Then

$$A_K(p;u) = -\frac{1}{2\pi}\int\limits_{\Omega_u} K(q)d\sigma_q\,.$$

Remark 2.2. To find an explicit formula for $A_K(p;\ \cdot\ )$ let $\Pi_p : \mathbb{S}^2 \setminus \{p\} \to \mathbb{R}^2$ be the stereographic projection from the pole p. If $u \in C^2_{\mathbb{S}^2\setminus\{p\}}$, then $\Pi_p \circ u$ is a curve in $\mathbb{R}^2$ and $(\Pi_p^{-1})^*(Kd\sigma_q) = (K\circ\Pi_p^{-1})\det J_{\Pi_p^{-1}}(z)dz$ is a 2-form on $\mathbb{R}^2$. Let $\tilde\beta^p_K$ be a 1-form on $\mathbb{R}^2$ such that $d\tilde\beta^p_K = (\Pi_p^{-1})^*(Kd\sigma_q)$. Then

$$A_K(p;u) = \fint\limits_{\mathbb{S}^1} u^*(\Pi_p^*\tilde\beta^p_K) = \fint\limits_{\mathbb{S}^1}(\Pi_p\circ u)^*\tilde\beta^p_K\,.$$

For instance, if $K \equiv 1$ is constant one can take

$$A_1(p;u) = \fint_{\mathbb{S}^1} \frac{p}{1-u\cdot p} \cdot u\wedge u'\,dx = 2\fint_{\mathbb{S}^1} \frac{p}{|u-p|^2}\cdot u\wedge u'\,dx.$$

The next lemma provides the predicted "local" variational approach to (2.1).

Lemma 2.3. *Let $K \in C^0(\mathbb{S}^2)$.*

i) For any $p \in \mathbb{S}^2$, the functional

$$E_{\varepsilon K}(p;u) = L(u) + \varepsilon A_K(p;u), \quad E_{\varepsilon K}(p;\ \cdot\ ) : C^2_{\mathbb{S}^2\setminus\{p\}} \setminus \mathbb{S}^2 \to \mathbb{R}$$

is of class C^1, with differential

$$L(u)E'_{\varepsilon K}(p;u)\varphi = \fint_{\mathbb{S}^1} \big(-u'' + L(u)\varepsilon K(u)u\wedge u'\big)\cdot\varphi\,dx, \quad \text{for any } \varphi \in T_uC^2_{\mathbb{S}^2}. \tag{2.7}$$

In particular, any critical point $u \in C^2_{\mathbb{S}^2\setminus\{p\}} \setminus \mathbb{S}^2$ for $E_{\varepsilon K}(p;\ \cdot\ )$ solves (2.1).

ii) If $R \in SO(3)$, $\xi \in \mathbb{S}^1$ and $p \in \mathbb{S}^2$, then $E_{\varepsilon K\circ^t R}(Rp; Ru\circ\xi) = E_{\varepsilon K}(p;u)$ for any nonconstant curve $u \in C^2_{\mathbb{S}^2\setminus\{p\}}$, and thus

$$E'_{\varepsilon K}(p;u)u' = 0 \quad \text{for any } u \in C^2_{\mathbb{S}^2\setminus\{p\}} \setminus \mathbb{S}^2. \tag{2.8}$$

iii) Let $u \in C^2_{\mathbb{S}^2} \setminus \mathbb{S}^2$. The function $E_{\varepsilon K}(\ \cdot\ ; u) : \mathbb{S}^2 \setminus u(\mathbb{S}^1) \to \mathbb{R}$ is locally constant.

iv) If $K \in C^1(\mathbb{S}^2)$ then the functional $E_{\varepsilon K}(p;\ \cdot\ )$ is of class C^2 on its domain.

Proof. Formula (2.4) and the property $A1$) of the area functional give the C^1 regularity of $E_{\varepsilon K}(p;\ \cdot\ )$ and (2.7). Let u be a critical point for $E_{\varepsilon K}(p;\ \cdot\ )$. Take any $\varphi \in C^2$ and put $\varphi^\top = \varphi - (\varphi\cdot u)u \in T_uC^2_{\mathbb{S}^2}$. We have $\varphi\cdot u\wedge u' = \varphi^\top\cdot u\wedge u'$ on $\mathbb{S}^1$, and $u'\cdot(\varphi^\top)' = u'\cdot\varphi' - (\varphi\cdot u)|u'|^2$ because $u'\cdot u \equiv 0$. Since

$$\begin{aligned} 0 = L(u)E'_{\varepsilon K}(p;u)\varphi^\top &= \fint_{\mathbb{S}^1} \big(u'\cdot(\varphi^\top)' + L(u)\varepsilon K(u)\varphi^\top\cdot u\wedge u'\big)dx \\ &= \fint_{\mathbb{S}^1} \big(u'\cdot\varphi' - (\varphi\cdot u)|u'|^2 + L(u)\varepsilon K(u)\varphi\cdot u\wedge u'\big)dx, \end{aligned}$$

and therefore u solves $u'' + |u'|^2u = L(u)\varepsilon K(u)u\wedge u'$ on $\mathbb{S}^1$. Since $u''\cdot u' \equiv 0$, we see that $|u'| \equiv L(u)$ is constant, and thus u solves (2.1).

Statements ii), iii) follow from (2.3), $A2$) and $A3$) (to check (2.8) take the derivative of the identity $E_{\varepsilon K}(p;u\circ\xi) = E_{\varepsilon K}(p;u)$ with respect to $\xi \in \mathbb{S}^1$ at $\xi = 1$). Finally, iv) can be proved via elementary arguments, starting from (2.7). □

3. Geodesics

For any rotation $R \in SO(3)$, the loop

$$\omega_R(x) = R\big(x_1, x_2, 0\big)\,, \quad x = x_1 + ix_2 \in \mathbb{S}^1\,,$$

is a parameterization of the boundary of $\mathcal{D}_{\frac{\pi}{2}}(Re_3)$ and solves

$$\omega_R'' + |\omega_R'|^2\omega_R = 0\,, \quad L(\omega_R) = |\omega_R'| = 1\,. \tag{3.1}$$

In order to simplify notations, from now on we write

$$\omega(x) = \omega_{\mathrm{Id}}(x) = \big(x_1, x_2, 0\big)\,, \quad x = x_1 + ix_2 \in \mathbb{S}^1\,.$$

The tangent space to the smooth 3-dimensional manifold

$$\mathcal{S} = \big\{\omega_R \mid R \in SO(3)\big\} \subset C^2_{\mathbb{S}^2}$$

at $\omega_R \in \mathcal{S}$ can be easily computed via formula (1.3). It turns out that

$$T_{\omega_R}\mathcal{S} = \{q \wedge \omega_R \mid q \in \mathbb{R}^3\} = \langle Re_1 \wedge \omega_R\ ,\ Re_2 \wedge \omega_R\ ,\ Re_3 \wedge \omega_R\ \rangle.$$

