# Solving the Periodic Scheduling Problem: An Assignment Approach in Non-Periodic Networks

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**Abstract**

The periodic event scheduling problem (PESP) is a well researched problem used for finding good periodic timetables in public transport. While it is based on a periodic network consisting of events and activities which are repeated every period, we propose a new periodic timetabling model using a *non-periodic* network. This is a first step towards the goal of integrating periodic timetabling with other planning steps taking place in the aperiodic network, e.g. passenger assignment or delay management. In this paper, we develop the new model, show how we can reduce its size and prove its equivalence to PESP. We also conduct computational experiments on close-to real-world data from Lower Saxony, a region in northern Germany, and see that the model can be solved in a reasonable amount of time.

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## 1 Introduction

An important aspect of optimising public transport is finding a good periodic timetable. From the passengers' point of view, short travelling times are desirable, which can be achieved by making the timetable as tight as possible. This problem is known as the *Periodic event Scheduling Problem (PESP)* and is well researched. It uses a periodic event-activity network in which each node represents many arrivals or departures, namely one per period. In this paper we develop a new model for the PESP in a (larger) aperiodic network. We first give our motivation why such a model is needed.

Tight periodic timetables minimise travelling times, but are very prone to delays which are inevitable in reality and highly dissatisfactory for the passengers. Hence, apart from short travelling times, a good timetable should also have some degree of delay resistance. Many concepts and ideas on how to increase robustness of a timetable against delays exist, see [15]. However, none of these approaches uses the promising concept of *recoverable robustness* introduced by [13]. The aim is to find a periodic timetable with small travelling times such that in every delay scenario from a given set it is possible to find a disposition timetable which fulfils some quality criteria. To this end, we have to integrate timetabling and delay management. Timetables are determined in a periodic network, but delay management is done in an aperiodic network, since in general delays do not occur periodically. In order to integrate delay management into timetabling, we hence have to find a way to solve both problems in the same network. The same holds for integrating passengers' assignment since also the demand does not occur periodically.

One way for such an integration is to develop a timetabling model which computes a periodic timetable in an aperiodic network, which is the goal of this paper. We call the new model ***P****eriodic* ***T****ime****t****abling in* ***A****periodic Network* (PTTA).


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Periodic timetabling is well studied in the literature. The PESP was first introduced in [27]. It aims at finding a feasible periodic timetable. Instead of only considering the feasibility problem, one can also consider different objective functions. In [17] this was done by minimising the waiting times of the transferring passengers. An alternative formulation which can be solved much faster uses cycle bases, see, e.g. [17, 9, 21]. The problem was solved with a branch-and-bound approach in [16] and with a genetic algorithm in [19]. The modulo simplex [18, 7] and a fast matching approach [20] are more recent heuristics for solving PESP. An approach running several solution methods in parallel was presented in [2]. Computing a periodic timetable in an aperiodic network was already considered in [28]. As opposed to our model, in [28] the decision on which transfer activities are needed is not part of the optimisation process but is fixed before by a simple heuristic rule. In [1] the problem is considered only for a single train line between two stations. A model putting an emphasis on passenger satisfaction and including the passenger routing is proposed in [22]. It uses the assumption that all drive and dwell times are fixed and does not consider track safety constraints. For a survey on timetabling we refer to [3, 6].

The remainder of this paper is structured as follows: The PESP is briefly reviewed in Section 2. In Section 3 we introduce the new timetabling model and make several modifications to the model such that it better meets our needs. In Section 4 we compare the new model PTTA2 to the established model PESP and show that they are equivalent. We present some computational results in Section 5 and conclude the paper with some final remarks and suggestions for further research in Section 6.

## 2 The Periodic Event Scheduling Problem

A model often used for periodic timetabling is the Periodic Event Scheduling Problem (PESP), which was introduced in [27]. In the PESP we are given a period $T$ together with a set of events $\underline{\mathcal{E}}$, which either correspond to the arrival or the departure of a traffic line at some station. Furthermore, we have activities $\underline{\mathcal{A}}$, which represent processes between the events. Together, we obtain an *event-activity-network* (EAN) $\mathcal{N} = (\underline{\mathcal{E}}, \underline{\mathcal{A}})$ in which the events are represented as nodes and the activities as arcs. We distinguish several different types of activities. *Driving activities* model a train line driving from one station to another, while *waiting activities* represent a line waiting at a station. Passengers have the possibility to transfer between different lines, which is included by the *transfer activities*. If a line has a frequency higher than one, i.e. the line is served several times in one period, we want to spread the rides equally over the period. This is done by *synchronisation activities*. *Headway activities* are used to model safety regulations requiring a minimal distance between two consecutive departures or arrivals, or the safety restriction on single-track lines. They usually come in pairs, since it is not clear beforehand in which order the two departures will take place. Given an EAN $\mathcal{N} = (\underline{\mathcal{E}}, \underline{\mathcal{A}})$, we want to find a periodic timetable with period $T$, which is a mapping $\pi\colon \underline{\mathcal{E}} \to \{0, \dots, T-1\}$ assigning a time to every event. To simplify notation we set $\pi_i := \pi(i)$ for $i \in \underline{\mathcal{E}}$. For every activity $a \in \underline{\mathcal{A}}$ a lower bound $L_a \in \mathbb{N}$ and an upper bound $U_a \in \mathbb{N}$ are given. $L_a$ is the minimal time necessary to perform the activity $a$, while $U_a$ is the maximal time allowed for $a$. A timetable is *feasible* if it respects the bounds on the activities, i.e. for every activity $a = (i, j) \in \underline{\mathcal{A}}$ we require $\pi_j - \pi_i + z_a T \in [L_a, U_a]$ for some $z_a \in \mathbb{Z}$. The modulo parameter $z_a$ takes the periodicity into account.

The PESP asks for a feasible timetable. In timetabling we additionally want to minimise the total travelling time summed over all passengers. For $a \in \underline{\mathcal{A}}$ let $w_a \in \mathbb{N}$ be the number of passengers using activity $a$. The following is the basic IP formulation for PESP:

$$\min \sum_{a=(i,j)\in\underline{\mathcal{A}}} w_a \cdot (\pi_j - \pi_i + z_a T) \tag{PESP}$$

$$\pi_j - \pi_i + z_a T \le U_a \qquad a=(i,j)\in\underline{\mathcal{A}} \tag{1}$$

$$\pi_j - \pi_i + z_a T \ge L_a \qquad a=(i,j)\in\underline{\mathcal{A}} \tag{2}$$

$$\pi_i \in \{0,\dots,T-1\} \qquad i\in\underline{\mathcal{E}} \tag{3}$$

$$z_a \in \mathbb{Z} \qquad a\in\underline{\mathcal{A}}. \tag{4}$$

Details about periodic timetabling can be found in the literature on PESP, a good introduction is given in [12, 17].

