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Research Article

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Blow-Up Phenomena and Asymptotic Profiles Passing from H^1-Critical to Super-Critical Quasilinear Schrödinger Equations

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Abstract: We study the asymptotic profile, as $\hbar \to 0$, of positive solutions to

$$-\hbar^2 \Delta u + V(x)u - \hbar^{2+\gamma} u\Delta u^2 = K(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N,$$

where $\gamma \geqslant 0$ is a parameter with relevant physical interpretations, V and K are given potentials and the dimension N is greater than or equal to 5, as we look for finite L^2-energy solutions. We investigate the concentrating behavior of solutions when $\gamma > 0$ and, differently from the case $\gamma = 0$ where the leading potential is V, the concentration is here localized by the source potential K. Moreover, surprisingly for $\gamma > 0$ we find a different concentration behavior of solutions in the case $p = \frac{2N}{N-2}$ and when $\frac{2N}{N-2} < p < \frac{4N}{N-2}$. This phenomenon does not occur when $\gamma = 0$.

Keywords: Semiclassical States, Concentration Phenomena, Finite Energy Solutions, Non-Autonomous Schrödinger Equations, Critical Growth

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1 Introduction

We are concerned with blow-up phenomena for positive solutions to the following class of quasilinear Schrödinger equations:

$$-\hbar^2 \Delta u + V(x)u - \hbar^{2+\gamma} u\Delta u^2 = K(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N,\ N \geqslant 5, \tag{1.1}$$

where $\hbar > 0$ is the adimensionalized Planck constant, $\gamma \in \mathbb{R}$ is a parameter which is relevant in several applications in Physics for which we refer to [21, 32], and which we assume here to be positive, V and K are given potentials, for the moment real continuous functions, and the nonlinearity is in the range $\frac{2N}{N-2} \leqslant p < \frac{4N}{N-2}$. The restriction on the Euclidean dimension is motivated by the fact that critical limit equations, related to (1.1) as $\hbar \to 0$, possess explicit solutions which fail to have finite L^2-energy in low dimension. Equations of the type (1.1) appear in the literature in the context of plasma physics and the continuum limit of discrete molecular structures; we refer to [6, 7, 24, 27] and the references therein for the more physics related context of (1.1).

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The existence of nontrivial solutions, in particular ground states for (1.1), has been intensively studied in recent years throughout a very extensive literature, among which let us mention [22, 23, 26]. Though it is not possible to give exhaustive references on the subject, let us recall a few results which are strictly related to our problem.

For semiclassical states of (1.1), namely $\gamma = 0$ and $\hbar \to 0$, assume $2 < p < \frac{4N}{N-2}$, $N \geqslant 3$, $K(x) \equiv 1$, and that $V : \mathbb{R}^N \to \mathbb{R}$ is Hölder continuous and satisfying the following conditions: $0 < V_0 < \inf_{x\in\mathbb{R}^N} V(x)$ and there is a bounded open set Λ such that $0 < a := \inf_{x\in\Lambda} V(x) < \min_{x\in\partial\Lambda} V(x)$. Then the existence of localized solutions concentrating near $\Omega := \{x \in \Lambda : V(x) = a\}$ has been obtained in [10, 19] and, by scaling properties, as $\hbar \to 0$, the limit equation turns out to be the following quasilinear autonomous Schrödinger equation:

$$-\Delta u + au - u\Delta u^2 = |u|^{p-2}u, \quad x \in \mathbb{R}^N. \tag{1.2}$$

We refer to [15, 16, 20, 31, 34] for related results.

Notice that the scaling invariance of (1.1) breaks down as soon as $\gamma > 0$. Recently in [11], it has been proved that in this context both the cases $\gamma = 0$ and $\gamma > 0$ have similar concentration behavior. However, the limit equation for $\gamma > 0$ is different form the case $\gamma = 0$ and turns out to be the following semilinear Schrödinger equation:

$$-\Delta u + au = |u|^{p-2}u, \quad x \in \mathbb{R}^N. \tag{1.3}$$

As we are going to see, this fact will play a crucial role in studying the blow-up profile of solutions to (1.1). Indeed, loosely speaking, one expects solutions can be localized along suitable normalized truncations and translations of ground states to the limit equation (1.2) or (1.3). Here the situation is completely different from the case $K \equiv 1$ and $\gamma = 0$, as a proper normalized, translated and rescaled solution will concentrate around critical points of the potential K.

It is well known from [3, 5, 30] that for the non-autonomous semilinear Schrödinger equation

$$-\hbar^2\Delta u + V(x)u = K(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N,$$

the function

$$\mathcal{A}(x) = [V(x)]^{\frac{p+2}{p}-\frac{N}{2}}[K(x)]^{-\frac{2}{p}}$$

retains important information for the concentrating behavior of solutions. Remarkably, for our problem (1.1) the external Schrödinger potential V does not play any role in the blow-up phenomenon which is governed by the source potential K.

Another interesting phenomenon addressed in this paper is the different concentrating behavior which occurs passing from critical to supercritical nonlinearities in (1.1). This is due to the fact that the limit equation, as $\hbar \to 0$, for (1.1) changes passing from $\frac{2N}{N-2} < p < \frac{4N}{N-2}$ to $p = \frac{2N}{N-2}$. Surprisingly, in the critical case the limit equation turns out to be the zero mass semilinear Schrödinger equation. To the best of our knowledge, this fact has not been observed before.

In order to state our main results, set

$$v(x) = \hbar^{\frac{\gamma}{2}}u(\hbar^{1+\frac{(p-2)\gamma}{4}}x).$$

Then equation (1.1) turns into

$$-\Delta v + \hbar^{\frac{(p-2)\gamma}{2}}V(\hbar^{1+\frac{(p-2)\gamma}{4}}y)v - v\Delta v^2 = K(\hbar^{1+\frac{(p-2)\gamma}{4}}y)|v|^{p-2}v, \quad x \in \mathbb{R}^N. \tag{1.4}$$

For simplicity, set

$$\kappa = \hbar^{\frac{(p-2)\gamma}{2}}, \quad \varepsilon = \hbar^{1+\frac{(p-2)\gamma}{4}}$$

and denote

$$V(\hbar^{1+\frac{(p-2)\gamma}{4}}y), \quad K(\hbar^{1+\frac{(p-2)\gamma}{4}}y)$$

by $V_\varepsilon(y)$, $K_\varepsilon(y)$, respectively. Thus, equation (1.4) can be written in the following form:

$$-\Delta v + \kappa V_\varepsilon(y)v - v\Delta v^2 = K_\varepsilon(y)|v|^{p-2}v, \quad x \in \mathbb{R}^N. \tag{1.5}$$

We assume the potential V and K satisfy the following conditions:
(V) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $0 < \inf V(x) \leqslant V(x) \leqslant \sup V(x) < +\infty$.
(K) $K : \mathbb{R}^N \to \mathbb{R}$ is Hölder continuous, $0 < \sup_{x\in\mathbb{R}^N} K(x) < K_0$ and there is a bounded open set $\mathcal{O}$ such that

$$\max_{x\in\partial\mathcal{O}} K(x) < m := \sup_{x\in\mathcal{O}} K(x).$$

Set $\mathcal{M} := \{x \in \mathcal{O} : K(x) = m\}$.

Our main results are the following.

Theorem 1.1. *Let $\gamma > 0$, assume* (V), (K) *and $\frac{2N}{N-2} \leqslant p < \frac{4N}{N-2}$, $N \geqslant 5$. Then, for sufficiently small $\varepsilon > 0$, there exists a positive solution v_ε of* (1.5).

The solution v_ε obtained in Theorem 1.1 is actually uniformly bounded with respect to ε. As a consequence, we will obtain the blow-up profile of solutions to the original equation (1.1).

In Section 2, we prove some preliminary results. In particular, we deal with the zero mass case and prove that the equation

$$-\Delta u - u\Delta u^2 = mu^p \tag{1.6}$$

has a unique positive radial solution U which belongs to $D^{1,2}(\mathbb{R}^N)$. Similarly to [1, Proposition 6.1], $v_\varepsilon \to U$ in $D^{1,2}(\mathbb{R}^N) \cap C^2_{\mathrm{loc}}(\mathbb{R}^N)$ as $\varepsilon \to 0$. That is,

$$\hbar^{\frac{\gamma}{2}} u_\hbar(\hbar^{1+\frac{(p-2)\gamma}{2}}(\cdot - x_\hbar)) \to U(\cdot)$$

in $D^{1,2}(\mathbb{R}^N) \cap C^2_{\mathrm{loc}}(\mathbb{R}^N)$ as $\hbar \to 0$.

Blow-up phenomena for the autonomous version of problem (1.1) (namely $V(x) = \lambda > 0$ and $K(x) \equiv 1$) have been studied in [1], where in order to get the asymptotic profile of the solution, uniform estimates of the rescaled ground state and energy estimates were established. However, their method can not be applied to deal with the non-autonomous problem (1.1). In [13], the Lyapunov–Schmidt reduction method has been used to deal with the problem

$$-\Delta u + \varepsilon V(x)u - u\Delta u^2 = u^p, \quad u > 0, \qquad \lim_{|x|\to+\infty} u(x) = 0, \quad x \in \mathbb{R}^N. \tag{1.7}$$

Assuming $V > 0$, $V \in L^\infty$ and $V(x) = o(|x|^{-2})$ as $|x| \to +\infty$, Cheng and Wei [13] proved that for ε sufficiently small problem (1.7) has a positive fast decaying solution provided $\frac{2N}{N-2} < p < \frac{4N}{N-2}$, $N \geqslant 3$.

Surprisingly, the limit equation for (1.1) changes again when $p = \frac{2N}{N-2}$. Precisely, let

$$v(x) = \hbar^{\frac{\alpha}{2}} u(\hbar^{1+\frac{(p-2)\alpha}{4}} x) \quad \text{for any } 0 < \alpha < \gamma,$$

let

$$\lambda = \hbar^{\frac{(p-2)\alpha}{2}}, \quad \zeta = \hbar^{\gamma-\alpha}, \quad \varepsilon = \hbar^{1+\frac{(p-2)\alpha}{4}}$$

and denote

$$V(\hbar^{1+\frac{(p-2)\alpha}{4}} y), \quad K(\hbar^{1+\frac{(p-2)\alpha}{4}} y)$$

by $V_\varepsilon(y)$, $K_\varepsilon(y)$, respectively. Then equation (1.1) turns into the following equation:

$$-\Delta v + \lambda V_\varepsilon(y)v - \zeta v\Delta v^2 = K_\varepsilon(y)|v|^{p-2}v, \quad x \in \mathbb{R}^N. \tag{1.8}$$

Note that $\lambda, \zeta \to 0$ as $\hbar \to 0$.

