

An Optimal Algorithm for Triangle Counting in the Stream

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Abstract

We present a new algorithm for approximating the number of triangles in a graph G whose edges arrive as an arbitrary order stream. If m is the number of edges in G , T the number of triangles, Δ_E the maximum number of triangles which share a single edge, and Δ_V the maximum number of triangles which share a single vertex, then our algorithm requires space:

$$\tilde{O}\left(\frac{m}{T} \cdot (\Delta_E + \sqrt{\Delta_V})\right)$$

Taken with the $\Omega\left(\frac{m\Delta_E}{T}\right)$ lower bound of Braverman, Ostrovsky, and Vilenchik (ICALP 2013), and the $\Omega\left(\frac{m\sqrt{\Delta_V}}{T}\right)$ lower bound of Kallaugher and Price (SODA 2017), our algorithm is optimal up to log factors, resolving the complexity of a classic problem in graph streaming.

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1 Introduction

Triangle counting is a fundamental problem in the study of graph algorithms, and one of the best studied in the field of graph streams. It arises in the analysis of social networks [5], web graphs [11], and spam detection [3], among other applications. From a theoretical perspective, it is of particular interest as the simplest subgraph counting problem that cannot be solved by considering only *local* information about individual vertices. In other words, counting triangles requires one to aggregate information between pairs of *non-incident* edges.

In this paper, we present an optimal algorithm for counting triangles in the *graph streaming* setting, settling a long line of work on this problem.



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1.1 Graph Streaming

In the (insertion-only) graph streaming setting, a graph $G = (V, E)$ is received as a stream of edges $(\sigma_t)_{t=1}^m$ from its edge set E in an arbitrary order, and an algorithm is required to output the answer to some problem at the end of the stream, using as little space as possible¹. Variants on this model include turnstile streaming (in which edges may be deleted as well as inserted), and models that restrict what kind of state the algorithm may maintain.

1.2 Triangle Counting in Graph Streams

The theoretical study of graph streaming was initiated by [2], who studied the problem of triangle counting – the problem of estimating the number of three-cliques in a graph. They demonstrated that, in general, sublinear space algorithms cannot exist for this problem; namely, in the worst case any algorithm for triangle counting in a stream must use $\Omega(n^2)$ bits of space. On the other hand, they also showed that, if one parameterizes in terms of the number of triangles T , one can often beat this pessimistic lower bound. In particular, they gave an algorithm that uses $\tilde{O}\left(\frac{mn}{T}\right)^3$ space to count triangles in a graph with m edges, n vertices, and T triangles, based on streaming algorithms for approximating frequency moments.² Of course, it is unreasonable to assume that an algorithm knows the number of triangles T in advance, as this would make counting superfluous. Instead, it will suffice to have constant factor bounds on the parameters in question.³

Several years later, the upper bound for this problem was improved to $\tilde{O}\left(\frac{mn}{T}\right)$ by [7], while [14] gave a (non-comparable) algorithm that samples edges and stores neighborhoods of their endpoints in order to find triangles, achieving $\tilde{O}\left(\frac{md^2}{T}\right)$ space in graphs with maximum degree d . Both algorithms were later subsumed by the $\tilde{O}\left(\frac{md}{T}\right)$ space algorithm of [23].

1.3 Additional Graph Parameters for Triangle Counting

Despite the large strides made by the aforementioned algorithms, none of them can achieve sublinear space, even for graphs guaranteed to have as many as $\Omega(m)$ triangles, without bounding parameters of the graph other than m and T . This feature was shown to be necessary by [6], who constructed a family of graphs with either 0 or $\Omega(m)$ triangles such that distinguishing between the two requires $\Omega(m)$ space. However, this “hard instance” is an unusual graph – every triangle in it shares a single edge. This motivated the introduction of a new graph parameter Δ_E , defined as the maximum number of triangles which share a single edge in G . When one parameterizes in terms of Δ_E , the lower bound of [6] becomes $\Omega\left(\frac{m\Delta_E}{T}\right)$. As it happens, the maximum degree of graphs in this family is also Δ_E , so in particular this proves [23] to be optimal among algorithms parametrized by only m , d , and T .

¹ Other properties, such as update time, are also of interest, but space has been the primary object of study in the theory of streaming.