## 3 A New Timetabling Model

As mentioned before, we want to compute a timetable in an aperiodic EAN. While in a periodic EAN the events represent the arrivals or departures of a *line* at some station (for unit line frequencies), in an aperiodic EAN they model the arrival or departure of a single *trip*. A trip is the journey of a vehicle from the beginning of a line to its end, i.e. one line can yield several trips (based on the number of periods and the frequency of the line). Hence, instead of only considering the lines, we consider all trips of the lines separately. This means we have to "roll out" the periodic EAN to an aperiodic one in a time interval $[t_{\min}, t_{\max}]$, a procedure which is also used in delay management, where a timetable is given and used for rolling out. Since we want to determine the timetable, we cannot use this roll-out procedure. Nevertheless, we first repeat how the roll-out is done for a given timetable (based on [14]) and then explain our procedure which leaves the timetable open.

**Rolling out with a given timetable.** For every $i\in\underline{\mathcal{E}}$ set

$$\pi_{\text{first}}(i) := \min\{\pi_i + kT : \pi_i + kT \ge t_{\min}, k\in\mathbb{Z}\},$$

$$\pi_{\text{last}}(i) := \max\{\pi_i + kT : \pi_i + kT \le t_{\max}, k\in\mathbb{Z}\}.$$

These are the first respectively last times the event $i$ occurs in the considered time horizon. The roll-out process then works as follows:

- For every $i\in\underline{\mathcal{E}}$ and $1\le s\le K_i := \left\lfloor \frac{\pi_{\text{last}}(i)-\pi_{\text{first}}(i)}{T}\right\rfloor + 1$ construct an aperiodic event $i_s$ with $\pi_{i_s} = \pi_{\text{first}}(i) + (s-1)T$. Let $\mathcal{E}(i) := \{i_s : 1\le s\le K_i\}$ be the set of aperiodic events corresponding to the periodic event $i$.
- For every $a=(i,j)\in\underline{\mathcal{A}}\backslash\underline{\mathcal{A}}_{\text{head}}$ (where $\underline{\mathcal{A}}_{\text{head}}$ is the set of headway activities) and $i_s\in\mathcal{E}(i)$ determine $j_t\in\mathcal{E}(j)$ (if it exists) such that $L_a\le \pi_{j_t}-\pi_{i_s}\le U_a$. We create an aperiodic activity $a_{st}=(i_s,j_t)$ and set $L_{a_{st}}=L_a$, $U_{a_{st}}=U_a$ and $w_{a_{st}}=w_a$. For each pair $a=(i,j), a'=(j,i)\in\underline{\mathcal{A}}_{\text{head}}$ of headway activities and $s\in\mathcal{E}(i), t\in\mathcal{E}(j)$ create two aperiodic activities $a_{st}=(i_s,j_t), a_{ts}=(j_t,i_s)$ with $L_{a_{st}}=L_a$ and $L_{a_{ts}}=T-U_a$. If $j_t$ does not exist we are at the end of $[t_{\min},t_{\max}]$ and nothing has to be done.

Note that in [14] the activities in the rolled out network do not have upper bounds, since these are ignored in delay management. Since we do timetabling, we want to respect the upper bounds and add them also in the rolled out EAN. Another particularity are the headway activities which ensure a security distance between two consecutive departures. Since it is not clear which of the two events will take place first, they come in pairs. For every pair of these headway arcs $a_{st}, a_{ts}$ exactly one of them is chosen for which the lower bound has to be respected, i.e. the pair $a = (i, j), a' = (j, i) \in \underline{\mathcal{A}}_{\text{head}}$ yields the following constraints:

$$\text{For all } 1 \leq s \leq K_i, 1 \leq t \leq K_j \text{ either } \pi_{j_t} - \pi_{i_s} \geq H_{ij} \text{ or } \pi_{i_s} - \pi_{j_t} \geq H_{ji},$$

where $H_{ij} = L_a$, $H_{ji} = L_{a'}$. For further details, we refer to [14]. (Note that the problem can be interpreted as a resource-constrained machine scheduling problem, see, e.g., [4, 26]). A common assumption is that

$$0 \leq L_a \leq T - 1 \text{ and } L_a \leq U_a \leq L_a + T - 1 \text{ for all } a \in \mathcal{A}. \tag{5}$$

In this case, the $j_t$ in the roll-out process is uniquely determined, if it exists. If we do not use this assumption, we may have to choose one of several possible $j_t$. We will later introduce a rule how to make this choice, but for now it is enough to choose an arbitrary one.

The goal of this paper is to compute the timetable in the rolled out EAN. Hence, we cannot use the timetable when rolling out. However, the timetable information is important for determining the activities between the correct arrival and departure events. This is shown in Figure 1 where in (c) and (d) two different timetables are used for the roll-out leading to two different aperiodic networks. Since we do not know beforehand which activities will be needed for the optimal timetable, we allow all possibilities (see part (b) of Figure 1) and leave it to the optimization to choose the correct activities together with the optimal timetable.

We hence adapt the procedure in the following way.

**Rolling out without knowing the timetable.**

- For every periodic event $i \in \underline{\mathcal{E}}$ and $1 \leq s \leq K := \lfloor \frac{t_{\max} - t_{\min}}{T} \rfloor + 1$ create an aperiodic event $i_s$. Let $\mathcal{E}(i) := \{i_s : 1 \leq s \leq K\}$ be the set of all aperiodic events corresponding to $i$. The set of all events is $\mathcal{E} := \cup_{i \in \underline{\mathcal{E}}} \mathcal{E}(i)$.
- For every periodic activity $a = (i, j) \in \underline{\mathcal{A}} \backslash \underline{\mathcal{A}}_{\text{head}}$, for exactly one arc $a = (i, j)$ of every pair of headway activities and for every $1 \leq s, t \leq K$ create a *possible (aperiodic) activity* $a_{st}$ with $L_{a_{st}} = L_a$, $U_{a_{st}} = U_a$ and $w_{a_{st}} = w_a$. Let $\mathcal{A}(a) := \{a_{st} = (i_s, j_t) : 1 \leq s, t \leq K\}$ be the set of possible activities corresponding to $a$. The set of all possible activities is

