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# Appendix to: Efficient European and American Option Pricing Under a Jump-diffusion Process

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# Appendix to: Efficient European and American Option Pricing Under a Jump-diffusion Process

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**Abstract**

This paper constitutes the Appendix of the article "Efficient European and American option pricing under a jump-diffusion process". Here are detailed the proofs that could not be part of the main sections of the article, for length and readability reasons. Every section is dedicated to a proof, starts with the recollection of the statement of the lemma, proposition or theorem involved and continues with its proof.

## 1. Proof of Lemma 5.10

**Lemma:**

$$Q_N^{\overline{k}}(k) \leq \sum_{i=1}^{N} \widetilde{Q}_N(2\overline{k} - k + 2i)$$

$$Q_{N\overline{l}}(k) \leq \sum_{i=1}^{N} \widetilde{Q}_N(2\overline{l} + k + 2i)$$

*for all* $-\overline{l} \leq k \leq \overline{k}$.

**Proof:**

The proof is analogous to that of Lemma 5.4. When the original path first trespasses the $\overline{k}$ level, it can reach level $\overline{k} + 1, \ldots, \overline{k} + N$. Therefore its reflection (defined as in Lemma 5.4) can end at level $2\overline{k} - k + 2$, $2\overline{k} - k + 4 \ldots, 2\overline{k} - k + 2N$. Likewise for the paths that cross the $-\overline{l}$ level.

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## 2. Proof of Proposition 5.11

**Proposition:** *Given* $G$ *and* $W_N$ *as defined in Equation (4.3) in the main article, for integers* $0 \leq k, \overline{k} \leq Nn$ *we have:*

$$\text{For } k \geq N\lceil 2W_N - 1\rceil \qquad \widetilde{Q}_N(k) \leq G\frac{W_N^{\left\lfloor \frac{k}{N} \right\rfloor}}{\left\lfloor \frac{k}{N} \right\rfloor!} \tag{1}$$

$$\text{For } \bar{k} \geq N\lceil 2W_N - 1\rceil \qquad \sum_{k=\bar{k}}^{Nn} \widetilde{Q}_N(k) \leq 2GN\frac{W_N^{\left\lfloor \frac{\bar{k}}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}}{N} \right\rfloor!} \tag{2}$$

$$\text{For } \bar{k} \geq N\lceil 2e^{Nh}W_N - 1\rceil \qquad \sum_{k=\bar{k}}^{Nn} e^{hk}\widetilde{Q}_N(k) \leq 2G\frac{(e^{hN}W_N)^{\left\lfloor \frac{\bar{k}}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}}{N} \right\rfloor!}\sum_{r=0}^{N-1} e^{hr} \tag{3}$$

$$\text{For } \bar{k} \geq N\lceil 2W_N - 1\rceil \qquad \sum_{k=\bar{k}}^{Nn} e^{-hk}\widetilde{Q}_N(-k) \leq 2G\frac{(e^{-hN}W_N)^{\left\lfloor \frac{\bar{k}}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}}{N} \right\rfloor!}\sum_{r=0}^{N-1} e^{-hr} \tag{4}$$

**Proof:**

We need an upper estimate of the probability $Q_N(k)$ of reaching level $k \geq 0$ in the jump dynamics. This will allow us to obtain an upper estimate of how much the value of the option in $(n, j, k)$ for some $j$ contributes to the current value.

We recall that for a fixed $N$, in a single timestep $\Delta t$ the possible jump moves are $-Nh, \dots, -h, 0, h, \dots, Nh$. For simplicity, in the following we will talk about $-N, \dots, -1, 0, 1, \dots, N$ jumps.

Level $k \geq 0$ at maturity can be reached with a variety of possible combinations of jumps. In order to consider all the possible paths that arrive at level $k$ in $n$ timesteps, exactly as we did in the $N = 1$ case, we distinguish between the positive and the negative jumps: if $k \geq 0$ is the total balance and the sum of all negative jumps is $-l$, then the sum of all positive jumps must be $k + l$, with $l \geq 0$. $Q_N(k)$ is the sum of all probabilities of reaching balance level $k$ with a negative balance of $-l$, over all possible non negative $l$, subject to the condition of a total of $n$ moves.

Let us denote by $e_j^-$ the number of $-j$ jumps and $e_j^+$ the number of $j$ jumps in a path, for $j = 1, \dots, N$.

With this notation, the probability $Q_N(k)$ of reaching at maturity level $k \geq 0$ for the jump dynamics is given by:

$$Q_N(k) = \sum_l \sum_{e_N^+} \cdots \sum_{e_1^+} \sum_{e_N^-} \cdots \sum_{e_1^-} C(e_N^+, e_N^-, e_{N-1}^+, e_{N-1}^-, \dots, e_1^+, e_1^-) q_{+N}^{e_N^+} \cdots q_{+1}^{e_1^+} q_{-N}^{e_N^-} \cdots q_{-1}^{e_1^-} q_0^{e_0}$$

where the $e_0$ exponent is given by $n - \sum_{i=1}^N e_i^+ - \sum_{i=1}^N e_i^-$, and $C(e_N^+, e_N^-, e_{N-1}^+, e_{N-1}^-, \dots, e_1^+, e_1^-)$ denotes the number of combinations of the $n$ factors, once the exponents are fixed, and is equal to

$$C(e_N^+, e_N^-, e_{N-1}^+, e_{N-1}^-, \dots, e_1^+, e_1^-) = \frac{n!}{e_N^+! e_N^-! e_{N-1}^+! e_{N-1}^-! \dots e_1^+! e_1^-! e_0!}.$$

While in the $N = 1$ setting, a $-l$ negative balance meant $l$ jumps of the $-h$ kind, and similarly a $k + l$ positive balance meant $k + l$ jumps of the $+h$ kind, here the situation is complicated by the possibility of different jump amplitudes, so extra care is needed in order to understand the relation between $l$ and the exponents $e_i^+, e_i^-$.

We use Euclidean division in order to write $l$ as a multiple of $N$ plus a remainder $0 \le r_N^- \le N - 1$: $l = Nz + r_N^-$. This means that the negative balance $-l$ is due to at most $z$ jumps of the $-N$ kind, and the difference between $Nz$ and $l$ shall be covered with smaller jumps.

Instead of summing over all possible $l$, then, it will be easier to consider the summation over all possible $z$ and $0 \le r_N^- \le N - 1$.

For any fixed $z$ and $r_N^-$, we will have at most $z$ jumps of the $-N$ kind, therefore we need to vary $e_N^-$ between 0 and $z$; the choice of $e_N^-$ sets additional constraints for $e_{N-1}^-$, and proceeding backwards the choice of every $e_i^-$ sets additional constraints for $e_{i-1}^-$. We apply the same idea to the positive balance $k + l$: given $k$, the values $t$ and $0 \le r_N \le N - 1$ such that $k = Nt + r_N$ are uniquely determined; therefore for any given pair of $z$ and $r_N^-$ the positive balance can be written as $N(t + z) + r_N + r_N^-$. This provides the limitation for $e_N^+$, and the choice of every $e_j^+$ imposes further conditions on the possible values for $e_{j-1}^+$.