² Here we assume the desired approximation is a multiplicative $(1 \pm \varepsilon)$ with success probability δ for some positive constants ε, δ . For most algorithms mentioned here, including our own, the dependence on non-constant ε, δ will go as $\varepsilon^{-2} \log \delta^{-1}$. We use $\tilde{O}(\cdot)$ to suppress logarithmic or polylogarithmic factors in the argument.

³ One might hope to use these parameters adaptively, giving an algorithm that uses more space the smaller T is without needing a lower bound at the start. However, this is in general impossible, as a graph stream with few triangles and a graph stream with many triangles may be indistinguishable until the last few updates.

The first algorithm to directly take advantage of the new parameter Δ_E was given by [24]. Their algorithm is simple: keep each edge in the stream independently with probability p , count the number of triangles T' in the resulting graph, and output $T'p^{-3}$. They show that setting $p = O(\frac{1}{T^{1/3}} + \frac{\Delta_E}{T})$ suffices for an accurate count, and thereby achieve $\tilde{O}(m(\frac{1}{T^{1/3}} + \frac{\Delta_E}{T}))$ space.

This algorithm has another important feature: it is a *non-adaptive sampling* algorithm – whether it keeps an edge it sees does not depend on the contents of the stream before the edge arrives. This means it can naturally handle *turnstile streams*, streams in which edges may be deleted as well as inserted. In fact, through the use of sketches for ℓ_0 sampling (see e.g. [10]) such algorithms may be converted into *linear sketches*, which are algorithms that store only a linear function of their input (when considered as a vector in $\{0, 1\}^{\binom{V}{2}}$).

An improved non-adaptive sampling algorithm was given in [22], which used the technique of coloring vertices with one of k colors, and keeping all monochromatic edges. This improved the space usage of the algorithm to $\tilde{O}(m(\frac{1}{\sqrt{T}} + \frac{\Delta_E}{T}))$. In [17], it was shown (in combination with the existing lower bound of [6]) that this is optimal, even for insertion-only algorithms – for every T up to $\Omega(m)$, a family of graphs exist with $\Delta_E \leq 1$ and either 0 or T triangles, such that $\Omega(\frac{m}{\sqrt{T}})$ space is required to distinguish the two.

However, as with the lower bound of [6], the hard instance from [17] is a rather strange graph: this time every triangle shares a single *vertex*. Also similarly to the lower bound of [6], the bound from [17] weakens as the maximum number of triangles sharing a single vertex, a parameter denoted by Δ_V , is restricted. In this case, when parameterized by Δ_V , the lower bound becomes $\Omega(\frac{m\sqrt{\Delta_V}}{T})$. This was accompanied in [17] by an algorithm that achieves $\tilde{O}(m(\frac{1}{T^{2/3}} + \frac{\sqrt{\Delta_V}}{T} + \frac{\Delta_E}{T}))$ space, improving on [22] for graphs with $\Delta_V = o(T)$.

Subsequently, it was shown in [15] that any linear sketching algorithm for counting triangles requires $\Omega(\frac{m}{T^{2/3}})$ space, even if every triangle is disjoint from every other and therefore $\Delta_E = \Delta_V \leq 1$, and so the [17] algorithm is optimal among linear sketches. By the turnstile streaming-linear sketching equivalence of [20], this suggests that [17] is also optimal among turnstile streaming algorithms.⁴

However, this leaves open the question of how hard triangle counting is for algorithms that are *not* required to handle deletions (i.e., the standard “insertion-only” model). We resolve this question (up to a log factor, as with previous optimality results), by giving an optimal algorithm for triangle counting in insertion-only streams.

1.4 Our Algorithm

We give a new algorithm for counting triangles in insertion-only graph streams. For every $\varepsilon, \delta \in (0, 1)$, there is an algorithm for insertion-only graph streams that approximates the number of triangles in a graph G to εT accuracy with probability $1 - \delta$, using

$$O\left(\frac{m}{T} \left(\Delta_E + \sqrt{\Delta_V}\right) \log n \frac{\log \frac{1}{\delta}}{\varepsilon^2}\right)$$

bits of space, where m is the number of edges in G , T the number of triangles, Δ_E the maximum number of triangles which share a single edge, and Δ_V the maximum number of triangles which share a single vertex.

⁴ However, the [20] equivalence depends on rather stringent conditions that a turnstile algorithm must satisfy. In [18], it was shown that relaxing these conditions allows turnstile streaming algorithms for triangle counting that are closer to the result of [14].

