Monads on Categories of Relational Structures

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Abstract

We introduce a framework for universal algebra in categories of relational structures given by finitary relational signatures and finitary or infinitary Horn theories, with the arity λ of a Horn theory understood as a strict upper bound on the number of premisses in its axioms; key examples include partial orders ($\lambda = \omega$) or metric spaces ($\lambda = \omega_1$). We establish a bijective correspondence between λ-accessible enriched monads on the given category of relational structures and a notion of λ-ary *algebraic theories* (i.e. with operations of arity $< \lambda$), with the syntax of algebraic theories induced by the relational signature (e.g. inequations or equations-up-to-ϵ). We provide a generic sound and complete derivation system for such *relational algebraic theories*, thus in particular recovering (extensions of) recent systems of this type for monads on partial orders and metric spaces by instantiation. In particular, we present an ω_1-ary algebraic theory of metric completion. The theory-to-monad direction of our correspondence remains true for the case of κ-ary algebraic theories and κ-accessible monads for $\kappa < \lambda$, e.g. for finitary theories over metric spaces.

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1 Introduction

Monads play an established role in the semantics of sequential and concurrent programming [21] – they encapsulate side-effects, such as statefulness, nontermination, nondeterminism, or probabilistic branching. The well-known correspondence between monads on the category of sets and algebraic theories [16] impacts accordingly on programming syntax, as witnessed, for example, in work on algebraic effects [24]: operations of the theory serve as syntax for computational effects such as non-deterministic or probabilistic choice. The comparative analysis of programs or systems beyond two-valued equivalence checking, e.g. under behavioural preorders, such as similarity, or behavioural distances, involves monads based on categories beyond sets, such as the categories **Pos** of partial orders or **Met** of (1-bounded) metric spaces. This has sparked recent interest in presentations of such monads using suitable variants of the notion of algebraic theory. While it is, in principle, possible to work with equational presentations that encapsulate the additional structure within the signature [14], it seems at least equally natural to represent the additional structure (e.g. distance or ordering) within the judgements of the theory. Indeed, Mardare, Panangaden, and Plotkin replace equations with equations-up-to-ϵ in their *quantitative algebraic theories* [20], which present monads on **Met**, and in our own previous work on behavioural preorders [9] as well as in our own recent work with Adámek [5], we have used *in*equational theories to present monads on **Pos**.

In the present paper, we introduce a generalized approach to such notions of algebraic theory: We work in categories of finitary relational structures (more precisely, the objects are sets interpreting a given signature of finitary relation symbols), axiomatized by Horn theories whose axioms are implications with possibly infinite sets of antecedents. We say that such a theory is λ-ary for a regular cardinal λ if all its axioms have less than λ antecedents. For instance, **Pos** can be presented by a finitary (i.e. ω-ary) Horn theory over a binary relation $\leq$, and **Met** by an ω_1-ary Horn theory over binary relations $=_\epsilon$ "equality up to ϵ" indexed over rational numbers ϵ. We exploit that the models of a λ-ary Horn theory form a locally λ-presentable category $\mathscr{C}$ [4] to give a syntactic characterization of λ-accessible monads on $\mathscr{C}$ in terms of a notion of relational algebraic theory, in the sense that we prove a monad-theory correspondence. Following Kelly and Power [14], we use λ-presentable objects of $\mathscr{C}$ as arities and as contexts of axioms; however, as indicated above, we provide a syntax by expressing axioms using the relational signature instead of necessarily using only equality. We give a sound and complete deduction system for the arising *relational logic* (which generalizes standard equational logic), thus obtaining an explicit description of the monad generated by a relational algebraic theory in the indicated sense. One consequence of our main result is that quantitative algebraic theories [20] induce ω_1-accessible monads. More generally, presentations of ω_1-presentable monads in our formalism may involve operations with countable non-discrete arities: indeed, we present an ω_1-ary relational algebraic theory that defines the metric completion monad. We also take a glimpse at the more involved setting of κ-accessible monads on $\mathscr{C}$ where $\kappa < \lambda$ (e.g. finitary monads on **Met**). We give a partial characterization of κ-presentable objects in this setting, and show that while the monad-to-theory direction of our correspondence fails for $\kappa < \lambda$, the theory-to-monad direction does hold. This implies that some salient quantitative algebraic theories induce finitary monads; e.g. the theory of quantitative join-semilattices [20].

Related Work. We have already mentioned work by Kelly and Power on finitary monads [14] and by Mardare et al. on quantitative algebraic theories [20], as well as our own previous work [9] and our joint work with Adámek [5].

Power and Nishizawa [22] have extended the approach of Kelly and Power to deal with different enrichments of a category and the monads thereon, and obtain a correspondence between enriched Lawvere theories [25] and finitary enriched monads. More recently, Power and Garner [10] have provided a more thorough understanding of the equivalence between enriched finitary monads and enriched Lawvere theories as an instance of a free completion of an enriched category under a class of absolute colimits. Rosický [26] establishes a monad-theory correspondence for λ-accessible enriched monads and a notion of λ-ary enriched theory *á la* Linton [17], where arities of operations are given by pairs of objects. Like in the setting of Kelly and Power, relations (inequations, distances) are encoded in the arities. So the syntactic notion of theory is different from (and more abstract than) ours. Lucyshyn-Wright [18] establishes a rather general correspondence between monads and abstract theories in symmetric monoidal closed categories, parametric in a choice of arities, which covers several notions of theory and correspondences in the categorical literature under one roof.

Kurz and Velebil [15] characterize classical ordered varieties [6] (which are phrased in terms of inequalities) as precisely the exact categories in an enriched sense with a "suitable" generator. In recent subsequent work, Adámek at al. [2] establish a correspondence of these varieties with enriched monads on **Pos** that are strongly finitary [13], i.e. their underlying functor is a left Kan-extension of the embedding of finite discrete posets into **Pos**.

The main distinguishing feature of our work relative to the above is the explicit syntactic description of the monad obtained from a theory via a sound and complete derivation system. A related derivation system is the partial Horn logic of Palmgren and Vickers [23], which reasons about partial operations with unrestricted domains of definition. In contrast, we consider (varieties of) algebras with partial operations whose domain of definition is specified by objects in the base category (cf. Section 4); an essential ingredient in our monad-theory correspondence.

2 Preliminaries

We review the basic theory of locally presentable categories (see [4] for more detail) and of monads. We assume a modest familiarity with the elementary concepts of category theory [3] and with ordinal and cardinal numbers [11]. We write $\mathsf{card}\, X$ for the cardinality of a set X and, where λ is a cardinal, we write $X' \subseteq_\lambda X$ to indicate that $X' \subseteq X$ and $\mathsf{card}\, X' < \lambda$.

Locally Presentable Categories. Fix a regular cardinal λ (i.e. an infinite cardinal which is not cofinal to any smaller cardinal). A poset $(I, \leq)$ is λ-*directed* if each subset $I_0 \subseteq_\lambda I$ has an upper bound: there exists $u \in I$ such that $i \leq u$ for all $i \in I_0$. A λ-*directed diagram* is a functor whose domain is a λ-directed poset (viewed as a category); colimits of such diagrams are also called λ-*directed*. An object X in a category $\mathscr{C}$ is λ-*presentable* if the covariant hom-functor $\mathscr{C}(X, -)$ preserves λ-directed colimits. That is, X is λ-presentable if for each λ-directed colimit $(D_i \xrightarrow{c_i} C)_{i\in I}$ in $\mathscr{C}$, every morphism $m\colon X \to C$ factors through one of the c_i essentially uniquely: there exists $i \in I$ and $g\colon X \to D_i$ such that $m = c_i \cdot g$, and for all $g'\colon X \to D_i$ such that $m = c_i \cdot g'$, there exists $j \geq i$ such that $D(i \to j) \cdot g = D(i \to j) \cdot g'$.

▶ **Definition 2.1.** A category $\mathscr{C}$ is *locally* λ-*presentable* if it is cocomplete, its full subcategory $\mathsf{Pres}_\lambda(\mathscr{C})$ given by the λ-presentable objects of $\mathscr{C}$ is essentially small, and every $C \in \mathscr{C}$ is a λ-directed colimit of objects in $\mathsf{Pres}_\lambda(\mathscr{C})$. When $\lambda = \omega$ (resp. ω_1), we speak of *locally finitely (resp. countably) presentable categories.* We call $\mathscr{C}$ *locally presentable* if it is locally λ-presentable for some cardinal λ. A functor F on a locally presentable category is λ-*accessible* if it preserves λ-directed colimits. When $\lambda = \omega$ or ω_1, we speak of *finitary* and *countably accessible functors*, respectively.