We introduce the function

$$J_0(u) := -u'' - |u'|^2u\,, \quad J_0 : C^2_{\mathbb{S}^2} \setminus \mathbb{S}^2 \to C^0,$$

so that $\mathcal{S} \subset \{J_0 = 0\}$. By (2.4) we have

$$L(u)L'(u)\varphi = \fint_{\mathbb{S}^1} J_0(u)\cdot\varphi\,dx \quad \text{for any } u \in C^2_{\mathbb{S}^2} \setminus \mathbb{S}^2,\ \varphi \in T_uC^2_{\mathbb{S}^2}. \tag{3.2}$$

Moreover, for $u \in C^2_{\mathbb{S}^2} \setminus \mathbb{S}^2$, $q \in \mathbb{R}^3$ and $R \in SO(3)$ it holds that

$$\fint_{\mathbb{S}^1} J_0(u)\cdot q \wedge u\,dx = 0\,, \quad J_0(Ru) = RJ_0(u)\,. \tag{3.3}$$

The first identity readily follows via integration by parts or can be obtained by differentiating the identity $L(Ru) = L(u)$ with respect to $R \in SO(3)$. The second one is immediate.

Clearly J_0 is of class C^2; for $R \in SO(3)$ and φ in the tangent space

$$T_{\omega_R}C^2_{\mathbb{S}^2} = \{\varphi = g_1\omega_R' + g_2\omega_R \wedge \omega_R' \mid g = (g_1, g_2) \in C^2(\mathbb{S}^1, \mathbb{R}^2)\ \}, \tag{3.4}$$

we have

$$J_0'(\omega_R)\varphi = -\varphi'' - 2(\omega_R'\cdot\varphi')\omega_R - \varphi$$

.

Further, the operator $J_0'(\omega_R)$ is self adjoint in $L^2(\mathbb{S}^1, \mathbb{R}^3)$, that is,

$$\fint_{\mathbb{S}^1} J_0'(\omega_R)\varphi\cdot\tilde\varphi\,dx = \fint_{\mathbb{S}^1} J_0'(\omega_R)\tilde\varphi\cdot\varphi\,dx \quad \text{for any } \varphi, \tilde\varphi \in T_{\omega_R}C^2_{\mathbb{S}^2}. \tag{3.5}$$

By differentiating the identity $J_0(\omega_R) = 0$ with respect to $R \in SO(3)$, we see that $T_{\omega_R}\mathcal{S} \subseteq \ker J_0'(\omega_R)$. Actually, equality holds, as shown in the next crucial lemma.

Lemma 3.1. *(Nondegeneracy) Let $R \in SO(3)$. Then*

i) $\ker J_0'(\omega_R) = T_{\omega_R}\mathcal{S}$;

ii) If $\varphi \in T_{\omega_R}C^2_{\mathbb{S}^2}$ and $J_0'(\omega_R)\varphi \in T_{\omega_R}\mathcal{S}$, then $\varphi \in T_{\omega_R}\mathcal{S}$;

iii) For any $u \in T_{\omega_R}\mathcal{S}^\perp$ there exists a unique $\varphi \in T_{\omega_R}C^2_{\mathbb{S}^2} \cap T_{\omega_R}\mathcal{S}^\perp$ such that $J_0'(\omega_R)\varphi = u$.

Proof. One can argue by adapting the computations in [18, Sect. 5]. We provide here a simpler argument.

Since $J_0'(\omega_R)(R\varphi) = R\big(J_0'(\omega)\varphi\big)$ for any $\varphi \in T_\omega C^2_{\mathbb{S}^2}$, it is not restrictive to assume that R is the identity matrix. By direct computations based on (3.1), one can check that

$$J_0'(\omega)(\psi\,\omega') = -\psi''\omega'\,, \quad J_0'(\omega)(\psi\,\omega\wedge\omega') = \big(-\psi''-\psi\big)\omega\wedge\omega'$$

for any $\psi \in C^2(\mathbb{S}^1,\mathbb{R})$. Since by (3.4) any function $\varphi \in T_\omega C^2_{\mathbb{S}^2}$ can be written as

$$\varphi = (\varphi\cdot\omega')\omega' + (\varphi\cdot\omega\wedge\omega')\omega\wedge\omega'\,,$$

we are led to introduce the differential operator $B : C^2(\mathbb{S}^1,\mathbb{R}^2) \to C^0(\mathbb{S}^1,\mathbb{R}^2)$,

$$B(g) = -g_1''e_1 + (-g_2''-g_2)e_2\,, \qquad g = (g_1,g_2) \in C^2(\mathbb{S}^1,\mathbb{R}^2)\,.$$

and the function transform

$$\Psi\varphi = (\varphi\cdot\omega')e_1 + (\varphi\cdot\omega\wedge\omega')e_2\,, \quad \Psi : T_\omega C^2_{\mathbb{S}^2} \to C^2(\mathbb{S}^1,\mathbb{R}^2)\,,$$

so that

$$J_0'(\omega)\varphi = \Psi^{-1}B(\Psi\varphi) \quad \text{for any } \varphi \in T_\omega C^2_{\mathbb{S}^2}, \quad \Psi(\ker J_0'(\omega)) = \ker B\,. \tag{3.6}$$

We proved that $\ker J_0'(\omega)$ and $T_\omega\mathcal{S}$ have both dimension 3, thus they must coincide because $T_\omega\mathcal{S} \subseteq \ker J_0'(\omega)$.

For future convenience we notice that Ψ is an isometry with respect to the L^2 norms, and in particular

$$\fint_{\mathbb{S}^1} \big(\Psi\varphi\big)\cdot\big(\Psi\tilde\varphi\big)\,dx = \fint_{\mathbb{S}^1} \varphi\cdot\tilde\varphi\,dx \quad \text{for any } \varphi,\tilde\varphi \in T_\omega C^2_{\mathbb{S}^2}. \tag{3.7}$$

Now we prove *ii*). If $\tau := J_0'(\omega)\varphi \in T_\omega\mathcal{S}$, then $J_0'(\omega)\tau = 0$, as $\ker J_0'(\omega) = T_\omega\mathcal{S}$. But then, using (3.5) we get

$$\fint_{\mathbb{S}^1} |J_0'(\omega)\varphi|^2\,dx = \fint_{\mathbb{S}^1} J_0'(\omega)\varphi\cdot\tau\,dx = \fint_{\mathbb{S}^1} J_0'(\omega)\tau\cdot\varphi\,dx = 0.$$

Thus $J_0'(\omega)\varphi = 0$, that means $\varphi \in T_\omega\mathcal{S}$.