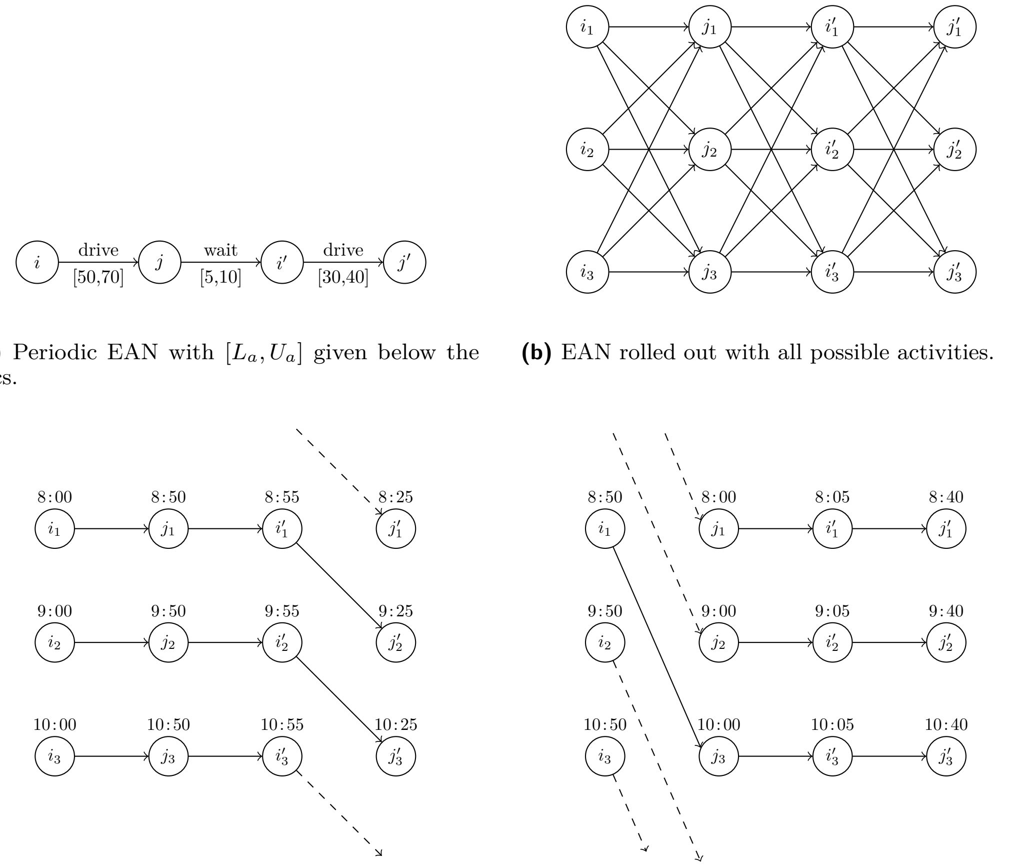
$$\mathcal{A} := \bigcup_{a \in \underline{\mathcal{A}}} \mathcal{A}(a). \tag{6}$$

The final network $(\mathcal{E}, \mathcal{A})$ is called the *rolled out network*.

We remark that when rolling out with a timetable, the number $K_i$ of aperiodic events corresponding to a periodic event $i$ depends on $i$. This is not the case when rolling out without knowing the timetable, where we have a constant $K$. However, this only makes a difference if our planning horizon $[t_{\min}, t_{\max}]$ covers a fractional number of periods. E.g. if we consider 3.5 periods, some events will take place three times and some four times. Since this depends on the timetable, we cannot make this distinction when rolling out without knowing the timetable, where we have to consider each event four times. If we assume that we only consider whole periods, $K_i$ is constant for all $i \in \underline{\mathcal{E}}$ and thus both procedures yield the same number of events.

The rolled out network contains not only the actual activities, but all possibilities for the activities. Thus, when fixing the timetable we have to simultaneously solve an assignment problem: for each periodic activity we have to choose exactly one of the corresponding arcs in every considered period. In order to do so we introduce a binary variable

$$u_a = \begin{cases} 1 & \text{if } a \text{ is chosen,} \\ 0 & \text{otherwise.} \end{cases}$$




**(a)** Periodic EAN with $[L_a, U_a]$ given below the arcs.

**(b)** EAN rolled out with all possible activities.

**(c)** Rolled out EAN after choosing a feasible timetable and the corresponding activities. The dashed arcs indicate the connections entering or leaving the planning horizon.

**(d)** Rolled out EAN with another feasible timetable, which results in different activities.

**Figure 1** Rolling out a periodic EAN without knowing the timetable for the time interval [8:00,10:59] with $T = 60$, i.e. $K = 3$.

Furthermore, for $a \in \mathcal{A}$ we set $b_a := \lceil \frac{U_a}{T} \rceil$ and $K_a := K - b_a$. It will become clear later why we need this notation. Below we give the first idea of the constraints we need. The first correct formulation will be given in PTTA1.

$$
\begin{aligned}
\min \quad & \sum_{a=(i_s,j_t)\in\mathcal{A}} w_a \cdot u_a(\pi_{j_t} - \pi_{i_s}) && && (7)\\
\text{s.t.} \quad & \pi_{j_t} - \pi_{i_s} + M(u_a - 1) \le U_a && a = (i_s, j_t) \in \mathcal{A} && (8)\\
& \pi_{j_t} - \pi_{i_s} + M(1 - u_a) \ge L_a && a = (i_s, j_t) \in \mathcal{A} && (9)\\
& \pi_{i_s} - \pi_{i_{s-1}} = T && i_s \in \mathcal{E}, 2 \le s \le K && (10)\\
& \sum_{t: a'=(i_s,j_t)\in\mathcal{A}} u_{a'} = 1 && a = (i,j) \in \underline{\mathcal{A}}, 1 \le s \le \ K_a && (11)\\
& \pi_i \ge t_{\min} && i \in \mathcal{E} && (12)\\
& \pi_{i_1} \le t_{\min} + T - 1 && i \in \underline{\mathcal{E}} && (13)\\
& \pi_i \in \mathbb{N} && i \in \mathcal{E} && (14)\\
& u_a \in \{0, 1\} && a \in \mathcal{A}. && (15)
\end{aligned}
$$

The objective function minimises the total travelling time over all passengers. In the case that an activity $a$ is chosen, i.e. $u_a = 1$, constraints (8) and (9) ensure that the upper and lower bounds for this activity are respected. If $a$ is not selected, the constraints become redundant for appropriately chosen $M$. Constraints (10) are called *synchronisation constraints* and ensure that the timetable has period $T$. For every periodic activity the assignment constraint (11) chooses exactly one of the corresponding aperiodic activities in every period in such a way that it fits to the timetable constraints (8) and (9). For the last $b_a$ periods it is possible that no feasible choice exists and hence, we omit the constraints for these periods. We will later explain why this problem cannot occur for the other periods. Constraints (12) and (13) enforce that no event is scheduled earlier than $t_{\min}$ and that the first event takes place in the first period we consider. Finally, to ensure that at least one period is considered in constraints (11) we assume $b_a < K$ for all $a \in \mathcal{A}$, i.e. the planning horizon is sufficiently large.

