Directed Non-Cooperative Tile Assembly Is Decidable

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Abstract

We provide a complete characterisation of all final states of a model called *directed non-cooperative tile self-assembly*, also called *directed temperature 1 tile assembly*, which proves that this model cannot possibly perform Turing computation. This model is a deterministic version of the more general *undirected* model, whose computational power is still open. Our result uses recent results in the domain, and solves a conjecture formalised in 2011. We believe that this is a major step towards understanding the full model.

Temperature 1 tile assembly can be seen as a two-dimensional extension of finite automata, where geometry provides a form of memory and synchronisation, yet the full power of these "geometric blockings" was still largely unknown until recently (note that nontrivial algorithms which are able to build larger structures than the naive constructions have been found).

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1 Introduction

Self-assembly is the process by which independent, unsynchronised components coalesce into complex forms and patterns, using geometry and local constraints to exchange information, and perform different sorts of computations. In particular, self-assembly is the process by which molecules, and in particular biomolecules, acquire their shape (and therefore their function).

A computational theory of self-assembly has a wealth of applications in a large range of fields and scales. At the molecular level, programming molecules would enable us to interact with living organisms, potentially defeating the geometric strategies used by nasty viruses to penetrate cells. Smart materials with new properties such as self-reproduction and self-repairing are another example. At a much larger scale, industrial processes could also benefit from a better understanding of self-assembly, as it could streamline processes and make industrial robots simpler.

This theory has already yielded experimental realisations such as DNA Origami [24], allowing anyone to make their own molecules of any prescribed shape of a diameter between 10nm and 500nm. DNA Self-Assembly has also been used to build fractal shapes [26], information retrieval circuits [23], cyclic machines using DNA as machine material *and* as fuel [30]. Another recent application has been the amplification of minuscule concentrations of a molecular compound in solution, by using it as a "seed" for self-assembling large structures [21]. DNA storage [2] has also been proposed and implemented as a technique to store a tremendous amount of information in a tiny space, with millions of years of potential durability.

These developments have happened in parallel to, and with interactions with work on the computer science theory of tile assembly. The most studied model in that direction is the abstract Tile Assembly Model (aTAM), created by Winfree [29, 25] with inspiration from Wang tilings [28]. This model studies assemblies made of *square tiles* with colours on their borders. Using a finite set of tile *types*, and an assumed infinite supply of each type, the assembly process starts with an initial "seed" assembly, and proceeds nondeterministically and asynchronously, one tile at a time. Unlike Wang tilings, which is mostly concerned with (potentially undecidable) full covers of the plane, the abstract Tile Assembly Model studies the *assembly sequence* of an assembly, which is the sequence of binding events necessary to build a shape.

In the fully general abstract Tile Assembly Model, tile borders have a *glue strength* on their border, and the model has a global assembly threshold called the "*temperature*": in order to remain stably attached, the sum of glue strengths on the attached borders of a tile must be at least equal to the temperature. One of the key complexity measures of this model is *program-size complexity*, meaning the number of tile types in the tileset. The fact that this model can simulate Turing machines has been used to encode complex shapes with a number of tile types logarithmic in the Kolmogorov complexity of the shapes [27]. Moreover, the aTAM model is also *intrinsically universal*, meaning that there is a single finite "universal" tileset capable of simulating any other tileset up to a constant scaling factor [6]. Over the years, a number of consequences and extensions of that result have also been studied [7, 4, 5], and intrinsic universality has also been used to classify models according to their simulation power [16].

1.1 Noncooperative self-assembly

Noncooperative self-assembly is a restriction of the aTAM to a temperature of 1, meaning that tiles always attach to an existing assembly as soon as at least one side has its colour matching the colour of the current assembly. In other words, the assembly cannot "wait" for two different "branches" to meet at a point in the plane before growing further. The restriction of this model to one-dimension is exactly equivalent to finite automata, where tiles map to the edges of the automaton, and border colours to states.

The only form of synchronisation in this model is by geometric "blocking", where two branches compete for a position in the plane, and the first one to get there can continue to grow. The fundamental question of noncooperative self-assembly is whether this rather weak form of communication is sufficient to achieve synchronisation. This has been an open problem since the early days of the field, and research in variants of the model has shown surprising results, in that every variation of the noncooperative model, however minor, seems to endow it with arbitrary computational capabilities. In the three-dimensional extension, for example, one can arrange little "bridges" and "tunnels" to block one branch of a test while allowing the other one to continue, which allows one to read and write bits [3]. In two dimensions, using random assembly sequences rather than asynchronous ones yields the same result [3], and so do negative glues [22], polyomino tiles (with at least three unit squares) [9], polygonal tiles, provided they have at least seven sides [12]. Separating the assembly process into stages with different sets of tiles available at each stage also makes the model Turing-universal [1], which is also the case for a model with detachable tiles [13].

On the negative side, no tileset is intrinsically universal at temperature 1 [20], meaning that no tileset can simulate all temperature 1 tile assembly systems, even when rescaled. Moreover, it has recently been shown that long enough *paths* built by a temperature-1 tile assembly system are *pumpable*, meaning that their growth can only be controlled within a finite radius, after which they degenerate into simple periodic paths [19]. Moreover, disallowing mismatches means that all assemblies are periodic [14] (note that the "no mismatches" condition is not known to be decidable).

One particularly puzzling fact about 2D noncooperative self-assembly is that even though it seems computationally weak, a handful of nontrivial algorithms have been designed, including assemblies of diameter $\Omega(n \log n)$, produced by a tileset of size n [15, 17]. In three-dimensions, recent results have also shown how to build thin rectangles [11, 10] with almost matching upper and lower bounds.

1.2 Main results

Our main result is that the assemblies producible by directed non-cooperative tile assembly are a finite union of ultimately periodic assemblies. We state this semi-formally here, even though not all terms have been defined yet:

▶ **Definition 1.** *The* complexity *of a finite assembly is* 0*. For* $i \geq 1$*, the complexity of an assembly* α *is* i *if* α *is either defined as* $\alpha = \bigcup_{\ell \in \mathbb{N}} (\beta + \ell \overrightarrow{v})$ *where* β *is a assembly of complexity* $i-1$*, or if* α *is a finite union of assemblies of complexity at most* i*.*

▶ **Theorem 2.** *Let* $\mathcal{T} = (T, \sigma, 1)$ *be a noncooperative tile assembly system. If* $\mathcal{T}$ *has a unique terminal assembly* α *(or otherwise said, if* $\mathcal{T}$ *is* directed*), then the complexity of* α *is less than* 2*.*

This result implies that a terminal assembly can be described by a finite number of finite assemblies and vectors. Along the proofs, we distinguish four different cases of terminal assemblies and for each class, we bound the size of the assemblies and the number of vectors used to describe a terminal assembly. Moreover, an algorithm which computes this description can be deduced from our work. Thus, directed tile assembly systems cannot perform Turing computation.

Our proof uses techniques similar to a theorem from 2011 [8] which achieves a complete characterisation of producible assemblies, but conditioned on an unproven conjecture (called the *pumpability conjecture*). In contrast to that proof, our work does not rely on any unproven hypothesis, and improve the previous characterisation of producible assemblies which implies the pumpability conjecture (this point is discussed for each case of the classification later). In order to do so, we use a result published in [19], showing that long enough paths are pumpable. That result is itself weaker than the original pumpability conjecture [8], in that the bound includes the size of the seed, whereas the pumpability conjecture is that any long enough subpath, even arbitrarily far from the seed, is pumpable. This subtle difference is important, since without our Theorem 2, seeds could be assumed to encode computation, for example by using complicated shapes. Theorem 2 shows that this is not the case.

2 Definitions

Some of these definitions come from [19]. As usual, let $\mathbb{R}$ be the set of real numbers, let $\mathbb{Z}$ be the set of all integers, let $\mathbb{N}$ be the set of all natural numbers including 0, and let $\mathbb{N}^*$ be the set of all natural numbers excluding 0. The domain of a function f is denoted $dom(f)$, and its range (or image) is denoted $f(dom(f))$.

2.1 Abstract tile assembly model

A *tile type* is a unit square with four sides, each consisting of a glue *type* and a nonnegative integer *strength*. Let T be a a finite set of tile types. The sides of a tile type are respectively called north, east, south, and west.

An *assembly* is a partial function $\alpha : \mathbb{Z}^2 \dashrightarrow T$ where T is a set of tile types and the domain of α (denoted $dom(\alpha)$) is connected.[1] The translation of an assembly α by a vector $\overrightarrow{v}$, written $\alpha + \overrightarrow{v}$, is the assembly β defined for all $(x, y) \in (dom(\alpha) + \overrightarrow{v})$ as $\beta(x, y) = \alpha((x, y) - \overrightarrow{v})$. We let $\mathcal{A}^T$ denote the set of all assemblies over the set of tile types T. In this paper, two tile types in an assembly are said to *bind* (or *interact*, or are *stably attached*), if the glue types on their abutting sides are equal, and have strength ≥ 1. An assembly α induces an undirected weighted *binding graph* $G_\alpha = (V, E)$, where $V = dom(\alpha)$, and there is an edge $\{a, b\} \in E$ if and only if the tiles at positions a and b interact, and this edge is weighted by the glue strength of that interaction. The assembly is said to be τ-stable if every cut of G_α has weight at least τ.

A *tile assembly system* is a triple $\mathcal{T} = (T, \sigma, \tau)$, where T is a finite set of tile types, σ is a τ-stable (hence connected) assembly called the *seed*, and $\tau \in \mathbb{N}$ is the *temperature*. Throughout this paper, $\tau = 1$. Note also that the seed may be large, and placed at an arbitrary position in the plane. And indeed, in this paper, we will sometimes define multiple "intuitively equivalent" tile assembly systems where only the position of the seed differs.