Reflective subcategories. A full subcategory $\mathscr{C}'$ of a category $\mathscr{C}$ is *reflective* if the embedding $\iota\colon \mathscr{C}' \hookrightarrow \mathscr{C}$ is a right adjoint. In this case, we write $r_X\colon X \to RX$ (or just r if X is clear from the context) for the universal arrows; we call RX *the reflection of* $X \in \mathscr{C}$, r_X the *reflective arrow*, and the left adjoint R the *reflector*. The universal property of $r_X\colon X \to RX$ is as follows: For each morphism $f\colon X \to Y$ in $\mathscr{C}$, where Y lies in $\mathscr{C}'$, there exists a unique morphism $f^\sharp\colon RX \to Y$ such that $f = f^\sharp \cdot r_X$. We call $\mathscr{C}'$ *epi-reflective* if r_X is epi for all $X \in \mathscr{C}$. We will employ the following reflection theorem:

▶ **Theorem 2.2** [4, Cor. 2.48]. *If $\mathscr{C}'$ is a full subcategory of a locally λ-presentable category $\mathscr{C}$ and $\mathscr{C}'$ is closed under limits and λ-directed colimits in $\mathscr{C}$, then $\mathscr{C}'$ is reflective and locally λ-presentable.*

Monads. A *monad* on a category $\mathscr{C}$ is a functor $T\colon \mathscr{C} \to \mathscr{C}$ equipped with natural transformations $\eta\colon \mathsf{Id} \to T$ (the *unit*) and $\mu\colon TT \to T$ (the *multiplication*) such that the diagrams below commute.

$$\begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, "\mathsf{id}"'] & TT \arrow[d, "\mu"] & T \arrow[l, "\eta T"'] \arrow[dl, "\mathsf{id}"] \\ & T & \end{tikzcd} \qquad \begin{tikzcd} TTT \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"] & T \end{tikzcd}$$

We call the monad (T, η, μ) λ-*accessible* if its underlying functor is λ-accessible.

▶ **Definition 2.3.** An *Eilenberg-Moore algebra* for the monad (T, η, μ) is a $\mathscr{C}$-morphism of the shape $a\colon TX \to X$ satisfying the following coherence laws:

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "\mathsf{id}"'] & TX \arrow[d, "a"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} TTX \arrow[r, "\mu"] \arrow[d, "Ta"'] & TX \arrow[d, "a"] \\ TX \arrow[r, "a"] & X \end{tikzcd}$$

A *homomorphism* from $a\colon TX \to X$ to an Eilenberg-Moore algebra $b\colon TY \to Y$ is a morphism $h\colon X \to Y$ in $\mathscr{C}$ such that $h \cdot a = Th \cdot b$.

▶ **Notation 2.4.** For a functor $F\colon \mathscr{C} \to \mathscr{C}$, we write $\mathsf{Alg}\,F$ for the category of F-algebras and homomorphisms, i.e. $\mathsf{Alg}\,F$ has $\mathscr{C}$-morphisms of the shape $a\colon FA \to A$ as objects, and a homomorphism $(A, a) \to (B, b)$ is a $\mathscr{C}$-morphism $h\colon A \to B$ such that $h \cdot a = b \cdot Fh$.

3 Categories of Relational Structures

As indicated previously, we will study monads over base categories consisting of (single-sorted) relational structures. Specifically, we will restrict the relational signature to be finitary but allow infinitary Horn axioms. We proceed to recall basic definitions, examples, and results, in particular on closed structure and (local) presentability. In Section 3.1, we present new results on the partial characterization of (internally) λ-presentable objects in cases where the overall local presentability index of the category is greater than λ.

▶ **Definition 3.1.**

1. A *relational signature* is a set Π of *relation symbols* $\alpha, \beta, \ldots$ together with a finite *arity* $0 < \mathsf{ar}(\alpha) \in \omega$ for all $\alpha \in \Pi$. A Π-*edge* in a set S is a pair $e = \alpha(f)$ where $\alpha \in \Pi$ and $f\colon \mathsf{ar}(\alpha) \to S$ is a function. For a map $g\colon S \to Y$, we write $g \cdot e = \alpha(g \cdot f)$. We extend this notation pointwise to sets E of edges: $g \cdot E = \{g \cdot e \mid e \in E\}$.
2. A Π-*structure* X consists of an underlying set $|X|$ (or just X when no confusion is likely) and a set $\mathsf{E}(X)$ of Π-edges in $|X|$. If $\alpha(f) \in \mathsf{E}(X)$, we write $\alpha_X(f)$ or even $X \models \alpha(f)$.
3. A *relation-preserving map* (or briefly a *morphism*) from X to a Π-structure Y is a function $g\colon |X| \to |Y|$ such that $g \cdot \mathsf{E}(X) \subseteq \mathsf{E}(Y)$. We call g an *embedding* if g is injective and *relation-reflecting*, i.e. if $Y \models g \cdot e$ for an edge $e = (\alpha, f\colon \mathsf{ar}(\alpha) \to X)$, then $X \models e$. We denote by $\mathbf{Str}(\Pi)$ the category of Π-structures and relation-preserving maps.

▶ **Notation 3.2.** Given an edge $\alpha(f)$ such that $f(i) := x_i$ for all $i \in \mathsf{ar}(\alpha)$, we sometimes write $\alpha(x_1, \ldots, x_{\mathsf{ar}(\alpha)})$ or even $\alpha(x_i)$ in lieu of $\alpha(f)$. We will pass between these presentations without further mention.

We are going to carve out full subcategories of $\mathbf{Str}(\Pi)$ by means of infinitary Horn axioms, whose **syntax** we recall next.

▶ **Definition 3.3.** Let Π be a relational signature, and λ a regular cardinal. We fix a set Var of variables such that $\mathsf{card}(\mathsf{Var}) = \lambda$. A *$\lambda$-ary Horn formula* over Π has the form

$$\Phi \implies \psi$$

where Φ is a set of Π-edges in Var such that $\mathsf{card}\,\Phi < \lambda$ and ψ is a $\Pi \cup \{=\}$-edge in Var, for a fresh binary relation symbol $=$. In case $\Phi = \{\varphi_1, \dots, \varphi_n\}$ is finite, we write $\varphi_1, \dots, \varphi_n \implies \psi$, and if $\Phi = \emptyset$, then we just write $\implies \psi$. A *λ-ary Horn theory* $\mathscr{H} = (\Pi, \mathcal{A})$ consists of a relational signature Π and a set $\mathcal{A}$ of λ-ary Horn formulae over Π, the *axioms* of $\mathscr{H}$.

We fix a λ-ary Horn theory $\mathscr{H} = (\Pi, \mathcal{A})$ for the rest of the paper. We define the **semantics** of Horn formulae in a Π-structure X as follows. We denote by $\overline{X}$ the $\Pi \sqcup \{=\}$-structure obtained from X by putting $=_X := \{(x, x) \mid x \in X\}$. A *valuation* is a map $\kappa\colon \mathsf{Var} \to |X|$. We say that X *satisfies* a Horn formula $\Phi \implies \psi$ if whenever κ is a valuation such that $X \models \kappa \cdot \phi$ for all $\phi \in \Phi$, then $\overline{X} \models \kappa \cdot \psi$. Finally, X is a *model* of $\mathscr{H}$, or of $\mathcal{A}$, if X satisfies all axioms of $\mathscr{H}$. The full subcategory of $\mathbf{Str}(\Pi)$ spanned by the models of $\mathcal{A}$ is $\mathbf{Str}(\Pi, \mathcal{A})$ (or $\mathbf{Str}\,\mathscr{H}$).

We have an obvious notion of *derivation* under $\mathscr{H}$ *over* a given set Z (e.g. of variables or points in a structure): We extend $\mathscr{H}$ to $(\Pi \cup \{=\}, \bar{\mathcal{A}})$ where $\bar{\mathcal{A}}$ consists of the axioms in $\mathcal{A}$ and additional axioms stating that $=$ is an equivalence and that all relations in Π are closed under $=$ in the obvious sense. Then we have a single (λ-ary) derivation rule for application of Horn axioms $(\Phi \implies \psi) \in \bar{\mathcal{A}}$ over Z:

$$\frac{\kappa \cdot \Phi}{\kappa \cdot \psi}\ (\kappa\colon \mathsf{Var} \to Z).$$

We say that a set E of edges over Z *entails* an edge e *over* Z (*under* $\mathscr{H}$) if e is derivable from edges in E in this system. In case $Z = \mathsf{Var}$ and $\mathsf{card}\,E < \lambda$, the expression $E \implies e$ is in fact a Horn formula, and we then also say that $\mathscr{H}$ *entails* $E \implies e$ if E entails e.

▶ **Assumption 3.4.** For technical convenience, we assume that the fixed Horn theory $\mathscr{H} = (\Pi, \mathcal{A})$ *expresses equality.* That is, there exists a set $Eq(x, y)$ of Π-edges in variables x, y such that $\mathscr{H}$ entails $Eq(x, y) \implies x = y$ as well as $\implies \psi$ for all edges $\psi \in Eq(x, x)$ (where we use obvious notation for substitution; formally, $Eq(x, x) = g \cdot Eq(x, y)$ where $g(x) = g(y) = x$). Moreover, we assume that $\mathcal{A}$ explicitly includes the (derivable) formulae $Eq(x_1, y_1) \cup \dots \cup Eq(x_{\mathsf{ar}(\alpha)}, y_{\mathsf{ar}(\alpha)}) \cup \{\alpha(x_1, \dots, x_{\mathsf{ar}(\alpha)})\} \implies \alpha(y_1, \dots, y_{\mathsf{ar}(\alpha)})$ saying that all relations $\alpha \in \Pi$ are closed under Eq (implying also that Eq is symmetric and transitive). This is without loss of generality as we can always extend a given Horn theory with an equality predicate axiomatized by the above conditions without changing its category of models; indeed we leave this predicate implicit in examples whose natural presentation does not include it.