In order to better express the relationships and mutual limitations between exponents, we need a change in perspective in the summations.

For any fixed $z$, let us define $b_{N-1} = z - e_N^-$. Of the negative balance $-(Nz + r_N^-)$, then, $-Ne_N^-$ will be covered by $-N$ jumps and the rest, $-(Nb_{N-1} + r_N^-)$, by jumps of smaller amplitude. Instead of summing over $e_N^-$ from 0 to $z$, we sum over $b_{N-1}$, that is over how many of the $-Nz$ are covered by jumps of amplitude smaller than $N$.

Once fixed $z$, $r_N^-$ and $e_N^-$, we have a negative balance of $-(Nb_{N-1} + r_N^-)$ to cover with negative jumps of amplitude at most $N - 1$: we compute the Euclidean division of $Nb_{N-1} + r_N^-$ by $N - 1$: the quotient $z_{N-1} = \lfloor \frac{Nb_{N-1} + r_N^-}{N-1} \rfloor$ is an upper bound (we shall consider the more stringent between this value and the condition of a total of $n$ moves), and we call $r_{N-1}^-$ the remainder. Once again, instead of summing over $e_{N-1}^-$, we sum over $b_{N-2} = z_{N-1} - e_{N-1}^-$.

We repeatedly use Euclidean division in order to find the upper bounds for all $e_j^-$, and operate in the same way for the positive jumps, where we similarly introduce the $a_j$ and $r_j^+$ values.

The probability $Q_N(k)$ of reaching at maturity level $k \geq 0$ for the jump dynamics can then be written as:

$$Q_N(k) = \sum_{r_N^-=0}^{N-1} \sum_z \sum_{a_{N-1}} \cdots \sum_{a_1} \sum_{b_{N-1}} \cdots \sum_{b_1} \frac{n!}{e_N^+! e_N^-! e_{N-1}^+! e_{N-1}^-! \ldots e_1^+! e_1^-! e_0!} q_{+N}^{e_N^+} \cdots q_{+1}^{e_1^+} q_{-N}^{e_N^-} \cdots q_{-1}^{e_1^-} q_0^{e_0}.$$

The indices $a_j$ ($b_j$) are indicators of how much of the total positive (respectively, negative) balance is due to moves of amplitude at most $j$, and are related to the exponents in the following way:

$$e_N^- = z - b_{N-1} \qquad e_N^+ = t + z + \left\lfloor \frac{r_N + r_N^-}{N} \right\rfloor - a_{N-1}$$

$$e_i^- = \left\lfloor \frac{(i+1)b_i + r_{i+1}^-}{i} \right\rfloor - b_{i-1} \text{ where } r_i^- \text{ is the remainder of } \frac{(i+1)b_i + r_{i+1}^-}{i} \text{ for } 1 < i < N$$

$$e_i^+ = \left\lfloor \frac{(i+1)a_i + r_{i+1}^+}{i} \right\rfloor - a_{i-1} \text{ where } r_i^+ \text{ is the remainder of } \frac{(i+1)a_i + r_{i+1}^+}{i} \text{ for } 1 < i < N$$

$$e_1^- = 2b_1 + r_2^- \qquad e_1^+ = 2a_1 + r_2^+$$

Substituting $c_{\pm i}$ with $w_i$, we obtain

$$\widetilde{Q}_N(k) = \sum_{r_N^-=0}^{N-1} \sum_z \sum_{a_{N-1}} \cdots \sum_{a_1} \sum_{b_{N-1}} \cdots \sum_{b_1} \frac{n!}{e_N^+! e_N^-! e_{N-1}^+! e_{N-1}^-! \ldots e_1^+! e_1^-! e_0!} \frac{w_N^{e_N^+} \cdots w_1^{e_1^+} w_N^{e_N^-} \cdots w_1^{e_1^-}}{n^{\sum_{i=1}^N e_i^+ + \sum_{i=1}^N e_i^-}} q_0^{e_0}$$

Since $q_0 \leq 1$ and $\frac{n!}{e_0! n^{\sum_{i=1}^N e_i^+ + \sum_{i=1}^N e_i^-}} \leq 1$:

$$\widetilde{Q}_N(k) \leq \sum_{r_N^-=0}^{N-1} \sum_z \sum_{a_{N-1}} \cdots \sum_{a_1} \frac{w_N^{e_N^+} \cdots w_1^{e_1^+}}{e_N^+! e_{N-1}^+! \ldots e_1^+!} \sum_{b_{N-1}} \cdots \sum_{b_1} \frac{w_N^{e_N^-} \cdots w_1^{e_1^-}}{e_N^-! e_{N-1}^-! \ldots e_1^-!}$$

We treat separately the positive and the negative parts, and we work from the inside outwards.

$$\sum_{b_{N-1}} \frac{w_N^{e_N^-}}{e_N^-!} \cdots \sum_{b_3} \frac{w_4^{e_4^-}}{e_4^-!} \sum_{b_2} \frac{w_3^{e_3^-}}{e_3^-!} \sum_{b_1} \frac{w_2^{e_2^-}}{e_2^-!} \frac{w_1^{e_1^-}}{e_1^-!} =$$

$$= \sum_{b_{N-1}} \frac{w_N^{e_N^-}}{e_N^-!} \cdots \sum_{b_3} \frac{w_4^{e_4^-}}{e_4^-!} \sum_{b_2} \frac{w_3^{e_3^-}}{e_3^-!} \sum_{b_1} \frac{w_2^{\left\lfloor \frac{3b_2 + r_3^-}{2} \right\rfloor - b_1}}{\left( \left\lfloor \frac{3b_2 + r_3^-}{2} \right\rfloor - b_1 \right)!} \frac{w_1^{2b_1 + r_2^-}}{(2b_1 + r_2^-)!}$$

$$\leq \sum_{b_{N-1}} \frac{w_N^{e_N^-}}{e_N^-!} \cdots \sum_{b_3} \frac{w_4^{e_4^-}}{e_4^-!} \sum_{b_2} \frac{w_3^{e_3^-}}{e_3^-!} w_1^{r_2^-} \frac{(w_2 + w_1^2)^{\left\lfloor \frac{3b_2 + r_3^-}{2} \right\rfloor}}{\left\lfloor \frac{3b_2 + r_3^-}{2} \right\rfloor !}$$