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■ **Table 1** Best known upper and lower bounds for triangle counting for insertion-only and linear sketching algorithms. m is the number of edges, T the number of triangles, d the maximum degree, and Δ_E, Δ_V are the maximum number of triangles sharing an edge or a vertex respectively. Note that linear sketching upper bounds imply insertion-only upper bounds, while lower bounds are the opposite.

Paper	Space	Model
[23]	$\tilde{O}\left(\frac{md}{T}\right)$	Insertion-only
[6]	$\Omega\left(\frac{m\Delta_E}{T}\right)$	Insertion-only
[17]	$\Omega\left(\frac{m\sqrt{\Delta_V}}{T}\right)$	Insertion-only
[22]	$\tilde{O}\left(m\left(\frac{1}{\sqrt{T}} + \frac{\Delta_E}{T}\right)\right)$	Linear Sketching
[17]	$\tilde{O}\left(m\left(\frac{1}{T^{2/3}} + \frac{\sqrt{\Delta_V}}{T} + \frac{\Delta_E}{T}\right)\right)$	Linear Sketching
[15]	$\Omega\left(\frac{m}{T^{2/3}}\right)$	Linear Sketching
This work	$\tilde{O}\left(\frac{m}{T}(\sqrt{\Delta_V} + \Delta_E)\right)$	Insertion-only

This matches, up to a log factor (and for constant ε, δ), the lower bounds of [6] and [17]. It subsumes both the algorithm of [17] and the $\tilde{O}\left(\frac{md}{T}\right)$ algorithm of [23], as in any graph with max degree d , we have $\Delta_E \leq d$ and $\Delta_V \leq \binom{d}{2}$. This closes the line of work discussed above on the complexity of triangle counting in insertion-only streams.

1.5 Other Related Work

In the *multi-pass* streaming setting, an algorithm is allowed to pass over the input stream more than once. [9] shows multipass algorithms take $\tilde{O}\left(m/\sqrt{T}\right)$ space for arbitrary graphs, giving an algorithm for two passes and a lower bound for a constant number of passes. [19] shows a three pass streaming algorithm using $O(\sqrt{m} + m^{3/2}/T)$ space. [4] gave a $O(m^{3/2}/T)$ four pass algorithm.

In the *adjacency-list* model, in which each vertex's list of neighbors is received as a block (and so in particular every edge is seen twice), [21] gave a $O\left(m/\sqrt{T}\right)$ space one-pass algorithm, while [16] gave $O(m/T^{2/3})$ space 2-pass algorithm, as well as tight (but conditional on open communication complexity conjectures) lower bounds for both.

The problem has also been studied in the query model, in which case rather than space the concern is minimizing time or query count. While this is a very different setting, similar concerns around mitigating the impact of “heavy” vertices or edges arise. [12] considered triangle counting in this setting, which was extended by [13] to general cliques and [1] to arbitrary constant-size subgraphs.

2 Overview of the Algorithm

At a high-level, many triangle counting algorithms in the literature adhere to the following template: **(1)** design a sampling scheme to sample triangles, **(2)** count the number of triangles which survive after this sampling process, **(3)** rescale the number of empirically sampled triangles by the expected fraction of surviving triangles to obtain an unbiased estimator for T .

As an example, one could sample each edge uniformly with probability q (this is the approach taken in [24]). Since for a triangle to survive all three of its edges must be sampled, the expected number of triangles that survive is Tq^3 . Thus, rescaling the number of empirically sampled triangles by $1/q^3$ yields an unbiased estimator. How large must q be to make this estimator accurate? In order to sample even a single triangle we need $Tq^3 \geq 1$, so clearly q must be at least $1/T^{1/3}$. Moreover, if Δ_E is the largest number of triangles that share an edge, there might be as few as T/Δ_E “heavy” edges such that sampling a triangle requires sampling at least one of them, and so q must be at least Δ_E/T . It turns out that, up to constant factors, this is also sufficient, and so the space needed by this algorithm is $\tilde{O}\left(m\left(\frac{1}{T^{1/3}} + \frac{\Delta_E}{T}\right)\right)$ bits.