The solution to (1.8) is closely related to the (unique) solution of the following zero mass mean field limit equation [4]:

$$-\Delta v = mv^{\frac{N+2}{N-2}}, \quad x \in \mathbb{R}^N, \quad v > 0, \quad v(0) = \max v(x). \tag{1.9}$$

It is well known since [29] that equation (1.9) possesses an explicit one-parameter family of solutions given by

$$U = (N(N-2)m)^{\frac{N-2}{4}} \left(\frac{\mu}{1+\mu^2|x|^2}\right)^{\frac{N-2}{2}}, \quad \mu > 0.$$

Notice that the above functions, sometimes called Talenti's functions, instantons as well as standard bubbles, do have finite L^2-energy provided $N \geqslant 5$.

Theorem 1.2. *Assume that $\gamma > 0$, that (V), (K) hold and that $\frac{2N}{N-2} < p < \frac{4N}{N-2}$, $N \geqslant 5$. Then, for sufficiently small $\hbar > 0$, there exists a local maximum point $x_\hbar$ of $u_\hbar$ such that $\lim_{\hbar\to 0} \operatorname{dist}(x_\hbar, \mathcal{M}) = 0$ and there exists a positive solution $u_\hbar$ of (1.1) satisfying*

$$u_\hbar(\cdot) = \hbar^{-\frac{\gamma}{2}} U(\hbar^{-1-\frac{(p-2)\gamma}{2}} \cdot - x_\hbar) + \omega_\hbar(\cdot),$$

where $\omega_\hbar(\cdot) \to 0$ in $D^{1,2}(\mathbb{R}^N) \cap C^2_{\mathrm{loc}}(\mathbb{R}^N)$ as $\hbar \to 0$, and U is the unique fast decay, positive and radial (least energy) solution of (1.6). Moreover, when $p = \frac{2N}{N-2}$, under the above hypotheses we have

$$u_\hbar(\cdot) = \hbar^{-\frac{\alpha}{2}} \left[\frac{N(N-2)\mu\sqrt{m}}{1 + \mu^2 |\hbar^{-1-\frac{(p-2)\alpha}{2}} \cdot - x_\hbar|^2} \right]^{\frac{N-2}{2}} + \omega_\hbar(\cdot) \quad \text{for all } 0 < \alpha < \gamma.$$

Remark 1.3. In Theorem 1.2, $\alpha = \gamma$ is not allowed. Indeed, there exist no fast decaying solutions to (1.6) if $p = \frac{2N}{N-2}$, as established in [13, Theorem 1.1].

Remark 1.4. Throughout this paper, we require $N \geqslant 5$. This assumption guarantees to obtain finite energy solutions, namely that the solutions of (1.6) and (1.9) belong to $L^2(\mathbb{R}^N)$. It seems out of reach at the moment to generalize the method of this paper to the case of $N \leqslant 4$ just assuming the mild condition (V). We mention that the same restriction on the dimension was used for instance in [12], where infinitely many nonradial solutions for the semilinear Schrödinger equation with critical growth were established by using a reduction argument.

Throughout this paper, C will denote a positive constant whose exact value may change from line to line without affecting the overall result.

2 Preliminaries

In this section, we collect a few results, which we will use in the sequel, on the following zero mass equation:

$$-\Delta u - u\Delta u^2 = m|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{2.1}$$

where $2^* := \frac{2N}{N-2} < p < 2(2^*) := \frac{4N}{N-2}$, $N \geqslant 5$.

Uniqueness and non-degeneracy of positive solutions to (2.1) have been completely solved in [2]; see also [13]. For the reader's convenience, below we recall a few results we need in the sequel.

The energy functional related to equation (2.1) is given by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (1 + 2u^2)|\nabla u|^2 \, dx - \frac{m}{p} \int_{\mathbb{R}^N} |u|^p \, dx$$

and it is well defined in the set

$$E = \Big\{ u \in D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2 |\nabla u|^2 \, dx < +\infty \Big\}.$$

Theorem 2.1 ([2, Theorem 1.1] or [13, Theorem 1.1]). *Equation (2.1) has a unique positive radial solution which belongs to $D^{1,2}(\mathbb{R}^N)$. In particular, the ground state of (2.1) is unique up to translations.*

Lemma 2.2 ([33, Lemma 2.1]). *Let $g(s) = \sqrt{1 + 2s^2}$ and $G(t) = \int_0^t g(s)\, ds$. Then $G(t)$ is an odd smooth function as well as the inverse function $G^{-1}(t)$. Moreover, the following properties hold:*

(i) *It holds*
$$\lim_{t\to 0} \frac{G^{-1}(t)}{t} = 1.$$
(ii) *It holds*
$$\lim_{t\to +\infty} \frac{G^{-1}(t)}{\sqrt{t}} = \sqrt[4]{2}.$$
(iii) $|G^{-1}(t)| \leqslant |t|$ *for all* $t \in \mathbb{R}$.
(iv) $|G^{-1}(t)|^2$ *is convex in* t.
(v) $|G^{-1}(t)| \leqslant \sqrt[4]{2}\sqrt{|t|}$ *for all* $t \in \mathbb{R}$.

Next consider the following semilinear elliptic equation, which in some sense is the dual problem of (2.1):

$$-\Delta v = m\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))}, \quad x \in \mathbb{R}^N. \tag{2.2}$$

The energy functional corresponding to (2.2) is defined by

$$L_m(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx - \frac{m}{p}\int_{\mathbb{R}^N} |G^{-1}(v)|^p\,dx,$$

which is well defined in $D^{1,2}(\mathbb{R}^N)$ by Lemma 2.2. Moreover, $L_m(v) \in C^1$.

Solutions to v of (2.2) satisfy the following Pohozaev identity:

$$\frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla v|^2\,dx - \frac{m}{p}\int_{\mathbb{R}^N} |G^{-1}(v)|^p\,dx = 0. \tag{2.3}$$

Moreover, the ground state has a mountain pass characterization, namely

$$L_m(U) = C_m = \inf_{\eta\in\Phi}\max_{t\in[0,1]} L_m(\eta(t)),$$

where

$$\Phi = \{\eta \in C([0,1], D^{1,2}(\mathbb{R}^N)) : \eta(0) = 0,\ L_m(\eta(1)) < 0\};$$

see [1, Proposition 4.3].

Theorem 2.3 ([2, Propositions 2.6 and 3.2]). *The following properties hold:*

(i) (2.2) *has a unique fast decay positive radial solution $v(r)$, namely*

$$\lim_{r\to+\infty} r^{N-2}v(r) = c \in (0,+\infty).$$

(ii) *Let $v \in D^{1,2}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ be a positive radially decreasing solution of* (2.2). *Then there exists $C > 0$ such that*

$$CA(r)(1 - O(r^{-2})) \leqslant v(r) \leqslant CA(r), \quad CA'(r) \leqslant v'(r) \leqslant CA'(r)(1 - O(r^{-2}))$$

for sufficiently large r. Here

$$A(r) = \frac{1}{(N-2)|S^{N-1}|r^{N-2}}$$

is the fundamental solution of $-\Delta$ on $\mathbb{R}^N$. In particular, we have that

$$\lim_{r\to+\infty} r^{N-2}v(r) = \frac{C}{(N-2)|S^{N-1}|}, \quad \lim_{r\to+\infty} r^{N-1}v'(r) = -\frac{C}{|S^{N-1}|}. \tag{2.4}$$

Theorem 2.4 ([2, Lemma 2.4]). *Suppose that*

$$v = \int_0^u g(s)\,ds.$$

Then the following assertions hold:

(i) *$u \in E \cap C^2(\mathbb{R}^N)$ if and only if $v \in D^{1,2}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$.*

(ii) *u is a positive solution of* (2.1) *if and only if v is a positive solution of* (2.2).

3 Proof of Theorems 1.1 and 1.2

We next consider the following quasilinear Schrödinger equation:

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + \kappa V_\varepsilon(x)u = K_\varepsilon(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{3.1}$$

where $g(s) = \sqrt{1+2s^2}$. Direct calculations show that (3.1) is equivalent to (1.5).

The energy functional corresponding to (3.1) is given by

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} [g^2(u)|\nabla u|^2 + \kappa V_\varepsilon(x)u^2]\,dx - \frac{1}{p}\int_{\mathbb{R}^N} K_\varepsilon(x)|u|^p\,dx.$$

Note that J_ε is not even well defined in $H^1(\mathbb{R}^N)$. However, it is well known since [14, 22] that a suitable dual approach, hidden in change of variables, turns the energy functional to be smooth and well defined in a proper function space setting; see [25], and also [10] for an Orlicz space approach. Here, the change of variables $u = G^{-1}(v)$ yields the following smooth energy:

$$P_\varepsilon(v) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + \kappa V_\varepsilon(x)|G^{-1}(v)|^2)\,dx - \frac{1}{p}\int_{\mathbb{R}^N} K_\varepsilon(x)|G^{-1}(v)|^p\,dx.$$

By Lemma 2.2, it is standard to check that $P_\varepsilon \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$.

The Euler–Lagrange equation associated to P_ε is

$$-\Delta v + \kappa V_\varepsilon(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} = K_\varepsilon(x)\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))}, \quad x \in \mathbb{R}^N. \tag{3.2}$$

If $v \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is a solution of (3.2), then it satisfies

$$\int_{\mathbb{R}^N} \Big[\nabla v\nabla\phi + \kappa V_\varepsilon(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\phi - K_\varepsilon(x)\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))}\phi\Big]\,dx = 0 \quad \text{for all } \phi \in H^1(\mathbb{R}^N).$$

Then $u = G^{-1}(v) \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. For any $\varphi \in C_0^\infty(\mathbb{R}^N)$, one has $\varphi g(G^{-1}(v)) \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} [\nabla u\nabla\varphi + g(u)g'(u)|\nabla u|^2\varphi + \kappa V_\varepsilon(x)u\varphi - K_\varepsilon(x)|w|^{p-2}w\varphi]\,dx = 0,$$

which implies that u is a weak solution of (3.1).

Therefore, in order to find nontrivial solutions to (3.1), we are compelled to find nontrivial solutions of (3.2). Since we are concerned with positive solutions, we actually consider the following truncated energy functional:

$$v \mapsto \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + \kappa V_\varepsilon(x)|G^{-1}(v)|^2)\,dx - \frac{1}{p}\int_{\mathbb{R}^N} K_\varepsilon(x)|G^{-1}(v^+)|^p\,dx.$$

However, in order to avoid cumbersome notations, hereafter we write v in place of v^+ in the last integral, when this does not yield confusion.

Set

$$\Gamma_\varepsilon(v) := P_\varepsilon(v) + Q_\varepsilon(v),$$

where

$$Q_\varepsilon(v) = \Big(\int_{\mathbb{R}^N} \chi_\varepsilon v^{\frac{p}{2}}\,dx - 1\Big)_+^2$$

with $\chi_\varepsilon(x) = 0$ for $x \in \mathcal{O}_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \mathcal{O}\}$ and $\chi_\varepsilon(x) = \varepsilon^{-\tau}$ for $x \notin \mathcal{O}_\varepsilon$, where $\tau > 0$ has to be determined later on. By inspection, $\Gamma_\varepsilon \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$. The functional Q_ε will act as a penalization to force the concentration phenomena to occur inside $\mathcal{O}$. This type of penalization was introduced in [8, 9].