Given two τ-stable assemblies α and β, we say that α is a *subassembly* of β, and write $\alpha \sqsubseteq \beta$, if $dom(\alpha) \subseteq dom(\beta)$ and for all $p \in dom(\alpha)$, $\alpha(p) = \beta(p)$. We also write $\alpha \rightarrow_1^{\mathcal{T}} \beta$ if we can obtain β from α by the binding of a single tile type, that is: $\alpha \sqsubseteq \beta$, $|dom(\beta) \setminus dom(\alpha)| = 1$ and the tile type at the position $dom(\beta) \setminus dom(\alpha)$ stably binds to α at that position. We say that γ is *producible* from α, and write $\alpha \rightarrow^{\mathcal{T}} \gamma$ if there is a (possibly empty) sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$ where $n \in \mathbb{N} \cup \{\infty\}$, $\alpha = \alpha_1$ and $\alpha_n = \gamma$, such that $\alpha_1 \rightarrow_1^{\mathcal{T}} \alpha_2 \rightarrow_1^{\mathcal{T}} \ldots \rightarrow_1^{\mathcal{T}} \alpha_n$. A sequence of $n \in \mathbb{Z}^+ \cup \{\infty\}$ assemblies $\alpha_0, \alpha_1, \ldots$ over $\mathcal{A}^T$ is a $\mathcal{T}$*-assembly sequence* if, for all $1 \leq i < n$, $\alpha_{i-1} \rightarrow_1^{\mathcal{T}} \alpha_i$.

Given two τ-stable assemblies α and β, the union of α and β, write $\alpha \cup \beta$, is an assembly defined if and only if and for all $p \in dom(\alpha) \cap dom(\beta)$, $\alpha(p) = \beta(p)$ and either at least one tile of α binds with a tile of β or $dom(\alpha) \cap dom(\beta) \neq \emptyset$. Then, for all $p \in dom(\alpha)$, we have $(\alpha \cup \beta)(p) = \alpha(p)$ and for all $p \in dom(\beta)$, we have $(\alpha \cup \beta)(p) = \beta(p)$.

The set of *productions*, or *producible assemblies*, of a tile assembly system $\mathcal{T} = (T, \sigma, \tau)$ is the set of all assemblies producible from the seed assembly σ and is written $\mathcal{A}[\mathcal{T}]$. An assembly α is called *terminal* if there is no β such that $\alpha \rightarrow_1^{\mathcal{T}} \beta$. The set of all terminal assemblies of $\mathcal{T}$ is denoted $\mathcal{A}_\Box[\mathcal{T}]$. If there is a unique terminal assembly, *i.e.* $|\mathcal{A}_\Box[\mathcal{T}]| = 1$, then $\mathcal{T}$ is *directed*. In this paper, this unique terminal assembly is denoted α.

[1] Intuitively, an assembly is a positioning of unit-sized tiles, each from some set of tile types T, so that their centers are placed on (some of) the elements of the discrete plane $\mathbb{Z}^2$ and such that those elements of $\mathbb{Z}^2$ form a connected set of points.

2.2 Paths

Let T be a set of tile types. A *tile* is a pair $((x, y), t)$ where $(x, y) \in \mathbb{Z}^2$ is a position and $t \in T$ is a tile type. Intuitively, a path is a finite or one-way-infinite simple (non-self-intersecting) sequence of tiles placed on points of $\mathbb{Z}^2$ so that each tile in the sequence interacts with the previous one, or more precisely:

▶ **Definition 3** (Path). *A* path *is a (finite or infinite) sequence $P = P_0P_1P_2\dots$ of tiles $P_i = ((x_i, y_i), t_i) \in \mathbb{Z}^2 \times T$, such that:*

- *for all P_j and P_{j+1} defined on P it is the case that t_j and t_{j+1} interact, and*
- *for all P_j, P_k such that $j \neq k$ it is the case that $(x_j, y_j) \neq (x_k, y_k)$.*

By definition, paths are simple (or self-avoiding), and this fact will be repeatedly used throughout the paper. For a tile P_i on some path P, its x-coordinate is denoted x_{P_i} and its y-coordinate is denoted y_{P_i}. The *concatenation* of two paths P and Q is the concatenation PQ of these two paths as sequences, and is a path if and only if (1) the last tile of P interacts with the first tile of Q and (2) P and Q do not intersect each other.

For a path $P = P_0 \dots P_iP_{i+1} \dots P_j \dots$, we define the notation $P_{i,i+1,\dots,j} = P_iP_{i+1} \dots P_j$, i.e. "the subpath of P between indices i and j, inclusive". Whenever P is finite, i.e. $P = P_0P_1P_2 \dots P_{n-1}$ for some $n \in \mathbb{N}$, n is termed the *length* of P and denoted by $|P|$. In the special case of a subpath where $i = 0$, we say that $P_{0,1,\dots,j}$ is a prefix of P and when $j = |P| - 1$, we say that $P_{i,\dots,|P|-1}$ is a suffix of P. For any path $P = P_0P_1P_2, \dots$ and integer $i \geq 0$, we write $\mathrm{pos}(P_i) \in \mathbb{Z}^2$, or $(x_{P_i}, y_{P_i}) \in \mathbb{Z}^2$, for the position of P_i and $\mathrm{type}(P_i)$ for the tile type of P_i. Hence if $P_i = ((x_i, y_i), t_i)$ then $\mathrm{pos}(P_i) = (\mathrm{x}_{P_i}, \mathrm{y}_{P_i}) = (x_i, y_i)$ and $\mathrm{type}(P_i) = t_i$. A "*position of* P" is an element of $\mathbb{Z}^2$ that appears in P (and therefore appears exactly once), and an *index* i of a path P of length $n \in \mathbb{N}$ is a natural number $i \in \{0, 1, \dots, n-1\}$. For a path $P = P_0P_1P_2 \dots$ we write $\mathrm{pos}(P)$ to mean "the sequence of positions of tiles along P", which is $\mathrm{pos}(P) = \mathrm{pos}(P_0)\mathrm{pos}(P_1)\mathrm{pos}(P_2)\dots$. For a finite path $P = P_0P_1P_2 \dots P_{|P|-1}$, we define $P^{\leftarrow}$ as the path $P_{|P|-1}P_{|P|-2} \dots P_0$. The vertical height of a path P is defined as $\max\{|y_{P_i} - y_{P_j}| : 0 \leq i \leq j \leq |P| - 1\}$ and its horizontal width is $\max\{|x_{P_i} - x_{P_j}| : 0 \leq i \leq j \leq |P| - 1\}$.

Although a path is not an assembly, we know that each adjacent pair of tiles in the path sequence interact implying that the set of path positions forms a connected set in $\mathbb{Z}^2$ and hence every path uniquely represents an assembly containing exactly the tiles of the path, more formally: for a path $P = P_0P_1P_2 \dots$ we define the set of tiles $\mathrm{asm}(P) = \{P_0, P_1, P_2, \dots\}$ which we observe is an assembly[2] and we call $\mathrm{asm}(P)$ a *path assembly.*

Given a tile assembly system $\mathcal{T} = (T, \sigma, 1)$ the path P is a *producible path of* $\mathcal{T}$ if $\mathrm{asm}(P)$ does not intersect[3] the seed σ and the assembly $(\mathrm{asm}(P) \cup \sigma)$ is producible by $\mathcal{T}$, i.e. $(\mathrm{asm}(P) \cup \sigma) \in \mathcal{A}[\mathcal{T}]$, and P_0 interacts with a tile of σ. Consider an assembly α (resp. a path Q), as a convenient abuse of notation we sometimes write $\sigma \cup P$ (resp. $P \cup Q$) as a shorthand for $\sigma \cup \mathrm{asm}(P)$ (resp. $\mathrm{asm}(P) \cup \mathrm{asm}(Q)$).

Note that producible paths may not necessarily result in producible assemblies: indeed, in this paper, we will need to reason on multiple translations of a single path, and only later prove that these translations are actually connected to the seed. Therefore, we must be able to talk about these "temporarily disconnected" paths, while proving that they actually result in producible assemblies.

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[2] I.e. $\mathrm{asm}(P)$ is a partial function from $\mathbb{Z}^2$ to tile types, and is defined on a connected set.

[3] Formally, the non-intersection of a path $P = P_0P_1, \dots$ and a seed assembly σ is defined as: $\forall t$ such that $t \in \sigma$, $\nexists i$ such that $\mathrm{pos}(P_i) = \mathrm{pos}(t)$.

Given a directed tile assembly system $\mathcal{T} = (T, \sigma, 1)$ and its unique terminal assembly α, the path P is *a path of* α if $\text{asm}(P)$ is a subassembly of α. We define the set of paths of α to be:

$$\mathbf{P}[\alpha] = \{P \mid P \text{ is a path and } \text{asm}(P) \text{ is a subassembly of } \alpha\}$$

Note that, for any tiles $((x, y), t) \in \alpha$ and $((x', y'), t') \in \alpha$ there is a path $P \in \mathbf{P}[\alpha]$ such that for some $P_0 = ((x, y), t)$ and $P_{|P|-1} = ((x', y'), t')$.