Since $r_2^-$ is the remainder of $\frac{3b_2+r_3^-}{2}$, it can only assume the values 0 or 1; therefore we can write:

$$\sum_{b_{N-1}} \frac{w_N^{e_N^-}}{e_N^-!} \cdots \sum_{b_3} \frac{w_4^{e_4^-}}{e_4^-!} \sum_{b_2} \frac{w_3^{e_3^-}}{e_3^-!} w_1^{r_2^-} \frac{(w_2 + w_1^2)^{\left\lfloor \frac{3b_2+r_3^-}{2} \right\rfloor}}{\left\lfloor \frac{3b_2+r_3^-}{2} \right\rfloor!} \leq$$

$$\leq \sum_{b_{N-1}} \frac{w_N^{e_N^-}}{e_N^-!} \cdots \sum_{b_3} \frac{w_4^{e_4^-}}{e_4^-!} \sum_{b_2} \frac{w_3^{\left\lfloor \frac{4b_3+r_4^-}{3} \right\rfloor - b_2}}{(\left\lfloor \frac{4b_3+r_4^-}{3} \right\rfloor - b_2)!} \max\{w_1, 1\} \frac{(w_2 + w_1^2)^{\frac{3b_2+r_3^- - r_2^-}{2}}}{\frac{3b_2+r_3^- - r_2^-}{2}!}$$

According to the definitions in Equation (4.3) in the main article, $\max\{w_1, 1\} = \max\{W_1^1, W_1^0\} = M_1$, and $w_2 + w_1^2 = W_2$.

In general, we take care of the sum over $b_{i-1}$, for $1 < i < N$, in the following way:

$$\sum_{b_{i-1}} \frac{w_i^{\left\lfloor \frac{(i+1)b_i + r_{i+1}^-}{i} \right\rfloor - b_{i-1}}}{\left(\left\lfloor \frac{(i+1)b_i + r_{i+1}^-}{i} \right\rfloor - b_{i-1}\right)!} \frac{W_{i-1}^{\left\lfloor \frac{ib_{i-1}+r_i^-}{i-1} \right\rfloor}}{\left\lfloor \frac{ib_{i-1}+r_i^-}{i-1} \right\rfloor!} =$$

$$= \sum_{b_{i-1}} \frac{w_i^{\left\lfloor \frac{(i+1)b_i + r_{i+1}^-}{i} \right\rfloor - b_{i-1}}}{(\left\lfloor \frac{(i+1)b_i + r_{i+1}^-}{i} \right\rfloor - b_{i-1})!} \frac{W_{i-1}^{\frac{ib_{i-1}+r_i^- - r_{i-1}^-}{i-1}}}{\left\lfloor \frac{ib_{i-1}+r_i^-}{i-1} \right\rfloor!} \leq$$

$$\leq \sum_{b_{i-1}} \frac{w_i^{\left\lfloor \frac{(i+1)b_i + r_{i+1}^-}{i} \right\rfloor - b_{i-1}}}{(\left\lfloor \frac{(i+1)b_i + r_{i+1}^-}{i} \right\rfloor - b_{i-1})!} \frac{(W_{i-1}^{\frac{i}{i-1}})^{b_{i-1}}}{b_{i-1}!} W_{i-1}^{\frac{r_i^- - r_{i-1}^-}{i-1}} \leq$$

$$\leq \sum_{b_{i-1}} \frac{w_i^{\left\lfloor \frac{(i+1)b_i + r_{i+1}^-}{i} \right\rfloor - b_{i-1}}}{(\left\lfloor \frac{(i+1)b_i + r_{i+1}^-}{i} \right\rfloor - b_{i-1})!} \frac{(W_{i-1}^{\frac{i}{i-1}})^{b_{i-1}}}{b_{i-1}!} \max\{W_{i-1}, W_{i-1}^{-\frac{i-2}{i-1}}\} =$$

$$= M_{i-1} \frac{(w_i + W_{i-1}^{\frac{i}{i-1}})^{\left\lfloor \frac{(i+1)b_i + r_{i+1}^-}{i} \right\rfloor}}{\left\lfloor \frac{(i+1)b_i + r_{i+1}^-}{i} \right\rfloor!} = M_{i-1} \frac{W_i^{\left\lfloor \frac{(i+1)b_i + r_{i+1}^-}{i} \right\rfloor}}{\left\lfloor \frac{(i+1)b_i + r_{i+1}^-}{i} \right\rfloor!}$$

and similarly for the sum over $a_{i-1}$, for $2 \leq i < N$. Proceeding in this way for both the negative and the positive balance parts of the summation, we get

$$\begin{aligned}
\widetilde{Q}_N(k) &\le \prod_{j=1}^{N-1} M_j^2 \sum_{r_N^-=0}^{N-1} \sum_z \frac{W_N^z}{z!} \frac{W_N^{t+z+\left\lfloor \frac{r_N+r_N^-}{N} \right\rfloor}}{(t+z+\left\lfloor \frac{r_N+r_N^-}{N} \right\rfloor)!} \\
&\le \prod_{j=1}^{N-1} M_j^2 \sum_z \frac{W_N^z}{z!} \frac{W_N^{t+z}}{(t+z)!} \sum_{r_N^-=0}^{N-1} W_N^{\left\lfloor \frac{r_N+r_N^-}{N} \right\rfloor} \\
&\le \prod_{j=2}^{N-1} M_j^2 \sum_z \frac{W_N^z}{z!} \sum_z \frac{W_N^{t+z}}{(t+z)!} N \max\{W_N, 1\} \\
&\le N \max\{W_N, 1\} \prod_{j=2}^{N-1} M_j^2 e^{W_N} \cdot 2 \frac{W_N^t}{t!}
\end{aligned}$$

for $t \ge 2W_N - 1$. Calling $G = 2N \max\{W_N, 1\} \prod_{j=2}^{N-1} M_j^2 e^{W_N}$ we have Equation (1) for $k \ge N\lceil 2W_N - 1 \rceil$.

Now we apply the previous inequality to the summation $\sum_{k=\bar{k}}^{Nn} \widetilde{Q}_N(k)$, obtaining

$$\begin{aligned}
\sum_{k=\bar{k}}^{+\infty} \widetilde{Q}_N(k) &\le \sum_{k=\bar{k}}^{+\infty} G \frac{W_N^{\lfloor \frac{k}{N} \rfloor}}{\lfloor \frac{k}{N} \rfloor !} \\
&\le 2GN \frac{W_N^{\left\lfloor \frac{\bar{k}}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}}{N} \right\rfloor !}
\end{aligned}$$

provided that $\bar{k} \ge N\lceil 2W_N - 1 \rceil$.