Let U be the unique fast decay positive radial solution of (2.2). Without loss of generality, we may assume $U(0) = \max U(x)$ and that $0 \in \mathcal{M}$. Set $U_t(x) := U(\frac{x}{t})$ for $t > 0$. By (2.3), there exists $t_0 > 1$ such that

$$L_m(U_t) = \Big(\frac{t^{N-2}}{2} - \frac{t^N}{2^*}\Big)\int_{\mathbb{R}^N} |\nabla U|^2\,dx < -2 \quad \text{for all } t \geqslant t_0. \tag{3.3}$$

Choose a positive number

$$\beta < \frac{\operatorname{dist}(\mathcal{M}, \mathbb{R}^N \setminus \mathcal{O})}{100}$$

and a cut-off function $\varphi(x) \in C_0^\infty(\mathbb{R}^N, [0,1])$ such that $\varphi(x) = 1$ for $|x| \leqslant \beta$ and $\varphi(x) = 0$ for $|x| \geqslant 2\beta$. Set $\varphi_\varepsilon(x) := \varphi(\varepsilon x)$ and then define

$$U_\varepsilon^y(x) := \varphi_\varepsilon\Big(x - \frac{y}{\varepsilon}\Big)U\Big(x - \frac{y}{\varepsilon}\Big) \quad \text{for each } y \in \mathcal{M}^\beta,$$

where $\mathcal{M}^\beta = \{x \in \mathbb{R}^N : \operatorname{dist}(x, \mathcal{M}) < \beta\}$.

We aim at finding a solution of (1.5) near the set $X_\varepsilon := \{U_\varepsilon^y(x) : y \in \mathcal{M}^\beta\}$ for sufficiently small $\varepsilon > 0$. Let $W_{\varepsilon,t}(x) = \varphi_\varepsilon(x)U_t(x)$. Note that for fixed $x \in \mathbb{R}^N$ we have $W_{\varepsilon,t}(x) \to 0$ as $t \to 0$. So, we set $W_{\varepsilon,0}(x) = 0$.

Next, we borrow some ideas from [11]. However, here the situation is quite different in particular for the decaying behavior of the ground state solution of the limit equation and the different concentrating behavior of the solution.

Lemma 3.1. *It holds*

$$\lim_{\varepsilon \to 0} \max_{t \in (0, t_0]} |\Gamma_\varepsilon(W_{\varepsilon,t}) - L_m(U_t)| \to 0.$$

Proof. Since $\operatorname{supp}(W_{\varepsilon,t}(x)) \subset \mathcal{O}_\varepsilon$, one has $Q_\varepsilon(W_{\varepsilon,t}(x)) = 0$. Thus, for $t \in (0, t_0]$, we have

$$\begin{aligned}\Gamma_\varepsilon(W_{\varepsilon,t}) - L_m(U_t) = {}& \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla W_{\varepsilon,t}|^2 - |\nabla U_t|^2)\,dx + \frac{\kappa}{2}\int_{\mathbb{R}^N} V_\varepsilon |G^{-1}(W_{\varepsilon,t})|^2\,dx \\ & - \frac{1}{p}\int_{\mathbb{R}^N} (K_\varepsilon |G^{-1}(W_{\varepsilon,t})|^p - m|G^{-1}(U_t)|^p)\,dx. \end{aligned} \tag{3.4}$$

By (2.4) and the Lebesgue dominated convergence theorem, we get

$$\Big|\int_{\mathbb{R}^N} (|\nabla W_{\varepsilon,t}|^2 - |\nabla U_t|^2)\,dx\Big| \leqslant C\varepsilon^2 \int_{\mathbb{R}^N} (|\nabla U|^2 + U^2)\,dx + Ct_0^N \int_{\mathbb{R}^N} |\varphi_\varepsilon^2(tx) - 1|(1 + |x|)^{2-2N}\,dx \to 0. \tag{3.5}$$

as $\varepsilon \to 0$. Clearly, by Lemma 2.2 (iii), we obtain

$$\int_{\mathbb{R}^N} V_\varepsilon |G^{-1}(W_{\varepsilon,t})|^2\,dx \leqslant C\int_{\mathbb{R}^N} |W_{\varepsilon,t}|^2\,dx \leqslant C\int_{\mathbb{R}^N} U^2\,dx < +\infty. \tag{3.6}$$

By the mean value theorem and dominated convergence again, we obtain

$$\begin{aligned}\Big|\int_{\mathbb{R}^N} (|G^{-1}(U_t)|^p - |G^{-1}(W_{\varepsilon,t})|^p)\,dx\Big| &\leqslant p\int_{\mathbb{R}^N} \frac{|G^{-1}(U_t + \theta W_{\varepsilon,t})|^{p-1}}{g(G^{-1}(U_t + \theta W_{\varepsilon,t}))}|U_t - W_{\varepsilon,t}|\,dx \\ &\leqslant C\int_{\mathbb{R}^N} (1 - \varphi_\varepsilon(tx))(U^2 + U^{\frac{p}{2}})\,dx \\ &\to 0 \quad \text{as } \varepsilon \to 0, \end{aligned} \tag{3.7}$$

where $0 < \theta < 1$. Similarly, we have

$$\Big|\int_{\mathbb{R}^N} (K_\varepsilon(x) - m)|G^{-1}(W_{\varepsilon,t})|^p\,dx\Big| \leqslant t_0^N \int_{\mathbb{R}^N} (K_\varepsilon(tx) - m)\varphi_\varepsilon(tx)U^{\frac{p}{2}}(x)\,dx \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.8}$$

The desired conclusion follows from (3.4)–(3.8). □

Now, from (3.3) and Lemma 3.1, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$,

$$\Gamma_\varepsilon(W_{\varepsilon,t_0}) \leqslant L_m(U_{t_0}) + 1 < -1.$$

Define the minimax level

$$C_\varepsilon = \inf_{\eta_\varepsilon \in \Phi_\varepsilon} \max_{s \in [0,1]} \Gamma_\varepsilon(\eta_\varepsilon(s)),$$

where

$$\Phi_\varepsilon = \{\eta_\varepsilon \in C([0,1], H^1(\mathbb{R}^N)) : \eta_\varepsilon(0) = 0,\ \eta_\varepsilon(1) = W_{\varepsilon,t_0}\}.$$

Lemma 3.2. *It holds* $\lim_{\varepsilon\to 0} C_\varepsilon = C_m$.

Proof. Let $\eta_\varepsilon(s) = W_{\varepsilon,st_0}$, $s \in [0,1]$, such that $\eta_\varepsilon(s) \in \Phi_\varepsilon$. Since $t_0 > 1$, from Lemma 3.1, we have

$$\limsup_{\varepsilon\to 0} C_\varepsilon \leqslant \limsup_{\varepsilon\to 0} \max_{t\in[0,t_0]} \Gamma_\varepsilon(W_{\varepsilon,t}) \leqslant \max_{t\in[0,t_0]} L_m(U_t) = C_m.$$

It remains to prove that $\liminf_{\varepsilon\to 0} C_\varepsilon \geqslant C_m$. By definition of C_ε, for any $\tilde{\varepsilon} > 0$, there exists $\tilde{\eta}_\varepsilon \in \Phi_\varepsilon$ such that

$$\max_{s\in[0,1]} \Gamma_\varepsilon(\tilde{\eta}_\varepsilon(s)) < C_\varepsilon + \tilde{\varepsilon}. \tag{3.9}$$

Since $P_\varepsilon(\tilde{\eta}_\varepsilon(0)) = 0$ and $P_\varepsilon(\tilde{\eta}_\varepsilon(1)) \leqslant \Gamma_\varepsilon(W_{\varepsilon,t_0}) < -1$, there exists $s_0 \in (0,1)$ such that

$$P_\varepsilon(\tilde{\eta}_\varepsilon(s_0)) = -1 \quad \text{and} \quad P_\varepsilon(\tilde{\eta}_\varepsilon(s)) > -1, \qquad s \in [0, s_0).$$

Then we have

$$Q_\varepsilon(\tilde{\eta}_\varepsilon(s)) \leqslant \Gamma_\varepsilon(\tilde{\eta}_\varepsilon(s)) + 1 < C_\varepsilon + \tilde{\varepsilon} + 1, \quad s \in [0, s_0].$$

By Lemma 2.2 (v), we have

$$\begin{aligned}\int_{\mathbb{R}^N\setminus\mathcal{O}_\varepsilon} |G^{-1}(\tilde{\eta}_\varepsilon(s))|^p\,dx &\leqslant \sqrt[4]{2p} \int_{\mathbb{R}^N\setminus\mathcal{O}_\varepsilon} |\tilde{\eta}_\varepsilon(s)|^{\frac{p}{2}}\,dx \\ &\leqslant \sqrt[4]{2p}\,\varepsilon^\tau\Big[\sqrt{Q_\varepsilon(\tilde{\eta}_\varepsilon(s))} + 1\Big] \leqslant \sqrt[4]{2p}\,\varepsilon^\tau\Big[\sqrt{C_\varepsilon + \tilde{\varepsilon} + 1} + 1\Big]\end{aligned}$$

for $s \in [0, s_0]$. Therefore, the following lower bound holds:

$$\begin{aligned}P_\varepsilon(\tilde{\eta}_\varepsilon(s)) &\geqslant L_m(\tilde{\eta}_\varepsilon(s)) + \frac{1}{p}(m - K_0) \int_{\mathbb{R}^N\setminus\mathcal{O}_\varepsilon} |G^{-1}(\tilde{\eta}_\varepsilon(s))|^p\,dx \\ &\geqslant L_m(\tilde{\eta}_\varepsilon(s)) + \frac{1}{p}(m - K_0)\sqrt[4]{2p}\,\varepsilon^\tau\Big[\sqrt{C_\varepsilon + \tilde{\varepsilon} + 1} + 1\Big], \quad s \in [0, s_0].\end{aligned} \tag{3.10}$$

In particular, we have

$$L_m(\tilde{\eta}_\varepsilon(s_0)) \leqslant \frac{1}{p}(K_0 - m)\sqrt[4]{2p}\,\varepsilon^\tau\Big[\sqrt{C_\varepsilon + \tilde{\varepsilon} + 1} + 1\Big] - 1 < 0 \quad \text{for small } \varepsilon > 0.$$

Hence, $\tilde{\eta}_\varepsilon(ts_0) \in \Phi$ and $\max_{t\in[0,1]} L_m(\tilde{\eta}_\varepsilon(ts_0)) \geqslant C_m$. So, by (3.9) and (3.10), we get

$$C_\varepsilon + \tilde{\varepsilon} > \max_{s\in[0,s_0]} \Gamma_\varepsilon(\tilde{\eta}_\varepsilon(s)) \geqslant C_m + \frac{1}{p}(m - K_0)\sqrt[4]{2p}\,\varepsilon^\tau\Big[\sqrt{C_\varepsilon + \tilde{\varepsilon} + 1} + 1\Big],$$

which yields $\liminf_{\varepsilon\to 0} C_\varepsilon \geqslant C_m$ since $\tilde{\varepsilon}$ is arbitrary. This completes the proof of the lemma. □

Remark 3.3. Let $D_\varepsilon = \max_{s\in[0,1]} \Gamma_\varepsilon(W_{\varepsilon,st_0})$. From the proof of Lemma 3.2 we get

$$\lim_{\varepsilon\to 0} D_\varepsilon = C_m.$$

Next, we consider the space $E_\varepsilon^R := H_0^1(B_{R/\varepsilon}(0))$ endowed with the norm

$$\|v\|_{\varepsilon,R} = \Big[\int_{B_{R/\varepsilon}(0)} (|\nabla v|^2 + v^2)\,dx\Big]^{\frac{1}{2}}.$$

Note that any $v \in E_\varepsilon^R$ can be regarded as an element of $H^1(\mathbb{R}^N)$ by defining $v = 0$ on $\mathbb{R}^N \setminus B_{R/\varepsilon}(0)$.