The starting point for our algorithm is the following simple observation, which can be seen as an optimization to the sampling algorithm above. Given three edges $uv, vw, wu \in E$ arriving in a stream in that order, once the first two edges uv, vw have been sampled and stored, upon seeing the “completing” edge wu , we will know that the triangle uvw exists in G , and may count it immediately – we get the closing edge of each triangle “for free”. Now for a single triangle to be sampled, we only need to sample the first two edges, and so the probability of finding any given triangle improves to q^2 , allowing a space complexity of $\tilde{O}\left(m\left(\frac{1}{\sqrt{T}} + \frac{m\Delta_E}{T}\right)\right)$. However, when $\Delta_V = o(T)$, this is still weaker than allowed by the $\Omega\left(\frac{m}{T}(\sqrt{\Delta_V} + \Delta_E)\right)$ lower bound that results from combining the results of [6, 17].

While the aforementioned algorithm is sub-optimal in general, notice that it does match the lower bounds in the extreme case when $\Delta_V = T$, and all triangles share a single vertex. On the other hand, when Δ_V is smaller, there are more “fully disjoint” triangles in the graph. Consequentially, we can afford to subsample by *vertices*, as now dropping a single vertex cannot lose too large a fraction of our triangles. We may sample vertices uniformly with some probability p , and deterministically store all edges adjacent to at least one sampled vertex, again counting a triangle whenever we observe an edge wu closing a sampled pair uv, vw . Each such triangle will be counted iff the “first” vertex v of the triangle is sampled, and these may be divided among as few as T/Δ_V “heavy” vertices, so p must be at least Δ_V/T . This again turns out to be sufficient, for a space usage of $\tilde{O}\left(\frac{m\Delta_V}{T}\right)$ (note that any pair of edges sharing an edge also share a vertex, so $\Delta_E \leq \Delta_V$, and thus this does not violate the known lower bounds). While this is an improvement on the aforementioned adaptive edge-sampling scheme for small Δ_V , it becomes worse once $\Delta_V > \sqrt{T}$.

The crucial insight behind our algorithm is to merge the two aforementioned algorithms with a careful choice of parameterization. Specifically, we sample both edges *and* vertices, before counting triangles that we see closing our sampled wedges. Specifically, we sample vertices $v \in V$ in the graph with probability $p \in (0, 1]$, and then “activate” each edge $e \in E$ with probability $q \in (0, 1]$. When an edge $uv \in E$ arrives in the stream, we store it iff uv is active *and* at least one of the vertices u or v was sampled. We denote by S the set of all edges stored by the algorithm. Finally, when a closing edge wu arrives that completes a triangle with edges uv, vw that were previously added to S , we check if the vertex v at the center of the wedge uv, vw was sampled, and if so we deterministically increment a counter \mathbf{C} .

Now observe that, for any given triangle uvw , the probability that uvw causes \mathbf{C} to be incremented is exactly pq^2 . Thus, if we output the quantity $\mathbf{C}/(pq^2)$ at the end of the stream, we obtain an unbiased estimator for the number of triangles in G .

Notice that when $p = 1$ our algorithm reduces to the simpler edge-sampling algorithm stated above. At the other extreme, when $q = 1$ our algorithm reduces to the vertex sampling algorithm. Intuitively, our choice of the parameters p and q are subject to the same constraints faced by the aforementioned edge- and vertex-sampling algorithms. Firstly, p

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must be at least Δ_V/T , otherwise the algorithm could miss a “heavy” vertex. Furthermore, the product pq must be at least Δ_E/T , to avoid missing “heavy” edges, and pq^2 must be at least $1/T$ to find any triangles at all. Putting these bounds together, it follows that q must be at least $\max\left\{\frac{\Delta_E}{\Delta_V}, \frac{1}{\sqrt{\Delta_V}}\right\}$.

As with all the algorithms discussed so far, this turns out to also be sufficient – we demonstrate that by fixing the sampling parameters⁵

$$p = \frac{\Delta_V}{T}, \quad q \geq \max\left\{\frac{\Delta_E}{\Delta_V}, \frac{1}{\sqrt{\Delta_V}}\right\}$$

we obtain an algorithm using space $O\left(\frac{m}{T}(\Delta_E + \sqrt{\Delta_V}) \log n\right)$ which yields an $O(T^2)$ variance estimator. We may therefore obtain a $(1 \pm \varepsilon)$ multiplicative estimate with probability $1 - \delta$ by using $O\left(\frac{1}{\varepsilon^2} \log \frac{1}{\delta}\right)$ copies of this algorithm.