▶ **Example 3.5.** We mention some key examples of Horn theories:

1. The category $\mathbf{Set}$ of sets and functions is specified by the trivial Horn theory $(\emptyset, \emptyset)$.
2. The category $\mathbf{Pos}$ of partially ordered sets (posets) and monotone maps is specified by the ω-ary Horn theory consisting of a single binary relation symbol $\leq$ and the axioms

$$x \leq x; \qquad x \leq y,\, y \leq z \implies x \leq z; \qquad \text{and} \qquad x \leq y,\, y \leq x \implies x = y.$$

   This theory expresses equality (Assumption 3.4) via $Eq(x, y) = \{x \leq y, y \leq x\}$.

3. The theory $\mathscr{H}_{\mathbf{Met}}$ of *metric spaces* is the ω_1-ary theory consisting of binary relation symbols $=_\epsilon$ for all $\epsilon \in \mathbb{Q} \cap [0,1]$, and the axioms

$$\begin{aligned} &\implies x =_0 x && (\mathbf{Refl}) \\ x =_0 y &\implies x = y && (\mathbf{Equal}) \\ x =_\epsilon y &\implies y =_\epsilon x && (\mathbf{Sym}) \\ x =_\epsilon y, y =_{\epsilon'} z &\implies x =_{\epsilon+\epsilon'} z && (\mathbf{Triang}) \\ x =_\epsilon y &\implies x =_{\epsilon+\epsilon'} y && (\mathbf{Up}) \\ \{x =_{\epsilon'} y \mid \mathbb{Q}_{\geq 0} \ni \epsilon' > \epsilon\} &\implies x =_\epsilon y && (\mathbf{Arch}) \end{aligned}$$

where ϵ, ϵ' range over $\mathbb{Q} \cap [0,1]$ (that is, the axioms mentioning ϵ, ϵ' are in fact axiom schemes representing one axiom for each ϵ, ϵ'). This theory expresses equality via $Eq(x,y) = \{x =_0 y\}$; in fact, even if we remove $=_0$, the remaining theory still expresses equality via $Eq(x,y) = \{x =_{1/n} y \mid n > 0\}$. The theory $\mathscr{H}_{\mathbf{Met}}$ specifies the category $\mathbf{Met}$ of 1-bounded metric spaces and non-expansive maps, in the sense that $\mathbf{Str}(\mathscr{H}_{\mathbf{Met}})$ and $\mathbf{Met}$ are concretely isomorphic: $X \in \mathbf{Str}(\mathscr{H}_{\mathbf{Met}})$ induces the 1-bounded metric space (X,d) given by $d(x,y) = \bigwedge\{\epsilon \mid x =_\epsilon y \in \mathsf{E}(X)\}$, and conversely a metric space (X,d) induces the $\mathscr{H}_{\mathbf{Met}}$-model on X with edges $\{x =_\epsilon y \mid x, y \in X, d(x,y) \leq \epsilon\}$.
4. Consider the theory obtained by taking two "copies" of the theory $\mathscr{H}_{\mathbf{Met}}$: its signature consists of binary relation symbols $=^0_\epsilon, =^1_\epsilon$ for all $\epsilon \in \mathbb{Q} \cap [0,1]$, each subject to (indexed variants of) the axiom schema above. This yields an ω_1-ary theory of *bi-metric spaces*: sets equipped with a pair of metrics. Morphisms are maps which are non-expansive with respect to both metrics. Further imposing axioms of the shape

$$x =^0_\epsilon y \implies x =^1_\epsilon y \qquad (\epsilon \in \mathbb{Q} \cap [0,1])$$

specifies bi-metric spaces in which one metric is always finer than the other. We aim to approach the problem of digital fingerprinting in future work on graded monads in precisely this setting.
5. Let L be a complete lattice (for simplicity), and let $L_0 \subseteq L$ be meet-dense in L in the sense that $l = \bigwedge\{p \in L_0 \mid p \geq l\}$ for each $l \in L$; whenever $q \geq \bigwedge P$ for $q \in L_0$ and $P \subseteq L_0$ such that $\bigwedge P \notin L_0$, then $q \geq p$ for some $p \in P$ (e.g. these conditions hold trivially for $L_0 = L$). Further, fix λ such that $|L_0| < \lambda$. Let $\mathscr{H}_L$ be the λ-ary Horn theory with binary relation symbols α_p for all $p \in L_0$ and axioms

$$\begin{aligned} \{\alpha_p(x,y) \mid p \in P\} &\implies \alpha_q(x,y) && (P \subseteq L_0, q = \textstyle\bigwedge P \in L_0) && (\mathbf{Arch}) \\ \alpha_p(x,y) &\implies \alpha_q(x,y) && (p, q \in L_0, p \leq q) && (\mathbf{Up}) \end{aligned}$$

where p, q range over L_0. Then $\mathbf{Str}(\mathscr{H}_L)$ is concretely isomorphic to the category of *L-valued relations*, whose objects X are sets X equipped with map $P\colon X \times X \to L$, and whose morphisms $(X,P) \to (Y,Q)$ are maps $X \to Y$ such that $Q(f(x), f(y)) \leq P(x,y)$. (Of course, $\mathbf{Met}$ is essentially the special case $L = [0,1]$, $L_0 = \mathbb{Q} \cap [0,1]$ with some additional axioms.)
6. A signature of partial operations is a set P of operation symbols f with assigned finite arities $\mathsf{ar}(f)$. A (partial) P-algebra is then a set A and, for each $f \in P$, a partial function $f_A\colon A^{\mathsf{ar}(f)} \to A$. A homomorphism of partial algebras is a map $h\colon A \to B$ such that whenever $f_A(x_1, \dots, x_{\mathsf{ar}(f)})$ is defined, then $f_B(h(x_1), \dots, h(x_{\mathsf{ar}(f)}))$ is defined and equals $h(f_A(x_1, \dots, x_{\mathsf{ar}(f)}))$. The category of partial P-algebras and their homomorphims is concretely isomorphic to the category of models of the ω-ary Horn theory consisting of

relational symbols α_f of arity $\mathsf{ar}(f)+1$ for all $f \in P$ (with $\alpha_f(x_1,\dots,x_{\mathsf{ar}(f)},y)$ being understood as $f(x_1,\dots,x_{\mathsf{ar}(f)})=y$), and axioms

$$\alpha_f(x_1,\dots,x_{\mathsf{ar}(f)},y),\alpha_f(x_1,\dots,x_{\mathsf{ar}(f)},z) \implies y=z.$$

We proceed to discuss some key aspects of the categorical structure of $\mathbf{Str}(\mathscr{H})$.

Reflection. We first note

▶ **Proposition 3.6.** *$\mathbf{Str}(\Pi,\mathcal{A})$ is a (full) epi-reflective subcategory of $\mathbf{Str}(\Pi)$.*

Since $\mathbf{Str}(\Pi)$ is easily seen to be complete and cocomplete, it follows that $\mathbf{Str}(\Pi,\mathcal{A})$ is cocomplete and moreover closed under limits in $\mathbf{Str}(\Pi)$, and hence complete. We write

$$R\colon \mathbf{Str}(\Pi) \to \mathbf{Str}(\Pi,\mathcal{A}) \quad \text{and} \quad r_X\colon X \to RX$$

for the left adjoint of the inclusion $\mathbf{Str}(\Pi,\mathcal{A}) \hookrightarrow \mathbf{Str}(\Pi)$ (the *reflector*) and the corresponding (surjective) reflection maps, respectively. Explicitly, RX is constructed as follows. We define an equivalence $\sim$ on X by $x \sim y$ if $\mathsf{E}(X)$ entails $x=y$ under $\mathscr{H}$ (in the sense defined above), and let $q\colon X \to X/{\sim}$ denote the quotient map; then RX has underlying set $X/{\sim}$, and contains precisely the edges $q\cdot e$ such that $\mathsf{E}(X)$ entails e; moreover, $r_X = q$ as a map.