Define also the level sets

$$\Gamma_\varepsilon^c := \{u \in E_\varepsilon^R : \Gamma_\varepsilon(u) \leqslant c\}$$

and

$$X^d := \{u \in E_\varepsilon^R : \inf_{v\in X}\|u - v\|_{\varepsilon,R} < d\}, \quad d > 0.$$

In what follows, for small $d > 0$, let $v_n \in X^d_{\varepsilon_n} \cap E^{R_n}_{\varepsilon_n}$ with $\varepsilon_n \to 0$ and $R_n \to +\infty$ be such that

$$\lim_{n\to\infty} \Gamma_{\varepsilon_n}(v_n) \leqslant C_m \quad \text{and} \quad \lim_{n\to\infty} \|\Gamma'_{\varepsilon_n}(v_n)\|_{(E^{R_n}_{\varepsilon_n})'} = 0.$$

From the definition of $X^d_{\varepsilon_n}$, we can find a sequence $\{y_n\} \subset \mathcal{M}^\beta$ such that

$$\Big\| v_n - \varphi_{\varepsilon_n}\Big(\cdot - \frac{y_n}{\varepsilon_n}\Big) U\Big(\cdot - \frac{y_n}{\varepsilon_n}\Big)\Big\|_{\varepsilon_n, R_n} \leqslant d. \tag{3.11}$$

This implies that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Since $\mathcal{M}^\beta$ is compact, we may assume, up to a subsequence, that $y_n \to y_0 \in \mathcal{M}^\beta$.

Lemma 3.4. *It holds*

$$\lim_{n\to\infty} \sup_{z \in \{z \in \mathbb{R}^N : \frac{1}{2}\beta \leqslant |\varepsilon_n z - y_n| \leqslant 3\beta\}} \int_{B_R(z)} v_n^2 \, dx = 0 \quad \textit{for all } R > 0.$$

Proof. Suppose by contradiction that there exist $R > 0$ and a sequence

$$\{z_n\} \subset \Big\{ z \in \mathbb{R}^N : \frac{1}{2}\beta \leqslant |\varepsilon_n z - y_n| \leqslant 3\beta \Big\}$$

such that

$$\lim_{n\to\infty} \int_{B_R(z_n)} v_n^2 \, dx > 0. \tag{3.12}$$

Assume

$$\varepsilon_n z_n \to z_0 \in \Big\{ z \in \mathbb{R}^N : \frac{1}{2}\beta \leqslant |z - y_0| \leqslant 3\beta \Big\}.$$

Let $\tilde{v}_n(\cdot) := v_n(\cdot + z_n)$ be such that $\tilde{v}_n \rightharpoonup \tilde{v}$ in $H^1(\mathbb{R}^N)$ and $\tilde{v}_n \to \tilde{v}$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$, $p \in [2, 2^*)$. Then, by (3.12), we get

$$\int_{B_R(0)} |\tilde{v}|^2 \, dx = \lim_{n\to\infty} \int_{B_R(0)} \tilde{v}_n^2 \, dx > 0,$$

which yields $\tilde{v} \not\equiv 0$.

Let $\phi \in C_0^\infty(\mathbb{R}^N)$. Then, for large n, we have $\phi(\cdot - z_n) \in E^{R_n}_{\varepsilon_n}$. Since

$$\lim_{n\to\infty} \|\Gamma'_{\varepsilon_n}(v_n)\|_{(E^{R_n}_{\varepsilon_n})'} = 0,$$

we obtain

$$\begin{aligned} o_n(1)\|\phi\|_{\varepsilon_n, R_n} = {} & \int_{\mathbb{R}^N} \Big[\nabla\tilde{v}_n \nabla\phi - K_{\varepsilon_n}(x + z_n) \frac{|G^{-1}(\tilde{v}_n)|^{p-2} G^{-1}(\tilde{v}_n)}{g(G^{-1}(\tilde{v}_n))} \phi \Big] \, dx + \kappa_n \int_{\mathbb{R}^N} V_{\varepsilon_n}(x + z_n) \frac{G^{-1}(\tilde{v}_n)}{g(G^{-1}(\tilde{v}_n))} \phi \, dx \\ & - p\Big(\int_{\mathbb{R}^N} \chi_{\varepsilon_n} |v_n|^{\frac{p}{2}} \, dx - 1 \Big)_+ \int_{\mathbb{R}^N} \chi_{\varepsilon_n}(x + z_n) |\tilde{v}_n|^{\frac{p}{2}-2} \tilde{v}_n \phi \, dx, \end{aligned} \tag{3.13}$$

where

$$\kappa_n = \varepsilon_n^{\frac{2(p-2)\gamma}{4+(p-2)\gamma}}.$$

Clearly,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \chi_{\varepsilon_n}(x + z_n) |\tilde{v}_n|^{\frac{p}{2}-2} \tilde{v}_n \phi \, dx = 0.$$

Since $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} V_{\varepsilon_n}(x + z_n) \frac{G^{-1}(\tilde{v}_n)}{g(G^{-1}(\tilde{v}_n))} \phi \, dx < +\infty. \tag{3.14}$$

By the Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K_{\varepsilon_n}(x + z_n) \frac{|G^{-1}(\tilde{v}_n)|^{p-2} G^{-1}(\tilde{v}_n)}{g(G^{-1}(\tilde{v}_n))} \phi \, dx = \int_{\mathbb{R}^N} K(z_0) \frac{|G^{-1}(\tilde{v})|^{p-2} G^{-1}(\tilde{v})}{g(G^{-1}(\tilde{v}))} \phi \, dx. \tag{3.15}$$

Combine (3.13), (3.14) and (3.15), to have

$$\int_{\mathbb{R}^N} \Big[\nabla\tilde{v} \nabla\phi - K(z_0) \frac{|G^{-1}(\tilde{v})|^{p-2} G^{-1}(\tilde{v})}{g(G^{-1}(\tilde{v}))} \phi \Big] \, dx = 0 \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^N),$$

which implies that $\tilde{v}$ is a positive solution of the following equation:

$$-\Delta\tilde{v} = K(z_0)\frac{|G^{-1}(\tilde{v})|^{p-2}G^{-1}(\tilde{v})}{g(G^{-1}(\tilde{v}))}, \quad x \in \mathbb{R}^N. \tag{3.16}$$

Recall that, on the right-hand side of the above equality, $\tilde{v}$ is actually $\tilde{v}^+$. Thus, by the maximum principle, we get $\tilde{v} > 0$. Because of $K(z_0) \leqslant m$, we get $C_{K(z_0)} \geqslant C_m$.

Choosing $R > 0$ sufficiently large, by Pohozaev's identity, we obtain

$$2\lim_{n\to\infty}\int_{B_R(z_n)}|\nabla v_n|^2\,dx \geqslant \int_{\mathbb{R}^N}|\nabla\tilde{v}|^2\,dx = NL_{K(z_0)}(\tilde{v}) \geqslant NC_{K(z_0)} \geqslant NC_m. \tag{3.17}$$

However, it follows from (3.11) that

$$\begin{aligned}\int_{B_R(z_n)}|\nabla v_n|^2\,dx &\leqslant d^2 + 2\varepsilon_n^2\int_{B_R(z_n)}\Big|\nabla\varphi_{\varepsilon_n}\Big(x-\frac{y_n}{\varepsilon_n}\Big)U\Big(x-\frac{y_n}{\varepsilon_n}\Big)\Big|^2\,dx + \int_{B_R(z_n)}\Big|\varphi_{\varepsilon_n}\Big(x-\frac{y_n}{\varepsilon_n}\Big)\nabla U\Big(x-\frac{y_n}{\varepsilon_n}\Big)\Big|^2\,dx\\ &\leqslant d + C\varepsilon_n^2 + C\int_{B_R(0)}\Big(1+\Big|x+z_n-\frac{y_n}{\varepsilon_n}\Big|\Big)^{2-2N}\,dx. \end{aligned}\tag{3.18}$$

Note that $\lim_{n\to\infty}|z_n - \frac{y_n}{\varepsilon_n}| = +\infty$. Thus, for n large enough, by (3.17) and (3.18), we get a contradiction for small $d > 0$. This completes the proof of Lemma 3.4. □

Now choose $\eta \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leqslant \eta \leqslant 1$ and

$$\eta(z) = \begin{cases} 1 & \text{if } z \in \{z \in \mathbb{R}^N : \beta \leqslant |z| \leqslant 2\beta\},\\ 0 & \text{if } z \in \mathbb{R}^N \setminus \{z \in \mathbb{R}^N : \frac{1}{2}\beta \leqslant |z| \leqslant 3\beta\}.\end{cases}$$

By setting $\eta_n(z) = \eta(\varepsilon_n z - y_n)v_n$, clearly, η_n is bounded in $H^1(\mathbb{R}^N)$. Thus, from Lemma 3.4, we have

$$\lim_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{\mathbb{R}^N}|\eta_n|^2\,dx = 0.$$

This fact together with Lions' concentration-compactness lemma gives $\eta_n \to 0$ in $L^q(\mathbb{R}^N)$, $q \in (2, 2^*)$. So, we obtain

$$\lim_{n\to\infty}\int_{\{x\in\mathbb{R}^N:\beta\leqslant|\varepsilon_n x - y_n|\leqslant 2\beta\}}|v_n|^q\,dx \leqslant \lim_{n\to\infty}\int_{\mathbb{R}^N}|\eta_n|^q\,dx = 0. \tag{3.19}$$

Set $v_{n,1}(\cdot) = \varphi_{\varepsilon_n}(\cdot - \frac{y_n}{\varepsilon_n})v_n(\cdot)$ and $v_{n,2} = v_n - v_{n,1}$. Then let us prove the following lemma.