It remains to prove *iii*). Since $\Psi(T_\omega\mathcal{S}) = \ker B$, from (3.6) and (3.7) we have that $u \in T_\omega\mathcal{S}^\perp$ if and only if $\Psi u \in \ker B^\perp$. In particular, if $u \in T_\omega\mathcal{S}^\perp$, then one can compute the unique solution $g_u \in \ker B^\perp$ to the system $Bg_u = \Psi u$. The function $\varphi := \Psi^{-1}g_u$ belongs to $T_\omega\mathcal{S}^\perp$; thanks to (3.6) it solves $J_0'(\omega)\varphi = u$, and is uniquely determined by u. The lemma is completely proved. □

Remark 3.2. For future convenience we compute

$$m_{hj} = \fint_{\mathbb{S}^1} (Re_h \wedge \omega_R) \cdot (Re_j \wedge \omega_R)\, dx = \fint_{\mathbb{S}^1} (e_h \wedge \omega) \cdot (e_j \wedge \omega)\, dx = \delta_{hj} - \fint_{\mathbb{S}^1} \omega_h \omega_j\, dx.$$

We see that the functions $Re_j \wedge \omega_R = R(e_j \wedge \omega)$ provide an orthogonal basis for $T_{\omega_R}\mathcal{S}$ endowed with the L^2 scalar product. More precisely, the matrix M associated to this scalar product with respect to the basis $\{Re_j \wedge \omega_R\}$ is given by

$$M = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3.1. Finite dimensional reduction

By the remarks at the beginning of Sect. 2, we are led to study problem (2.1) for $\varepsilon = c^{-1}$ close to 0. Further, since any solution u to (2.1) satisfies $|u'| \equiv L(u)$, we can rewrite (2.1) in the following, equivalent way,

$$u'' + |u'|^2 u = L(u)\varepsilon K(u)\, u \wedge u', \qquad u \in C^2_{\mathbb{S}^2} \setminus \mathbb{S}^2. \tag{3.8}$$

We will look for solutions to (3.8) by solving $J_\varepsilon(u) = 0$, where $J_\varepsilon : C^2_{\mathbb{S}^2} \setminus \mathbb{S}^2 \to C^0$,

$$J_\varepsilon(u) = J_0(u) + \varepsilon L(u) K(u)\, u \wedge u' = -u'' - |u'|^2 u + L(u)\varepsilon K(u)\, u \wedge u'. \tag{3.9}$$

Thanks to (2.7), we can write

$$L(u) E'_{\varepsilon K}(p; u)\varphi = \fint_{\mathbb{S}^1} J_\varepsilon(u) \cdot \varphi\, dx, \quad \text{for } u \in C^2_{\mathbb{S}^2} \setminus \mathbb{S}^2,\ p \notin u(\mathbb{S}^1),\ \varphi \in T_u C^2_{\mathbb{S}^2}. \tag{3.10}$$

The regularity assumption on K implies that J_ε is of class C^1 on its domain. In addition, $J_\varepsilon(u \circ \xi) = J_\varepsilon(u)$ for any $\xi \in \mathbb{S}^1$, and integration by parts gives

$$\fint_{\mathbb{S}^1} J_\varepsilon(u) \cdot u'\, dx = 0 \qquad \text{for any } u \in C^2_{\mathbb{S}^2} \setminus \mathbb{S}^2.$$

In general, the identities in (3.3) are not satisfied if $\varepsilon \neq 0$, because the perturbation term breaks the invariances of the operator J_0.

In the next lemma we provide the main step to obtain our multiplicity results.

Lemma 3.3. *There exist* $\overline{\varepsilon} > 0$ *and a* C^1 *function*

$$[-\overline{\varepsilon}, \overline{\varepsilon}] \times SO(3) \to C^2_{\mathbb{S}^2} \setminus \mathbb{S}^2 \quad (\varepsilon, R) \mapsto u_R^\varepsilon$$

such that u_R^ε *is an embedded loop, and moreover*

(i) $u_R^0 = \omega_R$;
(ii) $u_R^\varepsilon \in T_{\omega_R}\mathcal{S}^\perp$;
(iii) $J_\varepsilon(u_R^\varepsilon) \in T_{\omega_R}\mathcal{S}$;
(iv) *The function* $[-\overline{\varepsilon}, \overline{\varepsilon}] \times SO(3) \to \mathbb{R}$,

$$(\varepsilon, R) \mapsto \mathcal{E}^\varepsilon(R) := E_{\varepsilon K}(-Re_3; u_R^\varepsilon) = L(u_R^\varepsilon) + \varepsilon A_K(-Re_3; u_R^\varepsilon)$$

is well defined, of class C^1 *on its domain, and* $d_R\mathcal{E}^\varepsilon(R)(RT_3) = 0$.
(v) $R \in SO(3)$ *is critical for* $\mathcal{E}^\varepsilon : SO(3) \to \mathbb{R}$ *if and only if* $J_\varepsilon(u_R^\varepsilon) = 0$.
(vi) *Put* $\mathcal{E}_0^\varepsilon(R) = E_{\varepsilon K}(-Re_3; \omega_R) = 1 + \varepsilon A_K(-Re_3, \omega_R)$. *As* $\varepsilon \to 0$, *we have*

$$\mathcal{E}^\varepsilon(R) - \mathcal{E}_0^\varepsilon(R) = o(\varepsilon) \tag{3.11}$$

uniformly on $SO(3)$, *together with the derivatives with respect to* $R \in SO(3)$.

Proof. Consider the differentiable functions

$$\mathcal{F}_1 : \mathbb{R}\times SO(3)\times(C^2_{\mathbb{S}^2}\backslash\mathbb{S}^2)\times\mathbb{R}^3 \to C^0\ ,\ \mathcal{F}_1(\varepsilon, R, u; \zeta) = J_\varepsilon(u) - \sum_{j=1}^3 \zeta_j\,(Re_j \wedge \omega_R)$$

$$\mathcal{F}_2 : \mathbb{R}\times SO(3)\times(C^2_{\mathbb{S}^2}\backslash\mathbb{S}^2)\times\mathbb{R}^3 \to \mathbb{R}^3\ ,\ \mathcal{F}_2(\varepsilon, R, u; \zeta) = \sum_{j=1}^3\Big(\fint_{\mathbb{S}^1} u\cdot Re_j \wedge \omega_R\,dx\Big)e_j$$

where $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3$, and then let

$$\mathcal{F} : \mathbb{R} \times SO(3) \times (C^2_{\mathbb{S}^2}\backslash\mathbb{S}^2)\times\mathbb{R}^3 \to C^0\times\mathbb{R}^3\ , \quad \mathcal{F} = \big(\mathcal{F}_1, \mathcal{F}_2\big).$$

Fix $R \in SO(3)$. Since $J_0(\omega_R) = 0$ by (3.1), then $\mathcal{F}(0, R, \omega_R; 0) = 0$. Our first goal is to solve the equation $\mathcal{F}(\varepsilon, R, u; \zeta) = (0; 0)$ in a neighborhood of $(0, R, \omega_R; 0)$, via the implicit function theorem.