Local presentability. One easily checks

▶ **Lemma 3.7.** *An object $(X,E) \in \mathbf{Str}(\Pi)$ is λ-presentable iff $\mathsf{card}\,X < \lambda$ and $\mathsf{card}\,E < \lambda$; the category $\mathbf{Str}(\Pi)$ is locally finitely presentable.*

By Proposition 3.6 and since $\mathbf{Str}(\Pi,\mathcal{A})$ is easily seen to be closed under λ-directed colimits in $\mathbf{Str}(\Pi)$, we thus have

▶ **Proposition 3.8** [4, Example 5.27(3)]**.** *$\mathbf{Str}(\mathscr{H})$ is locally λ-presentable.*

The forgetful functor $\mathbf{Str}(\mathscr{H}) \to \mathbf{Set}$ preserves λ-directed colimits. Moreover, we have an easy characterization of λ-presentable objects:

▶ **Proposition 3.9.** *For an $\mathscr{H}$-model X, the following are equivalent.*

1. *X is λ-presentable in $\mathbf{Str}(\Pi,\mathcal{A})$;*
2. *$X \cong R(Y,E)$ for some λ-presentable $(Y,E) \in \mathbf{Str}(\Pi)$;*
3. *$\mathsf{card}\,|X| < \lambda$, and X is* λ-generated*, i.e. there exists $E \subseteq \mathsf{E}(X)$ such that $\mathsf{card}\,E < \lambda$ and E entails every edge in $\mathsf{E}(X)$ under $\mathscr{H}$ (equivalently, Ri is an isomorphism where $i\colon (|X|,E) \to X$ is the $\mathbf{Str}(\Pi)$-morphism carried by id_X).*

▶ **Remark 3.10.** For instance, every finite partial order is ω-presentable, and every countable metric space is ω_1-presentable. We emphasize that the situation is more complicated for κ-presentable objects where $\kappa < \lambda$; we treat this case in more detail in Section 3.1. For instance, every finite metric space with rational distances (cf. Example 3.5) is finitely generated in the sense of Proposition 3.9 but not finitely presentable.

Closed monoidal structure. The point-wise structure defines an *internal hom* functor:

▶ **Definition 3.11.** The *internal hom of* $X,Y \in \mathbf{Str}(\Pi)$ is the Π-structure $[X,Y]$ carried by $\mathbf{Str}(\Pi)(X,Y)$ with the edge set defined by

$$\mathsf{E}([X,Y]) := \{e \mid \forall x \in X.\, \pi_x \cdot e \in \mathsf{E}(Y)\},$$

where $\pi_x\colon \mathbf{Str}(\Pi)(X,Y) \to Y$ is defined by $\pi_x(g) = g(x)$. For each $X \in \mathbf{Str}(\Pi)$, the assignment $Y \mapsto [X,Y]$ defines a (covariant) *internal hom functor*

$$[X,-]\colon \mathbf{Str}(\Pi) \to \mathbf{Str}(\Pi)$$

with the action on a morphism $m\colon Y \to Z$ given by post-composition: $[X,m](g) := m \cdot g$.

One can show that $[-,-]$ endows $\mathbf{Str}(\Pi)$ with a symmetric closed structure. Using results of Day and LaPlaza [8] it follows that $\mathbf{Str}(\Pi)$ is a symmetric monoidal closed category. The ensuing monoidal product is given by the structure $X \otimes Y$ with underlying set $X \times Y$ and edges

$$\{e \mid (\pi_1 \cdot e \text{ constant } \wedge \pi_2 \cdot e \in \mathsf{E}(Y)) \vee (\pi_2 \cdot e \text{ constant } \wedge \pi_1 \cdot e \in \mathsf{E}(X))\}.$$

where an edge (α, f) is *constant* if f is a constant map, and $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the projection maps. The tensor unit is the trivial one point structure I_0 with no edges. To verify that this is the monoidal product arising from the symmetric closed structure $[-,-]$ it suffices to show that $(-) \otimes X$ is left adjoint to $[X,-]$. Indeed, we have $\mathbf{Str}(\Pi)$-morphisms $u_Y : Y \to [X, Y \otimes X]$, $u_Y(y) = \lambda x.(y,x)$. It is straightforward to check that u_Y is a universal arrow. That is:

▶ **Proposition 3.12.** *For every $X \in \mathbf{Str}(\Pi)$, $(-) \otimes X$ is a left adjoint of $[X,-]$.*

Moreover, one easily checks that $[X,-]$ restricts to a functor

$$[X,-]\colon \mathbf{Str}(\mathscr{H}) \to \mathbf{Str}(\mathscr{H}).$$

Hence, we can apply Day's reflection theorem [7, §1] to the reflector $R\colon \mathbf{Str}(\Pi) \to \mathbf{Str}(\Pi, \mathcal{A})$ (see the discussion immediately below Proposition 3.6) to obtain

▶ **Corollary 3.13.** *The category $\mathbf{Str}(\mathscr{H})$ is a closed symmetric monoidal category, with monoidal structure*

$$X \otimes_{\mathscr{H}} Y = R(X \otimes Y), \qquad I = RI_0$$

and internal hom given by $[X,-]\colon \mathbf{Str}(\mathscr{H}) \to \mathbf{Str}(\mathscr{H})$.

We briefly refer to $\otimes_{\mathscr{H}}$ as the *Manhattan product.*

▶ **Example 3.14.**
1. In **Pos** (Example 3.5.2), the Manhattan product coincides with binary Cartesian product (so **Pos** is Cartesian closed).
2. In **Met** (Example 3.5.3), the Manhattan product $(X, d_X) \otimes_{\mathscr{H}_{\mathbf{Met}}} (Y, d_Y)$ is $X \times Y$ equipped with the well-known Manhattan metric d given by $d((x_1, y_1), (x_2, y_2)) = \min(d_X(x_1, x_2) + d_Y(y_1, y_2), 1)$ (while Cartesian products carry the supremum metric).

▶ **Definition 3.15.** A functor $F\colon \mathbf{Str}(\mathscr{H}) \to \mathbf{Str}(\mathscr{H})$ that preserves the pointwise structure on morphisms is called *enriched.* That is, we call F enriched if for all $X, Y \in \mathbf{Str}(\mathscr{H})$ and all edges $f\colon \mathsf{ar}(\alpha) \to \mathbf{Str}(\mathscr{H})(X,Y)$ $(\alpha \in \Pi)$, if $[X,Y] \models \alpha(f_i)$, then $[FX, FY] \models \alpha(F(f_i))$.

Internal local presentability. For use of objects X as arities of operations, we will in fact need that the *internal* hom $[X,-]$ is λ-accessible. Using Kelly's results [12, (5.2) and (5.3)] this holds precisely for the λ-presentable objects since we have

▶ **Proposition 3.16.** *The λ-presentable objects of $\mathbf{Str}(\mathscr{H})$ are closed under the monoidal structure. That is, $I = RI_0$ is λ-presentable and $X \otimes_{\mathscr{H}} Y$ is λ-presentable whenever X and Y are so.*

In fact, this implies that $\mathbf{Str}(\mathscr{H})$ is *locally λ-presentable as a (symmetric monoidal) closed category* [12, (5.5)].

3.1 Compact Horn Models

We have seen above that the category $\mathbf{Str}(\mathscr{H})$ (where $\mathscr{H}$ is a λ-ary Horn theory) is (internally) locally λ-presentable, with a straightforward characterization of the (internally) λ-presentable objects (Propositions 3.9 and 3.16). We proceed to look at the rather less straightforward notion of internally κ-presentable objects in $\mathbf{Str}(\Pi, \mathcal{A})$ for $\kappa < \lambda$. The main scenario that motivates our interest in this case is that of finitary monads on categories that are internally locally λ-presentable only for some $\lambda > \omega$, such as metric spaces.

Further unfolding definitions, we have that an object X is internally κ-presentable if for every κ-directed colimit $(D_i \xrightarrow{c_i} C)_{i \in I}$, the canonical morphism

$$\mathsf{colim}[X, D_i] \to [X, \mathsf{colim}\, D_i]$$

is an isomorphism. We split this property into two parts: We say that X is *weakly κ-presentable* if the canonical morphism is always surjective, and *co-weakly κ-presentable* if the canonical morphism is always an embedding. Below, we give necessary and sufficient conditions for weak κ-presentability. Co-weak κ-presentability is a more elusive property; more concretely, it means roughly that X-indexed tuples of derivations in the given Horn theory can be synchronized into single derivations over X-indexed tuples of points. We give some examples below (Example 3.17).

▶ **Example 3.17.** We give some examples and non-examples of internally finitely presentable objects in locally ω_1-presentable categories $\mathbf{Str}(\Pi, \mathcal{A})$.

1. A metric space is internally finitely presentable iff it is finite and discrete. The "if" direction has surprisingly complicated reasons: It holds only because over the reals, finite joins distribute over directed infima. On the other hand, no non-empty metric space is *externally* finitely presentable, as its hom-functor will fail to preserve the colimit of the directed chain $(D_i)_{i<\omega}$ of spaces D_i with underlying set $\{0, 1\}$ and metric $d(0, 1) = 1/(i + 1)$.
2. In the category of L-valued relations for a complete lattice L in which binary joins fail to distribute over directed infima (such lattices exist), the two-element discrete space fails to be internally finitely presentable.
3. Let L be as in the previous item, and assume additionally that there is $l \in L$ such that in the downset of l, finite joins do distribute over directed infima (again, such L exist). Take the Horn theory of L-valued relations, extended with an additional (two-valued) relation α and axioms

$$\alpha(x, y) \wedge \alpha(x', y') \implies x =_l x' \qquad \alpha(x, y) \wedge \alpha(x', y') \implies y =_l y'.$$

   Then the set $\{0, 1\}$ equipped with the discrete L-valued relation and $\alpha(0, 1)$ is internally finitely presentable.