For $A, B \in \mathbb{Z}^2$, we define $\overrightarrow{AB} = B - A$ to be the vector from A to B, and for two *tiles* $P_i = ((x_i, y_i), t_i)$ and $P_j = ((x_j, y_j), t_j)$ we define $\overrightarrow{P_iP_j} = \text{pos}(P_j) - \text{pos}(P_i)$ to mean the vector from $\text{pos}(P_i) = (x_i, y_i)$ to $\text{pos}(P_j) = (x_j, y_j)$. The translation of a path P by a vector $\overrightarrow{v} \in \mathbb{Z}^2$, written $P + \overrightarrow{v}$, is the path Q such that $|P| = |Q|$ and for all indices $i \in \{0, 1, \ldots, |P| - 1\}$, $\text{pos}(Q_i) = \text{pos}(P_i) + \overrightarrow{v}$ and $\text{type}(Q_i) = \text{type}(P_i)$.

2.3 Intersections

If two paths, or two assemblies, or a path and an assembly, share a common position we say that they *intersect* at that position. Furthermore, we say that two paths, or two assemblies, or a path and an assembly, *agree* on a position if they both place the same tile type at that position and *conflict* if they place a different tile type at that position. We say that a path P is *fragile* to mean that there is a producible assembly α that conflicts with P. Intuitively, if we grow α first, then there is at least one tile that P cannot place. In directed tile assembly systems, which are the subject of our main result, since the terminal assembly is unique there are no fragile paths in $\mathbf{P}[\alpha]$.

Let P and Q be two paths. We say that Q *grows from* P at index i, if the only intersection between Q and P occurs at $\text{pos}(Q_0) = \text{pos}(P_i)$ and is an agreement. Note that if α is the terminal assembly of some tile assembly system $\mathcal{T}$, and $P \in \mathbf{P}[\alpha]$, the assertions "Q grows from P" or "Q is an arc of P" do not imply that $Q \in \mathbf{P}[\alpha]$, since Q might conflict with the seed. We say that Q is an *arc* of P between indices $i < j$ if and only if the only two intersections between Q and P, which occur at $\text{pos}(Q_0) = \text{pos}(P_i)$ and $\text{pos}(Q_{|Q|-1}) = \text{pos}(P_j)$ are both agreements and neither Q nor $Q^{\leftarrow}$ are subpaths of P[4] The *width of an arc* Q of P is defined by $|j - i|$.

2.4 Pumping a path, possibly in both directions

Next, for a path P, we define sequences of points and tile types (not necessarily a path, since these sequences might not be simple) called the *pumping of* P or the *bi-pumping of* P:

► **Definition 4** (Infinite and bi-infinite pumping of P). *Let* $\mathcal{T} = (T, \sigma, 1)$ *be a tile assembly system and a path* P *of length at least 2, such that* $\text{type}(P_0) = \text{type}(P_{|P|-1})$. *We say that the* "infinite pumping of P"*, denoted by* $(P)^*$*, is the infinite sequence* $\overline{q}$ *of elements from* $\mathbb{Z}^2 \times T$ *defined by:*

$$\overline{q}_k = P_{k \mod (|P|-1)} + \left\lfloor \frac{k}{|P| - 1} \right\rfloor \overrightarrow{P_0P_{|P|-1}} \text{ for } k \in \mathbb{N}$$

We say that the "bi-infinite pumping of P"*, denoted by* ${}^*(P)^*$*, is the bi-infinite sequence* $\overline{q}$ *of elements from* $\mathbb{Z}^2 \times T$ *defined by:*

$$\overline{q}_k = P_{k \mod (|P|-1)} + \left\lfloor \frac{k}{|P| - 1} \right\rfloor \overrightarrow{P_0P_{|P|-1}} \text{ for } k \in \mathbb{Z}$$

[4] The condition that neither Q nor $Q^{\leftarrow}$ are subpaths of P is only required when $|Q| = 2$, to avoid the cases where $j = i + 1$ or $j = i - 1$.

In this article, we will only consider cases where $\overline{q}$ is simple, *i.e.* where for any $s < t$, if $P + s\overrightarrow{P_0 P_{|P|-1}}$ intersects with $P + t\overrightarrow{P_i P_j}$, then $t = s + 1$ and the only intersection is an agreement between $P_0 + t\overrightarrow{P_0 P_{|P|-1}}$ and $P_{|P|-1} + s\overrightarrow{P_0 P_{|P|-1}}$. A sufficient condition for this is that the only intersection between P and $P + \overrightarrow{P_0 P_{|P|-1}}$ is an agreement between $P_0 + \overrightarrow{P_0 P_{|P|-1}}$ and $P_{|P|-1}$ (folklore, see [19] for example). If this condition is satisfied then P is called a *good candidate* and $(P)^*$ and ${}^*(P)^*$ are both paths. Note that, for all $k \in \mathbb{N}$ (resp. $k \in \mathbb{Z}$), we have $(P)^*_{k+|P|-1} = (P)^*_k + \overrightarrow{P_0 P_{|P|-1}}$ (resp. ${}^*(P)^*_{k+|P|-1} = {}^*(P)^*_k + \overrightarrow{P_0 P_{|P|-1}}$).

▶ **Definition 5** (Pumpable path). *Let $\mathcal{T} = (T, \sigma, 1)$ be a directed tile assembly system and let α be its unique terminal assembly. We say that a good candidate P is* pumpable *if $(P)^* \in \mathbf{P}[\alpha]$ and* bi-pumpable *if ${}^*(P)^* \in \mathbf{P}[\alpha]$. A good candidate that is pumpable but not bi-pumpable is called* simply pumpable.

An *ultimately periodic* path P can be written as $Q(R)^*$ where Q is a finite path and R is a good candidate. Q is called the *transient* part of P and $(R)^*$ is called the *periodic* part of P.

In our context, we will use the following version of the pumping lemma of [19] where there are no fragile path and the bound is replaced by a generic function $f(x, y)$ where x is the number of tile types and y is the size of the seed (the bound computed in [19] might not be optimal, and could be improved independently of the results presented here).

▶ **Theorem 6.** *There exists a function $f : \mathbb{N}^2 \to \mathbb{N}$ such that for any directed tile assembly system $\mathcal{T} = (T, \sigma, 1)$ and any of its producible path P, if P has vertical height or horizontal width at least $f(|T|, |\sigma|)$, then there exist $0 \le i < j \le |P| - 1$ such that $P_{i,\ldots,j}$ is pumpable.*

Here is the pumpability conjecture which will be a corollary of our result and which was stated in the study of [8], note that we do not consider here that the size of the seed could be reduced to 1 and we have to take it into account.

▶ **Theorem 7.** *There exists a function $f' : \mathbb{N}^2 \to \mathbb{N}$ such that for any directed tile assembly system $\mathcal{T} = (T, \sigma, 1)$ and a path P of α, if $|P| \ge f'(|T|, |\sigma|)$, then there exist $0 \le i < j \le |P| - 1$ such that for all $\ell \in \mathbb{N}$ either $P_{i,\ldots,j} + \ell\overrightarrow{P_i P_j}$ or $P_{i,\ldots,j} - \ell\overrightarrow{P_i P_j}$ is in α.*

3 Proof of our main theorem

3.1 Roadmap

An assembly α is $\overrightarrow{v}$*-periodic* if it is invariant by the translation of vector $\overrightarrow{v}$, *i.e.* $\alpha + \overrightarrow{v} = \alpha$. We say that an assembly α is *bi-periodic* if there exist two non-colinear vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ such that α is $\overrightarrow{u}$-periodic and $\overrightarrow{v}$-periodic. An assembly is *simply periodic* if it is not bi-periodic and if there exists a vector $\overrightarrow{v}$ such that α is $\overrightarrow{v}$-periodic. Assemblies that are neither bi-periodic nor simply periodic are called *nonperiodic*. Then, like in the original paper [8], we decompose terminal assemblies into four classes: finite, infinite with/without comb, periodic with/without comb and bi-periodic.

The complexity of a finite terminal assembly α of a tile assembly system $\mathcal{T} = (T, \sigma, 1)$ is 0. Moreover, the pumping lemma (Theorem 6) implies that α fits in a square of width $2f(|T|, |\sigma|) + |\sigma|$ and thus its size is bounded by $4(f(|T|, |\sigma|) + |\sigma|)^2$. In this case, the pumpability conjecture holds since we can claim that any path of length at least $4f(|T|, |\sigma|)^2 + |\sigma| + 1$ is pumpable.

To deal with the three remaining cases, we first show in Section 3.3 that "α is $\overrightarrow{v}$-periodic" is equivalent of "there exists a bi-pumpable path P in $\mathbf{P}[\alpha]$ such that $\overrightarrow{P_0 P_{|P|-1}} = \overrightarrow{v}$". Then, we proceed to characterise bi-periodic terminal assemblies in Section 3.4, the infinite nonperiodic ones in Section 3.5 and finally the simply periodic ones in Section 3.6 (a hybrid case of the two previous ones). Note that due to space constraint, we omit some details, in particular details of the original study [8] and focus on improving/patching it. A more self-contained version of this article is available on arXiv [18].

3.2 Relationship with the pumpability conjecture

The relationship between this result and the *pumpability conjecture* [8] is a bit subtle and deserves to be discussed. Indeed, the original statement of the conjecture is that in a directed tile assembly system, any *part of a path* long enough to have a repeated tile type is pumpable, meaning that this part can be repeated infinitely.

In contrast to that statement, [19] proved a weaker statement, namely that only the *initial* segment (starting from the seed) can be pumped, if that initial segment is long enough[5].

In this paper, we prove that the weaker statement actually implies the stronger one: indeed, we prove that the only terminal assembly that can be built by a directed system is made of pumped paths. Therefore, we prove that if a system is directed, any path P appearing in the terminal assembly is the concatenation of one, two or three (possibly infinite) fragments of periodic paths, which implies that any long enough segment of P contains at least one full period of one of these periodic paths, which is exactly the original pumpability conjecture.