Consider the differentiable function

$$\mathcal{F}(0, R,\ \cdot\ ;\ \cdot\ ) : (u; \zeta) \mapsto \mathcal{F}(0, R, u; \zeta), \quad (C^2_{\mathbb{S}^2}\backslash\mathbb{S}^2)\times\mathbb{R}^3 \to C^0\times\mathbb{R}^3$$

and let

$$\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2) : (T_{\omega_R}C^2_{\mathbb{S}^2})\times\mathbb{R}^3 \to C^0\times\mathbb{R}^3$$

be its differential evaluated at $(u; \zeta) = (\omega_R; 0)$. We need to prove that $\mathcal{L}$ is invertible.

Take $\varphi \in T_{\omega_R}C^2_{\mathbb{S}^2}$ and $p = (p_1, p_2, p_3) \in \mathbb{R}^3$. It is easy to compute

$$\mathcal{L}_1(\varphi; p) = J_0'(\omega_R)\varphi - \sum_{j=1}^3 p_j\,(Re_j \wedge \omega_R), \quad \mathcal{L}_2(\varphi; p) = \sum_{j=1}^3\Big(\fint_{\mathbb{S}^1} \varphi\cdot Re_j \wedge \omega_R\,dx\Big)e_j\,.$$

Next, recall that $T_{\omega_R}\mathcal{S}$ is spanned by the functions $Re_j \wedge \omega_R$. If $\mathcal{L}_1(\varphi; p) = 0$ then $J_0'(\omega_R)\varphi \in T_{\omega_R}\mathcal{S}$, hence $\varphi \in T_{\omega_R}\mathcal{S}$ by $ii)$ in Lemma 3.1; if $\mathcal{L}_2(\varphi; p) = 0$ then $\varphi \in T_{\omega_R}\mathcal{S}^\perp$. Therefore, the operator $\mathcal{L}$ is injective.

Before proving surjectivity we notice that

$$J_0'(\omega_R)\varphi \in T_{\omega_R}\mathcal{S}^\perp \quad \text{for any } \varphi \in T_{\omega_R}C^2_{\mathbb{S}^2} \tag{3.12}$$

because of (3.5) and since $T_{\omega_R}\mathcal{S} = \ker J_0'(\omega_R)$.

Now take arbitrary $\psi \in C^0$ and $q = (q_1, q_2, q_3) \in \mathbb{R}^3$. We have to find functions $\varphi^\top \in T_{\omega_R}\mathcal{S}$, $\varphi^\perp \in T_{\omega_R}\mathcal{S}^\perp$ and $p = (p_1, p_2, p_3) \in \mathbb{R}^3$ such that $\mathcal{L}(\varphi^\top + \varphi^\perp, p) = (\psi, q)$. Since $T_{\omega_R}\mathcal{S} = \ker J_0'(\omega_R)$ is spanned by the functions $Re_j \wedge \omega_R$, we only need to solve

$$\begin{cases} J_0'(\omega_R)\varphi^\perp = \psi + \sum_j p_j(Re_j \wedge \omega_R), & \varphi^\perp \in T_{\omega_R}\mathcal{S},\ \ p \in \mathbb{R}^3 \\ \displaystyle\fint_{\mathbb{S}^1} \varphi^\top \cdot Re_j \wedge \omega_R \, dx = q_j, & \varphi^\top \in T_{\omega_R}\mathcal{S}^\perp. \end{cases}$$

The tangential component $\varphi^\top \in T_{\omega_R}\mathcal{S}$ is uniquely determined. Thanks to (3.12), we see that the function $\sum_j p_j(Re_j \wedge \omega_R)$ must coincide with the projection of $-\psi$ on $T_{\omega_R}\mathcal{S}$. This gives the unknown p. More explicitly, we have

$$e_h \cdot Mp = \sum_{j=1}^{3} p_j \fint_{\mathbb{S}^1} (Re_h \wedge \omega_R) \cdot (Re_j \wedge \omega_R)\, dx = -\fint_{\mathbb{S}^1} \psi \cdot Re_h \wedge \omega_R \, dx\,,$$

where M is the invertible matrix in Remark 3.2. Once one knows p, the existence of $\varphi^\perp$ follows from $iii)$ in Lemma 3.1, and surjectivity is proved.

We are in position to apply the implicit function theorem for any fixed $R \in SO(3)$. Actually, by a compactness argument, we have that there exist $\varepsilon' > 0$ and uniquely determined differentiable functions

$$\begin{aligned} &u : (-\varepsilon', \varepsilon') \times SO(3) \to C^2_{\mathbb{S}^2}\backslash \mathbb{S}^2\,, && u : (\varepsilon, R) \mapsto u^\varepsilon_R \\ &\zeta : (-\varepsilon', \varepsilon') \times SO(3) \to \mathbb{R}^3\,, && \zeta : (\varepsilon, R) \mapsto \zeta^\varepsilon(R) = (\zeta^\varepsilon_1(R), \zeta^\varepsilon_2(R), \zeta^\varepsilon_3(R)) \end{aligned}$$

such that

$$\mathcal{F}(\varepsilon, R, u^\varepsilon_R; \zeta^\varepsilon(R)) = 0\,, \quad u^0_R = \omega_R\,, \qquad \zeta^0(R) = 0.$$

Clearly the function $(\varepsilon, R) \mapsto u^\varepsilon_R$ is differentiable. Since ω_R is embedded, then u^ε_R is embedded as well, provided that ε' is small enough.

Condition $i)$ in the Lemma is fulfilled; $ii)$ follows from $\mathcal{F}_2(\varepsilon, R, u^\varepsilon_R; \zeta^\varepsilon(R)) = 0$ while $\mathcal{F}_1(\varepsilon, R, u^\varepsilon_R; \zeta^\varepsilon(R)) = 0$ gives $iii)$.