We proceed to give the announced characterization of weakly finitely presentable objects.

▶ **Definition 3.18.** A *cover* (Y, E), or just E, of $X \in \mathbf{Str}(\Pi, \mathcal{A})$ is a set E of edges in some set $Y \supseteq |X|$ such that all edges of X are implied by those in E under the Horn theory $\mathcal{A}$. That is, the underlying map $r_{(Y,E)} \colon Y \to |R(Y, E)|$ of the reflection composes with the inclusion $i \colon |X| \hookrightarrow Y$ to yield a morphism $r_{(Y,E)} \cdot i \colon X \to R(Y, E)$ (in $\mathbf{Str}(\Pi, \mathcal{A})$). Then X is *κ-compact* if for each cover (Y, E) of X there exist $E' \subseteq_\kappa E$ and a morphism $f \colon X \to R(Y, E')$ such that $r_{(Y,E)} \cdot i = Rj \cdot f$ where $j \colon (Y, E') \to (Y, E)$ is the $\mathbf{Str}(\Pi)$-morphism carried by id_Y:

$$\begin{tikzcd} & R(Y,E') \arrow[d, "Rj"] \\ X \arrow[ur, "f"] \arrow[r, "r_{(Y,E)}\cdot i"'] & R(Y,E) \end{tikzcd} \tag{3.1}$$

▶ **Lemma 3.19.** *Every κ-compact object is κ-generated.*

▶ **Remark 3.20.** We will show that the weakly finitely presentable objects in $\mathbf{Str}(\Pi, \mathcal{A})$ are precisely the κ-compact objects with less than κ elements (Proposition 3.21). This characterization breaks under seemingly innocuous variations of the definition of κ-compactness:

1. It is essential that the edges of a cover live over a superset Y of X. If we were to restrict covers to consist of edges over X (call such a cover an *X-cover*), then finite ω-compact objects in the arising relaxed sense would in general fail to be finitely presentable. E.g. take $(\Pi, \mathcal{A})$ to be the theory of metric spaces additionally equipped with a transitive relation α. Then the set $X = \{0, 2\}$ equipped with the discrete metric and the edge $\alpha(0, 2)$ satisfies the relaxed definition of compactness (every X-cover must contain the edge $\alpha(0, 2)$) but fails to be weakly finitely presentable: The colimit of the ω-chain of objects D_i with underlying set $\{0, 1, 1', 2\}$, distances $d(0, 1) = d(1', 2) = 1$, $d(1, 1') = 1/i$, and edges $\alpha(0, 1)$ and $\alpha(1', 2)$ is not weakly preserved by the hom-functor $\mathbf{Str}(\Pi, \mathcal{A})(X, -)$ (the obvious inclusion of X into the colimit fails to factorize through any of the D_i).
2. Note that we do not require that the factorization f of $r_{(Y,E)} \cdot i$ in (3.1) equals $r_{(Y,E')} \cdot i$; i.e. f may rename elements of X into elements of Y that lie outside X. Let us refer to the natural-sounding strengthening of κ-compactness where we do require $f = r_{(Y,E')} \cdot i$ as *strong* κ-compactness; e.g. X is strongly ω-compact if every cover of X has a finite subcover. However, this notion is too strong, i.e. not every (weakly) finitely presentable object in $\mathbf{Str}(\Pi, \mathcal{A})$ is strongly ω-compact. As a counterexample, consider the same Horn theory as in the previous item but without the transitivity axiom for α. Then the same object X as in the previous item is weakly finitely presentable (even internally finitely presentable) but not strongly ω-compact, as witnessed by the cover $E = \{\alpha(0', 2)\} \cup \{0 =_{1/n} 0' \mid n > 0\}$.

▶ **Proposition 3.21.** *The following are equivalent for $X \in \mathbf{Str}(\Pi, \mathcal{A})$:*
1. *X is weakly κ-presentable;*
2. *X is κ-compact, and* $\mathsf{card}\,|X| < \kappa$.

4 Relational Algebraic Theories

We next describe a framework of universal algebra for enriched κ-accessible monads on the internally locally λ-presentable category $\mathbf{C} = \mathbf{Str}(\mathscr{H})$ of $\mathscr{H}$-models, for $\kappa \leq \lambda$. We etablish one direction of our theory-monad correspondence: We show that every theory in our framework induces a κ-accessible monad (Remark 4.12) whose algebras are precisely the models of the theory (Theorem 4.13). We address the converse direction in Section 5. We write $\mathbf{C}_0$ for the ordinary category underlying the closed monoidal category $\mathbf{C}$.

Following Kelly and Power [14], we use the internally λ-presentable objects in $\mathbf{C}$ as the arities of operation symbols. The full subcategory $\mathsf{Pres}_\lambda(\mathbf{C})$ of internally λ-presentable objects is essentially small (Proposition 3.16); *we fix a small subcategory $\mathscr{P}_\lambda$ of internally λ-presentable $\mathbf{C}$-objects representing all such objects up to isomorphism.* For all infinite $\kappa < \lambda$, the full subcategory $\mathscr{P}_\kappa \hookrightarrow \mathscr{P}_\lambda$ is given by the internally κ-presentable objects in $\mathscr{P}_\lambda$.

▶ **Definition 4.1.** Let $\kappa \leq \lambda$ be a regular cardinal. A *κ-ary signature* is a set Σ of *operation symbols* σ, each of which is equipped with an *arity* $\mathsf{ar}(\sigma) \in \mathscr{P}_\kappa$.

A *Σ-algebra* A consists of a $\mathbf{C}$-object, also denoted A, and a family of $\mathbf{C}$-morphisms

$$\sigma_A \colon [\mathsf{ar}(\sigma), A] \to A \qquad (\sigma \in \Sigma)$$

A *homomorphism* from A to a Σ-algebra B is a morphism $h\colon A \to B$ in $\mathbf{C}$ such that the diagram below commutes for all $\sigma \in \Sigma$.

$$\begin{tikzcd}
{[\mathsf{ar}(\sigma), A]} \arrow[r, "\sigma_A"] \arrow[d, "h\cdot(-)"'] & A \arrow[d, "h"] \\
{[\mathsf{ar}(\sigma), B]} \arrow[r, "\sigma_B"] & B
\end{tikzcd}$$

We write $\mathsf{Alg}\,\Sigma$ for the category of Σ-algebras and homomorphisms. By a *subalgebra* of the Σ-algebra A, we understand a Σ-algebra B equipped with a homomorphism $h\colon B \hookrightarrow A$ whose underlying $\mathbf{C}$-morphism is an embedding.

Signatures and their algebras. *Fix a κ-ary signature Σ for the remainder of this section.* The category $\mathsf{Alg}\,\Sigma$ can be presented as a category of functor algebras:

▶ **Definition 4.2.** The *signature functor associated to Σ*, $H_\Sigma \colon \mathbf{C} \to \mathbf{C}$, is given by

$$H_\Sigma = \textstyle\coprod_{\sigma \in \Sigma} [\mathsf{ar}(\sigma), -].$$

The categories $\mathsf{Alg}\,\Sigma$ and $\mathsf{Alg}\,H_\Sigma$ are clearly isomorphic as concrete categories over $\mathbf{C}$, so the forgetful functor $\mathsf{Alg}\,\Sigma \to \mathbf{C}_0$ inherits all properties of the forgetful functor $\mathsf{Alg}\,H_\Sigma \to \mathbf{C}_0$. We collect a few basic consequences of this observation:

▶ **Remark 4.3.**
1. In general, the forgetful functor $\mathcal{U}\colon\ \mathsf{Alg}\,F \to \mathscr{C}$ from the category $\mathsf{Alg}\,F$ of F-coalgebras for a functor F on a category $\mathscr{C}$ creates all limits in $\mathscr{C}$. It follows that $\mathsf{Alg}\,\Sigma$ has all limits, and the forgetful functor $\mathsf{Alg}\,\Sigma \to \mathbf{C}_0$ creates them.
2. Since H_Σ is a colimit of κ-accessible functors $[\mathsf{ar}(\sigma), -]$, it is itself κ-accessible, so that the forgetful functor $\mathsf{Alg}\,H_\Sigma \to \mathbf{C}_0$ creates κ-directed colimits, and the same holds for the forgetful functor $\mathsf{Alg}\,\Sigma \to \mathbf{C}_0$.
3. From the previous observation (which implies that H_Σ is also λ-accessible) and Proposition 3.8, we obtain by [4, Remark 2.75] (for λ-accessible functors F on locally λ-presentable categories, $\mathsf{Alg}\,F$ is locally λ-presentable) that $\mathsf{Alg}\,\Sigma$ *is locally λ-presentable.*
4. Adámek [1] shows that for a λ-accessible functor F on a cocomplete category $\mathscr{C}$, the forgetful functor $\mathsf{Alg}\,F \to \mathscr{C}$ is right adjoint. From 2 and cocompleteness of $\mathbf{C}_0$ (Section 3), we thus obtain that the forgetful functor $\mathsf{Alg}\,\Sigma \to \mathbf{C}_0$ is right adjoint; that is, *every object $X \in \mathbf{C}$ generates a free Σ-algebra $F_\Sigma X$.*

Varieties of Σ-Algebras. We now describe a syntax for specifying full subcategories of $\mathsf{Alg}\,\Sigma$. As a first step, we introduce a notion of Σ-term, defined as usual in universal algebra:

▶ **Definition 4.4** (Σ-Terms; substitution)**.** For $X \in \mathbf{C}$, we call its underlying set $|X|$ the set of *variables* in X. The set $T_\Sigma(X)$ of *Σ-terms in X* is defined inductively as follows:
1. Each variable in $|X|$ is a Σ-term in X;
2. For each $\sigma \in \Sigma$ and each map $f\colon |\mathsf{ar}(\sigma)| \to T_\Sigma(X)$, $\sigma(f)$ is a Σ-term in X.