3.3 Link between periodic assembly and bi-pumpable paths

In this subsection, Corollary 9 and Lemma 10 (see Appendix A for the proof of Lemma 10) show the equivalence between the statement "α is $\overrightarrow{v}$-periodic" and "there exists a bi-pumpable path P where $\overrightarrow{v} = \overrightarrow{P_0 P_{|P|-1}}$". Lemma 11 gives a sufficient condition for a pumpable path to be bi-pumpable. These results and the proofs of this section come from the original paper [8]. We have just reorganize the arguments to show the new stronger Lemma 8 which implies Corollary 9 (the first direction of the equivalence) and is later useful to improve the precision of the characterisations of the different classes. It stipulates that we can grow the same terminal assembly α starting from any tile of $^*(P)^*$ as the seed, and that the resulting tile assembly system is also directed. Later, this lemma will allow us to grow and pump paths easily. The proof is by contradiction: assuming the assembly weren't the same, had we started from a different seed, we show that we can get conflicts, contradicting the assumption that $\mathcal{T}$ is directed.

▶ **Lemma 8.** *Let $\mathcal{T} = (T, \sigma, 1)$ be a directed tile assembly system and α its unique terminal assembly. If a path $P \in \mathbf{P}[\alpha]$ is bi-pumpable then for any $i \in \mathbb{Z}$, the tile assembly system $(T, {}^*(P)^*_i, 1)$ (i.e. $\mathcal{T}$, with the seed σ replaced by the assembly made of a single tile defined as ${}^*(P)^*_i$), is directed and its terminal assembly is α.*

Proof. Since $^*(P)^*$ is in $\mathbf{P}[\alpha]$, let β be any finite assembly producible by $(T, \sigma, 1)$, such that ${}^*(P)^*_i$ is a tile of β. Since $\mathcal{T}$ is directed, $^*(P)^*$ and β cannot possibly conflict, hence $^*(P)^* \cup \beta$ is producible by $(T, \sigma, 1)$. Let therefore R be any path producible by $(T, {}^*(P)^*_i, 1)$. If R does

[5] That result also applies to nondirected tile assembly systems, in which case long paths can be either pumped or blocked, meaning that another assembly can be built first and prevent the path from growing.

not conflict with β nor with ${}^*(P)^*$, then $(R \cup {}^*(P)^* \cup \beta)$ is producible by $(T, \sigma, 1)$, and hence R is in $\mathbf{P}[\alpha]$. For the sake of contradiction suppose that such a conflict exists. We assume without loss of generality that the first such conflict along R happens between $R_{|R|-1}$ (i.e. at the last tile of R, which can always be achieved by considering a prefix of R) and either β or ${}^*(P)^*$. There are two cases:

- If this conflict is with ${}^*(P)^*_j$ for $j \neq i$, then since β and R are finite and $\overrightarrow{P_0P_{|P|-1}}$ is non-null, there exists $\ell \in \mathbb{Z}$ such that $R + \ell\overrightarrow{P_0P_{|P|-1}}$ does not intersect with β. Note that $R_0 + \ell\overrightarrow{P_0P_{|P|-1}} = {}^*(P)^*_{i+\ell(|P|-1)}$ and that $R_{|R|-1} + \ell\overrightarrow{P_0P_{|P|-1}}$ conflicts with ${}^*(P)^*_{j+\ell(|P|-1)}$. By definition of β and P, the assembly $\gamma = \beta \cup {}^*(P)^*_{i,\dots,i+\ell(|P|-1)}$ (or $\gamma = \beta \cup {}^*(P)^*_{i+\ell(|P|-1),\dots,i}$ if $\ell < 0$) is producible by $(T, \sigma, 1)$. By definition of ℓ, the tile $R_{|R|-1} + \ell\overrightarrow{P_0P_{|P|-1}}$ is not a tile of β and since $j \neq i$, we have $i + \ell(|P|-1) \neq j + \ell(|P|-1)$ thus $R_{|R|-1} + \ell\overrightarrow{P_0P_{|P|-1}}$ is not a tile of γ. Therefore, the assembly $\gamma \cup (R + \ell\overrightarrow{P_0P_{|P|-1}})$ is producible by $(T, \sigma, 1)$ and is in conflict with ${}^*(P)^* \in \mathbf{P}[\alpha]$, which is a contradiction.
- Otherwise, this conflict occurs with β. Since β and R are finite and $\overrightarrow{P_0P_{|P|-1}}$ is not null, there exists $\ell \in \mathbb{N}$ such that neither $\beta + \ell\overrightarrow{P_0P_{|P|-1}}$ not $R + \ell\overrightarrow{P_0P_{|P|-1}}$ intersect with β. Since ${}^*(P)^*$ is $\overrightarrow{P_0P_{|P|-1}}$-periodic then ${}^*(P)^*$ does not conflict with $\beta + \ell\overrightarrow{P_0P_{|P|-1}}$ nor with $R + \ell\overrightarrow{P_0P_{|P|-1}}$. Then the two assemblies $(\beta \cup {}^*(P)^* \cup (\beta + \ell\overrightarrow{P_0P_{|P|-1}}))$ and $(\beta \cup {}^*(P)^* \cup (R + \ell\overrightarrow{P_0P_{|P|-1}}))$ are both producible by $(T, \sigma, 1)$, but these two assemblies conflict, which contradicts the hypothesis that $(T, \sigma, 1)$ is directed.

Thus, any path R producible by $(T, {}^*(P)^*_i, 1)$ is producible by $(T, \sigma, 1)$. Therefore, if two assemblies producible by $(T, {}^*(P)^*_i, 1)$ conflicted, then the same conflict can be achieved in $(T, \sigma, 1)$, contradicting the hypothesis that $(T, \sigma, 1)$ is directed. Thus $(T, {}^*(P)^*_i, 1)$ is directed. The terminal assembly α contains ${}^*(P)^*_i$ therefore, α is the unique terminal assembly of $(T, {}^*(P)^*_i, 1)$. ◀

As a corollary of this result, any path Q that grows on ${}^*(P)^*$ is in $\mathbf{P}[\alpha]$. Moreover, since for any $\ell \in \mathbb{Z}$, ${}^*(P)^* + \ell\overrightarrow{P_0P_{|P|-1}} = {}^*(P)^*$ then $Q + \ell\overrightarrow{P_0P_{|P|-1}}$ also grows on ${}^*(P)^*$ and is in $\mathbf{P}[\alpha]$ which leads to the following corollary:

▶ **Corollary 9.** *Let $\mathcal{T} = (T, \sigma, 1)$ be a directed tile assembly system and let α be the unique terminal assembly of $\mathcal{T}$. If $P \in \mathbf{P}[\alpha]$ is bi-pumpable then α is $\overrightarrow{P_0P_{|P|-1}}$-periodic.*

▶ **Lemma 10.** *Let $\mathcal{T} = (T, \sigma, 1)$ be a directed tile assembly system and let α be the unique terminal assembly of $\mathcal{T}$. If α is periodic then there exists a path $P \in \mathbf{P}[\alpha]$ that is bi-pumpable.*

▶ **Lemma 11.** *Let $\mathcal{T} = (T, \sigma, 1)$ be a directed tile assembly system and let α be the unique terminal assembly of $\mathcal{T}$. Consider a pumpable path P of $\mathbf{P}[\alpha]$ and a path Q growing on $(P)^*$ at index $i \geq |P| - 1$ such that Q and $Q + \overrightarrow{P_0P_{|P|+1}}$ intersect then P is bi-pumpable.*

Proof. Since P is in $\mathbf{P}[\alpha]$, there is a finite producible subassembly β of α such that P_0 is a tile of β. For the sake of contradiction suppose that there is a conflict between ${}^*(P)^*$ and β otherwise P would be bi-pumpable.

Let R be the largest prefix of Q which does not intersect with ${}^*(P)^*$ then for all $\ell \in \mathbb{N}$, $R + \ell\overrightarrow{P_0P_{|P|-1}}$ grows on $(P)^*$. Moreover, if $R \neq Q$ then R still intersects with $R + \overrightarrow{P_0P_{|P|-1}}$. Indeed, without loss of generality, suppose that Q and ${}^*(P)^*$ agree (the following reasoning does not rely on the tile type) then there exists $j < 0$ such that RP_j is an arc of ${}^*(P)^*$ of width greater than $|P|$. Then RP_j and $P_{j,\dots,i}$ form a cycle which delimits a finite area of the 2D plane. The arc $(RP_j) + \overrightarrow{P_0P_{|P|-1}}$ starts in $P_{i+|P|-1}$, a tile which is not in the finite area since $i + |P| - 1 > i$, and ends in $P_{j+|P|-1}$, a tile which is in the finite area since $j < j + |P| - 1 < i$. Then $R + \overrightarrow{P_0P_{|P|-1}}$ must cross R to reach P_j.

Note that $R + \overrightarrow{P_0P_{|P|-1}}$ intersects with both R and $R + 2\overrightarrow{P_0P_{|P|-1}}$. Moreover all these intersections are agreements (because for ℓ large enough the translations of these three paths by $\ell\overrightarrow{P_0P_{|P|-1}}$ do not intersect with β and thus are in $\mathbf{P}[\alpha]$). Then, the assembly $\gamma = R \cup (R_{1,2,\ldots,|R|-1} + \overrightarrow{R_0R_{|P|-1}}) \cup (R + 2\overrightarrow{R_0R_{|R|-1}})$ is well-defined. By definition of growing, the only intersection between $(P)^*$ and γ is R_0 and $R_0 + 2\overrightarrow{R_0R_{|R|-1}}$. Thus there exists an arc A growing on $(P)^*$ of width $2(|P| - 1) > |P|$ and such that $\text{asm}(A)$ is a subassembly of γ. By definition of R, for all $\ell \in \mathbb{N}$, the arc $A + \ell\overrightarrow{P_0P_{|P|-1}}$ also grows on $(P)^*$.