Consequently one obtains an algorithm matching, up to a log factor, the lower bounds of [6, 17], with optimal space usage in terms of m, T, Δ_E, Δ_V .

3 The Triangle Counting Algorithm

Let $G = (V, E)$ be a graph on n vertices, received as a stream of undirected edges, adversarially ordered. Let m be the number of edges in the stream. We write the stream as $\sigma = (\sigma_i)_{i=1}^m$, with each $\sigma_i \in E$. We use T to refer to the number of triangles in G , Δ_E to refer to the maximum number of them sharing a single edge, and Δ_V the maximum number sharing a single vertex.

► **Remark 1.** As with all streaming triangle counting algorithms, our algorithm will need to be parametrized by statistics of the graph that cannot be known exactly without trivializing the problem – in our case T , Δ_E , and Δ_V . However, it will not be necessary to know these exactly – an upper bound on Δ_E , Δ_V and a lower bound on T will be sufficient. If these bounds are tight up to a constant, the complexity of our algorithm will be unchanged, otherwise replace the parameters T , Δ_E , Δ_V with the respective upper and lower bounds.

3.1 Description of the Algorithm

We begin by choosing two hash functions $\mathbf{f} : V \rightarrow \{0, 1\}$ and $\mathbf{g} : E \rightarrow \{0, 1\}$, which will serve as our “vertex sampling” and “edge sampling” functions, respectively. We choose \mathbf{f} to be pair-wise independent. \mathbf{g} will only be evaluated at most once for each edge, and so we may choose it to be fully independent. We pick the two functions \mathbf{f}, \mathbf{g} such that

$$\mathbb{E}[\mathbf{f}(v)] = p$$

for each $v \in V$ and

$$\mathbb{E}[\mathbf{g}(e)] = q$$

for each $e \in E$, where p, q are parameters to be set later. Such a hash function \mathbf{f} can be generated by taking a two-wise independent function $\mathbf{h} : V \rightarrow [M]$, where $M = \text{poly}(n)$ is a sufficiently large multiple of $1/p$, and setting $\mathbf{f}(v) = 1$ whenever $\mathbf{h}(v) \leq pM$ (one can construct \mathbf{g} similarly using a four-wise independent hash function). Such functions can be generated and stored in at most $O(\log n)$ bits of space [8].

⁵ As mentioned earlier, $\Delta_E \leq \Delta_V$, while $\Delta_V \leq T$ holds trivially. Thus p, q are valid probabilities.

The algorithm will be simple: sample vertices with probability p , sample incident edges with probability q . The formal description is given below in Algorithm 1.

■ **Algorithm 1** Triangle Counting Algorithm.

```

1: procedure TRIANGLECOUNTING( $p, q$ )
2:    $S \leftarrow \emptyset$ 
3:    $\bar{T} \leftarrow 0$ 
4:   for each update  $uv$  do
5:     for  $u \in V$  do
6:       if  $f(u) > 0 \wedge uv, uw \in S$  then
7:          $\bar{T} += 1/pq^2$ 
8:       end if
9:     end for
10:    if  $g(uv)(f(u) + f(v)) > 0$  then
11:       $S \leftarrow S \cup \{uv\}$ 
12:    end if
13:  end for
14:  return  $\bar{T}$ .
15: end procedure

```

3.2 Analysis of the Algorithm

► **Lemma 2.** *This algorithm uses $O(mpq \log n)$ bits of space.*

Proof. Besides an $O(\log n)$ sized counter and the hash function f (g is never evaluated more than once for an edge and thus does not need to be stored), the algorithm maintains a set of edges. Each edge will be kept with probability at most $2pq$ and takes $O(\log n)$ space to store, so the result follows. ◀