We apply again Equation (1) to the summation $\sum_{k=\bar{k}}^{+\infty} e^{hk} \widetilde{Q}_N(k)$; for $\bar{k} \ge N\lceil 2e^{Nh} W_N - 1 \rceil$ we have:

$$\sum_{k=\bar{k}}^{+\infty} e^{hk} \widetilde{Q}_N(k) \le \sum_{k=\bar{k}}^{Nn} e^{hk} G \frac{W_N^{\lfloor \frac{k}{N} \rfloor}}{\lfloor \frac{k}{N} \rfloor !} \le G \sum_{t=\left\lfloor \frac{\bar{k}}{N} \right\rfloor}^{+\infty} \sum_{r=0}^{N-1} e^{hNt+hr} \frac{W_N^t}{t!} \le 2G \frac{(e^{hN} W_N)^{\left\lfloor \frac{\bar{k}}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}}{N} \right\rfloor !} \sum_{r=0}^{N-1} e^{hr}. \tag{5}$$

Similarly, we obtain the analogous inequality for $\sum_{k=\bar{k}}^{+\infty} e^{-hk} \widetilde{Q}_N(-k)$ with $\bar{k} \ge N\lceil 2W_N - 1 \rceil$.

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## 3. Proof of Theorem 4.1

**Theorem:** *Given* $\varepsilon > 0$, *considering* $V$ *the HS European call option value, taking*

$$\bar{k} \ge \max\{N \left\lceil e^{hN+1} W_N - \ln \varepsilon + \ln(4S_0 G) + (\alpha - r)\tau + \ln k_+ \right\rceil - 1, N \left\lceil 2e^{hN} W_N - 1 \right\rceil - 1\} \tag{6}$$

$$\bar{l} \ge \max\{N \left\lceil e^{-hN+1} W_N - \ln \varepsilon + \ln(4S_0 G) + (\alpha - r)\tau + \ln k_- \right\rceil - 1, N \left\lceil 2e^{hN} W_N - 1 \right\rceil - 1\} \tag{7}$$

*with $k_+$ and $k_-$ the following constants,*

$$k_+ = \sum_{r=0}^{N-1} e^{hr} + N\max\{W_N^2, 1\}e^{2hN}\sum_{r=0}^{N-1} e^{-hr}$$
$$k_- = \sum_{r=0}^{N-1} e^{-hr} + N\max\{W_N^2, 1\}\sum_{r=0}^{N-1} e^{hr},$$

*we have that the European call option value $V^{TT}$ obtained via truncation of the tree at levels $\bar{k}$ and $-\bar{l}$ satisfies:*

$$V - V^{TT} < \varepsilon.$$

**Proof:**

Combining Equation (5.3) in the main article,

$$V - V^{PT} \leq e^{(\alpha-r)\tau} S_0 \left( \sum_{k=\bar{k}+1}^{Nn} e^{hk} \widetilde{Q}_N(k) + \sum_{k=\bar{l}+1}^{Nn} e^{-hk} \widetilde{Q}_N(k) \right)$$

and Equation (5.7) in the main article, to which we apply Lemma 5.10,

$$\begin{aligned} V^{PT} - V^{TT} &\leq e^{(\alpha-r)\tau} S_0 \sum_{k=-\bar{l}}^{\bar{k}} e^{hk} (Q_N^{\bar{k}}(k) + Q_{N\bar{l}}(k)) \\ &\leq e^{(\alpha-r)\tau} S_0 \left( \sum_{s=\bar{k}+2}^{2\bar{k}+\bar{l}+2} e^{h(2\bar{k}-s+2)} \sum_{i=0}^{N-1} \widetilde{Q}_N(s+2i) + \sum_{s=\bar{l}+2}^{2\bar{l}+\bar{k}+2} e^{h(s-2\bar{l}-2)} \sum_{i=0}^{N-1} \widetilde{Q}_N(s+2i) \right) \end{aligned}$$

the difference between $V$ and $V^{TT}$ is less or equal than the sum of four discarded parts:

$$V - V^{TT} \leq e^{(\alpha-r)\tau} S_0 \left( \sum_{k=\bar{k}+1}^{Nn} e^{hk} \widetilde{Q}_N(k) + \sum_{k=\bar{l}+1}^{Nn} e^{-hk} \widetilde{Q}_N(k) + e^{h(2\bar{k}+2)} \sum_{s=\bar{k}+2}^{Nn} e^{-hs} \sum_{i=0}^{N-1} \widetilde{Q}_N(s+2i) + e^{h(-2\bar{l}-2)} \sum_{s=\bar{l}+2}^{Nn} e^{hs} \sum_{i=0}^{N-1} \widetilde{Q}_N(s+2i) \right)$$

By Proposition 5.11:

$$V - V^{TT} \leq e^{(\alpha-r)\tau} S_0 G \left( 2 \frac{(e^{hN} W_N)^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor!} \sum_{r=0}^{N-1} e^{hr} + 2 \frac{(e^{-hN} W_N)^{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor!} \sum_{r=0}^{N-1} e^{-hr} + \right. \tag{8}$$

$$\left. + e^{h(2\bar{k}+2)} \sum_{s=\bar{k}+2}^{Nn} e^{-hs} \sum_{i=0}^{N-1} \frac{W_N^{\left\lfloor \frac{s+2i}{N} \right\rfloor}}{\left\lfloor \frac{s+2i}{N} \right\rfloor!} + e^{h(-2\bar{l}-2)} \sum_{s'=\bar{l}+2}^{Nn} e^{hs'} \sum_{i=0}^{N-1} \frac{W_N^{\left\lfloor \frac{s'+2i}{N} \right\rfloor}}{\left\lfloor \frac{s'+2i}{N} \right\rfloor!} \right) \tag{9}$$

where we operated the substitutions $s = 2\bar{k} - k + 2$, $s' = 2\bar{l} + k + 2$ and $G = 2N \max\{W_N, 1\} e^{W_N} \prod_{i=1}^{N-1} M_i^2$, and considered $\bar{k} \geq N\lceil 2e^{hN} W_N - 1\rceil - 1$ and $\bar{l} \geq N\lceil 2W_N - 1\rceil - 1$.