### Can we disregard $s > K_a$ in the assignment constraints?

As mentioned above, if we have a timetable $\pi$ and an $s > K_a$ there may be no $t$ such that $\pi_{j_t} - \pi_{i_s} \in [L_a, U_a]$, since the time of the event we would theoretically have to choose exceeds the planning horizon, as already seen in Figure 1 for the dashed arcs. Hence, we disregard the last $b_a$ periods for the assignment. We will show in Lemma 3 that this indeed does not exclude optimal solutions.

However, disregarding the last $b_a$ periods causes a problem with the objective function. Since setting $u_a = 1$ increases the objective value and we minimise, for every $a = (i, j) \in \mathcal{A}, s > K_a$ we will always have $u_{(i_s, j_t)} = 0$ for every $t$ in an optimal solution. Hence, the passengers in the last $b_a$ periods are (falsely) not considered. Fortunately, we can use the following trick to overcome this problem: Due to periodicity the contribution to the objective function of these passengers is the same as in all other periods. This means that we can correct this mistake in the objective function by replacing it by

$$\min \sum_{a=(i_1, j_t) \in \mathcal{A}} w_a \cdot u_a(\pi_{j_t} - \pi_{i_1}) \cdot K.$$

We obtain the following formulation:

$$\begin{aligned} \min & \sum_{a=(i_1, j_t) \in \mathcal{A}} w_a \cdot u_a(\pi_{j_t} - \pi_{i_1}) \cdot K \qquad \text{(PTTA1)} \\ \text{s.t.} \ & (8) - (15) \end{aligned}$$

### Analysis of the headway constraints

Note that when rolling out with a timetable we handled the headway activities differently than when rolling out without knowing a timetable. For the PESP it is known that even without knowing the order of the events, one headway constraint suffices to cover a pair of headway activities. This is also true in our case, i.e. both ways of handling the headways are equivalent. The proof can be found in the appendix.

▶ **Lemma 1.** *Let $a = (i, j), a' = (j, i) \in \underline{\mathcal{A}}_{\text{head}}$. The following statements are equivalent:*
**(a)** *For all $1 \le s, t \le K$ we have either $\pi_{j_t} - \pi_{i_s} \ge L_a = H_{ij}$ or $\pi_{i_s} - \pi_{j_t} \ge L_{a'} = H_{ji}$.*
**(b)** *For all $1 \le s \le K_a$ there is some $1 \le t \le K$ such that $\pi_{j_t} - \pi_{i_s} \in [L_a, U_a] = [H_{ij}, T - H_{ji}]$.*

In the following, for simplicity, we will always handle the headways as given by the constraints in $(b)$, regardless whether we roll out with or without using a given timetable. In the following we analyse and strengthen PTTA1.

### Can we linearise the quadratic objective function?

This can be done using standard techniques. We introduce a new variable $F_a$ for $a = (i_1, j_t) \in \mathcal{A}$ to obtain the following equivalent formulation:

$$\min \sum_{a=(i_1,j_t)\in\mathcal{A}} w_a F_a \cdot K \tag{PTTA2}$$

$$\text{s.t. } (8) - (15)$$

$$F_a \geq M(u_a - 1) + \pi_{j_t} - \pi_{i_1} \qquad a = (i_1, j_t) \in \mathcal{A} \tag{16}$$

$$F_a \in \mathbb{N} \qquad a = (i_1, j_t) \in \mathcal{A}. \tag{17}$$

It is straightforward to prove that the linearisation is correct, i.e. PTTA1 and PTTA2 are equivalent.

### How to choose $M$?

▶ **Lemma 2.** *$M := t_{\max} + T - 1 + \max_{a\in\mathcal{A}} L_a$ is sufficiently large.*

**Proof.** We have to show that for every $a = (i_s, j_t) \in \mathcal{A}$ the following inequalities hold:

- $M \geq \pi_{i_s} - \pi_{j_t} + L_a$
- $M \geq \pi_{j_t} - \pi_{i_s} - U_a$
- $M \geq \pi_{j_t} - \pi_{i_s} - F_a$

In order to see this we use the following observations. First, using constraints (10) inductively yields $\pi_{i_s} = \pi_{i_1} + (s-1)T$. Second, by constraints (13) we know that $\pi_{i_1} \leq t_{\min} + T - 1$. And finally, by choice of $K$ we have $KT \leq t_{\max} - t_{\min} + T$. Putting all this together we obtain

$$\pi_{i_s} = \pi_{i_1} + (s-1)T \leq \pi_{i_1} + (K-1)T \leq t_{\min} + KT - 1 \leq t_{\max} + T - 1.$$

Thus, we have $\pi_{i_s} - \pi_{j_t} + L_a \leq \pi_{i_s} + L_a \leq M$, which shows the first inequality. Similarly, we obtain the other two. ◀

### Reducing the number of variables and constraints

So far, we have considered every combination $(i_s, j_t)$ for $(i,j) \in \underline{\mathcal{A}}$ and $1 \leq s, t \leq K$. However, for some of these we can show that they cannot be selected in a feasible solution.

▶ **Lemma 3.** *Let $(i,j) \in \underline{\mathcal{A}}$ and $1 \leq s \leq K$. Then for $a = (i_s, j_t)$ with $t \geq s + 1 + b_a$ or $t \leq s - 1$ we have $u_a = 0$ in any feasible solution.*

**Proof.** We have $t_{\min} \leq \pi_{i_1}, \pi_{j_1} \leq t_{\min} + T - 1$, which implies $1 - T \leq \pi_{j_1} - \pi_{i_1} \leq T - 1$. By periodicity we obtain for $t \geq s + 1 + b_a$:

$$\begin{aligned}\pi_{j_t} - \pi_{i_s} &= (\pi_{j_1} + (t-1)T) - (\pi_{i_1} + (s-1)T) \geq 1 - T + (t-s)T \\ &\geq 1 - T + (1 + b_a)T \geq 1 + U_a > U_a.\end{aligned}$$

Similarly, for $t \leq s-1$ we have:

$$\begin{aligned}\pi_{j_t} - \pi_{i_s} &= (\pi_{j_1} + (t-1)T) - (\pi_{i_1} + (s-1)T) \\ &\leq T - 1 + (t-s)T \leq T - 1 - T = -1 < L_a\end{aligned}$$

By constraints (8) and (9) it follows $u_a = 0$. ◀

Hence, we only have to consider $(i_s, j_t)$ for $s \leq t \leq s + b_a$. In particular, for $s \leq K_a$ we only have to consider $t \leq K$, i.e. all relevant $j_t$ are in the planning horizon. We adapt $\mathcal{A}(a)$ in (6) and now use the smaller sets

$$\mathcal{A}(a) \coloneqq \{a_{st} = (i_s, j_t) : 1 \leq s \leq K, s \leq t \leq \min\{s + b_a, K\}\}. \tag{18}$$

Note that this may be a significant reduction, e.g. under the assumption (5) we have $U_a \leq L_a + T - 1 \leq 2(T-1)$ and hence $b_a \leq 2$.