We will write T_{uvw} for the variable that is 1 if uvw is a triangle in G with its edges arriving in the order (uv, uw, vw) , and 0 otherwise, and so

$$T = \sum_{(u,v,w) \in V^3} T_{uvw}.$$

We will write \bar{T}_{uvw} for the random variable that is $1/pq^2$ if $T_{uvw} = 1$ and $f(u)g(uv)g(uw) = 1$, and 0 otherwise. We will therefore have

$$\bar{T} = \sum_{(u,v,w) \in V^3} \bar{T}_{uvw}.$$

► **Lemma 3.**

$$\mathbb{E}[\bar{T}] = T.$$

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Proof. For any (u, v, w) , $\mathbf{f}(u)\mathbf{g}(uv)\mathbf{g}(uw) = 1$ with probability pq^2 , so $\mathbb{E}[\overline{\mathbf{T}}_{uvw}] = T_{uvw}$. Therefore,

$$\begin{aligned}\mathbb{E}[\overline{\mathbf{T}}] &= \sum_{(u,v,w) \in V^3} \mathbb{E}[\overline{\mathbf{T}}_{uvw}] \\ &= \sum_{(u,v,w) \in V^3} T_{uvw} \\ &= T\end{aligned}$$

► **Lemma 4.**

$$\text{Var}(\overline{\mathbf{T}}) \leq T/pq^2 + T\Delta_E/pq + T\Delta_V/p.$$

Proof. Consider any (ordered) pair of triples $(u, v, w), (x, y, z) \in V^3$ such that $T_{uvw}T_{xyz} = 1$. If $(u, v, w) = (x, y, z)$, $\overline{\mathbf{T}}_{uvw}\overline{\mathbf{T}}_{xyz} = 1/p^2q^4$ with probability pq^2 and 0 otherwise, so

$$\mathbb{E}[\overline{\mathbf{T}}_{uvw}\overline{\mathbf{T}}_{xyz}] = \mathbb{E}[\overline{\mathbf{T}}_{uvw}^2] = 1/pq^2.$$

At most T such pairs of triples can exist.

Now, if $|\{uv, uw\} \cap \{xy, xz\}| = 1$, then $u = x$ and so $\overline{\mathbf{T}}_{uvw}\overline{\mathbf{T}}_{xyz} = 1/p^2q^4$ iff $\mathbf{f}(u) = 1$ and $\mathbf{g}(e) = 1$ for all e in the size-3 set $\{uv, uw, xy, xz\}$, which happens with probability pq^3 , and so

$$\mathbb{E}[\overline{\mathbf{T}}_{uvw}\overline{\mathbf{T}}_{xyz}] = 1/pq.$$

Each triangle has at most Δ_E other triangles it shares an edge with, so there are at most $T\Delta_E$ such pairs.

If $\{uv, uw\} \cap \{xy, xz\} = \emptyset$ but $u = x$, then $\overline{\mathbf{T}}_{uvw}\overline{\mathbf{T}}_{xyz} = 1/p^2q^4$ iff $\mathbf{f}(u) = 1$ and $\mathbf{g}(e) = 1$ for all e in the size-4 set $\{uv, uw, xy, xz\}$, which happens with probability pq^4 , and so

$$\mathbb{E}[\overline{\mathbf{T}}_{uvw}\overline{\mathbf{T}}_{xyz}] = 1/p.$$

Each triangle has at most Δ_V other triangles it shares a vertex with, so there are at most $T\Delta_V$ such pairs.

Finally, if $\{u, v, w\} \cap \{x, y, z\} = \emptyset$, then $\overline{\mathbf{T}}_{uvw}\overline{\mathbf{T}}_{xyz} = 1/p^2q^4$ iff $\mathbf{f}(u) = 1$, $\mathbf{f}(x) = 1$, and $\mathbf{g}(e) = 1$ for all e in the size-4 set $\{uv, uw, xy, xz\}$, which happens with probability p^2q^4 , and so

$$\mathbb{E}[\overline{\mathbf{T}}_{uvw}\overline{\mathbf{T}}_{xyz}] = 1.$$

At most T^2 such pairs can exist. Therefore,

$$\begin{aligned}\mathbb{E}[\overline{\mathbf{T}}^2] &= \sum_{(u,v,w) \in V^3} \sum_{(x,y,z) \in V^3} \mathbb{E}[\overline{\mathbf{T}}_{uvw}\overline{\mathbf{T}}_{xyz}] \\ &= \sum_{(u,v,w) \in V^3} \mathbb{E}[\overline{\mathbf{T}}_{uvw}^2] + \sum_{(u,v,w) \in V^3} \left(\sum_{\substack{(x,y,z) \in V^3 \\ |\{uv, uw\} \cap \{xy, xz\}|=1}} \mathbb{E}[\overline{\mathbf{T}}_{uvw}\overline{\mathbf{T}}_{xyz}] + \right. \\ &\quad \left. \sum_{\substack{(x,y,z) \in V^3 \\ \{uv, uw\} \cap \{xy, xz\} = \emptyset \\ u=x}} \mathbb{E}[\overline{\mathbf{T}}_{uvw}\overline{\mathbf{T}}_{xyz}] + \sum_{\substack{(x,y,z) \in V^3 \\ \{u, v, w\} \cap \{x, y, z\} = \emptyset}} \mathbb{E}[\overline{\mathbf{T}}_{uvw}\overline{\mathbf{T}}_{xyz}] \right) \\ &\leq T/pq^2 + T\Delta_E/pq + T\Delta_V/p + T^2\end{aligned}$$