Since $\left\lfloor \frac{s}{N} \right\rfloor \leq \left\lfloor \frac{s+2i}{N} \right\rfloor \leq \left\lfloor \frac{s}{N} \right\rfloor + 2$ for $0 \leq i < N$, we have that $\frac{W_N^{\left\lfloor \frac{s+2i}{N} \right\rfloor}}{\left\lfloor \frac{s+2i}{N} \right\rfloor !} \leq \frac{W_N^{\left\lfloor \frac{s}{N} \right\rfloor}}{\left\lfloor \frac{s}{N} \right\rfloor !} \cdot \max\{W_N^2, 1\}$:

$$
\begin{aligned}
V - V^{TT} \leq & 2e^{(\alpha - r)\tau} S_0 G \left( \frac{(e^{hN} W_N)^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor !} \sum_{r=0}^{N-1} e^{hr} + \frac{(e^{-hN} W_N)^{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor !} \sum_{r=0}^{N-1} e^{-hr} \right) + \\
& + e^{(\alpha - r)\tau} S_0 G N \max\{W_N^2, 1\} \left( e^{h(2\bar{k}+2)} \sum_{s=\bar{k}+2}^{Nn} e^{-hs} \frac{W_N^{\left\lfloor \frac{s}{N} \right\rfloor}}{\left\lfloor \frac{s}{N} \right\rfloor !} + e^{h(-2\bar{l}-2)} \sum_{s=\bar{l}+2}^{Nn} e^{hs} \frac{W_N^{\left\lfloor \frac{s}{N} \right\rfloor}}{\left\lfloor \frac{s}{N} \right\rfloor !} \right) \\
\leq & 2e^{(\alpha - r)\tau} S_0 G \left( \frac{(e^{hN} W_N)^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor !} \sum_{r=0}^{N-1} e^{hr} + \frac{(e^{-hN} W_N)^{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor !} \sum_{r=0}^{N-1} e^{-hr} \right) + \\
& + 2e^{(\alpha - r)\tau} S_0 G N \max\{W_N^2, 1\} \left( e^{2h(\bar{k}+1)} \frac{(e^{-hN} W_N)^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor !} \sum_{r=0}^{N-1} e^{-hr} + e^{-2h(\bar{l}+1)} \frac{(e^{hs} W_N)^{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor !} \sum_{r=0}^{N-1} e^{hr} \right)
\end{aligned}
$$

for $\bar{k}, \bar{l} \geq N\lceil 2e^{hN} W_N - 1\rceil - 1$. Since we also have $hs \leq hN \left\lfloor \frac{s}{N} \right\rfloor + hN$ and $-hs \leq -hN \left\lfloor \frac{s}{N} \right\rfloor$, we can write:

$$
\begin{aligned}
V - V^{TT} \leq & 2e^{(\alpha - r)\tau} S_0 G \left[ \frac{(e^{hN} W_N)^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor !} \sum_{r=0}^{N-1} e^{hr} + \frac{(e^{-hN} W_N)^{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor !} \sum_{r=0}^{N-1} e^{-hr} \right. \\
& \left. + N \max\{W_N^2, 1\} \left( e^{2hN} \frac{(e^{hN} W_N)^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor !} \sum_{r=0}^{N-1} e^{-hr} + \frac{(e^{-hN} W_N)^{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor !} \sum_{r=0}^{N-1} e^{hr} \right) \right] \\
\leq & 2e^{(\alpha - r)\tau} S_0 G \left[ \frac{(e^{hN} W_N)^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor !} \left( \sum_{r=0}^{N-1} e^{hr} + N \max\{W_N^2, 1\} e^{2hN} \sum_{r=0}^{N-1} e^{-hr} \right) + \frac{(e^{-hN} W_N)^{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor !} \left( \sum_{r=0}^{N-1} e^{-hr} + N \max\{W_N^2, 1\} \sum_{r=0}^{N-1} e^{hr} \right) \right]
\end{aligned}
$$

In order to have the desired inequality, $V - V^{TT} < \varepsilon$, we ask:

$$
\begin{aligned}
\frac{(e^{hN} W_N)^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor !} \left( \sum_{r=0}^{N-1} e^{hr} + N \max\{W_N^2, 1\} e^{2hN} \sum_{r=0}^{N-1} e^{-hr} \right) &< \frac{\varepsilon}{4e^{(\alpha - r)\tau} S_0 G} \\
\frac{(e^{-hN} W_N)^{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{l}+1}{N} \right\rfloor !} \left( \sum_{r=0}^{N-1} e^{-hr} + N \max\{W_N^2, 1\} \sum_{r=0}^{N-1} e^{hr} \right) &< \frac{\varepsilon}{4e^{(\alpha - r)\tau} S_0 G}.
\end{aligned}
$$

Let us call

$$k_+ = \sum_{r=0}^{N-1} e^{hr} + N\max\{W_N^2, 1\}e^{2hN}\sum_{r=0}^{N-1} e^{-hr}$$
$$k_- = \sum_{r=0}^{N-1} e^{-hr} + N\max\{W_N^2, 1\}\sum_{r=0}^{N-1} e^{hr}.$$

Using Lemma 5.3 we impose:

$$e^{hN+1}W_N - \left\lfloor \frac{\bar{k}+1}{N} \right\rfloor \leq \ln\varepsilon - \ln(4S_0G) - (\alpha - r)\tau - \ln k_+$$

$$e^{-hN+1}W_N - \left\lfloor \frac{\bar{l}+1}{N} \right\rfloor \leq \ln\varepsilon - \ln(4S_0G) - (\alpha - r)\tau - \ln k_-$$

which means

$$\bar{k} \geq N\left\lceil e^{hN+1}W_N - \ln\varepsilon + \ln(4S_0G) + (\alpha - r)\tau + \ln k_+ \right\rceil - 1$$

$$\bar{l} \geq N\left\lceil e^{-hN+1}W_N - \ln\varepsilon + \ln(4S_0G) + (\alpha - r)\tau + \ln k_- \right\rceil - 1$$

Adding the conditions for Proposition 5.11, we have:

$$\bar{k} \geq \max\{N\left\lceil e^{hN+1}W_N - \ln\varepsilon + \ln(4S_0G) + (\alpha - r)\tau + \ln k_+ \right\rceil - 1, N\left\lceil 2e^{hN}W_N - 1 \right\rceil - 1\} \tag{10}$$
$$\bar{l} \geq \max\{N\left\lceil e^{-hN+1}W_N - \ln\varepsilon + \ln(4S_0G) + (\alpha - r)\tau + \ln k_- \right\rceil - 1, N\left\lceil 2e^{hN}W_N - 1 \right\rceil - 1\} \tag{11}$$

◇

## 4. Proof of Theorem 4.2

**Theorem:** *Given $\varepsilon > 0$, considering $V$ the HS European put option value, taking $\bar{k} \geq \max\{N\lceil 2W_N - 1\rceil - 1, N\lceil W_N e - \ln\varepsilon - r\tau + \ln(4N(N+1)KG)\rceil - 1\}$, we have that the European put option value $V^{TT}$ obtained via truncation of the tree at levels $\bar{k}$ and $-\bar{l}$ with $\bar{l} = \bar{k}$ satisfies*

$$V - V^{TT} < \varepsilon.$$

**Proof:**

Taking $\bar{l} = \bar{k}$ in Equation (5.34) in the main article, we have

$$V - V^{TT} \leq 2e^{-r\tau}K(N+1)\sum_{k=\bar{k}+1}^{Nn}\widetilde{Q}_N(k) \tag{12}$$

Applying Proposition 5.11 to Equation (12) we obtain:

$$V - V^{TT} \leq 4e^{-r\tau}K(N+1)GN\frac{W_N^{\left\lfloor\frac{\bar{k}+1}{N}\right\rfloor}}{\left\lfloor\frac{\bar{k}+1}{N}\right\rfloor!}$$

for $\bar{k} \geq N\lceil 2W_N - 1\rceil - 1$.