We usually omit the signature Σ from the notation and speak simply of *terms* (in X). We employ standard syntactic notions: A *substitution* is a map $\tau\colon |Y| \to T_\Sigma(X)$, for $X, Y \in \mathbf{C}$. We extend τ to a map $\bar{\tau}$ on terms $t \in T_\Sigma(X)$ as usual. Formally, we define $\bar{\tau}(t)$ inductively by $\bar{\tau}(x) = \tau(x)$ for $x \in X$, and $\bar{\tau}\sigma(f) = \sigma(\bar{\tau} \cdot f)$ for $f\colon \mathsf{ar}(\sigma) \to T_\Sigma(X)$. We will not further distinguish between τ and $\bar{\tau}$ in the notation, writing $\tau(t) = \bar{\tau}(t)$ and $\tau \cdot f = \bar{\tau} \cdot f$ for t, f as above. Moreover, the set $\mathsf{sub}(t)$ of *subterms* of a term $t \in T_\Sigma(X)$ is defined as usual; formally, we simultaneously define $\mathsf{sub}(t)$ and $\mathsf{sub}(f)$ for $f\colon I \to T_\Sigma(X)$ (with I some index set or object) inductively by $\mathsf{sub}(x) = \{x\}$ for $x \in X$; $\mathsf{sub}(\sigma(f)) = \{\sigma(f)\} \cup \mathsf{sub}(f)$ for $f\colon \mathsf{ar}(\sigma) \to T_\Sigma(X)$; and $\mathsf{sub}(f) = \bigcup_{i \in I} \mathsf{sub}(f(i))$.

Note that term formation operates without regard for the relational structure. Consequently, the evaluation of terms in a given Σ-algebra may fail to be defined:

► **Definition 4.5.** Let A be a Σ-algebra. For an object $X \in \mathbf{C}$ and a relation-preserving assignment $e\colon X \to A$, the partial *evaluation map* $e^\#\colon T_\Sigma(X) \to A$ is inductively defined by

1. $e^\#(x) = e(x)$ for $x \in X$, and
2. $e^\#(\sigma(f))$ is defined for $\sigma \in \Sigma$ and $f\colon |\mathsf{ar}(\sigma)| \to T_\Sigma(X)$ iff the following hold:
   a. $e^\#(f(i))$ is defined for all $i \in \mathsf{ar}(\sigma)$, and
   b. if $\alpha(g)$ is a Π-edge in $\mathsf{ar}(\sigma)$, then $A \models \alpha(e^\# \cdot (f \cdot g))$.

   In case $e^\#(\sigma(f))$ is defined, we put $e^\#(\sigma(f)) = \sigma_A(e^\# \cdot f)$.

As indicated previously, we phrase theories using the relations in Π:

► **Definition 4.6.** A Σ-*relation* $X \vdash \alpha(f)$ consists of a *context* $X \in \mathbf{C}$ and a Π-edge $\alpha(f)$ in $T_\Sigma(X)$. We say that $X \vdash \alpha(f)$ is κ-*ary* if $X \in \mathscr{P}_\kappa$. A Σ-algebra A *satisfies* $X \vdash \alpha(f)$ if, for each relation preserving assignment $e\colon X \to A$, $e^\# \cdot f(i)$ is defined for all $i \in X$, and $\alpha_A(e^\# \cdot f)$. A (κ-*ary*) *relational algebraic theory* $(\Sigma, \mathcal{E})$ consists of the (κ-ary) signature Σ and a set $\mathcal{E}$ of κ-ary Σ-relations. It determines the subcategory $\mathsf{Alg}(\Sigma, \mathcal{E})$ of $\mathsf{Alg}\,\Sigma$ consisting of those Σ-algebras which satisfy each Σ-relation in $\mathcal{E}$. We refer to categories of the shape $\mathsf{Alg}(\Sigma, \mathcal{E})$ as *varieties* of Σ-algebras.

► **Remark 4.7.** For $\mathbf{C}_0 = \mathbf{Pos}$, the above notion of variety of Σ-algebras corresponds precisely to what we have termed “varieties of coherent algebras” in earlier work with Adámek [5].

► **Example 4.8.** Recall that a (1-bounded) metric space X is *complete* if every Cauchy sequence $(x_i)_{i\in\omega}$ of points in X has a limit in X. That is, if (x_i) satisfies the Cauchy property

$$\forall \epsilon > 0.\ \exists N_\epsilon \in \omega.\ \forall n, m \geq N_\epsilon\ (d(y_n, y_m) < \epsilon), \tag{4.1}$$

then there is a point $\mathsf{lim}(x_i) \in X$ with the property of a limit: for all $\epsilon > 0$ there is $N \in \omega$ such that $x_n =_\epsilon \mathsf{lim}(x_i)$ for all $n \geq N$. The full subcategory $\mathbf{CMS} \hookrightarrow \mathbf{Met}$ of complete metric spaces is specified by the relational algebraic theory described below. Thus, by Theorem 4.13 below, we recover the fact that $\mathbf{CMS}$ is monadic over $\mathbf{Met}$. Furthermore, we obtain a completely syntactic ω_1-ary description of the metric completion monad via the deduction system introduced later in this section.

The *theory* $\mathbb{T}_{\mathbf{CMS}}$ *of complete metric spaces* has Γ-ary *limit operations* lim_Γ for all spaces $\Gamma \in \mathscr{P}_{\omega_1}$ of the form $\Gamma = \{x_i \mid i \in \omega\}$ where $(x_i)_{i\in\omega}$ is a Cauchy sequence in Γ. The axioms of $\mathbb{T}_{\mathbf{CMS}}$ then say precisely that $\mathsf{lim}_\Gamma(x_i)$ is a limit of (x_i). Explicitly, for all Γ as above, we impose all axioms of the shape

$$\Gamma \vdash \mathsf{lim}_\Gamma(x_n) =_\epsilon x_k \quad (k \geq N_\epsilon) \qquad \text{where } N_\epsilon \text{ is as in (4.1).}$$

We fix a variety $\mathcal{V} = \mathsf{Alg}(\Sigma, \mathcal{E})$ for the remainder of this section. We are going to see that $\mathcal{V}$ is a reflective subcategory of $\mathsf{Alg}\,\Sigma$ by application of Theorem 2.2, i.e. we show that $\mathcal{V}$ is closed under limits and κ-directed colimits in $\mathsf{Alg}\,\Sigma$. We state the second property separately:

▶ **Proposition 4.9.** *$\mathcal{V}$ is closed under κ-directed colimits in $\mathsf{Alg}\,\Sigma$.*

Combining this with Remark 4.3.2, we obtain

▶ **Corollary 4.10.** *The forgetful functor $V\colon \mathcal{V} \to \mathbf{C}$ is κ-accessible.*

It is fairly straightforward to show that $\mathcal{V}$ is also closed under products and subobjects, and hence under limits (in $\mathsf{Alg}\,\Sigma$). Thus, as announced, we have:

▶ **Proposition 4.11.** *$\mathcal{V}$ is a reflective subcategory of $\mathsf{Alg}\,\Sigma$.*

▶ **Remark 4.12.** It follows that the forgetful functor $\mathcal{V} \to \mathbf{C}_0$ has a left adjoint, namely the composite $\mathbf{C} \xrightarrow{F_\Sigma} \mathsf{Alg}\,\Sigma \xrightarrow{R_\mathcal{V}} \mathcal{V}$, where F_Σ is the left adjoint of the forgetful functor $\mathsf{Alg}\,\Sigma \to \mathbf{C}$ (Remark 4.3.4) and $R_\mathcal{V}$ is the reflector according to Proposition 4.11. We call the ensuing monad $\mathbb{T}_\mathcal{V}$ the *free-algebra monad* of $\mathcal{V}$; by Corollary 4.10, $\mathbb{T}_\mathcal{V}$ is κ-accessible.