We can reduce the activities we have to consider even further with the following reasoning: Because of the periodicity of the timetable, the choice of $u_{(i_1,j_t)}$ already determines the value of $u$ for later periods. Hence, we only need to consider variables $u_{(i_1,j_t)} \in \mathcal{A}$ with $i_1$ being the event in the first period instead of $u_{(i_s,j_t)} \in \mathcal{A}$ for all $i_s$ with $(i_s, i_t) \in \mathcal{A}$. This affects constraints (8), (9), (11), and (15) in PTTA2 and reduces the number of variables and constraints in our formulation considerably leading to the following IP. Note that we also use the reduced set $\mathcal{A}$ resulting from (18).

$$\min \sum_{a=(i_1,j_t)\in\mathcal{A}} w_a F_a \cdot K \tag{PTTA3}$$

$$\begin{aligned}
&\pi_{j_t} - \pi_{i_1} + M(u_a - 1) \leq U_a && a = (i_1, j_t) \in \mathcal{A} && (19)\\
&\pi_{j_t} - \pi_{i_1} + M(1 - u_a) \geq L_a && a = (i_1, j_t) \in \mathcal{A} && (20)\\
&\pi_{i_s} - \pi_{i_{s-1}} = T && i_s \in \mathcal{E}, 2 \leq s \leq K && (21)\\
&\sum_{t:a=(i_1,j_t)\in\mathcal{A}} u_a = 1 && (i,j) \in \underline{\mathcal{A}} && (22)\\
&F_a \geq M(u_a - 1) + \pi_{j_t} - \pi_{i_1} && a = (i_1, j_t) \in \mathcal{A} && (23)\\
&\pi_i \geq t_{\min} && i \in \mathcal{E} && (24)\\
&\pi_{i_1} \leq t_{\min} + T - 1 && i \in \underline{\mathcal{E}} && (25)\\
&\pi_i \in \mathbb{N} && i \in \mathcal{E} && (26)\\
&u_a \in \{0,1\} && a = (i_1, j_t) \in \mathcal{A} && (27)\\
&F_a \in \mathbb{N} && a = (i_1, j_t) \in \mathcal{A}. && (28)
\end{aligned}$$

▶ **Lemma 4.** *PTTA2 and PTTA3 are equivalent.*

The proof is in the appendix.

## 4 Comparison of PTTA2 and PESP

We now want to compare the new assignment-based model with the established model PESP. We consider the version PTTA2. Let an instance of PESP be given. We roll out the EAN without knowing a timetable. Suppose we can solve either PESP or PTTA2 quickly. Does this help to find a solution of the other problem? More precisely, we are interested in the following questions:

**(a)** Let $(\tilde{\pi}, z)$ be a feasible (optimal) solution for PESP. Can we use it to construct a feasible (optimal) solution for PTTA2?

**(b)** Let $(\pi, u, F)$ be a feasible solution for PTTA2. Can we use it to construct a feasible (optimal) solution for PESP?

We start with (a). Let a periodic timetable be given. As an intermediate step we consider the roll-out w.r.t this timetable. The following lemma ensures that for any realization $i_s$ of event $i$ (except for those at the end of the planning horizon) we can choose a corresponding realization $j_t$ feasible for the rolled out constraint $(i_s, j_t)$.

▶ **Lemma 5.** *Let $(\tilde{\pi}, z)$ be a feasible solution for PESP and $\pi$ the solution constructed in the roll-out process. Let $a = (i, j) \in \underline{\mathcal{A}}$ and $k, l \in \mathbb{Z}$ such that $\pi_{\text{first}}(i) = \tilde{\pi}_i + kT$ and $\pi_{\text{first}}(j) = \tilde{\pi}_j + lT$. For any choice of $1 \leq s \leq K$ and $t := z_a + k - l + s$ with $t \leq K$, the bounds on activity $(i_s, j_t)$ are fulfilled, i.e. $\pi_{j_t} - \pi_{i_s} \in [L_a, U_a]$.*

**Proof.** By definition of $\pi$ we have $\pi_{i_s} = \pi_{\text{first}}(i) + (s-1)T = \tilde{\pi}_i + (k+s-1)T$ and $\pi_{j_t} = \pi_{\text{first}}(j) + (t-1)T = \tilde{\pi}_j + (l+t-1)T$. Hence, it follows

$$\pi_{j_t} - \pi_{i_s} = \tilde{\pi}_j - \tilde{\pi}_i + (l - k - s + t)T = \tilde{\pi}_j - \tilde{\pi}_i + z_a T \in [L_a, U_a].$$ ◀

▶ **Corollary 6.** *In the situation of Lemma 5 for $1 \leq s \leq K_a$ there exists an $s \leq t \leq s + b_a$ with $\pi_{j_t} - \pi_{i_s} \in [L_a, U_a]$.*

**Proof.** We remark that by Lemma 3 it follows that for $t$ as chosen in Lemma 5 we have $s \leq t \leq s + b_a$. Since $s \leq K_a$, this implies $t \leq s + b_a \leq K_a + b_a = K$, so by Lemma 5 we obtain $\pi_{j_t} - \pi_{i_s} \in [L_a, U_a]$. ◀

As mentioned already for the roll-out process for a given timetable, the choice of $t$ has not to be unique in the general case and we could choose one of the possibilities arbitrarily. From now on, we will choose $t$ as in Lemma 5.

We can use these results to construct a solution for the rolled out network. Let an instance of PESP $(\underline{\mathcal{E}}, \underline{\mathcal{A}})$ be given and $(\mathcal{E}, \mathcal{A})$ be the EAN received by rolling out without knowing a solution. Let $(\tilde{\pi}, z)$ be a solution for PESP. We define $\pi$ as in the roll-out process with the timetable given, i.e. $\pi_{i_s} = \pi_{\text{first}}(i) + (s-1)T$. Furthermore, for $a = (i_s, j_t) \in \mathcal{A}$ we choose $k, l$ as in Lemma 5 and set

$$u_a = \begin{cases} 1 & \text{if } t = z_a + k - l + s, \\ 0 & \text{otherwise,} \end{cases}$$

and for $a = (i_1, j_t)$ we set

$$F_a = \begin{cases} \pi_{j_t} - \pi_{i_1} & \text{if } u_a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This construction gives us a feasible solution for PTTA2 in the rolled out network as the following lemma shows. The proof can be found in the appendix.