In order for it to be less than an arbitrary $\varepsilon$, we impose $\bar{k} \geq N\lceil W_N e - \ln\varepsilon - r\tau + \ln(4N(N+1)KG)\rceil - 1$.

Collecting all requirements on $\bar{k}$, we get

$\bar{k} \geq \max\{N\lceil 2W_N - 1\rceil - 1, N\lceil W_N e - \ln\varepsilon - r\tau + \ln(4N(N+1)KG)\rceil - 1\}$.

$\diamond$

## 5. Proof of Lemma 6.1

**Lemma:** $V_E^0(0,0,0) = V^{TT}$.

**Proof:** We want to show that the value $V^{TT}$ coincides with the value $V_E^0(0,0,0)$ obtained via backward procedure according to the following formula: $V_E^0(i,j,k) = e^{-r\Delta t}\sum_{l=-N}^{N}(V_E^0(i+1,j+1,k+l)p + V_E^0(i+1,j,k+l)(1-p))q_l$ if $k \in [-\bar{l},\bar{k}]$, 0 otherwise; with initial data $V_E^0(n,j,k) = 0$ for $j$ integer between 0 and $n$ and $k$ integer such that $-nN \leq k \leq -\bar{l} - 1$ or $\bar{k} + 1 \leq k \leq nN$, and $V_E^0(n,j,k) = (S(n,j,k) - K)^+$ for the call option, $V_E^0(n,j,k) = (K - S(n,j,k))^+$ for the put option, for $j$ integer between 0 and $n$ and $k$ integer such that $-\bar{l} \leq k \leq \bar{k}$.

Let us denote as $B$ the class of all paths on the tree that go from the node $(0,0,0)$ to one of the nodes $(n,j,k)$ at maturity $\tau$. For any $\beta \in B$ we will denote by prob($\beta$) the probability of following $\beta$ and value($\beta$) the value of the option at the end of the path $\beta$. Let us denote $B_{[-\bar{l},\bar{k}]}$ the class of all the paths on the tree that go from the node $(0,0,0)$ to one of the nodes at maturity without trespassing the $-\bar{l}$ and $\bar{k}$ boundaries, that is, where every node $(i,j,k)$ of the path has $-\bar{l} \leq k \leq \bar{k}$.

The expression

$$e^{-r\tau}\sum_{\beta\in B_{[-\bar{l},\bar{k}]}}\text{prob}(\beta)\cdot\text{value}(\beta) \tag{13}$$

coincides with $V^{TT}$, since they identify the same sum: every path that does not go out of the borders needs to end at a level $-\bar{l}\le k\le\bar{k}$; all the paths ending in a node $(n,j,k)$ share the same value for the option, so if we collect all the addenda in (13) that end in the same node we obtain $(K-S_0e^{(-n+2j)\sigma\sqrt{\Delta t}+hk})^{+}P(j)Q_N^T(k)$ in the put case and $(S_0e^{(-n+2j)\sigma\sqrt{\Delta t}+hk}-K)^{+}P(j)Q_N^T(k)$ in the call case.

We will show that the $V^{TT}$ as in (13) coincides with $V_E^0(0,0,0)$ for induction on the number of steps $n$.

Let us start with $n=1$. Our tree has only one step, which means that the values at maturity of the option are given by the $2(2N+1)$ children of $(0,0,0)$. In this case $\Delta t=\tau$. Let $0\le\bar{l},\bar{k}\le N$, that means that $(0,0,0)$ is surely in the allowed zone, while some of its children may not. Since the value of the option on the nodes $(1,j,k)$ with $k\notin[-\bar{l},\bar{k}]$ is 0, we can write:

$$\begin{aligned}V_E^0(0,0,0) &= e^{-r\tau}\sum_{l=-N}^{N}(V_E^0(1,j+1,l)p+V_E^0(1,j,l)(1-p))q_l =\\ &= e^{-r\tau}\sum_{l=-\bar{l}}^{\bar{k}}V_E^0(1,j+1,l)pq_l+V_E^0(1,j,l)(1-p)q_l =\\ &= e^{-r\tau}\sum_{\beta\in B_{[-\bar{l},\bar{k}]}}\text{prob}(\beta)\cdot\text{value}(\beta)=V^{TT}\end{aligned}$$

where the last equality is due to the fact that in a single step the paths that trespass are those that end outside the boundary.

Let us now suppose the thesis is true for all trees with $n-1$ steps. Let us consider a tree of $n$ steps. In this case $\Delta t=\tau/n$. We focus on the first step and compute the value of the option in $(0,0,0)$, with the backward procedure: $V_E^0(0,0,0)=e^{-r\Delta t}\sum_{l=-N}^{N}(V_E^0(1,1,l)p+V_E^0(1,0,l)(1-p))q_l$.

If $l\notin[-\bar{l},\bar{k}]$, $V_E^0(1,1,l)=V_E^0(1,0,l)=0$. Otherwise, we consider the $n-1$ trees that start at $(1,j,l)$ with $j=0,1$ and $l\in[-\bar{l},\bar{k}]$ and end at $\tau$. For such $j,l$, let us denote $B_{[-\bar{l},\bar{k}]}^{(1,j,l)}$ the class of all the paths on the tree that go from the node $(1,j,l)$ to one of the nodes $(n,j,k)$ at maturity without going out of the $[-\bar{l},\bar{k}]$ zone. On these smaller trees we apply induction and write the values $V_E^0(1,j,l)$ as

$$V_E^0(1,j,l)=e^{-r\tau'}\sum_{\beta'\in B_{[-\bar{l},\bar{k}]}^{(1,j,l)}}\text{prob}(\beta')\cdot\text{value}(\beta')$$

where we indicated with $\tau'$ the time interval $\tau' = \Delta t(n-1)$.