Indeed, $\mathcal{V}$ is essentially the category of Eilenberg-Moore algebras of $\mathbb{T}_\mathcal{V}$: Using Beck's monadicity theorem, one can show that

▶ **Theorem 4.13.** *The forgetful functor $\mathcal{V} \to \mathbf{C}_0$ is monadic.*

▶ **Corollary 4.14.** *Every κ-ary relational algebraic theory may be translated into an enriched κ-accessible monad, preserving categories of models.*

Relational Logic. We proceed to set up a system of rules for deriving relations among terms. The calculus will involve two forms of judgements, both mentioning a context $X \in \mathbf{Str}(\Pi)$ (not necessarily κ-presentable). By a *relational judgement*

$$X \vdash \alpha(t_1, \dots, t_{\mathsf{ar}(\alpha)}),$$

where $t_1, \dots, t_{\mathsf{ar}(\alpha)} \in T_\Sigma(X)$, we indicate that for every valuation of X that is *admissible*, i.e. satisfies the relational constraints specified in X, the terms t_i are defined, and the resulting tuple of values is in relation α. We treat expressions $\alpha(t_1, \dots, t_{\mathsf{ar}(\alpha)})$ notationally as edges over $T_\Sigma(X)$, in particular sometimes write them in the form $\alpha(f)$ for $f\colon \mathsf{ar}(\alpha) \to T_\Sigma(X)$. Moreover, a *definedness judgement* of the form

$$X \vdash {\downarrow} t$$

states that t is defined for all admissible valuations of X. (We could encode ${\downarrow}t$ as $\phi(t,t)$ for any $\phi \in Eq(x,y)$ but for technical reasons we prefer to keep definedness judgement distinct from relational judgements.)

The rules of the arising system of *relational logic* are shown below:

$$(\mathsf{Var})\ \frac{}{X \vdash {\downarrow} x}\ (x \in X) \qquad (\mathsf{Ctx})\ \frac{}{X \vdash \alpha(x_1, \dots, x_{\mathsf{ar}(\alpha)})}\ (X \models \alpha(x_1, \dots, x_{\mathsf{ar}(\alpha)}))$$

$$(\mathsf{Mor})\ \frac{\{X \vdash \alpha(f_i(j)) \mid j \in \mathsf{ar}(\sigma)\} \cup \{X \vdash {\downarrow}\sigma(f_i) \mid i \in \mathsf{ar}(\alpha)\}}{X \vdash \alpha(\sigma(f_i))}\ ((f_i\colon \mathsf{ar}(\sigma) \to T_\Sigma(X))_{i \in \mathsf{ar}(\alpha)})$$

$$(\mathsf{E\text{-}Ar})\ \frac{\{X \vdash \alpha(f \cdot g) \mid \alpha(g) \in \mathsf{ar}(\sigma)\} \cup \{X \vdash {\downarrow} f(i) \mid i \in \mathsf{ar}(\sigma)\}}{X \vdash {\downarrow}\sigma(f)}\ (f\colon \mathsf{ar}(\sigma) \to T_\Sigma(X))$$

$$\text{(I-Ar)}\ \frac{\{X \vdash \alpha(\tau \cdot f) \mid \alpha(f) \in \Delta\} \cup \{X \vdash {\downarrow}\tau(y) \mid y \in \Delta\}}{X \vdash \beta(c)}\ (+)$$

$$\text{(RelAx)}\ \frac{\{X \vdash \tau \cdot \varphi \mid \varphi \in \Phi\} \cup \{X \vdash {\downarrow}\tau(f(i)) \mid i \in \mathsf{ar}(\alpha)\}}{X \vdash \alpha(\tau \cdot f)}\ \begin{array}{l}(\Phi \implies \alpha(f) \in \mathcal{A},\\ \tau\colon \mathsf{Var} \to T_\Sigma(X))\end{array}$$

$$\text{(Ax)}\ \frac{\{X \vdash \alpha(\tau \cdot f) \mid \alpha(f) \in \Delta\} \cup \{X \vdash {\downarrow}\tau(y) \mid y \in \Delta\}}{X \vdash \beta(\tau \cdot g)}\ (\Delta \vdash \beta(g) \in \mathcal{E})$$

Recall that both the arities of operations in Σ and the contexts of the κ-ary Σ-relations in $\mathcal{E}$ are in $\mathscr{P}_\kappa$. We assume such a $\Delta \in \mathscr{P}_\kappa$ to be specified as $\Delta = R(Y, E)$ by a κ-presentable object $(Y, E) \in \mathbf{Str}(\Pi)$ (cf. Lemma 3.7, Proposition 3.9, Lemma 3.19, Proposition 3.21); by writing $\phi \in \Delta$ for an edge ϕ, we indicate that $\phi \in E$ (rather than just $\phi \in \mathsf{E}(\Delta)$). The rules (E-Ar) and (I-Ar) apply to every $\sigma \in \Sigma$, and rule (Mor) applies to every $\sigma \in \Sigma$ and every $\alpha \in \Pi$. The side condition $(+)$ of (I-Ar) is the following: for some axiom $\Delta \vdash \gamma(g)$ of $\mathcal{V}$ there is $\sigma(h) \in \mathsf{sub}(g)$, where $h\colon \mathsf{ar}(\sigma) \to T_\Sigma(\Delta)$, such that $\mathsf{ar}(\sigma) \models \beta(k)$ and

$$c = \mathsf{ar}(\beta) \xrightarrow{k} \mathsf{ar}(\sigma) \xrightarrow{h} T_\Sigma(\Delta) \xrightarrow{\tau} T_\Sigma(X).$$

Rule (Mor) captures the fact that operations σ are interpreted as morphisms of type $[\mathsf{ar}(\sigma), A] \to A$, a condition that relates to enrichment of the induced monad. Rule (E-Ar) states that operations are defined when all the constraints given by their arity are satisfied. Rules (RelAx) and (Ax) allow application of the axioms of the Horn theory and the variety, respectively, in both cases instantiated with a substitution. A general substitution rule is not included but admissible. Rule (I-Ar) captures that every axiom of the variety is understood as implying that (under the constraints of the context) all subterms occurring in it are defined, in the sense that the constraints in the arities of the relevant operations hold.

▶ **Remark 4.15.** Instantiating the above system of rules to the theory of partial orders yields essentially the ungraded version of our previous deduction system for graded monads on **Pos** [9], up to the above-mentioned coding of definedness judgements. At first glance, the instantiation to the theory of metric spaces appears to yield a system that differs in several respects from the existing system of quantitative algebra [20]; besides the mentioned absence of a general substitution rule, this concerns most prominently the absence of a cut rule (included in [20]) in our system. These distinctions are only superficial: as mentioned above, the more general substitution rule is admissible in our system, and it follows from completeness (Theorem 4.19) that the cut rule is admissible as well.

▶ **Lemma 4.16.** *The following rules are admissible:*

$$\text{(Arity)}\ \frac{X \vdash {\downarrow}\sigma(m)}{X \vdash \alpha(m \cdot f)}\ \begin{array}{l}(\mathsf{ar}(\sigma) \models \alpha(f),\\ m\colon |\mathsf{ar}(\sigma)| \to T_\Sigma(X))\end{array} \qquad \text{(Subterm)}\ \frac{X \vdash \alpha(f)}{X \vdash {\downarrow}u}\ (u \in \mathsf{sub}(f))$$

Constructing free algebras. We now show that relational logic gives rise to a syntactic construction of free algebras in the variety $\mathcal{V}$.

The set $\mathscr{T}_\mathcal{V}(X)$ of *derivably $\mathcal{V}$-defined terms in X* consists of those terms $t \in T_\Sigma(X)$ such that $X \vdash {\downarrow}t$ is derivable. We equip $\mathscr{T}_\mathcal{V}(X)$ with the relations

$$\mathscr{T}_\mathcal{V}(X) \models \alpha(f) \iff X \vdash \alpha(f) \text{ is derivable} \qquad (\alpha \in \Pi, f\colon \mathsf{ar}(\alpha) \to \mathscr{T}_\mathcal{V}(X))$$

making it into a Π-structure. We write $\sim$ for the relation on $\mathscr{T}_\mathcal{V}(X)$ given by *derivable equality*: that is, for all $s, t \in \mathscr{T}_\mathcal{V}(\Gamma)$ we put $s \sim t$ iff $X \vdash \varphi$ is derivable for all $\varphi \in Eq(s, t)$, which is clearly an equivalence relation. The $\sim$-equivalence class of $t \in \mathscr{T}_\mathcal{V}(X)$ is denoted by $[t]$. We pick a *splitting* $u\colon \mathscr{T}_{\mathcal{V}}(X)/{\sim} \to \mathscr{T}_{\mathcal{V}}(X)$ of the canonical quotient map $q\colon \mathscr{T}_{\mathcal{V}}(X) \to \mathscr{T}_{\mathcal{V}}(X)/{\sim}$, i.e. $q \cdot u = \mathsf{id}$, so u picks representatives of $\sim$-equivalence classes. Then $\mathscr{T}_{\mathcal{V}}(X)/{\sim}$ carries the structure of a $\mathbf{C}$-object, with edges defined by $\mathscr{T}_{\mathcal{V}}(X)/{\sim} \models \alpha(f)$ iff $\mathscr{T}_{\mathcal{V}}(X) \models \alpha(u \cdot f)$. ("Only if" means that u is relation preserving.)