by adding the previously established bounds for all four kinds of pair. The lemma then follows from the fact that $\text{Var}(\mathbf{T}) = \mathbb{E}[\mathbf{T}^2] - \mathbb{E}[\mathbf{T}]^2 = \mathbb{E}[\mathbf{T}^2] - T^2$. ◀

We may now prove Theorem 1.4. For every $\varepsilon, \delta \in (0, 1)$, there is an algorithm for insertion-only graph streams that approximates the number of triangles in a graph G to εT accuracy with probability $1 - \delta$, using

$$O\left(\frac{m}{T}(\Delta_E + \sqrt{\Delta_V}) \log n \frac{\log \frac{1}{\delta}}{\varepsilon^2}\right)$$

bits of space, where m is the number of edges in G , T the number of triangles, Δ_E the maximum number of triangles which share a single edge, and Δ_V the maximum number of triangles which share a single vertex.

Proof. We may assume Δ_V (more specifically, the upper bound we have on it) is at least 1, as otherwise we already know G to be triangle-free. By Lemmas 3 and 4, we can set $p = \Delta_V/T$, $q = \max\{\Delta_E/\Delta_V, 1/\sqrt{\Delta_V}\}$ and run Algorithm 1 to obtain an estimator with expectation T and variance at most $3T^2$. (These will give valid probabilities, as $\Delta_V \leq T$ by definition, and Δ_E is at least Δ_V , as any pair of triangles sharing an edge also share a vertex.) By Lemma 2, this will take $O\left(\frac{m}{T}(\Delta_E + \sqrt{\Delta_V}) \log n\right)$ space.

Repeating this $36/\varepsilon^2$ times and taking the mean will give an estimator with expectation T and variance at most $\varepsilon T^2/2$. We can then repeat *this* $O(\log \frac{1}{\delta})$ times and take the median to get an estimator that will be within εT of T with probability $1 - \delta$. ◀

4 Conclusion

We resolve the complexity of triangle counting in the insertion-only streaming model, in terms of the well-studied natural graph parameters m, T, Δ_E, Δ_V . The results of [15] resolved this problem for the *linear sketching* model, and a result of [20] states that, under certain conditions, turnstile streaming algorithms are equivalent to linear sketches, suggesting that the algorithm of [17] is optimal for turnstile streams as well. However, [18] showed that an insertion-only algorithm of [14] can be converted into a turnstile streaming algorithm provided that, for instance, the length of the stream is reasonably constrained (with the number of insertions and deletions no more than $O(1)$ times the final size of the graph). It remains open whether this algorithm can be converted into a turnstile algorithm under such constraints, or whether the bounded-stream turnstile complexity of triangle counting is somewhere between insertion-only and linear sketching.

Another natural question is about the choice of parameters – the algorithm of [22] is optimal in terms of m, T , and Δ_E , but not when the parameter Δ_V is considered. Are there natural extensions of the parametrization that allow for better results? The results of [17] include a proof of instance-optimality for a restricted subclass of non-adaptive sampling algorithms, but for more general algorithms it is clear that there are at least *unnatural* extensions of the parametrization that help. For instance, if all the edges of a graph are guaranteed to belong to high-degree vertices, but all the triangles belong to low-degree vertices, a simple filtering strategy allows an improvement.

In particular, the lower bound instances of [6, 17] are both sparse graphs, and so cannot be constructed if n is constrained to be small relative to m or T . For the most dense graphs (with $\Theta(n^2)$ edges and $\Theta(n^3)$ triangles) our algorithm and the algorithm of [17] are already trivially optimal up to log factors, since they use only $\text{polylog}(n)$ bits. However, the complexity landscape for more general dense graphs remains open.

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