Lemma 3.5. *It holds*

$$\Gamma_{\varepsilon_n}(v_n) \geqslant \Gamma_{\varepsilon_n}(v_{n,1}) + \Gamma_{\varepsilon_n}(v_{n,2}) + o_n(1).$$

Proof. Since $\operatorname{supp}(v_{n,1}) \subset \mathcal{O}_\varepsilon$, we have

$$Q_{\varepsilon_n}(v_{n,1}) = 0 \quad \text{and} \quad Q_{\varepsilon_n}(v_{n,2}) = Q_{\varepsilon_n}(v_n).$$

Therefore, by Lemma 2.2 (iv), (v) and $G^{-1}(0) = 0$, for large n, we deduce that

$$\begin{aligned}\Gamma_{\varepsilon_n}(v_{n,1}) + \Gamma_{\varepsilon_n}(v_{n,2}) &= \Gamma_{\varepsilon_n}(v_n) + \int_{\mathbb{R}^N}\varphi_{\varepsilon_n}\Big(x-\frac{y_n}{\varepsilon_n}\Big)\Big[\varphi_{\varepsilon_n}\Big(x-\frac{y_n}{\varepsilon_n}\Big)-1\Big]|\nabla v_n|^2\,dx\\ &\quad + \frac{\kappa_n}{2}\int_{\mathbb{R}^N}V_{\varepsilon_n}(x)\big[|G^{-1}(v_{n,1})|^2 + |G^{-1}(v_{n,2})|^2 - |G^{-1}(v_n)|^2\big]\,dx\\ &\quad + \frac{1}{p}\int_{\mathbb{R}^N}K_{\varepsilon_n}(x)\big[|G^{-1}(v_n)|^p - |G^{-1}(v_{n,1})|^p - |G^{-1}(v_{n,2})|^p\big]\,dx + o_n(1)\\ &\leqslant \Gamma_{\varepsilon_n}(v_n) + C\int_{\{x\in\mathbb{R}^N:\beta\leqslant|\varepsilon_n x-y_n|\leqslant 2\beta\}}|v_n|^{\frac{p}{2}}\,dx + o_n(1).\end{aligned}\tag{3.20}$$

By (3.19) and (3.20), we get the result. □

In what follows, we use the following notation:

$$\mathcal{A}_0 = \{x \in \mathbb{R}^N : |\varepsilon_n x - y_n| \leqslant \beta\},$$
$$\mathcal{A}_1 = \{x \in \mathbb{R}^N : |\varepsilon_n x - y_n| \geqslant 2\beta\},$$
$$\mathcal{A}_2 = \{x \in \mathbb{R}^N : \beta \leqslant |\varepsilon_n x - y_n| \leqslant 2\beta\}.$$

Lemma 3.6. *It holds* $\Gamma_{\varepsilon_n}(v_{n,2}) > 0$.

Proof. From (3.11), we get

$$\int_{\mathcal{A}_1} (|\nabla v_{n,2}|^2 + v_{n,2}^2)\,dx = \int_{\mathcal{A}_1} (|\nabla v_n|^2 + v_n^2)\,dx \leqslant d^2.$$

Similarly, we get

$$\int_{\mathcal{A}_2} (|\nabla v_{n,2}|^2 + v_{n,2}^2)\,dx \leqslant C \int_{\mathcal{A}_2} (|\nabla v_n|^2 + v_n^2)\,dx \leqslant Cd^2 + o_n(1).$$

For n large enough, we have $\|v_{n,2}\|_{H^1(\mathbb{R}^N)} \leqslant Cd$ for small $d > 0$. On the other hand, by Lemma 2.2 (i) and (ii), we get $|G^{-1}(v_{n,2})|^p \leqslant C|v_{n,2}|^{2^*}$. Hence,

$$\begin{aligned}\Gamma_{\varepsilon_n}(v_{n,2}) &\geqslant \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_{n,2}|^2\,dx - C\int_{\mathbb{R}^N} |v_{n,2}|^{2^*}\,dx\\ &\geqslant \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_{n,2}|^2\,dx - Cd^{\frac{2^*-2}{2}}\int_{\mathbb{R}^N} |\nabla v_{n,2}|^2\,dx\\ &\geqslant \frac{1}{4}\int_{\mathbb{R}^N} |\nabla v_{n,2}|^2\,dx > 0. \end{aligned} \tag{3.21}$$

This concludes the proof of Lemma 3.6. □

Denote the usual norm in $D_0^{1,2}(B_{R/\varepsilon}(0))$ as follows:

$$\|v\|_{\varepsilon,R}^* = \Big(\int_{B_{R/\varepsilon}(0)} |\nabla v|^2\,dx\Big)^{\frac{1}{2}}.$$

Lemma 3.7. *For small $d > 0$, there exist a sequence $\{z_n\} \subset \mathbb{R}^N$ and $y_0 \in \mathcal{M}$ with $\varepsilon_n \to 0$ and $R_n \to +\infty$ satisfying, up to a subsequence,*

$$\lim_{n\to\infty} |\varepsilon_n z_n - y_0| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|v_n(\cdot) - \varphi_{\varepsilon_n}(\cdot - z_n)U(\cdot - z_n)\|_{\varepsilon_n,R_n}^* = 0.$$

Proof. Let

$$w_n(\cdot) := v_{n,1}\Big(\cdot + \frac{y_n}{\varepsilon_n}\Big).$$

Then $\{w_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Thus, up to a subsequence if necessary, we may assume $w_n \rightharpoonup w$ in $H^1(\mathbb{R}^N)$, $w_n \to w$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$, $q \in [2, 2^*)$, and $w_n \to w$ a.e. in $\mathbb{R}^N$. From (3.11), for a given $R > 0$, as n is large enough we get

$$d^2 \geqslant \int_{\mathcal{A}_0} \Big|v_{n,1} - \varphi_{\varepsilon_n}\Big(x - \frac{y_n}{\varepsilon_n}\Big)U\Big(x - \frac{y_n}{\varepsilon_n}\Big)\Big|^2\,dx \geqslant \int_{B_R(0)} |w_n - \varphi_{\varepsilon_n}U|^2\,dx.$$

Thus, we have

$$\int_{B_R(0)} w^2\,dx = \lim_{n\to\infty}\int_{B_R(0)} w_n^2\,dx \geqslant C - d^2,$$

which yields $w \neq 0$.

Let $\phi \in C_0^\infty(\mathbb{R}^N)$. Note that

$$w_n(x) = v_{n,1}\Big(x + \frac{y_n}{\varepsilon_n}\Big) = v_n\Big(x + \frac{y_n}{\varepsilon_n}\Big)$$

for $x \in \operatorname{supp}(\phi)$ and large n. Moreover,

$$\operatorname{supp}(w_n(x)) \subset \{x \in \mathbb{R}^N : |\varepsilon_n x| \leqslant 2\beta\} \subset \mathcal{O}.$$

Thus, from

$$\Big\langle \Gamma'_{\varepsilon_n}(v_n), \phi\Big(\cdot - \frac{y_n}{\varepsilon_n}\Big)\Big\rangle = o_n(1)\|\phi\|_{\varepsilon_n,R_n}$$

and analogously to the proof of (3.14) and (3.15), w is a positive solution of the following equation:

$$-\Delta w = K(y_0)\frac{|G^{-1}(w)|^{p-2}G^{-1}(w)}{g(G^{-1}(w))}, \quad x \in \mathbb{R}^N. \tag{3.22}$$

Claim:

$$\lim_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B_1(z)} |w_n - w|^2\,dx = 0. \tag{3.23}$$

Indeed, if (3.23) does not occur, then there exists a sequence $\{z_n\} \subset \mathbb{R}^N$ with $|z_n| \to +\infty$ such that

$$\lim_{n\to\infty}\int_{B_1(z_n)} |w_n - w|^2\,dx > 0.$$

Thus, we have

$$\lim_{n\to\infty}\int_{B_1(z_n)} |w|^2\,dx = 0, \quad \lim_{n\to\infty}\int_{B_1(z_n)} |w_n|^2\,dx > 0.$$

We have $|\varepsilon_n z_n| \leqslant \frac{1}{2}\beta$. In fact, if $|\varepsilon_n z_n| > \frac{1}{2}\beta$, by Lemma 3.4, we have

$$0 < \lim_{n\to\infty}\int_{B_1(z_n)} |w_n|^2\,dx \leqslant \lim_{n\to\infty}\sup_{z\in\{z\in\mathbb{R}^N:\frac{1}{2}\beta\leqslant|\varepsilon_n z - y_n|\leqslant 3\beta\}}\int_{B_1(z)} |v_n|^2\,dx = 0,$$

which is impossible. Thus, up to a subsequence, we may assume $\varepsilon_n z_n \to z_0 \in \{z \in \mathbb{R}^N : |z| \leqslant \frac{1}{2}\beta\}$. Suppose $v_{n,1}(\cdot + z_n + \frac{y_n}{\varepsilon_n}) \rightharpoonup v_1(\cdot)$ in $H^1(\mathbb{R}^N)$. As in the proof of (3.22), we have

$$-\Delta v_1 = K(y_0 + z_0)\frac{|G^{-1}(v_1)|^{p-2}G^{-1}(v_1)}{g(G^{-1}(v_1))}, \quad x \in \mathbb{R}^N. \tag{3.24}$$

By the maximum principle, $v_1 > 0$.

Thus, for large R, we obtain

$$\begin{aligned}\frac{1}{2}NC_m &\leqslant \int_{B_R(0)} \Big|\nabla v_{n,1}\Big(x + z_n + \frac{y_n}{\varepsilon_n}\Big)\Big|^2\,dx\\ &= \int_{B_R(z_n+\frac{y_n}{\varepsilon_n})} |\nabla v_{n,1}(x)|^2\,dx\\ &\leqslant C\varepsilon_n^2 + C\int_{B_R(z_n+\frac{y_n}{\varepsilon_n})} |\nabla v_n|^2\,dx\\ &\leqslant C\varepsilon_n^2 + Cd + C\int_{B_R(0)} (1 + |x + z_n|)^{2-2N}\,dx.\end{aligned}$$

We get a contradiction for large n and small d since $|z_n| \to +\infty$.

Therefore, (3.23) holds and the claim is proved.