Since β is finite and $\overrightarrow{P_0P_{|P|-1}}$ is not null, there is an integer $L \in \mathbb{N}$ such that for all $\ell \geq L$, neither $A + \ell\overrightarrow{P_0P_{|P|-1}}$ nor $\beta + \ell\overrightarrow{P_0P_{|P|-1}}$ intersect β. Since the width of A is strictly greater than $|P|$, we can find $\ell, \ell' > L$ and $0 < a < b < c$ such that $A + \ell$ is an arc of $(P)^*$ between $(P)^*_c$ and $(P)^*_a$ and there is conflict between $\beta + \ell'\overrightarrow{P_0P_{|P|-1}}$ and $(P)^*_b$. Then there exists a path S such that $\text{asm}(S)$ is a subassembly of $\beta + \ell'\overrightarrow{P_0P_{|P|-1}}$, $S_0 = P_0 + \ell'\overrightarrow{P_0P_{|P|-1}}$ and the only conflict between $(P)^*$ and S occurs between $S_{|S|+1}$ and P_b. By definition of ℓ and ℓ', the paths $A + \ell\overrightarrow{P_0P_{|P|-1}}$ and $S_{0,\ldots,|S|-2}$ are both in $\mathbf{P}[\alpha]$ and thus cannot conflict. Consider the following assembly $\delta = \beta \cup (P)^* \cup (A + \ell\overrightarrow{P_0P_{|P|-1}}) \cup S_{0,\ldots,|S|-2}$. Removing the tile P_b disconnects $(P)^*$ in two parts, but adding $A + \ell\overrightarrow{P_0P_{|P|-1}}$ reconnects them (a<b<c), and it is therefore possible to remove the tile P_b in δ and to replace it by the tile $S_{|S|-1}$ which can bind with $S_{|S|-2}$ contradicting the hypothesis that $\mathcal{T}$ is directed. ◀

3.4 Characterisation of the bi-periodic terminal assemblies

The characterisation of the bi-periodic terminal assemblies does not rely on the pumping lemma. Thus the result of the original paper [8] still holds for this case. Here is a summary, if the terminal assembly α of a tile assembly system $\mathcal{T} = (T, \sigma, 1)$ is bi-periodic, by Lemma 10 there exists two paths P of Q of $\mathbf{P}[\alpha]$ which are bi-pumpable and such that $\overrightarrow{P_0P_{|P|-1}}$ is not colinear with $\overrightarrow{Q_0Q_{|Q|-1}}$. Moreover for all $\ell \in \mathbb{Z}$, ${}^*(P)^* + \ell\overrightarrow{Q_0Q_{|Q|}}$ and ${}^*(Q)^* + \ell\overrightarrow{P_0P_{|P|}}$ are in $\mathbf{P}[\alpha]$. All these paths can be used to "tile the plane" with a periodic grid-like structure (see Figure B.1). By considering the assembly β which is the restriction of α to a "cell" of this grid, we obtain that $\alpha = \bigcup_{\ell,\ell' \in \mathbb{Z}}(\beta + \ell\overrightarrow{u} + \ell'\overrightarrow{v})$. For the main Theorem 2, α is an assembly of complexity 2 defined by the union of the four assemblies of complexity 2: $\bigcup_{\ell,\ell' \in \mathbb{N}}(\beta + \ell\overrightarrow{u} + \ell'\overrightarrow{v})$, $\bigcup_{\ell,\ell' \in \mathbb{N}}(\beta + \ell\overrightarrow{u} - \ell'\overrightarrow{v})$, $\bigcup_{\ell,\ell' \in \mathbb{N}}(\beta - \ell\overrightarrow{u} + \ell'\overrightarrow{v})$ and $\bigcup_{\ell,\ell' \in \mathbb{N}}(\beta - \ell\overrightarrow{u} - \ell'\overrightarrow{v})$.

Here we improve this result by introducing paths *without redundancy* where there are no repetition of a tile type along the path (except at its extremities), see Definition 12. Of course, the length of such a path is bounded by $|T| + 1$. Lemma 8 allow us to extract bi-pumpable paths without redundancy from bi-pumpable paths and then the size of the cell (and thus of β) of the periodic grid-like structure becomes bounded by $O(|T|^2)$.

▶ **Definition 12.** *A path P is without redundancy if for all $0 \leq i < j \leq |P| - 1$, $\text{type}(P_i) = \text{type}(P_j)$ implies that $i = 0$ and $j = |P| - 1$.*

▶ **Theorem 13.** *Let $\mathcal{T} = (T, \sigma, 1)$ be a directed tile assembly system, and let α be its unique terminal assembly. If α is bi-periodic, then there exists an assembly β and two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ such that $|\beta| \leq 4|T|^2$ and $\alpha = \bigcup_{\ell,\ell' \in \mathbb{Z}}(\beta + \ell\overrightarrow{u} + \ell'\overrightarrow{v})$.*

Proof. If α is bi-periodic then by Lemma 10, there exists a path P of $\mathbf{P}[\alpha]$ which is bi-pumpable. By definition of a bi-pumpable path, P_0 and $P_{|P|-1}$ have the same tile type, then there exists $0 \leq i < j \leq |P| - 1$ such that the path $R = P_{i,\ldots,j}$ is without redundancy. Consider $0 \leq i', j' \leq |P| - 1$ such that $\text{pos}(R_{i'}) = \text{pos}(R_{j'}) + \overrightarrow{R_0R_{|R|-1}}$, since $\overrightarrow{R_0R_{|R|-1}}$ is not null then $i' \neq j'$. By definition, R is in $\mathbf{P}[\alpha]$ and by Lemma 8, we can consider that $R_{|R|-1}$ is the seed, then $R + \overrightarrow{R_0 R_{|R|+1}}$ is in $\mathbf{P}[\alpha]$ and $R_{i'}$ and $R_{j'}$ have the same tile type which implies $i' = 0$ and $j' = |R| - 1$, *i.e.* R is a good candidate. Since we can consider that $R_{|R|-1}$ is the seed then ${}^*(R)^*$ is a bi-infinite path of $\mathbf{P}[\alpha]$ and R is bi-pumpable.

Since α is bi-periodic there exists a non null vector $\overrightarrow{v}$ which is not colinear with $\overrightarrow{R_0 R_{|R|-1}}$ and such that α is $\overrightarrow{v}$-periodic. Then ${}^*(R)^* + \overrightarrow{v}$ is a path of $\mathbf{P}[\alpha]$ and since R is without redundancy, by a reasoning similar to the one of the previous paragraph, ${}^*(R)^*$ and ${}^*(R)^* + \overrightarrow{v}$ cannot intersect. Then we consider the shortest path Q such that $Q_0 = R_0$ and $Q_{|Q|-1} = R_0 + \overrightarrow{v} + \ell\overrightarrow{R_0 R_{|R|-1}}$ for some $\ell \in \mathbb{Z}$, *i.e.* $Q_{|Q|-1}$ has the same tile type than R_0 and belongs to ${}^*(R)^* + \overrightarrow{v}$. Again, it is possible to find $0 \leq i' \leq j' \leq |Q| - 1$ such that $S = Q_{i',\ldots,j'}$ is a good candidate without redundancy but proving that S is bi-pumpable is more tricky. By Lemma 8, we consider that the seed is $Q_0 = R_0$. From this seed, we grow $Q_{1,\ldots,j'}$ and then we pump S is both direction until either assembling ${}^*(S)^*$ or to obtain a path S' whose growth was blocked by $Q_{1,\ldots,j'}$. In the second case, the path S' is in $\mathbf{P}[\alpha]$ and thus cannot conflict with $Q_{i',\ldots,|Q|-1}$ and we obtain a contradiction by using Lemma 8 and considering that the seed is $Q_{|Q|-1}$(a tile of ${}^*(R)^* + \overrightarrow{v}$) this time: from this seed, we grow $Q_{i',\ldots,|Q|-1}$ and S'. In this case, $Q_{0,\ldots,i'-1}$ is not here to block the growth of S' and at least one more tile can be added, creating a conflict with $Q_{0,\ldots,i'-1}$ which is a contradiction. Then S is bi-pumpable and for the sake of contradiction, if $\overrightarrow{S_0 S_{|S|-1}}$ and $\overrightarrow{R_0 R_{|R|-1}}$ are colinear then α is $\overrightarrow{S_0 S_{|S|-1}}$-periodic and ${}^*(R)^*$ intersect with ${}^*(R)^* + \overrightarrow{S_0 S_{|S|-1}}$, since R is without redundancy then $\overrightarrow{S_0 S_{|S|-1}} = \ell\overrightarrow{R_0 R_{|R|-1}}$ for some $\ell \in \mathbb{N}$. Using the assembly $\mathrm{asm}(Q_{1,\ldots,i'}) \cup \mathrm{asm}(Q_{j',\ldots,|Q|-1} - \overrightarrow{S_0 S_{|S|-1}})$ we can find a path Q' such that $|Q'| < |Q|$, $Q'_0 = Q_0$ and $Q'_{|Q'|-1} = Q_{|Q|-1} - \ell\overrightarrow{R_0 R_{|R|-1}}$, contradicting the definition of Q.