▶ **Definition 4.17.** The *algebra $\mathscr{F}X$ of defined terms in X* is the Σ-algebra obtained by equipping $\mathscr{T}_{\mathcal{V}}(X)/{\sim}$ with the operations $\sigma_{\mathscr{F}X}\colon [\mathsf{ar}(\sigma), \mathscr{T}_{\mathcal{V}}(X)/{\sim}] \to \mathscr{T}_{\mathcal{V}}(X)/{\sim}$ well-defined by $f \mapsto [\sigma(u \cdot f)]$, where $u\colon \mathscr{F}\Gamma \to \mathscr{T}_{\mathcal{V}}(X)$ is the chosen splitting of $q\colon \mathscr{T}_{\mathcal{V}}(X) \to \mathscr{F}X$.

▶ **Theorem 4.18.** *For every $X \in \mathbf{C}$, $\mathscr{F}X$ is a free algebra in $\mathcal{V}$.*

Thus, we see that the free-algebra monad $\mathbb{T}_{\mathcal{V}}$ (Remark 4.12) of a variety $\mathcal{V}$ maps each $X \in \mathbf{Str}(\mathscr{H})$ to the carrier of the algebra $\mathscr{F}X$. We note that the reflection $\mathsf{Alg}\,\Sigma \to \mathcal{V}$ (see Proposition 4.11) need not be epi: the rule (Ax) generally "adds" new defined terms in the presence of axioms; see Adámek et al. [5, Ex. 3.25] for a more detailed view on this point.

▶ **Theorem 4.19** (Soundness and Completeness)**.** *$X \vdash \alpha(f)$ is derivable iff every $A \in \mathcal{V}$ satisfies $X \vdash \alpha(f)$.*

▶ **Remark 4.20.** Consequently, our system instantiated to the theory of metric spaces and the system of quantitative algebra [20], which is also sound and complete, are deductively equivalent. Hence, our results thus far imply that every quantitative algebraic theory induces an ω_1-accessible monad. Indeed this remains true if one admits operations of countable arity, as in our theory of complete metric spaces (Example 4.8). Due to non-discrete contexts in axioms, monads induced by quantitative algebraic theories (such as $x =_{1/2} y \vdash x =_0 y$) in general fail to be finitary. However, our results do imply that the induced monad is finitary if only discrete contexts are used; e.g. this holds for the theories of left-invariant barycentric algebras and of quantitative semi-lattices, respectively [20] (note for the latter that axiom (S4) can be omitted in [20, Def. 9.1]). We conjecture that monads induced by *continuous equation schemes* [20] are also finitary.

5 Enriched Accessible Monads

We proceed to establish the monad-to-theory direction of our correspondence; as already indicated, given our fixed λ-ary Horn theory $\mathscr{H}$, this works only for λ-accessible monads and λ-ary theories, but not for accessibility degrees $\kappa < \lambda$ as in the theory-to-monad direction. So let $\mathbb{T} = (T, \eta, \mu)$ be an enriched λ-accessible monad on $\mathbf{C}$. We proceed to extract a λ-ary relational algebraic theory from $\mathbb{T}$. We first review the equivalence between monads and *Kleisli triples* (see, e.g., Moggi [21], and originally Manes [19, Exercise 12]).

▶ **Definition 5.1.** A *Kleisli triple* in $\mathbf{C}_0$ is a triple $(T, \eta, (-)^*)$ consisting of a mapping $T\colon \mathbf{C}_0 \to \mathbf{C}_0$ (of objects), a morphism $\eta_X\colon X \to TX$ for all $X \in \mathbf{C}_0$, and an assignment of a morphism $f^*\colon TX \to TY$ to every morphism $f\colon X \to TY$. This data is subject to the laws below for all $X \in \mathbf{C}_0$ and all morphisms $f\colon X \to TY$ and $g\colon Y \to TZ$:

$$\eta_X^* = \mathsf{id}_X, \qquad f^* \cdot \eta_X = f, \qquad \text{and} \qquad g^* \cdot f^* = (g^* \cdot f)^*. \tag{5.1}$$

▶ **Remark 5.2.** The mapping which assigns to each monad (T, η, μ) the Kleisli triple $(T, \eta, (-)^*)$ with $(-)^*$ defined by $f^* = TX \xrightarrow{Tf} TTY \xrightarrow{\mu_Y} TY$ for $f \in \mathbf{C}_0(X, TY)$ yields a bijective correspondence between monads and Kleisli triples on $\mathbf{C}_0$.

▶ **Notation 5.3.** For each operation σ in a signature Σ, we have a term $\sigma(u_{\mathsf{ar}(\sigma)})$, where $u_{\mathsf{ar}(\sigma)}$ is the inclusion $\mathsf{ar}(\sigma) \hookrightarrow T_\Sigma(\mathsf{ar}(\sigma))$. By abuse of notation, we also write σ for $\sigma(u_{\mathsf{ar}(\sigma)})$.

▶ **Definition 5.4.** The λ-ary signature $\Sigma_{\mathbb{T}}$ *induced by* $\mathbb{T}$ is the disjoint union of the sets $|T\Gamma|$ ($\Gamma \in \mathscr{P}_\lambda$), where elements of $|T\Gamma|$ have arity Γ. The *variety* $\mathcal{V}_{\mathbb{T}}$ *induced by* $\mathbb{T}$ is $\mathcal{V}_{\mathbb{T}} = \mathsf{Alg}(\Sigma_{\mathbb{T}}, \mathcal{E}_{\mathbb{T}})$ where $\mathcal{E}_{\mathbb{T}}$ contains all axioms of the following shape, with $\Gamma \in \mathscr{P}_\lambda$:

1. $\Gamma \vdash \alpha(\sigma_1, \dots, \sigma_{\mathsf{ar}(\alpha)})$ for all $\sigma_i \in T\Gamma$ such that $T\Gamma \models \alpha(\sigma_1, \dots, \sigma_{\mathsf{ar}(\alpha)})$
2. $\Gamma \vdash f^*(\sigma) = \sigma(f)$ for all $\Delta \in \mathscr{P}_\lambda$, morphisms $f\colon \Delta \to T\Gamma$, and $\sigma \in |T\Delta|$.
3. $\Gamma \vdash \eta_\Gamma(x) = x$ for every $x \in \Gamma$.

Note that in the second item above, for every $x \in \Delta$ the operation symbol $f(x) \in |T\Gamma|$ is considered as a term according to Notation 5.3. Hence $\sigma(f)$ is a term, too.

We now show that $\mathbb{T}$ is the free-algebra monad of its induced variety $\mathcal{V}_{\mathbb{T}}$. For each $X \in \mathbf{C}$, the $\mathbf{C}$-object TX carries a canonical Σ-algebra structure with each operation σ_{TX} being defined by $\sigma_{TX}(f) := f^*(\sigma)$. We call TX the *canonical algebra over* X.

▶ **Lemma 5.5.** *Every canonical algebra lies in* $\mathcal{V}_{\mathbb{T}}$.

▶ **Theorem 5.6.** *Each enriched λ-accessible monad $\mathbb{T}$ is the free-algebra monad of its induced variety $\mathcal{V}_{\mathbb{T}}$, with the free algebra on X given by the canonical algebra TX.*

▶ **Remark 5.7.** We have thus shown that given a λ-ary Horn theory $\mathscr{H}$, we we can translate λ-accessible monads on $\mathbf{Str}(\mathscr{H})$ back into λ-ary theories, preserving the notion of model. For example, every ω_1-accessible monad on $\mathbf{Met}$ is induced by an ω_1-ary theory, as illustrated in Example 4.8. The situation is more complicated for κ-ary monads where $\kappa < \lambda$. E.g. we can generate a finitary monad on $\mathbf{Met}$ from a single binary operation of type $\{(x, y) \in A^2 \mid d(x, y) < 1/2\} \to A$. This monad is not induced by any theory with operations of internally finitely presentable (i.e. discrete) arity, in particular neither by an ω-ary theory in our framework nor by a quantitative algebraic theory [20].

6 Conclusions

We have introduced the framework of *relational logic* for reasoning about algebraic structure on categories of (finitary) relational structures axiomatized by possibly infinitary Horn theories, such as partial orders or metric spaces. We have proved soundness and completeness of a generic algebraic deduction system, and we have shown that λ-ary relational algebraic theories are in correspondence with λ-accessible enriched monads when the underlying Horn theory is also λ-ary (where "λ-ary" refers to the arity of operations for relational algebraic theories, and to the number of premisses in axioms for Horn theories). Our results allow for a straightforward specification also of infinitary constructions such as metric completion.

The theory-to-monad direction of the above-mentioned correspondence remains true for κ-ary relational algebraic theories and κ-accessible monads on categories of models of λ-ary Horn theories for $\kappa < \lambda$, e.g. when looking at monads and theories on metric spaces. One open end that we leave for future research is to obtain a more complete coverage of this case, which will require a substantial generalization of both the way arities of operations are defined (these can no longer be taken to be objects of the base category) and in the way the axioms of the theory are organized, likely using more topologically-minded approaches.

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