Again by Lions' concentration-compactness lemma, we have $w_n \to w$ in $L^q(\mathbb{R}^N)$, $q \in (2, 2^*)$. As a consequence

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K_{\varepsilon_n}\Big(x + \frac{y_n}{\varepsilon_n}\Big)|G^{-1}(w_n)|^p\,dx = \int_{\mathbb{R}^N} K(y_0)|G^{-1}(w)|^p\,dx. \tag{3.25}$$

By Fatou's lemma, Lemmas 3.5 and 3.6 and by (3.21), up to a subsequence, we have

$$\begin{aligned} C_m &\geqslant \lim_{n\to\infty} \Gamma_{\varepsilon_n}(v_n) \\ &\geqslant \lim_{n\to\infty} \Gamma_{\varepsilon_n}(v_{n,1}) + \lim_{n\to\infty} \Gamma_{\varepsilon_n}(v_{n,2}) \\ &\geqslant \lim_{n\to\infty} \Gamma_{\varepsilon_n}(v_{n,1}) \\ &= \lim_{n\to\infty} \frac{1}{2} \int_{\mathbb{R}^N} \Big(|\nabla w_n|^2 + \kappa_n V_{\varepsilon_n}\Big(x + \frac{y_n}{\varepsilon_n}\Big) |G^{-1}(w_n)|^2 \Big)\, dx - \frac{1}{p} \int_{\mathbb{R}^N} K_{\varepsilon_n}\Big(x + \frac{y_n}{\varepsilon_n}\Big) |G^{-1}(w_n)|^p \, dx \\ &\geqslant \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 \, dx - \frac{1}{p} \int_{\mathbb{R}^N} K(y_0) |G^{-1}(w)|^p \, dx \\ &\geqslant C_{K(y_0)} \geqslant C_m. \end{aligned} \tag{3.26}$$

Hence, we get $\lim_{n\to\infty} \Gamma_{\varepsilon_n}(v_{n,1}) = C_m$. Moreover, we get $K(y_0) = m$ and we see that w is a ground state to (2.2). Thus, there exists some $z \in \mathbb{R}^N$ such that $w(\cdot + z) = U(\cdot)$. By (3.25) and (3.26), we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla w_n|^2 \, dx = \int_{\mathbb{R}^N} |\nabla w|^2 \, dx.$$

Let $z_n = z + \frac{y_n}{\varepsilon_n}$. Then

$$\|v_{n,1}(\cdot) - \varphi_{\varepsilon_n}(\cdot - z_n) U(\cdot - z_n)\|^*_{\varepsilon_n, R_n} \to 0.$$

Finally, by (3.21) and (3.26), we have

$$0 = \lim_{n\to\infty} \Gamma_{\varepsilon_n}(v_{n,2}) \geqslant \frac{1}{4} \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_{n,2}|^2 \, dx,$$

which yields $\lim_{n\to\infty} \|v_{n,2}\|^*_{\varepsilon_n, R_n} = 0$, and the lemma is proved. □

Let $d \in (0, d_0)$ such that Lemmas 3.4–3.7 hold and define

$$\widetilde{X}^d_\varepsilon := \{u \in E^R_\varepsilon : \inf_{v \in X_\varepsilon} \|u - v\|^*_{\varepsilon, R} < d\}, \quad d > 0.$$

Lemma 3.8. *For any $d \in (0, d_0)$, there exist positive constants δ_d, R_d and ε_d such that*

$$\|\Gamma'_\varepsilon(v)\|_{(E^R_\varepsilon)'} \geqslant \delta_d$$

for any $v \in E^R_\varepsilon \cap \Gamma^{D_\varepsilon}_\varepsilon \cap (X^{d_0}_\varepsilon \setminus \widetilde{X}^d_\varepsilon)$, $R \geqslant R_d$ and $\varepsilon \in (0, \varepsilon_d)$.

Proof. By contradiction, we assume that for some $d \in (0, d_0)$ there exist $\varepsilon_n < \frac{1}{n}$, $R_n > n$ and

$$v_n \in E^{R_n}_{\varepsilon_n} \cap \Gamma^{D_\varepsilon}_{\varepsilon_n} \cap (X^{d_0}_{\varepsilon_n} \setminus \widetilde{X}^d_{\varepsilon_n})$$

such that

$$\|\Gamma'_{\varepsilon_n}(v_n)\|_{(E^{R_n}_{\varepsilon_n})'} < \frac{1}{n}.$$

By Lemma 3.7, there exist a sequence $\{z_n\} \subset \mathbb{R}^N$ and $y_0 \in \mathcal{M}$ satisfying

$$\lim_{n\to\infty} |\varepsilon_n z_n - y_0| = 0, \qquad \lim_{n\to\infty} \|v_n - \varphi_{\varepsilon_n}(\cdot - z_n) U(\cdot - z_n)\|^*_{\varepsilon_n, R_n} = 0$$

up to a subsequence. Thus, for large n, $\varepsilon_n z_n \in \mathcal{M}^\beta$, $\varphi_{\varepsilon_n}(\cdot - z_n) U(\cdot - z_n) \in X_{\varepsilon_n}$ and $v_n \in \widetilde{X}^d_{\varepsilon_n}$, which contradicts the fact that $v_n \in X^{d_0}_{\varepsilon_n} \setminus \widetilde{X}^d_{\varepsilon_n}$. □

Lemma 3.9. *For any given $\delta > 0$, there exist small positive constants ε_1 and $d \leqslant d_0$ such that $\Gamma_\varepsilon(v) > C_m - \delta$ for any $v \in X^d_\varepsilon$ and $\varepsilon \in (0, \varepsilon_1)$.*

Proof. For $v \in X^d_\varepsilon$, there exists $y \in \mathcal{M}^\beta$ such that

$$U^y_\varepsilon(x) := \varphi_\varepsilon\Big(x - \frac{y}{\varepsilon}\Big) U\Big(x - \frac{y}{\varepsilon}\Big) \in X_\varepsilon \quad \text{and} \quad \|v - U^y_\varepsilon(x)\|_\varepsilon \leqslant d.$$

Thus, we get

$$\Gamma_\varepsilon(U_\varepsilon^y) - C_m \geqslant \frac{1}{2}\int_{\mathbb{R}^N} [|\nabla(\varphi_\varepsilon U)|^2 - |\nabla U|^2]\,dx + \frac{m}{p}\int_{\mathbb{R}^N} (|G^{-1}(U)|^p - |G^{-1}(\varphi U)|^p)\,dx.$$

Similarly to the proof of Lemma 3.1, for small $\varepsilon > 0$, we have

$$\Gamma_\varepsilon(U_\varepsilon^y) \geqslant C_m - \frac{\delta}{2}. \tag{3.27}$$

On the other hand, for $v \in X_\varepsilon^d$, by choosing d small enough, we have

$$\Gamma_\varepsilon(v) - \Gamma_\varepsilon(U_\varepsilon^y) \geqslant -\frac{\delta}{2}. \tag{3.28}$$

The result follows from (3.27) and (3.28). □

Lemma 3.10. *For sufficiently small $\varepsilon > 0$ and large $R > 0$, there exists a sequence $\{v_{\varepsilon,n}^R\} \subset E_\varepsilon^R \cap \Gamma_\varepsilon^{D_\varepsilon} \cap X_\varepsilon^{d_0}$ such that*

$$\|\Gamma_\varepsilon'(v_{\varepsilon,n}^R)\|_{(E_\varepsilon^R)'} \to 0 \quad \text{as } n \to \infty.$$

Proof. The proof is similar to [18, 19]. For the reader's convenience, let us give a detailed proof. By contradiction, for small $\varepsilon > 0$ and large $R > 0$, there exists $C(\varepsilon, R) > 0$ such that

$$\|\Gamma_\varepsilon'(v)\|_{(E_\varepsilon^R)'} \geqslant C(\varepsilon, R), \quad v \in E_\varepsilon^R \cap \Gamma_\varepsilon^{D_\varepsilon} \cap X_\varepsilon^{d_0}.$$

On the other hand, by Lemma 3.8, there exists $\delta > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ and $R > R_0$ such that

$$\|\Gamma_\varepsilon'(v)\|_{(E_\varepsilon^R)'} \geqslant \delta, \quad v \in E_\varepsilon^R \cap \Gamma_\varepsilon^{D_\varepsilon} \cap (X_\varepsilon^{d_0} \setminus \widetilde{X}_\varepsilon^{d_1}).$$

Thus, there exists a pseudo-gradient vector field Υ_ε^R in a neighborhood $N_\varepsilon^R \subset E_\varepsilon^R$ of $E_\varepsilon^R \cap \Gamma_\varepsilon^{D_\varepsilon} \cap X_\varepsilon^{d_0}$. Let $\widetilde{N}_\varepsilon^R \subset N_\varepsilon^R$ such that

$$\|\Gamma_\varepsilon'(v)\|_{(E_\varepsilon^R)'} \geqslant \frac{1}{2}C(\varepsilon, R), \quad v \in \widetilde{N}_\varepsilon^R.$$

We choose two positive Lipschitz continuous functions ζ_ε^R and ξ satisfying

$$\zeta_\varepsilon^R(v) = \begin{cases} 1 & \text{if } v \in E_\varepsilon^R \cap \Gamma_\varepsilon^{D_\varepsilon} \cap X_\varepsilon^{d_0}, \\ 0 & \text{if } v \in E_\varepsilon^R \setminus \widetilde{N}_\varepsilon^R,\ 0 \leqslant \zeta_\varepsilon^R \leqslant 1, \end{cases}$$

and $\xi \leqslant 1$,

$$\xi(a) = \begin{cases} 1 & \text{if } |a - C_m| \leqslant \frac{1}{2}C_m, \\ 0 & \text{if } |a - C_m| \geqslant C_m. \end{cases}$$

Define

$$\Psi_\varepsilon^R = \begin{cases} -\zeta_\varepsilon^R(v)\xi(\Gamma_\varepsilon(v))\Upsilon_\varepsilon^R & \text{if } v \in N_\varepsilon^R, \\ 0 & \text{if } v \notin E_\varepsilon \setminus N_\varepsilon^R. \end{cases}$$

Then the initial value problem

$$\begin{aligned} &\frac{d}{dt}F_\varepsilon^R(v, t) = \Psi_\varepsilon^R(F_\varepsilon^R(v, t)), \\ &F_\varepsilon^R(v, 0) = v, \end{aligned}$$

yields a unique global solution $F_\varepsilon^R : E_\varepsilon \times [0, +\infty) \to E_\varepsilon^R$. For the properties of F_ε^R, we refer to, e.g., [19, 28]. Let $\eta_\varepsilon(s) = W_{\varepsilon,st_0} = \varphi_\varepsilon U_{st_0}$, $s \in [0, 1]$, as before. Then, for small $d_1 > 0$, there exists some $\mu > 0$ such that if $|st_0 - 1| \leqslant \mu$, then

$$\|\eta_\varepsilon(s) - \varphi_\varepsilon U\| = \|\varphi_\varepsilon(U_{st_0} - U)\| \leqslant C\|U_{st_0} - U\| \leqslant d_1,$$

which implies $\eta_\varepsilon(s) \in X_\varepsilon^{d_1} \subset \widetilde{X}_\varepsilon^{d_1}$ since $0 \in \mathcal{M}$. On the other hand, if $|st_0 - 1| \geqslant \mu$, since $t = 1$ is the unique maximum point of $L_m(U_t)$ and $\max_{t \geqslant 0} L_m(U_t) = L_m(U_1) = C_m$, there exists $\rho > 0$ such that $L_m(U_{st_0}) < C_m - 2\rho$

for $|st_0 - 1| \geqslant \mu$. From Lemma 3.1, there exists $\varepsilon_1 > 0$ such that

$$\max_{s\in(0,1]} |\Gamma_\varepsilon(W_{\varepsilon,st_0}) - L_m(U_{st_0})| < \rho, \quad \varepsilon \in (0, \varepsilon_1).$$

So, for $|st_0 - 1| \geqslant \mu$, we have

$$\begin{aligned}\Gamma_\varepsilon(\eta_\varepsilon(s)) &= \Gamma_\varepsilon(W_{\varepsilon,st_0}) \\ &\leqslant |\Gamma_\varepsilon(W_{\varepsilon,st_0}) - L_m(U_{st_0})| + L_m(U_{st_0}) \\ &< C_m - \rho, \quad \varepsilon \in (0, \varepsilon_1).\end{aligned} \tag{3.29}$$

Define $\eta_\varepsilon^R(s,t) := F_\varepsilon^R(\eta_\varepsilon(s), t)$, $(s,t) \in [0,1] \times [0,+\infty)$. Since

$$\Gamma_\varepsilon(\eta_\varepsilon(0)), \Gamma_\varepsilon(\eta_\varepsilon(1)) \notin (0, 2C_m),$$

we get $\eta_\varepsilon^R(s,t) \in \Phi_\varepsilon$ for any $t > 0$.