As explained in the beginning of this section, these two bi-pumpable paths R and S create a periodic grid-like structure and α can be characterised by its restriction to a "cell" of this grid. In our case, the cell is delimited by four paths of length bounded by $|T|$ and thus the size of the assembly is bounded by $4|T|^2$. ◀

Any path P of length $O(|T|^3)$ would have to pass by at least $O(|T|)$ cells of the periodic grid-like structure and thus P must intersect the translations of one bi-pumpable path R at least $|R|$ times, among these intersections two have the same tile type. Using Lemma 10, the subpath of P between these two tiles can be pumped and the pumpability conjecture holds in this case.

3.5 Characterisation of the infinite nonperiodic terminal assemblies

We present here a summary of the analysis relying on the pumpability conjecture of the infinite nonperiodic terminal assemblies done in [8] before explaining how to patch this result when replacing the pumpability conjecture (Theorem 7) by the pumping lemma (Theorem 6).

Note that if α is nonperiodic then Lemma 10 implies that all pumpable paths of $\mathbf{P}[\alpha]$ are simply pumpable. Any nonperiodic assembly can be decomposed in three parts (see Figure B.2 for an example): the first part is a finite assembly which contains the seed, the second part is made of some simply pumpable paths growing from this finite assembly called *combs* used to generate periodic paths called the *backbone* of the combs, and the third part is made of paths growing on the backbone of a comb called the *teeth* of the comb. More formally a comb C is a pumpable path of α which is linked to the seed by a producible path containing no pumpable subpath, the backbone of C is $(C)^*_{|C|,\ldots,+\infty}$ and a tooth t is a path growing on the backbone of a comb C. It was shown in [8] that if an infinite ultimately periodic tooth t is in $\mathbf{P}[\alpha]$ then for any $\ell \in \mathbb{N}$, $t + \ell\overrightarrow{C_0C_{|C|-1}}$ also grow on the backbone of the comb and also belongs to $\mathbf{P}[\alpha]$. Also, only finite path can grow on the periodic part of a comb. For the simple example of Figure B.2, the path P of Figure B.3 allow us to describe the terminal assembly with a finite amount of information.

Lemma 11 is the key to obtain this result: a tooth cannot intersect with its translation by $\overrightarrow{C_0C_{|C|-1}}$ otherwise the comb would be bi-pumpable. Moreover, if an infinite ultimately periodic path P grows on the periodic part of an ultimately periodic tooth t then P would either intersect $t + \overrightarrow{C_0C_{|C|-1}}$ (or $t - \overrightarrow{C_0C_{|C|-1}}$) and C would be bi-pumpable or P would intersect with one of its copy growing on the periodic part of the tooth t and then the periodic part of the tooth would be bi-pumpable in this case. Thus only finite path can grow on the periodic part of a tooth and the pumpability conjecture (Theorem 7) allows us to bound their size. As stated in [8], the pumpability conjecture is needed only three times: the first time to locate the combs, the second time to create an ultimately periodic tooth and a last time to bound the length of the paths growing on the periodic part of a tooth[6].

To obtain a similar result we have to explain how to use the pumping lemma instead of the pumpability conjecture. Consider again the finite producible path P of Figure B.3: there are five indices $0 \leq i_1 < j_1 < t < i_2 < j_2 \leq |P| - 1$ such that $P_{i_1,\dots,j_1}$ is a comb, P_k is the first tile of a tooth, $P_{i_2,\dots,j_2}$ is pumpable and belongs to the tooth and $P_{j_2+1,\dots,|P|-1}$ is a path growing on the periodic part of the tooth. The pumping lemma of [19] is able to find one pumpable subpath $P_{i_1,\dots,j_1}$ of P but it may seem too weak to find another, different pumpable subpath $P_{i_2,\dots,j_2}$ and too weak to bound the size of $P_{j_2+1,\dots,|P|-1}$. However, the pumping Lemma of [19] can be applied to $P_{k+1,\dots,|P|-1}$ (where P_t is the first tile of the tooth) considered as a path producible by the directed tile assembly system $(T, \sigma \cup P_{0,1,\dots,k}, 1)$, whose terminal assembly is also α (see Figure B.4). This remark shows that the following result is a direct corollary of the pumping lemma of [19].

▶ **Corollary 14.** *There exists a function $f : \mathbb{N}^2 \to \mathbb{N}$ such that for any directed tile assembly system $\mathcal{T} = (T, \sigma, 1)$ and for any producible path P and $0 \leq i \leq |P| - 1$, if $P_{i,\dots,|P|-1}$ has vertical height or horizontal width at least $f(|T|, |\sigma| + i)$, then there exist $i \leq j < k \leq |P| - 1$ such that $P_{j,\dots,k}$ is pumpable.*

The result of [8] still holds with this corollary and this is why the authors of [19] claimed that their result allows to solve the conjecture with the proof sketch of [8]. Nevertheless, to provide a bound on the size of the assemblies needed to characterise α, we need to bound for a comb where the first ultimately periodic tooth in $\mathbf{P}[\alpha]$ appears on the backbone of a comb. Indeed a tooth growing at the beginning of the backbone may be blocked by the seed or a previous assembly and will not belong to $\mathbf{P}[\alpha]$. To locate this tooth, we show that there is an index for any periodic path such that any path growing after this index is in $\mathbf{P}[\alpha]$.

▶ **Lemma 15.** *Let $\mathcal{T} = (T, \sigma, 1)$ be a directed tile assembly system whose terminal assembly is α. If there is a simply pumpable path P in $\mathbf{P}[\alpha]$ and a producible finite assembly β such that P_0 is a tile of β, then there is an index i such that any path growing on $(P)^*$ at index $j \geq i$ is in $\mathbf{P}[\alpha]$. Moreover, i only depends on $|\beta|$, $|P|$ and $|T|$.*

Proof. Without loss of generality we assume that P_0 is the only intersection between β and $(P)^*$. Let $j = (4|\beta| + 2)(|P| - 1) + 1$ and let γ be the assembly defined as $\gamma = \beta \cup (P)^*_{0,1,\dots,j}$, note that γ is a subassembly of α. Since β is finite and $\overrightarrow{P_0P_{|P|-1}}$ is not null, there is an

[6] In the extended version of this article [18], we improve this result and show a more efficient way which avoid using the conjecture for the last case. This result is omitted due to space constraint.

integer $i > j + (|P| - 1)$ such that the distance between any tile of $(P)^*_{i,i+1,..,+\infty}$ and any tile of β is at least $f(|T|, |\gamma|) + 1$ (see Theorem 6 for the definition of f). Note that i only depends on $|\beta|$, $|P|$ and $|T|$. See Figure B.5 for an illustration of the following reasoning.

Let Q be a path growing on $(P)^*$ at position P_k with $k \geq i$ (Q is in red in Subfigure a of Figure B.5). For the sake of contradiction, assume that Q is not in $\mathbf{P}[\alpha]$, which implies that Q conflicts with β and by the definition of i the vertical height or the horizontal width of Q is at least $f(|T|, |\gamma|) + 1$. Let $m = \max\{n : Q_{0,\ldots,n}$ does not intersect with $\beta\}$ and by definition of m, $Q_{0,1,\ldots,m}$ is in $\mathbf{P}[\alpha]$.

Consider the finite area A of $2D$ plane (in light red in Subfigure b of Figure B.5), whose border is delimited by $(P)^*_{0,1,\ldots,k}$, $Q_{0,1,\ldots,m}$ and β. Let R be the translation of Q by $\ell\overrightarrow{P_0P_{|P|-1}}$ for some $\ell \in \mathbb{N}$ such that R_0 is a tile of $P + \overrightarrow{P_0P_{|P|-1}}$ (R grows on $(P)^*$ at an index between $|P|$ and $2|P| - 1$). Note that for all $0 \leq \ell \leq 4|\beta|$, the path $R + \ell\overrightarrow{P_0P_{|P|-1}}$ starts to grow in the finite area A. Since there are at most $4|\beta|$ positions that are neighbors of a tile of β, this implies that if for all $0 \leq \ell \leq 4|\beta|$, the paths $R + \ell\overrightarrow{P_0P_{|P|-1}}$ intersect with β then there exists $\ell \in \mathbb{N}$ such that $R + \ell\overrightarrow{P_0P_{|P|-1}}$ and $R + (\ell + 1)\overrightarrow{P_0P_{|P|-1}}$ intersect each other before intersecting β, which by Lemma 11 would imply that P is bi-periodic. Therefore, there is at least one $0 \leq \ell \leq 4|\beta|$ such that $R + \ell\overrightarrow{P_0P_{|P|-1}}$ does not intersect β and thus is in $\mathbf{P}[\alpha]$. Moreover $R + \ell\overrightarrow{P_0P_{|P|-1}}$ grows on γ, thus we can consider that $R + \ell\overrightarrow{P_0P_{|P|-1}}$ is producible by $(T, \gamma, 1)$ and since the vertical height or the horizontal width of R is at least $f(|T|, |\gamma|)$, by the Pumping Lemma (Theorem 6), we can find an ultimately periodic path S of $\mathbf{P}[\alpha]$ growing on $(P)^*_{0,1,\ldots,(4|\beta|+1)(|P|-1)}$ and which does not intersect with γ.