Now we prove that $iv)$ holds for any $\overline{\varepsilon} \in (0, \varepsilon')$, provided that ε' is small enough. Since $|\omega + e_3| \geq 1$ and $u^\varepsilon_R \to \omega_R$ uniformly on $\mathbb{S}^1$ as $\varepsilon \to 0$, we can assume that

$$|u^\varepsilon_R(x) + Re_3| \geq \frac{1}{2} \quad \text{for any } x \in \mathbb{S}^1, (\varepsilon, R) \in (-\varepsilon', \varepsilon') \times SO(3).$$

In particular, Lemma 2.3 guarantees that the function $\mathcal{E}^\varepsilon(R) = E_{\varepsilon K}(-Re_3; u^\varepsilon_R)$ is well defined and of class C^1 on $SO(3)$, for any $\varepsilon \in (-\varepsilon', \varepsilon')$. By $iii)$ in Lemma

2.3 we have that the derivative of $p \mapsto E_{\varepsilon K}(p; u^\varepsilon_R)$ vanishes for $p \in \mathbb{S}^2 \setminus u^\varepsilon_R(\mathbb{S}^1)$, and we can compute

$$d_R\mathcal{E}^\varepsilon(R)(RT_h) = E'_{\varepsilon K}(-Re_3; u^\varepsilon_R)(d_R u^\varepsilon_R(RT_h)) \quad \text{for } h \in \{1, 2, 3\}, \tag{3.13}$$

where $E'_{\varepsilon K}(-Re_3; \;\cdot\;)$ is the differential of the energy with respect to curves running in $C^2_{\mathbb{S}^2\setminus\{-Re_3\}}$. The C^1 dependence of $\mathcal{E}^\varepsilon(R)$ on ε and thus on the pair (ε, R) is evident.

Next, notice that $R^\xi_3\omega = \omega \circ \xi$ for any rotation $\xi \in \mathbb{S}^1$ (recall that R^ξ_3 rotates $\mathbb{S}^2$ about the $\langle e_3\rangle$ axis). Hence $RR^\xi_3\omega = \omega_R \circ \xi$ and $T_{RR^\xi_3\omega}\mathcal{S} = \big\{\tau \circ \xi \mid \tau \in T_{\omega_R}\mathcal{S}\,\big\}$ for any $R \in SO(3)$. Taking also $ii)$, $iii)$ into account, we have that

$$u^\varepsilon_R \circ \xi \in (T_{RR^\xi_3\omega}\mathcal{S})^\perp\,, \quad J_\varepsilon(u^\varepsilon_R \circ \xi) = J_\varepsilon(u^\varepsilon_R) \circ \xi \in T_{RR^\xi_3\omega}\mathcal{S}.$$

Since in addition $u^\varepsilon_R \circ \xi$ is close to $\omega_R \circ \xi = RR^\xi_3\omega$ in the C^2-norm by $i)$, we see that

$$u^\varepsilon_{RR^\xi_3} = u^\varepsilon_R \circ \xi \tag{3.14}$$

by the uniqueness of the function $\varepsilon \mapsto u^\varepsilon_R$ given by the implicit function theorem. By differentiating (3.14) with respect to ξ at $\xi = 1$ we obtain $d_R u^\varepsilon_R(RT_3) = (u^\varepsilon_R)'$, that compared with (2.8) gives $E'_{\varepsilon K}(-Re_3; u^\varepsilon_R)(d_R u^\varepsilon_R(RT_3)) = E'_{\varepsilon K}(-Re_3; u^\varepsilon_R)(u^\varepsilon_R)' = 0$. Thus $d_R\mathcal{E}^\varepsilon(R)(RT_3) = 0$ by (3.13), and $iv)$ is proved.

To prove that $v)$ holds for $\overline{\varepsilon}$ small enough, first take $R \in SO(3)$, $h \in \{1, 2, 3\}$ and notice that the condition $u^\varepsilon_R \in T_{\omega_R}\mathcal{S}^\perp$ trivially gives

$$d_R\Big(\fint_{\mathbb{S}^1} u^\varepsilon_R \cdot R(e_j \wedge \omega)\,dx\Big)(RT_h) = 0.$$

We compute $d_R R(e_j \wedge \omega)(RT_h) = Re_h \wedge (R(e_j \wedge \omega)) = R\big(e_h \wedge (e_j \wedge \omega)\big)$. Since in addition $u^\varepsilon_R \cdot R(e_h \wedge (e_j \wedge \omega)) = -(Re_h \wedge u^\varepsilon_R) \cdot (Re_j \wedge \omega_R)$ we obtain

$$m^\varepsilon_{hj}(R) := \fint_{\mathbb{S}^1} d_R u^\varepsilon_R(RT_h) \cdot Re_j \wedge \omega_R\,dx = \fint_{\mathbb{S}^1} (Re_h \wedge u^\varepsilon_R) \cdot (Re_j \wedge \omega_R)\,dx. \tag{3.15}$$

Since $u^\varepsilon_R \to \omega_R$ uniformly for $R \in SO(3)$, from (3.15) we obtain

$$m^\varepsilon_{hj}(R) = \fint_{\mathbb{S}^1} (Re_h \wedge \omega_R) \cdot (Re_j \wedge \omega_R)\,dx + o(1) = m_{hj} + o(1),$$

where m_{hj} are the entries of the invertible matrix M in Remark 3.2. It follows that the 3×3 matrix $M^\varepsilon_R = (m^\varepsilon_{hj}(R))_{j,h=1,2,3}$ is invertible for any $R \in SO(3)$, if ε is small enough.

We are in position to conclude the proof of v). We know that there exists a differentiable function $(\varepsilon, R) \mapsto \zeta^\varepsilon(R) \in \mathbb{R}^3$ such that

$$J_\varepsilon(u_R^\varepsilon) = \sum_{j=1}^{3} \zeta_j^\varepsilon(R)(Re_j \wedge \omega_R). \tag{3.16}$$

On the other hand, (3.13) and (3.10) give

$$L(u_R^\varepsilon) d_R \mathcal{E}^\varepsilon(R)(RT_h) = \fint_{\mathbb{S}^1} J_\varepsilon(u_R^\varepsilon) \cdot d_R u_R^\varepsilon(RT_h)\, dx, \tag{3.17}$$

by (3.16) and recalling (3.15) we obtain

$$L(u_R^\varepsilon) d_R \mathcal{E}^\varepsilon(R)(RT_h) = \sum_{j=1}^{3} m_{hj}^\varepsilon(R)\zeta_j^\varepsilon(R) = e_h \cdot M_R^\varepsilon(\zeta^\varepsilon(R)).$$

If $\varepsilon \approx 0$ so that the matrix M_R^ε is invertible, then R is a critical matrix for $\mathcal{E}^\varepsilon$ if and only if $\zeta^\varepsilon(R) = 0$, which is equivalent to say that $J_\varepsilon(u_R^\varepsilon) = 0$.