Therefore we can write

$$
\begin{aligned}
V_E^0(0,0,0) &= e^{-r\Delta t} \sum_{\substack{l=-N \\ l\in[-\bar{l},\bar{k}]}}^{N} (V_E^0(1,1,l)p + V_E^0(1,0,l)(1-p))q_l \\
&= e^{-r\tau} \sum_{\substack{l=-N \\ l\in[-\bar{l},\bar{k}]}}^{N} \left( \sum_{\beta'\in B^{(1,1,l)}_{[-\bar{l},\bar{k}]}} \text{prob}(\beta')\cdot\text{value}(\beta')pq_l + \sum_{\beta'\in B^{(1,0,l)}_{[-\bar{l},\bar{k}]}} \text{prob}(\beta')\cdot\text{value}(\beta')(1-p)q_l \right) = \\
&= e^{-r\tau} \sum_{\beta\in B_{[-\bar{l},\bar{k}]}} \text{prob}(\beta)\cdot\text{value}(\beta)
\end{aligned}
$$

where we used the fact that $\Delta t + \tau' = \tau$, and we considered that if a path $\beta$ that connects the node $(0,0,0)$ to a node at maturity $\tau$ (without trespassing) visits node $(1,0,l)$ and is afterwards identical to $\beta'$, we will have value($\beta$) =value($\beta'$) and prob($\beta$) $= (1-p)q_l\cdot$ prob($\beta'$), while if a path $\beta$ that connects the node $(0,0,0)$ to a node at maturity $\tau$ (without trespassing) visits node $(1,1,l)$ and is afterwards identical to $\beta'$, we will have value($\beta$) = value($\beta'$) and prob($\beta$) = $pq_l\cdot$ prob($\beta'$).

$\diamond$

## 6. Proof of Lemma 6.2

**Lemma:** $V_E^b(0,0,0) = \widehat{V^b}$.

**Proof:** The proof, similar to that of Lemma 6.1, is written for induction on the number of steps $n$.

In this situation, in order to understand the contribution of every path to the value of the option, we are interested in when a path, going from $(0,0,0)$ to maturity, first crosses the boundaries. Given any $\beta \in B \setminus B_{[-\bar{l},\bar{k}]}$, we will denote with i($\beta$) the time index $0 \le i \le n$ of the first exit of $\beta$ from the allowed zone $[-\bar{l},\bar{k}]$.

When $n = 1$, the tree has only one step, which means that the values at maturity of the option are given by the $2(2N+1)$ children of $(0,0,0)$. In this case $\Delta t = \tau$. Let $0 \le \bar{l},\bar{k} \le N$, that means that $(0,0,0)$ is surely in the allowed zone, while some of its children may be not. Since the value of the option is $b$ on the nodes $(1,j,k)$ with $k \notin [-\bar{l},\bar{k}]$, we can write:

$$
\begin{aligned}
V_E^b(0,0,0) =& e^{-r\tau}\sum_{l=-N}^{N}(V_E^b(1,j+1,l)p + V_E^b(1,j,l)(1-p))q_l = \\
=& e^{-r\tau}\sum_{l=-\bar{l}}^{\bar{k}}(V_E^b(1,j+1,l)pq_l + V_E^b(1,j,l)(1-p)q_l) + e^{-r\tau}\sum_{l=-N}^{-\bar{l}-1} b + e^{-r\tau}\sum_{l=\bar{k}+1}^{N} b = \\
=& e^{-r\tau}\sum_{\beta\in B_{[-\bar{l},\bar{k}]}}\text{prob}(\beta)\cdot\text{value}(\beta) + \sum_{\beta\in B\setminus B_{[-\bar{l},\bar{k}]}}\text{prob}(\beta)\cdot be^{-r\Delta t\,\mathrm{i}(\beta)} \\
=& V^{TT} + \sum_{\beta\in B\setminus B_{[-\bar{l},\bar{k}]}}\text{prob}(\beta)\cdot be^{-r\Delta t\,\mathrm{i}(\beta)} = \widehat{V^b}
\end{aligned}
$$

where we take into account the fact that in a single step the paths that trespass are those that end outside the boundaries.

Let us now suppose the thesis is true for all trees with $n-1$ steps. Let us consider a tree of $n$ steps. In this case $\Delta t = \tau/n$. We focus on the first step and compute the value of $V_E^b(0,0,0)$ with the backward procedure: $V_E^b(0,0,0) = e^{-r\Delta t}\sum_{l=-N}^{N}(V_E^b(1,1,l)p + V_E^b(1,0,l)(1-p))q_l$.

If $l \notin [-\bar{l},\bar{k}]$, $V_E^b(1,1,l) = V_E^b(1,0,l) = b$.

$$
V_E^b(0,0,0) = e^{-r\Delta t}\sum_{\substack{l=-N\\ l\in[-\bar{l},\bar{k}]}}^{N}(V_E^b(1,1,l)p + V_E^b(1,0,l)(1-p))q_l + e^{-r\Delta t}\sum_{\substack{l=-N\\ l\notin[-\bar{l},\bar{k}]}}^{N} bq_l
$$

If $l \in [-\bar{l},\bar{k}]$, we can consider the $n-1$ trees that start at $(1,j,l)$ for $j = 0,1$ and end at maturity $\tau$. For any such $j,l$, we will denote as $B^{(1,j,l)}$ the class of all paths starting from $(1,j,l)$ and ending at maturity. For any $\beta' \in B^{(1,j,l)} \setminus B^{(1,j,l)}_{[-\bar{l},\bar{k}]}$, $\mathrm{i}(\beta')$ is the time index $0 \le i \le n$ of the first exit of $\beta'$ from the allowed zone $[-\bar{l},\bar{k}]$.

We apply induction and write that the value $V_E^b(1,j,l)$ for this smaller trees is given by

$$
V_E^b(1,j,l) = e^{-r\tau'}\sum_{\beta'\in B^{(1,j,l)}_{[-\bar{l},\bar{k}]}}\text{prob}(\beta')\cdot\text{value}(\beta') + \sum_{\beta'\in B^{(1,j,l)}\setminus B^{(1,j,l)}_{[-\bar{l},\bar{k}]}}\text{prob}(\beta')\cdot be^{-r\Delta t\,\mathrm{i}(\beta')}
$$

where $\tau'$ indicates $\tau' = \tau - \Delta t$, $\Delta t' = \tau'/(n-1)$.