Since S is infinite, it cannot fit into the finite area A and thus S must either intersect $(P)^*_{j+1,j+2,\ldots,k}$ or $Q_{0,1,\ldots,m}$ (see Subfigure c of Figure B.5). In the first case, a subpath of S is an arc of $(P)^*$ of width at least $j - (4|\beta| + 1)(|P| - 1) > |P|$ and in the second case an arc of $(P)^*$ of width $k - (4|\beta| + 1)(|P| - 1) > |P|$ is a subassembly of $\mathrm{asm}(S) \cup \mathrm{asm}(Q_{0,\ldots,m})$. As explained in the proof of Lemma 11, an arc of width at least $|P|$ must intersect with its translation by $\overrightarrow{P_0P_{|P|-1}}$ and then by Lemma 11, P should be bi-pumpable which is a contradiction. ◀

▶ **Theorem 16.** *Let $\mathcal{T} = (T, \sigma, 1)$ be a directed tile assembly system, and let α be its unique terminal assembly, if α is nonperiodic then the complexity of α is bounded by 2 and the size of characterisation depend only of $|\sigma|$ and $|T|$.*

Proof. The pumping Lemma (Theorem 6) shows that the restrictions of α to a square of width $2f(|T|, |\sigma|) + |\sigma|$ can contain the seed, all the combs and all the paths linking the seed to the comb, this finite assembly is of complexity 0.

Let C be a comb. Then its backbone is an assembly of complexity 1, which can be characterised by C and $\overrightarrow{C_0C_{|C|-1}}$. Moreover, let j be the index associated to $(C)^*$ by Lemma B.5 (j depends only of $|T|$ and $|\sigma|$ in this case).

Now, consider a finite tooth t of $\mathbf{P}[\alpha]$ growing on $(C)^*$ at index $i > |C| - 1$. If $i \leq j + |C| - 1$, we add t (an assembly of complexity 0) to α. Else, $i > j + |C| - 1$, and there exists a tooth t' in $\mathbf{P}[\alpha]$ such that $T = T' + \ell\overrightarrow{C_0C_{|C|-1}}$ for some $\ell \in \mathbb{N}$ and T' grow on $(C)^*_k$ with $j \leq k \leq j + |P| - 1$. Therefore, by adding $\bigcup_{\ell\in\mathbb{N}}(t' + \ell\overrightarrow{C_0C_{|C|-1}})$ to the resulting assembly, we also add the tooth t, and this union's characterisation has complexity 1.

Now if the tooth t is ultimately periodic, we can apply the same reasoning. In this case, we only need to prove that t and all the paths growing on the periodic part of t form an assembly of complexity 1 (the same reasoning will produce an assembly of complexity 2). Corollary 14 and Lemma 15 allow us to bound the length of the transient part of t, which depends only on $|\sigma|$ and $|T|$. Any path growing on the periodic part of the tooth is finite and by using the same reasoning with the comb and its finite tooth (which requires using Corollary 14 and Lemma 15 again), we obtain that the tooth t and all the paths growing on its periodic part form an assembly of complexity 1 and the length of the finite path growing on the periodic part of the tooth is bounded by a function depending only of $|\sigma|$ and $|T|$. We do not give the exact characterisation size, since this technique is unlikely to yield a tight bound. ◀

3.6 The simply periodic terminal assembly

In the original paper [8], the simply periodic terminal assembly were not studied in details. By Lemma 10, there exists a path P which is bi-pumpable and then ${}^*(P)^*$ cuts the $2D$ plane into two parts: its left and right side. Some ultimately periodic paths may grow on ${}^*(P)^*$, stay in one of the two sides and behave as the teeth of the previous section. This class of terminal assembly is a mix of the two previous ones.

We go in further details here, as in Section 3.4, we consider bi-pumpable paths without redundancy (see Definition 12). If P and Q are two bi-pumpable paths without redundancy, then the following Lemma (see Appendix A for the proof) shows that, by potentially reversing one of the two paths, we can consider that $\overrightarrow{P_0 P_{P_{|P|-1}}} = \overrightarrow{Q_0 Q_{Q_{|Q|-1}}}$. This remark allow us to introduce an order on the bi-pumpable paths without redundancy of $\mathbf{P}[\alpha]$ in Definition 18.

▶ **Lemma 17.** *Let $\mathcal{T} = (T, \sigma, 1)$ be a directed tile assembly system and let α be its unique simply periodic terminal assembly, if P and Q are two bi-pumpable paths of $\mathbf{P}[\alpha]$ without redundancy then either $\overrightarrow{P_0 P_{P_{|P|-1}}} = \overrightarrow{Q_0 Q_{Q_{|Q|-1}}}$ or $\overrightarrow{P_0 P_{P_{|P|-1}}} = -\overrightarrow{Q_0 Q_{Q_{|Q|-1}}}$.*

▶ **Definition 18.** *If P and Q are two bi-pumpable paths without redundancy of $\mathbf{P}[\alpha]$ such that $\overrightarrow{P_0 P_{P_{|P|-1}}} = \overrightarrow{Q_0 Q_{Q_{|Q|-1}}}$, we say P* is greater or equal to *Q, denoted by $P \geq Q$, if and only if ${}^*(P)^*$ is inside the left-hand side of the (directed) curve defined by ${}^*(Q)^*$ (as in [19], the "left-hand side" is considered as if we were walking on the curve). Moreover, if ${}^*(P)^* \neq {}^*(Q)^*$, we say that P* is strictly greater than *Q, denoted by $P > Q$.*

Lemma 19 and Lemma 20 aim to show that there is maximum path P^+ and a minimum path P^- for the order defined in 18 which means that there is no infinite sequence $P^{(0)} < P^{(1)} < ....$ To achieve this goal we show in Lemma 19 that the tile type which appear in a bi-pumpable path P without redundancy can only appear in ${}^*(P)^*$ otherwise α would be bi-periodic. We conclude in Lemma 20 by showing that if there would be an infinite sequence of increasing paths, one of them would have to use again a tile type of a previous path leading to a contradiction.

Lemma 21 implies that the paths growing in the left side of P^+ or in the right side of P^- behave as the teeth of Section 3.5 and we conclude in Lemma 22 by showing that a simply periodic terminal assembly can be described as follow: ${}^*(P^+)^*$ and ${}^*(P^-)^*$ cut the $2D$ plane in three parts, one them is a *stripe* which can be characterised by an assembly of size $O(|T^2|)$, as in the analysis of bi-periodic terminal assembly, while the paths growing in the left side of ${}^*(P^+)^*$ or the right side of ${}^*(P^-)^*$ behave as teeth (Lemma 21), as in the analysis of nonperiodic terminal assembly.

▶ **Lemma 19.** *Let $\mathcal{T} = (T, \sigma, 1)$ be a directed tile assembly system and let α be its unique terminal assembly. If there is a tile A of α and a bi-pumpable path P without redundancy of $\mathbf{P}[\alpha]$ such that A is not a tile of ${}^*(P)^*$ and such that A and P_0 have the same tile type then α is bi-periodic.*

Proof. If $\overrightarrow{P_0A}$ and $\overrightarrow{P_0P_{|P|-1}}$ are colinear, then we have $\overrightarrow{P_0A} = \ell\overrightarrow{P_0P_{|P|-1}}$ for some $\ell \in \mathbb{Z}$ (see the proof of Lemma 17) and then A is tile of $^*(P)^*$ which is a contradiction. If $\overrightarrow{P_0A}$ and $\overrightarrow{P_0P_{|P|-1}}$ are not colinear then consider a path Q of $\mathbf{P}[\alpha]$ such that $Q_0 = P_0$ and $Q_{|Q|-1} = A$. By Lemma 8, we consider that the seed is P_0 and then $Q - \overrightarrow{P_0A}$ is a path of $\mathbf{P}[\alpha]$ which implies that there is no conflict from $Q - \overrightarrow{P_0A}$ and $^*(P)^*$. Then from P_0, the paths Q followed by $^*(P)^* + \overrightarrow{P_0A}$ can grow and these two paths are in $\mathbf{P}[\alpha]$. By iterating this reasoning, for all $\ell \in \mathbb{Z}$, the path $Q + \ell\overrightarrow{P_0A}$ is in $\mathbf{P}[\alpha]$ which implies that α is $\overrightarrow{P_0A}$-periodic by a reasoning similar to the one of Corollary 9. The assembly α would be bi-periodic which is a contradiction. ◀

▶ **Lemma 20.** *Let $\mathcal{T} = (T, \sigma, 1)$ be a directed tile assembly system whose terminal assembly α is simply periodic. Then there are two paths P^+ and P^- without redundancy that are bi-pumpable, such that P^+ is maximum and P^- is minimum for the order defined in Definition 18.*

Proof. Since α is simply periodic then by Lemma 10, there is a bi-pumpable path in $\mathbf{P}[\alpha]$ and one of its subpath is bi-pumpable without redundancy. Then there is at least one bi-pumpable path $P^{(0)}$ of $\mathbf{P}[\alpha]$ without redundancy.