To prove the last claim of the lemma we take $R \in SO(3)$ and compute the Taylor expansion formula of the function

$$f_R(\varepsilon) = \mathcal{E}^\varepsilon(R) - \mathcal{E}_0^\varepsilon(R) = L(u_R^\varepsilon) - 1 + \varepsilon\big(A_K(-Re_3; u_R^\varepsilon) - A_K(-Re_3; \omega_R)\big)$$

at $\varepsilon = 0$. Clearly $f_R(0) = 0$. Now we recall that $L'(\omega_R) = 0$ because ω_R is a geodesic, and we write

$$\begin{aligned} f_R'(\varepsilon) = {} & \big(L'(u_R^\varepsilon) - L'(\omega_R)\big)(\partial_\varepsilon u_R^\varepsilon) + \varepsilon A_K'(-Re_3; u_R^\varepsilon)(\partial_\varepsilon u_R^\varepsilon) \\ & + \big(A_K(-Re_3; u_R^\varepsilon) - A_K(-Re_3; \omega_R)\big). \end{aligned}$$

To take the limit as $\varepsilon \to 0$, we notice that $\partial_\varepsilon u_R^\varepsilon$ is uniformly bounded in $C^2_{\mathbb{S}^2}$ because the function $(\varepsilon, R) \mapsto u_R^\varepsilon$ is of class C^1. Further, $L'(u_R^\varepsilon) \to L'(\omega_R)$ in the norm operator, $A_K'(-Re_3; u_R^\varepsilon)(\partial_\varepsilon u_R^\varepsilon)$ remains bounded and $A_K(-Re_3; u_R^\varepsilon) \to A_K(-Re_3; \omega_R)$. In conclusion, we have that $f_R'(0) = 0$, and therefore $f_R(\varepsilon) = o(\varepsilon)$ as $\varepsilon \to 0$, uniformly on $SO(3)$. That is, (3.11) holds true "at the zero order".

To conclude the proof we have to handle the derivatives of $\mathcal{E}^\varepsilon(R) - \mathcal{E}_0^\varepsilon(R)$ with respect to R, along any direction $RT_h \in T_R SO(3)$. We use (3.16), the second equality in (3.15) and then (3.16) again to obtain

$$\begin{aligned} \fint_{\mathbb{S}^1} J_\varepsilon(u_R^\varepsilon) \cdot (d_R u_R^\varepsilon(RT_h))\, dx &= \sum_{j=1}^{3} \zeta_j^\varepsilon(R) \fint_{\mathbb{S}^1} (d_R u_R^\varepsilon(RT_h)) \cdot (Re_j \wedge \omega_R)\, dx \\ &= \sum_{j=1}^{3} \zeta_j^\varepsilon(R) \fint_{\mathbb{S}^1} (Re_h \wedge u_R^\varepsilon) \cdot (Re_j \wedge \omega_R)\, dx \\ &= \fint_{\mathbb{S}^1} J_\varepsilon(u_R^\varepsilon) \cdot (Re_h \wedge u_R^\varepsilon)\, dx. \end{aligned}$$

By (3.9), the last integral can be written as

$$\begin{aligned}\fint_{\mathbb{S}^1} J_0(u_R^\varepsilon)\cdot(Re_h\wedge u_R^\varepsilon)dx + \varepsilon L(u_R^\varepsilon)A'_K(-Re_3;u_R^\varepsilon)(Re_h\wedge u_R^\varepsilon)\\ =\varepsilon L(u_R^\varepsilon)A'_K(-Re_3;u_R^\varepsilon)(Re_h\wedge u_R^\varepsilon)\end{aligned}$$

because of (3.3). Thus (3.17) leads to the new formula

$$d_R\mathcal{E}^\varepsilon(R)(RT_h)=\varepsilon A'_K(-Re_3;u_R^\varepsilon)(Re_h\wedge u_R^\varepsilon).$$

On the other hand, it is easy to see that

$$d_R\mathcal{E}_0^\varepsilon(R)(RT_h)=\varepsilon A'_K(-Re_3;\omega_R)(d_R(\omega_R)(RT_h))=\varepsilon A'_K(-Re_3;\omega_R)(Re_h\wedge\omega_R),$$

because $A_K(\,\cdot\,;\omega_R)$ is locally constant, and we can conclude that

$$\begin{aligned}d_R\big(\mathcal{E}^\varepsilon(R)-\mathcal{E}_0^\varepsilon(R)\big)(RT_h)\\ =\varepsilon\big(A'_K(-Re_3;u_R^\varepsilon)(Re_h\wedge u_R^\varepsilon)-A'_K(-Re_3;u_R^\varepsilon)(Re_h\wedge\omega_R)\big)=o(\varepsilon),\end{aligned}$$

because $u_R^\varepsilon\to\omega_R$. The lemma is completely proved. □

4. Two solutions

In the present section we use Lemma 3.3 together with the local variational approach in Sect. 2 to provide a more direct, self-contained and analytical treatment to Viterbo's and Bottkoll's result which avoids the deep and general theories of characteristics and symplectic actions.

We stress the fact that, differently from [11], [18] and [16], in the next theorem we do not make any sign assumptions on K. For instance, K might vanish on some geodesic circle of radius $\pi/2$ about a point $z\in\mathbb{S}^2$ and thus $\partial\mathcal{D}_{\frac{\pi}{2}}(z)$ can be parameterized by two K-magnetic geodesics that coincide up to orientation. This is the reason why, in that case, we have to add an extra assumption to obtain two distinct solutions.

Theorem 4.1. *Let $K\in C^1(\mathbb{S}^2)$ be given. For every $c>0$ large enough, Problem $(\mathcal{P}_{K,c})$ has at least a solution γ. If in addition K does not vanish on any closed geodesic, or*

$$\int_{\mathcal{D}_{\frac{\pi}{2}}(z)} K(q)\,d\sigma_q=\int_{\mathcal{D}_{\frac{\pi}{2}}(-z)} K(q)\,d\sigma_q\quad \textit{whenever } K\equiv 0 \textit{ on } \partial\mathcal{D}_{\frac{\pi}{2}}(z), \tag{4.1}$$

then for every $c>0$ large enough, Problem $(\mathcal{P}_{K,c})$ has at least two embedded, distinct solutions.

Recall that changing the orientation of a curve only changes the sign of its curvature.

Proof. Let $\bar{\varepsilon}$ be given by Lemma 3.3. For any $c > \bar{\varepsilon}^{-1}$, let $\varepsilon := c^{-1} < \bar{\varepsilon}$ and $(\varepsilon, R) \mapsto u^{\varepsilon}_R$, $(\varepsilon, R) \mapsto \mathcal{E}^{\varepsilon}(R)$ be the functions in Lemma 3.3. To every critical point R^{ε} for $\mathcal{E}^{\varepsilon}$ corresponds a curve $u^{\varepsilon}_{R^{\varepsilon}}$ that solves $J_{\varepsilon}(u^{\varepsilon}_{R^{\varepsilon}}) = 0$. Hence $u^{\varepsilon}_{R^{\varepsilon}}$ solves (3.8) and, as explained at the beginning of Sect. 2, yields a solution to $(\mathcal{P}_{K,\varepsilon^{-1}}) = (\mathcal{P}_{K,c})$.