▶ **Lemma 7.** *Let $(\tilde{\pi}, z)$ be a solution for PESP with objective value $\tilde{f}$. Then $(\pi, u, F)$ as defined above is a feasible solution for PTTA2 and the corresponding objective value is $f = K\tilde{f}$.*

We now turn to (b). Again, let an instance of PESP $(\underline{\mathcal{E}}, \underline{\mathcal{A}})$ be given and $(\mathcal{E}, \mathcal{A})$ be the EAN received by rolling out without knowing a solution. Let $(\pi, u, F)$ be a feasible solution to PTTA2. For $i \in \underline{\mathcal{E}}$ we set

$$\tilde{\pi}_i := \pi_{i_1} \mod T,$$

i.e. there is some $r_i \in \mathbb{Z}$ such that $\pi_{i_1} = \tilde{\pi}_i + r_i T$. For $a = (i, j) \in \underline{\mathcal{A}}$ there is some $t$ such that $u_{(i_1, j_t)} = 1$. Set

$$z_a := r_j - r_i + t - 1.$$

Also this construction works, i.e. we get a feasible solution for PESP with bounded objective function value. Again, the proof can be found in the appendix.

▶ **Lemma 8.** *Let $(\pi, u, F)$ be a feasible solution to PTTA2 with objective value $f$. Then $(\tilde{\pi}, z)$ as defined above is a feasible solution for PESP and for its objective value $\tilde{f}$ we have $\tilde{f} \leq f \cdot \frac{1}{K}$.*

Putting the two constructions together, we finally conclude that we can in fact construct an optimal solution for PESP if we know an optimal solution for PTTA2 and vice versa. In particular, it makes no difference whether one computes a solution with PTTA2 or rolls out a solution obtained with PESP, i.e. in this sense, PTTA2 and PESP are equivalent. The proof directly follows from Lemma 7 and Lemma 8 (see appendix).

▶ **Corollary 9.** *If $(\tilde{\pi}, z)$ is an optimal solution for PESP, the solution $(\pi, u, F)$ constructed in Lemma 7 is optimal for PTTA2. On the other hand, if $(\pi, u, F)$ is an optimal solution for PTTA2, the solution $(\tilde{\pi}, z)$ constructed in Lemma 8 is optimal for PESP.*

## 5 Computational Experiments

In this section, we test the performance of the new models when solving the IP formulations with Gurobi and compare them to PESP. We use data of the regional railway network in a region of Lower Saxony in northern Germany, since they have a size for which our integer programs can still be solved in reasonable time. The dataset is part of the open-source software framework LinTim, see [24, 23]. We use LinTim to generate different line concepts and the resulting EANs. An overview of the number of lines $|\mathcal{L}|$ and the size of the EANs is given in Table 1. We solve PTTA2 and PTTA3 for different time horizons (we vary the number of periods from $K = 3, 4, \ldots, 15$), observe the run time and compare it to the run time when solving PESP. We implemented the IP models in Python and ran them on an Lenovo laptop with Intel(R) Core(TM) i5-10310U CPU @ 1.70GHz, 2.21 GHz and 16 GB RAM using the solver Gurobi 9.1.1 ([10]). The results are shown in Figure 2.

**Table 1** Size of the periodic EAN for the used line concepts.

| Line concept | $\lvert\mathcal{L}\rvert$ | $\lvert\underline{\mathcal{E}}\rvert$ | $\lvert\underline{\mathcal{A}}\rvert$ |
|---|---|---|---|
| `line concept 1` | 5 | 180 | 262 |
| `line concept 2` | 6 | 196 | 314 |
| `line concept 3` | 6 | 212 | 372 |

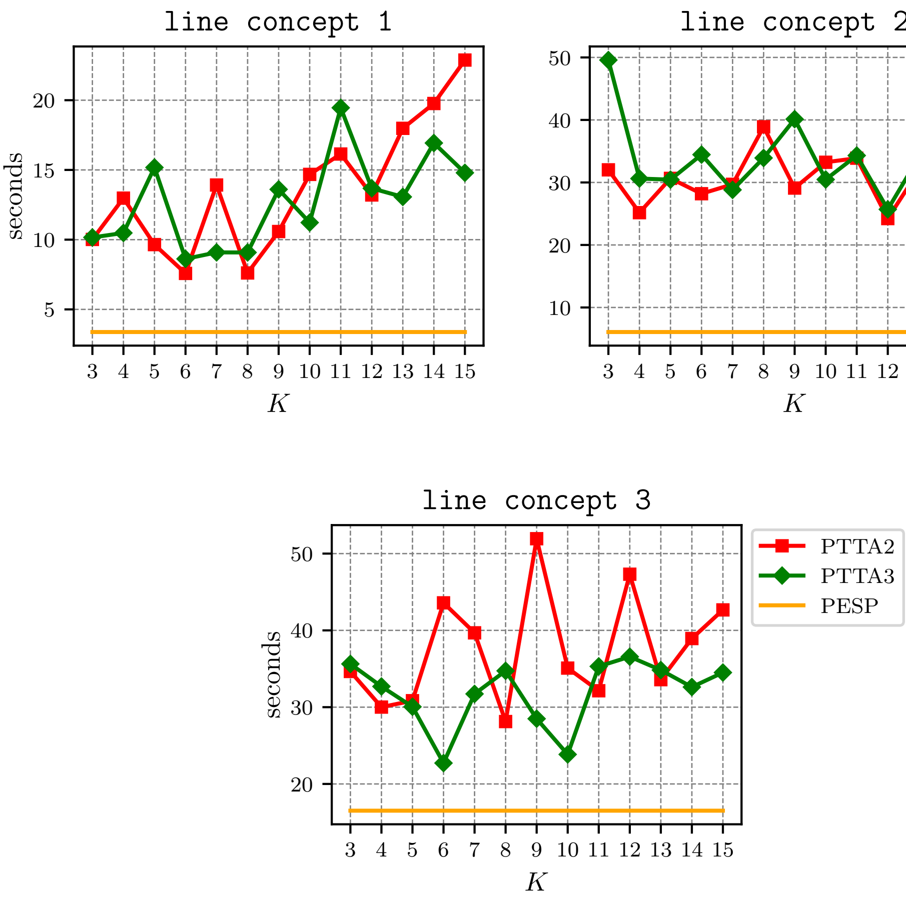



**Figure 2** Average run time for different line concepts with varying $K$.

We first note that, as expected due to the higher number of variables and the additional assignment constraints, for all versions of PTTA the solver takes much longer than for PESP. However, recall that our motivation was to integrate delay management – a task the PESP is not suited for – so we do not have the aspiration to beat the PESP when doing pure timetabling. Since PTTA3 only solves the assignment for the first period, while PTTA2 does this for all periods, one would expect it to be faster solvable than PTTA2. Indeed, we can see this behaviour in the instance `line concept 3`. For `line concept 2` both models perform quite similar. In the instance `line concept 1` we can observe that for larger $K$ the run time of PTTA2 increases more than for PTTA3, which can again be explained with PTTA3 only solving the assignment in the first period. An exception is the peak of PTTA3 at $K = 11$. However, inspecting the progress of the solver shows that the optimal solution was actually found much earlier and the most part of the run time was dedicated to proving optimality, so we treat this as an random outlier. The instance `line concept 3`, which is the largest one, shows the largest variance. Investigating the solving process shows that also here the solver often has difficulties to determine that the incumbent solution is indeed optimal, a well known phenomenon for many integer problems. Thus, providing dual bounds has the potential to speed up the solving process significantly.