If $|st_0 - 1| \geqslant \mu$, by (3.29), we have

$$\Gamma_\varepsilon(\eta_\varepsilon^R(s,t)) \leqslant \Gamma_\varepsilon(\eta_\varepsilon(s)) < C_m - \rho,$$

which is impossible by Lemma 3.2.

If $|st_0 - 1| \leqslant \mu$, we get $\eta_\varepsilon(s) \in \tilde{X}_\varepsilon^{d_1}$. In this case one of the following alternatives holds:

(a) $\eta_\varepsilon^R(s,t) \in X_\varepsilon^{d_0}$ for all $t > 0$.

(b) There exists some $t_s > 0$ such that $\eta_\varepsilon^R(s,t_s) \notin X_\varepsilon^{d_0}$.

If (a) holds, we have

$$\begin{aligned}\Gamma_\varepsilon(\eta_\varepsilon^R(s,t)) &= \Gamma_\varepsilon(\eta_\varepsilon(s)) + \int_0^t \frac{d}{d\tau}\Gamma_\varepsilon(\eta_\varepsilon^R(s,\tau))\,d\tau \\ &\leqslant D_\varepsilon - \min\{\delta^2, C(\varepsilon,R)^2\}t.\end{aligned}$$

Thus, $\lim_{t\to+\infty} \Gamma_\varepsilon(\eta_\varepsilon^R(s,t)) = -\infty$, which contradicts Lemma 3.9. So, we have that (b) holds. For any fixed s with $|st_0 - 1| \leqslant \mu$, we find $t_s^1, t_s^2 > 0$ such that $\eta_\varepsilon^R(s,t) \in X_\varepsilon^{d_0} \setminus \tilde{X}_\varepsilon^{d_1}$ for $t \in [t_s^1, t_s^2] \subset (0, t_s)$ for $|t_s^1 - t_s^2| > \sigma$ for some $\sigma > 0$ dependent of d_0 and d_1. Thus, by Remark 3.3, we get

$$\begin{aligned}\Gamma_\varepsilon(\eta_\varepsilon^R(s,t_{s_0})) &\leqslant \Gamma_\varepsilon(\eta_\varepsilon(s)) + \int_{t_s^1}^{t_s^2} \frac{d}{dt}\Gamma_\varepsilon(\eta_\varepsilon^R(s,\tau))\,d\tau \\ &\leqslant D_\varepsilon - \delta^2(t_\varepsilon^2 - t_\varepsilon^1) \\ &< C_m - \frac{1}{2}\delta^2\sigma, \quad t \in [t_s^1, t_s^2], \text{ and if } |st_0 - 1| \leqslant \mu.\end{aligned}$$

Therefore, since $[0,1]$ is compact, by the covering theorem, for all $s \in [0,1]$ with $|st_0 - 1| \leqslant \mu$, we can find t_ε^R such that

$$\Gamma_\varepsilon(\eta_\varepsilon^R(s,t_\varepsilon^R)) < C_m - \frac{1}{2}\delta^2\sigma,$$

which is a contradiction to Lemma 3.2 since $\eta_\varepsilon^R(s,t_\varepsilon^R) \in \Phi_\varepsilon$. □

Lemma 3.11. *For sufficiently small $\varepsilon > 0$, there exists a critical point $v_\varepsilon \in X_\varepsilon^{d_0} \cap \Gamma_\varepsilon^{D_\varepsilon}$ of Γ_ε.*

Proof. By Lemma 3.10, there exist ε_0 and $R_0 > 0$ such that there exists a sequence $\{v_{\varepsilon,n}^R\} \subset E_\varepsilon^R \cap \Gamma_\varepsilon^{D_\varepsilon} \cap X_\varepsilon^{d_0}$ such that

$$\|\Gamma'_\varepsilon(v_{\varepsilon,n}^R)\|_{(E_\varepsilon^R)'} \to 0 \quad \text{as } n \to \infty$$

for $\varepsilon \in (0, \varepsilon_0)$ and $R \in (R_0, +\infty)$. Clearly, $\{v_{\varepsilon,n}^R\}$ is bounded in $H_0^1(B_{R/\varepsilon}(0))$ since $v_{\varepsilon,n}^R \in X_\varepsilon^{d_0}$. Up to a subsequence if necessary, we may assume

$$\begin{aligned}&v_{\varepsilon,n}^R \rightharpoonup v_\varepsilon^R \quad \text{in } H_0^1(B_{R/\varepsilon}(0)), \\ &v_{\varepsilon,n}^R \to v_\varepsilon^R \quad \text{in } L^p(B_{R/\varepsilon}(0)),\ p \in [1, 2^*), \\ &v_{\varepsilon,n}^R \to v_\varepsilon^R \quad \text{a.e. in } \mathbb{R}^N.\end{aligned}$$

Thus v_ε^R is a solution of

$$-\Delta v = \frac{K_\varepsilon(x)|G^{-1}(v)|^{p-2}G^{-1}(v) - \kappa V_\varepsilon(x)G^{-1}(v)}{g(G^{-1}(v))} - p\Big(\int_{B_{R/\varepsilon(0)}} \chi_\varepsilon |v|^{\frac{p}{2}}\,dx - 1\Big)_+ \chi_\varepsilon |v|^{\frac{p}{2}-2}v, \quad x \in B_{R/\varepsilon}(0). \tag{3.30}$$

From (3.30) we have $v_{\varepsilon,n}^R \to v_\varepsilon^R$ in $H_0^1(B_{R/\varepsilon}(0))$ and $v_\varepsilon^R \in X_\varepsilon^{d_0} \cap \Gamma_\varepsilon^{D_\varepsilon}$. By the maximum principle, $v_\varepsilon^R > 0$. Note that any positive solution of (3.30) satisfies

$$-\Delta v \leqslant C v^{\frac{p}{2}-1}, \quad x \in B_{R/\varepsilon}(0),$$

where $C > 0$ is independent of ε and R. In particular, $-\Delta v_\varepsilon^R \leqslant C(v_\varepsilon^R)^{\frac{p}{2}-1}$, $x \in B_{R/\varepsilon}(0)$. By applying the standard Moser iteration (see [17]), $\{v_\varepsilon^R\}$ is bounded in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ uniformly on $R \geqslant R_0$ and $\varepsilon \in (0, \varepsilon_0)$ for any $q < \infty$. Moreover, for any $y \in \mathbb{R}^N$, we have

$$\|v_\varepsilon^R\|_{L^q(B_3(y))} \leqslant C\|v_\varepsilon^R\|_{L^{\frac{p}{2}}(B_4(y))}. \tag{3.31}$$

By [17, Theorem 8.17] and (3.31), we have

$$\sup_{B_1(y)} v_\varepsilon^R \leqslant C(\|v_\varepsilon^R\|_{L^{\frac{p}{2}}(B_2(y))} + \|(v_\varepsilon^R)^{\frac{p}{2}-1}\|_{L^q(B_3(y))}) \leqslant C\|v_\varepsilon^R\|_{L^{\frac{p}{2}}(B_4(y))}.$$

In particular, this implies that v_ε^R stays bounded in $L^\infty(\mathbb{R}^N)$. Since $\|v_\varepsilon^R\|_\varepsilon$ and $\{\Gamma_\varepsilon(v_\varepsilon^R)\}$ are bounded, we get that $\{Q_\varepsilon(v_\varepsilon^R)\}$ is uniformly bounded on $R \geqslant R_0$ and $\varepsilon \in (0, \varepsilon_0)$. So, we have

$$\int_{\mathbb{R}^N \setminus B_{\frac{R_0}{\varepsilon}}(0)} |v_\varepsilon^R|^{\frac{p}{2}}\,dx \leqslant \int_{\mathbb{R}^N \setminus \mathcal{O}_\varepsilon} |v_\varepsilon^R|^{\frac{p}{2}}\,dx = \varepsilon^\tau \int_{\mathbb{R}^N} \chi_\varepsilon |v_\varepsilon^R|^{\frac{p}{2}}\,dx \leqslant \varepsilon^\tau C$$

for any $R \geqslant R_0$ and $\varepsilon \in (0, \varepsilon_0)$. Thus, for $|x| \geqslant \frac{R_0}{\varepsilon} + 4$ and $R \geqslant R_0$, we have $(v_\varepsilon^R)^{\frac{p}{2}-1} \leqslant \varepsilon^\tau C v_\varepsilon^R$. By the comparison principle, similarly to the proof of [9, Proposition 3], we get

$$\lim_{A \to +\infty} \int_{\mathbb{R}^N \setminus B_A(0)} [|\nabla v_\varepsilon^R|^2 + \kappa V_\varepsilon(x)|G^{-1}(v_\varepsilon^R)|^2]\,dx = 0 \tag{3.32}$$

uniformly on $R \geqslant R_0$. Let $v_k = v_\varepsilon^{R_k}$ and $R_k \to +\infty$ as $k \to \infty$. Then $\{v_k\}$ is bounded in $H^1(\mathbb{R}^N)$, and we may assume $v_k \rightharpoonup v_\varepsilon$ in $H^1(\mathbb{R}^N)$ and $v_k \to v_\varepsilon$ a.e. in $\mathbb{R}^N$. Since v_k satisfies (3.30) and by using (3.32), we get $\|v_k - v_\varepsilon\|_\varepsilon \to 0$ as $k \to \infty$. Thus, $v_\varepsilon \in X_\varepsilon^{d_0} \cap \Gamma_\varepsilon^{D_\varepsilon}$ and $\Gamma'_\varepsilon(v_\varepsilon) = 0$. □

We are now in the position to prove Theorem 1.1 and Theorem 1.2.