Now, if $\mathcal{E}^{\varepsilon}$ is constant, then u^{ε}_R solves (3.8) for every $R \in SO(3)$ and the conclusions in Theorem 4.1 hold. Otherwise, take $\underline{R}^{\varepsilon}, \overline{R}^{\varepsilon} \in SO(3)$ achieving the minimum and the maximum value of $\mathcal{E}^{\varepsilon}$, respectively. Then $\underline{u}^{\varepsilon} := u^{\varepsilon}_{\underline{R}^{\varepsilon}}$ and $\overline{u}^{\varepsilon} := u^{\varepsilon}_{\overline{R}^{\varepsilon}}$ solve (3.8) and this concludes the proof of the existence part.

Next, assume that $\mathcal{E}^{\varepsilon}$ is not constant, and that $\underline{u}^{\varepsilon} = \overline{u}^{\varepsilon} \circ g$ for a diffeomorphism g of $\mathbb{S}^1$. To conclude the proof we have to show that (4.1) can not hold.

We have $E_{\varepsilon K}(\underline{z}^{\varepsilon}, \underline{u}^{\varepsilon}) < E_{\varepsilon K}(\overline{z}^{\varepsilon}, \overline{u}^{\varepsilon})$, that is,

$$L(\underline{u}^{\varepsilon}) + \varepsilon A_K(\underline{z}^{\varepsilon}, \underline{u}^{\varepsilon}) < L(\overline{u}^{\varepsilon}) + \varepsilon A_K(\overline{z}^{\varepsilon}, \overline{u}^{\varepsilon}) \tag{4.2}$$

where $\underline{z}^{\varepsilon} = -\underline{R}^{\varepsilon} e_3$, $\overline{z}^{\varepsilon} = -\overline{R}^{\varepsilon} e_3$. Since $|(\underline{u}^{\varepsilon})'|$, $|(\overline{u}^{\varepsilon})'|$ are constant, then $|g'|$ is constant as well. Thus $|g'| = 1$ and $L(\underline{u}^{\varepsilon}) = L(\overline{u}^{\varepsilon})$. Therefore, (4.2) implies

$$A_K(\underline{z}^{\varepsilon}, \underline{u}^{\varepsilon}) \neq A_K(\overline{z}^{\varepsilon}, \overline{u}^{\varepsilon}) \tag{4.3}$$

for any $\varepsilon \neq 0$. In particular, g can not be a positive rotation of the circle by the property $A2$) of the area functional. Thus g is a counterclockwise rotation of $\mathbb{S}^1$. Recall that $\underline{u}^{\varepsilon}$ has curvature $\varepsilon K(\underline{u}^{\varepsilon})$ and $\overline{u}^{\varepsilon}$ has curvature $\varepsilon K(\overline{u}^{\varepsilon})$. Since changing the orientation of a curve changes the sign of its curvature, we have that at any point $p \in \Gamma := \underline{u}^{\varepsilon}(\mathbb{S}^1) = \overline{u}^{\varepsilon}(\mathbb{S}^1)$ we have $K(p) = -K(p)$. It follows that $K \equiv 0$ on Γ, and hence Γ is the boundary of a half-sphere $\mathcal{D}_{\frac{\pi}{2}}(w^{\varepsilon})$. We can assume that $\underline{u}^{\varepsilon}$ is a positive parameterization of $\partial \mathcal{D}_{\frac{\pi}{2}}(w^{\varepsilon})$. Then $\underline{z}^{\varepsilon} \notin \overline{\mathcal{D}_{\frac{\pi}{2}}(w^{\varepsilon})}$ because $\underline{u}^{\varepsilon} \approx \omega_{\underline{R}^{\varepsilon}}$, see i) in Lemma 3.3. Next, since $\overline{u}^{\varepsilon}$ parameterizes the same geodesic with opposite direction, then $\overline{u}^{\varepsilon}$ a positive parameterization of $\partial \mathcal{D}_{\frac{\pi}{2}}(-w^{\varepsilon})$ and $\overline{z}^{\varepsilon} \notin \overline{\mathcal{D}_{\frac{\pi}{2}}(-w^{\varepsilon})}$. In particular, from the properties $A3$) and $A4$) of the area functional we infer

$$A_K(\underline{z}^{\varepsilon}, \underline{u}^{\varepsilon}) = A_K(-w^{\varepsilon}, \underline{u}^{\varepsilon}) = -\frac{1}{2\pi} \int\limits_{\mathcal{D}_{\frac{\pi}{2}}(w^{\varepsilon})} K(q)\, d\sigma_q$$

$$A_K(\overline{z}^{\varepsilon}, \overline{u}^{\varepsilon}) = A_K(w^{\varepsilon}, \overline{u}^{\varepsilon}) = -\frac{1}{2\pi} \int\limits_{\mathcal{D}_{\frac{\pi}{2}}(-w^{\varepsilon})} K(q)\, d\sigma_q\,,$$

that compared with (4.3) shows that (4.1) is violated. The theorem is completely proved. □

5. Many solutions

In this section we suggest a way to obtain more and more distinct K-magnetic geodesics. It involves the C^1 Mel'nikov-type function

$$F_K(z) = \int\limits_{\mathcal{D}_{\frac{\pi}{2}}(z)} K(p)\, d\sigma_p\,, \quad F_K : \mathbb{S}^2 \to \mathbb{R}\,, \tag{5.1}$$

where $K \in C^1(\mathbb{S}^2)$ is given. We start by recalling the definition of stable critical point proposed in [3, Chapter 2], see also [14].

Definition 5.1. Let $\Omega \subset \mathbb{S}^2$ be open. We say that F_K has a stable critical point in Ω if there exists $r > 0$ such that any function $G \in C^1(\overline{\Omega})$ satisfying $\|G - F_K\|_{C^1(\overline{\Omega})} < r$ has a critical point in Ω.

If F_K is not constant, then it has at least two distinct stable critical points, namely, its minimum and its maximum. Different sufficient conditions to have the existence of (possible multiple) stable critical points $z \in \Omega$ for F_K are easily given via elementary calculus. For instance, one can assume that one of the following conditions holds:

(i) $\nabla F_K(z) \neq 0$ for any $z \in \partial\Omega$, and $\deg(\nabla F_K, \Omega, 0) \neq 0$, where "deg" is Browder's topological degree;

(ii) $\min\limits_{\partial\Omega} F_K > \min\limits_{\Omega} F_K$ or $\max\limits_{\partial\Omega} F_K < \max\limits_{\Omega} F_K$;

(iii) F_K is of class C^2 on Ω, it has a critical point $z_0 \in \Omega$, and the Hessian matrix of F_K at z_0 is invertible.

In the next result we show that any stable critical point z_0 for F_K gives rise, for any $c > 0$ large enough, to a solution γ^c to Problem $(\mathcal{P}_{K,c})$ which is a perturbation of the closed geodesic about z_0. Taking advantage of the remarks at the beginning of Sect. 2, we only need to show that for any stable critical point z_0 for F_K and for any $\varepsilon = c^{-1} \approx 0^+$, there exists a solution u^ε to (3.8), such that u^ε is close to the closed geodesic about z_0.