Therefore

$$V_E^b(1,j,l) = e^{-r\tau}\sum_{l=-N}^{N}\Bigg(pq_l \sum_{\beta'\in B^{(1,1,l)}_{[-\bar{l},\bar{k}]}} \text{prob}(\beta')\cdot\text{value}(\beta') + pq_l \sum_{\beta'\in B^{(1,1,l)}\setminus B^{(1,1,l)}_{[-\bar{l},\bar{k}]}} \text{prob}(\beta')\cdot be^{-r\Delta t\,\text{i}(\beta')} +$$

$$+(1-p)q_l \sum_{\beta'\in B^{(1,0,l)}_{[-\bar{l},\bar{k}]}} \text{prob}(\beta')\cdot\text{value}(\beta') + (1-p)q_l \sum_{\beta'\in B^{(1,0,l)}\setminus B^{(1,j,l)}_{[-\bar{l},\bar{k}]}} \text{prob}(\beta')\cdot be^{-r\Delta t\,\text{i}(\beta')}\Bigg) +$$

$$+ e^{-r\Delta t}\sum_{\substack{l=-N\\ l\notin[-\bar{l},\bar{k}]}}^{N} bq_l$$

Applying Lemma 6.1, we can rewrite the previous expression introducing the values $V_E^0(1, j, l)$.

$$V_E^b(0,0,0) = e^{-r\tau}\sum_{\substack{l=-N\\ l\in[-\bar{l},\bar{k}]}}^{N}\Big(pq_l V_E^0(1,j,l) + (1-p)q_l V_E^0(1,0,l) +$$

$$+pq_l \sum_{\beta'\in B^{(1,1,l)}\setminus B^{(1,1,l)}_{[-\bar{l},\bar{k}]}} \text{prob}(\beta')\cdot be^{-r\Delta t\,\text{i}(\beta')} + (1-p)q_l \sum_{\beta'\in B^{(1,0,l)}\setminus B^{(1,j,l)}_{[-\bar{l},\bar{k}]}} \text{prob}(\beta')\cdot be^{-r\Delta t\,\text{i}(\beta')}\Bigg) +$$

$$+ e^{-r\Delta t}\sum_{\substack{l=-N\\ l\notin[-\bar{l},\bar{k}]}}^{N} bq_l$$

Now we consider a path $\beta$ starting from the node $(0, 0, 0)$, visiting node $(1, j, l)$ and reaching maturity trespassing the boundaries. We call $\beta'$ the path going from $(1, j, l)$ to maturity which visits the same nodes as $\beta$. If $j = 0$ then $\text{prob}(\beta) = (1-p)q_l\,\text{prob}(\beta')$, while if $j = 1$ $\text{prob}(\beta) = pq_l\,\text{prob}(\beta')$. If $l \notin [-\bar{l}, \bar{k}]$, then $\text{i}(\beta) = 1$, otherwise $\text{i}(\beta) = \text{i}(\beta') + 1$. This means we can write

$$\begin{aligned} V_E^b(0,0,0) &= V_E^0(0,0,0) + \\ &+ \sum_{\substack{\beta\in B\setminus B_{[-\bar{l},\bar{k}]}\\ \text{i}(\beta)>1}} \text{prob}(\beta)\cdot be^{-r\Delta t\,\text{i}(\beta)} + \\ &+ \sum_{\substack{\beta\in B\setminus B_{[-\bar{l},\bar{k}]}\\ \text{i}(\beta)=1}} \text{prob}(\beta)\cdot be^{-r\Delta t\,\text{i}(\beta)} = \\ &= \widehat{V^b} \end{aligned}$$

$\diamond$

## 7. Proof of Lemma 6.3

**Lemma:** *Given* $\varepsilon > 0$*, taking* $G = 2N\max\{W_N, 1\}\prod_{i=1}^{N-1} M_i^2 e^{W_N}$*, the values* $\widehat{V^K}$ *and* $V^{TT}$ *obtained via truncation of the tree at levels* $\bar{k}$ *and* $-\bar{k}$*, with* $\bar{k}$ *the smallest integer which satisfies:*

$\bar{k} \geq \max\{N\lceil 2W_N - 1\rceil - 1, N\lceil W_N e - \ln\varepsilon + \ln(4N(N+1)KG)\rceil - 1\}$*, we have*

$$\left|\widehat{V^K} - V^{TT}\right| < \varepsilon$$

**Proof:**

$$\widehat{V^K} - V^{TT} = \sum_{\beta \in B \setminus B_{[-\bar{l},\bar{k}]}} \text{prob}(\beta) \cdot Ke^{-r\Delta t \mathbf{i}(\beta)}$$

For brevity, let us call $B^k$ the set of all paths in $B \setminus B_{[-\bar{l},\bar{k}]}$ which reach a node $(n, j, k)$, with $0 \leq j \leq n$, at maturity. We have:

$$\begin{aligned}
\widehat{V^K} - V^{TT} \leq & K \sum_{k=-Nn}^{Nn} \sum_{\beta \in B^k} \text{prob}(\beta) \\
\leq & K \sum_{k=-Nn}^{-\bar{l}-1} \sum_{\beta \in B^k} \text{prob}(\beta) + K \sum_{k=-\bar{l}}^{\bar{k}} \sum_{\beta \in B^k} \text{prob}(\beta) + K \sum_{k=\bar{k}+1}^{Nn} \sum_{\beta \in B^k} \text{prob}(\beta) \\
\leq & K \sum_{k=-Nn}^{-\bar{l}-1} Q_N(k) + K \sum_{k=\bar{k}+1}^{Nn} Q_N(k) + \\
& + K \sum_{k=-\bar{l}}^{\bar{k}} \sum_{\substack{\beta \in B^k \\ \text{first trespassing } -\bar{l}}} \text{prob}(\beta) + K \sum_{k=-\bar{l}}^{\bar{k}} \sum_{\substack{\beta \in B^k \\ \text{first trespassing } \bar{k}}} \text{prob}(\beta) \\
\leq & K \sum_{k=\bar{l}+1}^{Nn} \widetilde{Q}_N(k) + K \sum_{k=\bar{k}+1}^{Nn} \widetilde{Q}_N(k) + K \sum_{k=-\bar{l}}^{\bar{k}} Q_{\bar{l}}(k) + K \sum_{k=-\bar{l}}^{\bar{k}} Q^{\bar{k}}(k).
\end{aligned}$$

Therefore we have

$$
\begin{aligned}
\widehat{V^K} - V^{TT} &\leq K(N+1)\left(\sum_{k=\bar{l}+1}^{Nn} \widetilde{Q}_N(k) + \sum_{k=\bar{k}+1}^{Nn} \widetilde{Q}_N(k)\right) \\
&\leq 2K(N+1)\sum_{k=\bar{k}+1}^{Nn} \widetilde{Q}_N(k) \\
&\leq 4K(N+1)GN\frac{W_N^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor !}
\end{aligned}
$$

for $\bar{l} = \bar{k} \geq N\lceil 2W_N - 1\rceil - 1$ and applying Equation (2).

We ask $\bar{k} \geq N\lceil W_N e - \ln\varepsilon + \ln(4N(N+1)KG)\rceil - 1$, in order to have

$$
4e^{-r\tau}K_0(N+1)GN\frac{W_N^{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor}}{\left\lfloor \frac{\bar{k}+1}{N} \right\rfloor !} < \varepsilon.
$$

Collecting all the requirements on $\bar{k}$, we get that for

$\bar{k} \geq \max\{N\lceil 2W_N - 1\rceil - 1, N\lceil W_N e - \ln\varepsilon + \ln(4N(N+1)KG)\rceil - 1\}$

we have

$$
\left|\widehat{V^K} - V^{TT}\right| < \varepsilon.
$$

◇