## 6 Conclusion

We have developed a new model for periodic timetabling which uses a non-periodic network as basis. We have shown that the new model is equivalent to PESP and that – although this was not our main focus – the achieved run times are acceptable. We also derived a streamlined version which uses significantly less variables and constraints.

The new model opens many possibilities for future research. An obvious line of research is to strengthen its IP formulation, e.g. by using dual bounds, to speed up the solving process. A possible extension of our model could be to allow more flexibility in the synchronisation constraints, e.g. to allow that the differences between repetitions of events are not exactly $T$ but in some interval $[T - \epsilon, T + \epsilon]$. Our main interest, however, is to use the model for integration purposes. Here, the following topics are of particular interest.

First, we plan to use the new aperiodic model for integrating timetabling and delay management in a two-stage model. This is necessary if the practically relevant concept of recovery robustness [13, 8] is to be used in which we look for a timetable that can be recovered by a suitable delay management strategy (see [11, 5] for an overview on delay management). Note that the reduced model PTTA3 cannot be used in this context since for delay management all periods need to be considered separately. Second, the new model can also be used for dealing with timetabling problems with different line frequencies. This topic is only scarcely treated in the literature on PESP, its main difficulty being to distribute passengers on the different possible transfer activities before knowing the timetable. We currently use PTTA to get an optimal distribution of passengers even if the frequencies between incoming and outgoing trains differ from each other.

Finally, we suppose that the model can also be used to integrate timetabling and passenger routing as done in [25].

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# A Proofs

## Proof of Lemma 1

**Proof.** First, note that $U_a = T - H_{ji} \leq T$ for $a = (i,j) \in \mathcal{A}_{\text{head}}$, i.e. $K_a = K - \lceil \frac{U_a}{T} \rceil = K - 1$.
"$(a) \Rightarrow (b)$" Let $1 \leq s \leq K-1$. We consider the event $j_K$ in the last period. Since the event $i_s$ takes place in the $s$-th period, we have $\pi_{i_s} < \pi_{j_K}$. In particular, $\pi_{i_s} - \pi_{j_K} < 0 \leq H_{ji}$ and hence, by $(a)$, we have $\pi_{j_K} - \pi_{i_s} \geq H_{ij}$. Let now $t$ be minimal such that $\pi_{j_t} - \pi_{i_s} \geq H_{ij} = L_a$. It remains to show that $\pi_{j_t} - \pi_{i_s} \leq T - H_{ji} = U_a$.
First case: $t > 1$. By minimality of $t$ we have $\pi_{j_{t-1}} - \pi_{i_s} < H_{ij}$ and hence, $\pi_{i_s} - \pi_{j_{t-1}} \geq H_{ji}$. This yields $\pi_{j_t} - \pi_{i_s} = \pi_{j_{t-1}} + T - \pi_{i_s} \leq T - H_{ji} = U_a$.
Second case: $t = 1$. Assume $\pi_{j_1} - \pi_{i_s} > T - H_{ji}$. Then $\pi_{j_1} - \pi_{i_{s+1}} = \pi_{j_1} - \pi_{i_s} - T > -H_{ji}$, i.e. $\pi_{i_{s+1}} - \pi_{j_1} < H_{ji}$. Hence, we must have $\pi_{j_1} - \pi_{i_{s+1}} \geq H_{ij}$, which in particular means that $\pi_{j_1} \geq \pi_{i_{s+1}}$. Since $j_1$ takes place in the first period and $i_{s+1}$ in the $s+1$-th period, this is a contradiction. Thus, our assumption was false and we have $\pi_{j_1} - \pi_{i_s} \leq T - H_{ji} = U_a$.
"$(b) \Rightarrow (a)$" We first consider $1 \leq s \leq K-1$. By assumption there is some $t'$ such that $\pi_{j_{t'}} - \pi_{i_s} \in [H_{ij}, T - H_{ji}]$. For $t \geq t'$ we have $\pi_{j_t} - \pi_{i_s} \geq \pi_{j_{t'}} - \pi_{i_s} \geq H_{ij}$. On the other hand, for $t < t'$ we have $\pi_{j_t} \leq \pi_{j_{t'}} - T$ and hence $\pi_{i_s} - \pi_{j_t} \geq \pi_{i_s} - \pi_{j_{t'}} + T \geq H_{ji}$. Thus, for every $t$ one of the conditions is fulfilled.
It remains to show the claim for $s = K$. Using the assumption for $s' = K-1$ yields the existence of some $t'$ such that $\pi_{j_{t'}} - \pi_{i_{K-1}} \in [H_{ij}, T - H_{ji}]$. In particular, $\pi_{j_{t'}} \geq \pi_{i_{K-1}}$, which implies $t' \geq K-1$.
First case: $t' = K-1$. We have $\pi_{j_K} - \pi_{i_K} = (\pi_{j_{K-1}} + T) - (\pi_{i_{K-1}} + T) = \pi_{j_{K-1}} - \pi_{i_{K-1}} \geq H_{ij}$. Furthermore, for $t \leq K-1$ it follows $\pi_{i_K} - \pi_{j_t} = \pi_{i_{K-1}} + T - \pi_{j_t} \geq \pi_{i_{K-1}} + T - \pi_{j_{K-1}} \geq H_{ji}$, where the last inequality follows from $\pi_{j_{K-1}} - \pi_{i_{K-1}} \leq T - H_{ji}$.
Second case: $t' = K$. For every $t \leq K$ we have $\pi_{j_t} - \pi_{i_K} \leq \pi_{j_K} - \pi_{i_K} = \pi_{j_K} - \pi_{i_{K-1}} - T \leq -H_{ji}$, which implies $\pi_{i_K} - \pi_{j_t} \geq H_{ji}$. ◀

## Proof of Lemma 4

**Proof.** "$\Rightarrow$" Let $(\pi, u, F)$ be a solution for PTTA2. For $a = (i_1, j_t)$ set $u'_a := u_a$. Clearly, $(\pi, u', F)$ is a feasible solution for PTTA3 and the objective values coincide. "$\Leftarrow$" Let $(\pi, u', F)$ be a solution for PTTA3. For $a = (i_s, j_t) \in \mathcal{A}$ set $u_a := u'_{(i_1, j_{t-s+1})}$. Note that since $a \in \mathcal{A}$ we have $s \leq t \leq s + b_a$ and therefore $1 \leq t - s + 1 \leq 1 + b_a$, which implies that also $(i_1, j_{t-s+1}) \in \mathcal{A}$. We show that $(\pi, u, F)$ is a feasible solution for PTTA2:

- Let $a = (i_s, j_t) \in \mathcal{A}$. We have

$$\begin{aligned} &\pi_{j_t} - \pi_{i_s} + M(u_a - 1) = (\pi_{j_{t-s+1}} + (s-1)T) - (\pi_{i_1} + (s-1)T) + M(u_a - 1) \\ =&\pi_{j_{t-s+1}} - \pi_{i_1} + M(u'_{(i_1, j_{t-s+1})} - 1) \leq U_{(i_1, j_{t-s+1})} = U_a, \end{aligned}$$

  which shows constraints (8). Analogously be obtain (9).
- Let $(i,j) \in \underline{\mathcal{A}}, 1 \leq s \leq K$. We have

$$\sum_{t: a = (i_s, j_t) \in \mathcal{A}} u_a = \sum_{t: a = (i_1, j_{t-s+1}) \in \mathcal{A}} u'_a = 1$$

  and hence, (11) holds.
- Constraints (10) and (12) to (17) follow immediately.

Consequently, $(\pi, u, F)$ is a feasible solution for PTTA2 with the same objective value as $(\pi, u', F)$. ◀

## Proof of Lemma 7

**Proof.** We check that $(\pi, u, F)$ fulfils all constraints:
- (8) and (9) are fulfilled by choice of $u$ and Lemma 5.
- Let $i_s \in \mathcal{E}$, $2 \leq s \leq K$. By definition of $\pi$ it follows

$$\pi_{i_s} - \pi_{i_{s-1}} = (\pi_{\text{first}}(i) + (s-1)T) - (\pi_{\text{first}}(i) + (s-2)T) = T,$$

  which proves (10).
- Let $a = (i,j) \in \underline{\mathcal{A}}, 1 \leq s \leq K_a$. By Lemma 5 we have $\pi_{j_t} - \pi_{i_s} \in [L_a, U_a]$ for $t = z_a + k - l + s$, which by Lemma 3 implies $t \leq s + b_a$. In particular, $(i_s, j_t) \in \mathcal{A}$. By choice of $u$ it follows $\sum_{t: a=(i_s,j_t)\in\mathcal{A}} u_a = 1$, i.e. constraints (11) are fulfilled.
- Constraints (12) to (15) are obviously fulfilled.
- Let $a = (i_1, j_t) \in \mathcal{A}$.
  First case: $u_a = 1$. $F_a = \pi_{j_t} - \pi_{i_1} = M(u_a - 1) + \pi_{j_t} - \pi_{i_1}$.
  Second case: $u_a = 0$. $F_a = 0 > -M + \pi_{j_t} - \pi_{i_1} = M(u_a - 1) + \pi_{j_t} - \pi_{i_1}$.
  Hence, constraints (16) are fulfilled.
- For (17), $F_a \in \mathbb{Z}$ is clear. Note that by (9) $u_a = 1$ is only possible if $\pi_{j_t} \geq \pi_{i_1}$, which in particular means that $F_a \geq 0$ and therefore $F_a \in \mathbb{N}$.

Hence, $(\pi, u, F)$ is indeed a feasible solution. For the objective value we obtain:

$$\begin{aligned} f &= K \cdot (\sum_{a=(i_1,j_t)\in\mathcal{A}} w_a F_a) = K \cdot (\sum_{a=(i_1,j_t)\in\mathcal{A}: u_a = 1} w_a(\pi_{j_t} - \pi_{i_1})) \\ &\overset{(*)}{=} K \cdot (\sum_{a=(i,j)\in\underline{\mathcal{A}}} w_a(\tilde{\pi}_j - \tilde{\pi}_i + z_a T)) = K \cdot \tilde{f}, \end{aligned}$$

where $(*)$ follows from the proof of Lemma 5. ◀

## Proof of Lemma 8

**Proof.** Let $a = (i,j) \in \underline{\mathcal{A}}$. The following holds:

$$\begin{aligned} \tilde{\pi}_j - \tilde{\pi}_i + z_a T &= (\pi_{j_1} - r_j T) - (\pi_{i_1} - r_i T) + z_a T \\ &= (\pi_{j_t} - (t-1)T - r_j T) - (\pi_{i_1} - r_i T) + z_a T \\ &= \pi_{j_t} - \pi_{i_1} - (r_j - r_i + t - 1)T + z_a T \\ &= \pi_{j_t} - \pi_{i_1} \in [L_a, U_a]. \end{aligned}$$

Hence, $(\tilde{\pi}, z)$ is a feasible solution to PESP. For the objective value we have:

$$\begin{aligned} \tilde{f} &= \sum_{a=(i,j)\in\underline{\mathcal{A}}} w_a(\tilde{\pi}_j - \tilde{\pi}_i + z_a T) \\ &= \sum_{a=(i_1,j_t)\in\mathcal{A}: u_a=1} w_a(\pi_{j_t} - \pi_{i_1}) \\ &\overset{(*)}{\leq} \sum_{a=(i_1,j_t)\in\mathcal{A}: u_a=1} w_a F_a \\ &\overset{(**)}{\leq} \sum_{a=(i_1,j_t)\in\mathcal{A}} w_a F_a = f \cdot \frac{1}{K}. \end{aligned}$$

Here, $(*)$ follows from (16) and $(**)$ from $F_a \geq 0$. ◀

## Proof of Corollary 9

**Proof.** Let $(\tilde{\pi}, z)$ be an optimal solution for PESP with objective value $\tilde{f}$. By Lemma 7 we obtain a feasible solution $(\pi, u, F)$ for PTTA with objective value $f = K\tilde{f}$. Assume this is not optimal, i.e. there is a solution $(\pi', u', F')$ with objective value $f' < f$. By Lemma 8 we get a solution $(\bar{\pi}, \bar{z})$ for PESP with objective value $\bar{f} \leq f' \cdot \frac{1}{K} < f \cdot \frac{1}{K} = \tilde{f}$, which is a contradiction to $(\tilde{\pi}, z)$ being an optimal solution.
On the other hand, let $(\pi, u, F)$ be an optimal solution to PTTA with objective value $f$. Lemma 8 yields a feasible solution $(\tilde{\pi}, z)$ for PESP with objective value $\tilde{f} \leq f \cdot \frac{1}{K}$. Assume $(\tilde{\pi}, z)$ is not optimal, i.e. there is a solution $(\bar{\pi}, \bar{z})$ with objective value $\bar{f} < \tilde{f}$. By Lemma 7 we receive a solution $(\pi', u', F')$ for PTTA with objective value $f' = K\bar{f} < K\tilde{f} \leq f$, a contradiction. ◀