Theorem 5.2. *Let $K \in C^1(\mathbb{S}^2)$ be given. Assume that F_K has a stable critical point in an open set $\Omega \subset \mathbb{S}^2$, such that $\overline{\Omega} \subsetneq \mathbb{S}^2$.*

Then for every $\varepsilon \in \mathbb{R}$ close enough to 0, there exists a point $z_\varepsilon \in \Omega$, an embedding $\omega^\varepsilon : \mathbb{S}^1 \to \mathbb{S}^2$ parameterizing the boundary of a circle of geodesic radius $\pi/2$ about z_ε, and a solution u^ε to $(\mathcal{P}_{K,\varepsilon^{-1}})$, such that $\|u^\varepsilon - \omega^\varepsilon\|_{C^2} = O(\varepsilon)$.

Proof. We can assume $-e_3 \notin \overline{\Omega}$. Otherwise, take any rotation $R \in SO(3)$ such that $-e_3 \notin R\overline{\Omega}$, and look for a solution $\tilde{u}^\varepsilon$ to

$$u'' + |u'|^2 u = L(u)\varepsilon(K \circ {}^t R)(u)\, u \wedge u' \qquad \text{on } \mathbb{S}^1,$$

in a C^2-neighborhood of a geodesic circle about some point $\tilde{z}^\varepsilon \in R\Omega$. Conclude by noticing that $u^\varepsilon :={}^t R\tilde{u}^\varepsilon$ solves (3.8) and approaches the geodesic circle about $R^{\mathrm{t}}\tilde{z}^\varepsilon \in \Omega$.

Next, for $z \in \mathbb{S}^2 \setminus \{-e_3\}$ consider the rotation

$$N(z) = \begin{pmatrix} 1 - \frac{z_1^2}{1+z_3} & -\frac{z_1 z_2}{1+z_3} & z_1 \\ -\frac{z_1 z_2}{1+z_3} & 1 - \frac{z_2^2}{1+z_3} & z_2 \\ -z_1 & -z_2 & z_3 \end{pmatrix},$$

that maps e_3 to z. Clearly the function $N : \mathbb{S}^2 \setminus \{-e_3\} \to SO(3)$ is differentiable; its differential $dN(z)$ at any $z \in \mathbb{S}^2 \setminus \{-e_3\}$ is a linear map $T_z\mathbb{S}^2 \to T_{N(z)}SO(3)$. We have

$$T_z\mathbb{S}^2 = \langle N(z)e_1, N(z)e_2\rangle \tag{5.2}$$

$$T_{N(z)}SO(3) = \langle dN(z)\big(N(z)e_1\big), dN(z)\big(N(z)e_2\big)\rangle \oplus \langle N(z)T_3\rangle\,. \tag{5.3}$$

Equality (5.2) and the inclusion $\supseteq$ in (5.3) are trivial. To conclude the proof of (5.3) we need to show that the matrices

$$dN(z)\big(N(z)e_1\big)\,, \quad dN(z)\big(N(z)e_2\big)\,, \quad N(z)T_3$$

are linearly independent. By differentiating the identity $N(z)e_3 = z$ one gets

$$dN(z)\tau \cdot e_3 = \tau\,, \quad \tau \in T_z\mathbb{S}^2\,.$$

By choosing $\tau = N(z)e_h$, $h = 1, 2$ we infer that the third columns of the matrices $dN(z)\big(N(z)e_1\big), dN(z)\big(N(z)e_2\big)$ are linearly independent. Thus the matrices $dN(z)\big(N(z)e_1\big)$, $dN(z)\big(N(z)e_2\big)$ are linearly independent as well. On the other hand, the third column on $N(z)T_3$ is identically zero, that concludes the proof of (5.3).

Now, take the differentiable functions $(\varepsilon, R) \mapsto u^\varepsilon_R \in C^2_{\mathbb{S}^2}$, $(\varepsilon, R) \mapsto \mathcal{E}^\varepsilon(R) \in \mathbb{R}$ given by Lemma 3.3. To simplify notations, for $z \in \mathbb{S}^2 \setminus \{-e_3\}$ we write

$$\widetilde{\mathcal{E}}^\varepsilon(z) = \mathcal{E}^\varepsilon(N(z)) = E_{\varepsilon K}(-z; u^\varepsilon_{N(z)})\,, \quad \widetilde{\mathcal{E}}^\varepsilon_0(z) = \mathcal{E}^\varepsilon_0(N(z)) = E_{\varepsilon K}(-z; N(z)\omega)\,.$$

Notice that $N(z)\omega$ parameterizes $\partial\mathcal{D}_{\pi/2}(z)$. Therefore, using $ii)$ in Lemma 2.3, property $A4$) and elementary computations we get

$$\begin{aligned}\widetilde{\mathcal{E}}^\varepsilon_0(z) =& L(N(z)\omega) + \varepsilon A_K(-z; N(z)\omega)\\ =& L(\omega) - \frac{\varepsilon}{2\pi}\int\limits_{D_{\pi/2}(z)} K(q)d\sigma_q = L(\omega) - \frac{\varepsilon}{2\pi}F_K(z).\end{aligned} \tag{5.4}$$

Next, for any small $\varepsilon \neq 0$ consider the function

$$G^\varepsilon(z) = \frac{2\pi}{\varepsilon}(\widetilde{\mathcal{E}}^\varepsilon(z) - L(\omega))$$

and use (5.4) together with iv) in Lemma 3.3 to get

$$\|G^\varepsilon + F_K\|_{C^1(\overline{\Omega})} = \frac{2\pi}{|\varepsilon|}\left\| E_{\varepsilon K}(-z; u^\varepsilon_{N(z)}) - E_{\varepsilon K}(-z; N(z)\omega)\right\|_{C^1(\overline{\Omega})} = o(1)$$

as $\varepsilon \to 0$. We see that for ε small enough the function G^ε has a critical point $z^\varepsilon \in \Omega$. Thus, for any $\tau \in T_{z^\varepsilon}\mathbb{S}^2$ we have

$$0 = d_z\widetilde{\mathcal{E}}^\varepsilon(z^\varepsilon)\tau = d_R\mathcal{E}^\varepsilon(N(z^\varepsilon))\big(d_zN(z^\varepsilon)\tau\big)\,.$$

Taking (5.3) and iv) in Lemma 3.3 into account, we infer that the matrix $N(z^\varepsilon)$ is critical for $\mathcal{E}^\varepsilon$. Thus, by arguing as for Theorem 4.1 we have that the curve $u^\varepsilon := u^\varepsilon_{N(z_\varepsilon)}$ is a solution to $(\mathcal{P}_{K,\varepsilon^{-1}})$. $\square$

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