By Lemma 3.11, for small $\varepsilon > 0$, there exists a positive solution v_ε to the following equation:

$$-\Delta v + \kappa V_\varepsilon(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} = K_\varepsilon(x)\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))} - p\Big(\int_{\mathbb{R}^N} \chi_\varepsilon v^{\frac{p}{2}}\,dx - 1\Big)_+ \chi_\varepsilon v^{\frac{p}{2}-1}. \tag{3.33}$$

Since $v_\varepsilon \in X_\varepsilon^{d_0}$, by the Moser iteration [17], $\{v_\varepsilon\}$ is uniformly bounded in $L^\infty(\mathbb{R}^N)$ for small $\varepsilon > 0$.

By Lemma 3.4, for small $d > 0$, there exist a sequence $\{z_\varepsilon\} \subset \mathbb{R}^N$ and $y_0 \in \mathcal{M}$ satisfying

$$\lim_{\varepsilon \to 0}|\varepsilon z_\varepsilon - y_0| = 0, \quad \lim_{\varepsilon \to 0}\|v_\varepsilon(\cdot) - \varphi_\varepsilon(\cdot - z_\varepsilon)U(\cdot - z_\varepsilon)\|_\varepsilon^* = 0$$

up to a subsequence. Then

$$\lim_{n \to \infty}\|v_\varepsilon(\cdot + z_\varepsilon) - U(\cdot)\|^* = 0. \tag{3.34}$$

From (3.33), there exist some $C_1 > 0$ and $C_2 > 0$ such that

$$-\Delta v_\varepsilon + C_1 \kappa v_\varepsilon \leqslant C_2 v_\varepsilon^{2^*-1}.$$

Thus, for given $\sigma > 0$, there exist $R > 0$ and $\varepsilon_0 > 0$ such that

$$\sup_{\varepsilon \in (0,\varepsilon_0)} \kappa \int_{\mathbb{R}^N \setminus B_R(0)} v_\varepsilon^2(x + z_\varepsilon)\,dx \leqslant \sigma. \tag{3.35}$$

Setting $w_\varepsilon(\cdot) = v_\varepsilon(\cdot + z_\varepsilon)$, we have $-\Delta w_\varepsilon \leqslant C w_\varepsilon$. Hence, from [17, Theorem 8.17], there exists a constant $C_0 = C_0(N, C)$ such that

$$\sup_{B_1(y)} w_\varepsilon \leqslant C_0 \|w_\varepsilon\|_{L^2(B_2(y))} \quad \text{for all } y \in \mathbb{R}^N.$$

In view of (3.35), we conclude that $w_\varepsilon(x) \to 0$ as $|x| \to \infty$. Let y_ε be a maximum point of $w_\varepsilon(x)$. Then $\{y_\varepsilon\}$ is bounded. Otherwise, $\|w_\varepsilon(y_\varepsilon)\|_{L^\infty} \to 0$ as $\varepsilon \to 0$, which would contradict (3.34).

Now, fix $\varepsilon > 0$ sufficiently small. By Lemma 2.2 (i), we choose $R_0 > 0$ such that

$$\frac{G^{-1}(v)}{g(G^{-1}(v))} \geqslant \frac{1}{2} v \quad \text{for } |x| \geqslant R_0$$

Thus, from (3.33), we have

$$-\Delta v + \frac{1}{2}\kappa V_0 v \leqslant K_0 v^{p-1} \quad \text{for } |x| \geqslant R_0.$$

Let

$$\phi(y) = \kappa^{\frac{1}{2-p}} v\Big(\frac{y}{\sqrt{\kappa}}\Big), \quad |y| \geqslant \sqrt{\kappa} R_0.$$

Then

$$-\Delta\phi + \frac{1}{2} V_0 \phi \leqslant K_0 \phi^{p-1}.$$

Since

$$\frac{1}{\sqrt{\kappa}}|y| \to +\infty,$$

we get $\phi(y) \to 0$. Thus, there exists $R_1 > 0$ such that

$$K_0 |w|^{p-2} \leqslant \frac{1}{4} V_0.$$

Thus, for $|y| \geqslant \sqrt{\kappa} R_1$, we have

$$-\Delta\phi + \frac{1}{4} V_0 \phi \leqslant 0.$$

Now define the function

$$\psi(x) = M \exp(-\xi |y|),$$

where ξ and M are such that $4\xi^2 < V_0$ and for all $|y| = \sqrt{\kappa} R_1$,

$$M \exp(-\xi \sqrt{\kappa} R_1) > \phi(y)$$

It is straightforward to check that for all $x \neq 0$,

$$\Delta\psi \leqslant \xi^2 \psi \leqslant \frac{1}{4} V_0 \psi.$$

Thus,

$$-\Delta(\psi - \phi) + \frac{1}{4} V_0 (\psi - \phi) \geqslant 0 \quad \text{for } |y| \geqslant \sqrt{\kappa} R_1.$$

By the maximum principle, we have that

$$\phi(y) \leqslant M \exp(-\xi |y|) \quad \text{for } |y| \geqslant \sqrt{\kappa} R_1,$$

which yields that

$$w_\varepsilon(x) \leqslant M \kappa^{\frac{1}{p-2}} \exp(-\xi \sqrt{\kappa} |x|) \quad \text{for } |x| \geqslant R_1. \tag{3.36}$$

and in turn

$$w_\varepsilon(x) \leqslant C \exp(-\xi \sqrt{\kappa} |x|) \quad \text{for } x \in \mathbb{R}^N.$$

Set $x_\varepsilon = y_\varepsilon + z_\varepsilon$. Then x_ε is a maximum point of $v_\varepsilon(x)$ and

$$v_\varepsilon(x) = w_\varepsilon(x - z_\varepsilon) \leqslant C \exp(-\xi \sqrt{\kappa} |x - x_\varepsilon|) \quad \text{for } x \in \mathbb{R}^N.$$

As a consequence, we have that

$$\begin{aligned}\int_{\mathbb{R}^N} \chi_\varepsilon v_\varepsilon^{\frac{p}{2}}\,dx &= \varepsilon^{-\tau} \int_{\mathbb{R}^N\setminus\mathcal{O}_\varepsilon} v_\varepsilon^{\frac{p}{2}}\,dx \\ &\leqslant C\varepsilon^{-\tau} \int_{\mathbb{R}^N\setminus\mathcal{O}_\varepsilon} \exp(-c\sqrt{\kappa}|x-x_\varepsilon|)\,dx \\ &\leqslant C\varepsilon^{-\tau-N} \int_{\mathbb{R}^N\setminus\mathcal{O}} \exp(-c\sqrt{\kappa}|x/\varepsilon - x_\varepsilon|)\,dx \to 0 \quad \text{as } \varepsilon \to 0\end{aligned}$$

(notice that here we use that fact $\sqrt{\kappa}/\varepsilon = 1/\hbar \to 0$ as $\varepsilon \to 0$). Thus, $Q_\varepsilon(v_\varepsilon) = 0$ for small $\varepsilon > 0$ and v_ε is a positive critical point of P_ε. So, $u_\varepsilon = G^{-1}(v_\varepsilon)$ is a positive solution of (1.5). Furthermore,

$$\|u_\varepsilon(\cdot + x_\varepsilon) - G^{-1}(U(\cdot + y_0))\|_{D^{1,2}(\mathbb{R}^N)} \to 0.$$

Finally, let us prove the last part of Theorem 1.2 which concerns the critical case $p = \frac{2N}{N-2}$. Set

$$G_\varepsilon^{-1}(t) = \int_0^t g_\varepsilon(s)\,ds,$$

where

$$g_\varepsilon(s) = \sqrt{1 + 2\zeta s^2}.$$

Then equation (1.7) turns into the following equation:

$$-\Delta v + \kappa V_\varepsilon(x) \frac{G_\varepsilon^{-1}(v)}{g_\varepsilon(G_\varepsilon^{-1}(v))} = K_\varepsilon(x) \frac{|G_\varepsilon^{-1}(v)|^{p-2} G_\varepsilon^{-1}(v)}{g_\varepsilon(G_\varepsilon^{-1}(v))}, \quad x \in \mathbb{R}^N.$$

Here we just stress the differences with respect to the previous case. First, the unique fast decay positive radial solution of (2.2) should be replaced by the unique positive radial solution (ground state) of (1.9). Besides,

$$L_m(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2\,dx - \frac{m}{p} \int_{\mathbb{R}^N} |v|^p\,dx.$$

Since $g_\varepsilon(t) \to 1$ and $G_\varepsilon^{-1}(t) \to t$ for any $t \in \mathbb{R}$ as $\varepsilon \to 0$, equation (3.15) turns into

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K_{\varepsilon_n}(x + z_n) \frac{|G_\varepsilon^{-1}(\tilde{v}_n)|^{\frac{4}{N-2}} G_\varepsilon^{-1}(\tilde{v}_n)}{g_\varepsilon(G_\varepsilon^{-1}(\tilde{v}_n))} \phi\,dx = \int_{\mathbb{R}^N} K(z_0)|\tilde{v}|^{\frac{4}{N-2}} \tilde{v}\phi\,dx.$$

So, (3.16) turns into

$$-\Delta\tilde{v} = K(z_0)|\tilde{v}|^{\frac{4}{N-2}}\tilde{v}, \quad x \in \mathbb{R}^N.$$

Similarly, (3.24) becomes

$$-\Delta v_1 = K(y_0 + z_0)|v_1|^{\frac{4}{N-2}} v_1, \quad x \in \mathbb{R}^N.$$

The rest can be discussed in a similar fashion and following the analysis carried out in Lemmas 3.1–3.11; the proofs are thus complete.

Remark 3.12. Inequality (3.36) is not true for all $x \in \mathbb{R}^N$. In fact, from (3.33), we deduce that

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2\,dx &< \int_{\mathbb{R}^N} \Big[1 + \frac{g'(G^{-1}(w_\varepsilon))G^{-1}(w_\varepsilon)}{g(G^{-1}(w_\varepsilon))}\Big]|\nabla w_\varepsilon|^2\,dx + \kappa \int_{\mathbb{R}^N} V_\varepsilon(x + z_\varepsilon)|G^{-1}(w_\varepsilon)|^2\,dx \\ &\leqslant \int_{\mathbb{R}^N} K_\varepsilon(x + z_\varepsilon)|G^{-1}(w_\varepsilon)|^p\,dx \leqslant C\|w_\varepsilon\|_{L^\infty(\mathbb{R}^N)}^{p-2^*} \Big[\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2\,dx\Big]^{\frac{2^*}{2}}.\end{aligned}$$

By (3.34), we get $\|w_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \geqslant C$, where C is independent of κ.

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