For the sake of contradiction, suppose that there are more than $|T|^2$ bi-pumpable paths without redundancy of $\mathbf{P}[\alpha]$ such that $P^{(0)} < P^{(1)} < P^{(2)} < ... < P^{(|T|^2)}$. Let $A^{(0)}$ be any tile of $P^{(0)}$. Since the length of a path without redundancy is at least 2 and less than $|T| + 1$ then the case where for some $0 \leq i \leq |T|^2$, $|P^{(i+1)}| < |P^{(i)}|$ occurs at most $|T|$ times consecutively. Since we have $|T|^2$ paths, the case where $|P^{(i+1)}| \geq |P^{(i)}|$ occurs at least $|T|$ times. In such a case, since $^*(P^{(i)})^* \neq {}^*(P^{(i+1)})^*$ there exists a tile of $P^{(i+1)}$ which is not a tile of $^*(P^{(i)})^*$ and which is in the left side of $^*(P^{(i)})^*$ and thus this tile is not a tile of $^*(P^{(0)})^*, {}^*(P^{(1)})^*, \ldots, {}^*(P^{(i)})^*$. Then we can create a sequence of tiles $A^{(0)}$, $A^{(1)}$, ... $A^{(|T|+1)}$ such that each tile of the sequence belongs to a bi-pumpable path without redundancy whose pumping does not pass by the other tiles. Two tiles of this sequence share a common tile type which by Lemma 19 means that α is bi-periodic. Then the sequence $P^{(0)} < P^{(1)} < P^{(2)} < ...$ is finite and let P^+ be the last bi-pumpable path without redundancy of this sequence and P^+ is maximum. A similar reasoning shows that there is a minimum path P^-. ◀

▶ **Lemma 21.** *Let $\mathcal{T} = (T, \sigma, 1)$ be a directed tile assembly system whose terminal assembly α is simply periodic. If there is a bi-pumpable path $P \in \mathbf{P}[\alpha]$ without redundancy, and an arc Q that grows in the left (resp. right) side of P, then there exists a bi-pumpable path $R \in \mathbf{P}[\alpha]$ without redundancy such that $R > P$ ($R < P$).*

Proof. If Q and $Q + \overrightarrow{P_0P_{|P|-1}}$ intersect, both paths are in $\mathbf{P}[\alpha]$ (since α is $\overrightarrow{P_0P_{|P|-1}}$-periodic by Corollary 9) and then the tiles at their intersection have the same tile type. Since $\overrightarrow{P_0P_{|P|-1}}$ is not null, there are two indices $0 \leq i < j \leq |Q| - 1$ such that $R = Q_{i,i+1,\ldots,j}$ is a path without redundancy. By Lemma 8 and since Q_0 and $Q_{|Q|-1}$ are tiles of $^*(P)^*$, we can consider that either $Q_{0,\ldots,i}$ or $Q_{j,\ldots,|Q|-1}$ is the seed and if R would not bi-pumpable, we can obtain a conflict with one of these two paths as done in the proof of Theorem 13. By Lemma 17, we can consider that $\overrightarrow{P_0P_{|P|-1}} = \overrightarrow{R_0R_{|R|-1}}$ and hence, since Q grows in the left side of $^*(P)^*$, R and $^*(R)^*$ are in the left side of $^*(P)^*$. Therefore, $R \geq P$ but $^*(P)^* = {}^*(R)^*$ would contradict the definition of an arc. Hence, $R > P$.

In the second case, Q and $Q + \overrightarrow{P_0P_{|P|-1}}$ do not intersect which implies that the width of Q is less than $|P| - 1$ (an arc of width greater than $|P|$ intersect with its translation by $\overrightarrow{P_0P_{|P|-1}}$). Without loss of generality we can assume that there are two indices $0 \leq i < j \leq |P| - 1$ such that Q starts at P_i and ends at P_j. Let R be the path defined by

$R = P_{0,1,\ldots,i}Q_{1,2,\ldots,|Q|-2}P_{j,j+1,\ldots|P|-1}$. By definition of an arc, $R \neq P$. If R has a redundancy, there are two integers $0 \leq k \leq |P|-1$ and $1 \leq k' \leq |Q|-2$ such that $\text{type}(P_k) = \text{type}(Q_{k'})$. By definition of an arc, $Q_{k'}$ is not a tile of $^*(P)^*$, and hence, by Lemma 19, α is bi-periodic, which contradicts the hypotheses of this lemma. Therefore, R is without redundancy, moreover since Q does not intersect with $Q + \overrightarrow{P_0P_{|P|-1}}$ then R is a good candidate. Since $\overrightarrow{R_0R_{|R|-1}} = \overrightarrow{P_0P_{|P|-1}}$ and α is $\overrightarrow{P_0P_{|P|-1}}$-periodic then R is bi-pumpable. By definition of R and Q, R and $^*(R)^*$ is in the left side of $^*(P)^*$ and by definition of an arc $R \neq P$, thus $R > Q$. ◀

▶ **Theorem 22.** *Let $\mathcal{T} = (T, \sigma, 1)$ be a directed tile assembly system, and let α be its unique terminal assembly, if α is simply periodic then its complexity is bounded by 2 and the size of characterisation depend only of $|\sigma|$ and $|T|$.*

Proof. By Lemma 20, there exists two bi-pumpable paths without redundancy P^+ and P^- of $\mathbf{P}[\alpha]$ which are maximum and minimum for the order defined in 18. By Lemma 21, the paths growing in left side of $^*(P^+)^*$ cannot intersect with their translation by $\overrightarrow{P_0^+P_{|P|-1}^+}$, then they behave as the teeth of Section 3.5. Let C be the union of a ultimately periodic tooth t and the paths growing on its periodic part, then C admits a characterisation of complexity 1 (see the proof of Theorem 16). By adding all the translations by $\ell\overrightarrow{P_0^+P_{|P|-1}^+}$ for $\ell \in \mathbb{N}$ to a copy of C and adding all the translations by $-\ell\overrightarrow{P_0^+P_{|P|-1}^+}$ for $\ell \in \mathbb{N}$ to an other copy, all the translations of t and the paths growing on its periodic part are an assembly of complexity 2. Thus, all the paths growing on left side of $^*(P^+)^*$ is an assembly of complexity 2. A similar reasoning holds for the right side of $^*(P^-)^*$.

Consider the shortest path between a tile of $^*(P^+)^*$ and a tile of $^*(P^-)^*$ (Q is empty if the two paths intersect). If there exist $0 \leq i < j \leq |Q|-1$ such that $R = Q_{i,\ldots,j}$ is without redundancy then R is bi-pumpable (with a reasoning similar to the proof of 13, since both Q_0 and $Q_{|Q|-1}$ belong to bi-pumpable path). By Lemma 17, we can consider that $\overrightarrow{P_0^+P_{|P^+|-1}^+} = \overrightarrow{R_0R_{|R|-1}} = \overrightarrow{P_0^-P_{|P^-|-1}^-}$ and we can find a path strictly shorter than Q which is a subassembly of $\text{asm}(Q_{0,\ldots,i}) \cup \text{asm}(Q_{j,\ldots,|Q|-1} - \overrightarrow{R_0R_{|R|-1}})$ with the same property than Q which is a contradiction. The paths P^+,Q, P^- and $Q + \overrightarrow{P_0^+P_{|P^+|-1}^+}$ define a cycle and let the finite assembly β be the restriction of α to the finite area of this cycle. The restriction of α to the "stripe" defined by the intersection of the right side of P^+ and the left side of P^- is $\bigcup_{\ell\in\mathbb{Z}} \beta + \ell\overrightarrow{P_0^+P_{|P^+|-1}^+}$ which is an assembly of complexity 1. Note that if Q is not the empty path then by Lemma 19, P^+, P^- and Q cannot share a common tile type except at the extremities of Q, then the periodic assembly $^*(P^+)^* \cup {}^*(P^-)^* \cup (\bigcup_{\ell\in\mathbb{Z}} Q + \ell\overrightarrow{P_0^+P_{|P^+|-1}^+})$ that delimits the stripe can be described by a path of length less than $|T|$. [7] ◀

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[7] The same reasoning could be done for the bi-periodic terminal assembly and the 2D periodic grid can be described by a path of length less than $|T|$: there is no other way than to hardcode the 2D periodic grid.

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A Omitted Proof

Here is the proof of Lemma 10.

Proof. By hypothesis, there exists a non-null vector $\overrightarrow{v}$ such that α is $\overrightarrow{v}$-periodic. Let A be any tile of α, then $A + \overrightarrow{v}$ is also a tile of α (because α is $\overrightarrow{v}$-periodic) and there exists a finite path $P \in \mathbf{P}[\alpha]$ such that $P_0 = A$ and $P_{|P|-1} = A + \overrightarrow{v}$.

Let Q be the shortest such path of $\mathbf{P}[\alpha]$, i.e. the shortest path such that $Q_0 + s\overrightarrow{v} = Q_{|Q|-1}$ for $s = 1$ or $s = -1$. The path Q exists since P itself satisfies the criterion $P + s\overrightarrow{v} = P$ for $s = 1$. Since α is $\overrightarrow{v}$-periodic, then for $\ell \in \mathbb{Z}$, $Q + \ell\overrightarrow{v}$ is in $\mathbf{P}[\alpha]$. There are two cases:

- If Q and $Q + \overrightarrow{v}$ intersect only in one point at their ends (either at $Q_{|Q|-1}$ and $Q_0 + \overrightarrow{v}$ or at Q_0 and $Q_{|Q|-1} + \overrightarrow{v}$, depending on whether $s = 1$ or $s = -1$), then Q is a good candidate and $^*(Q)^*$ is a simple bi-infinite path, and is in $\mathbf{P}[\alpha]$, meaning that Q is bi-pumpable, which is our conclusion (with $P = Q$).

- Otherwise, there exists another intersection between Q and $Q + \overrightarrow{v}$, i.e. there exists $0 \leq i, j \leq |Q| - 1$ such that $Q_i = Q_j + \overrightarrow{v}$ and if $j > i$ (resp. $i > j$) then $(i, j) \neq (0, |Q| - 1)$ (resp. $(j, i) \neq (0, |Q| - 1)$). Thus, $Q_{i,\ldots,j}$ (resp. $Q_{j,\ldots,i}$) contradicts our assumption that Q is the shortest path intersecting $Q + s\overrightarrow{v}$. ◀

Here is the proof of Lemma 17.

Proof. By Corollary 9, α is $\overrightarrow{P_0P_{|P|-1}}$-periodic and $\overrightarrow{Q_0Q_{|Q|-1}}$-periodic then both vectors are colinear since α is simply periodic. Then ${}^*(P)^* + \overrightarrow{Q_0Q_{|Q|-1}}$ intersects with ${}^*(P)^*$ and since P is without redundancy, we have $\overrightarrow{Q_0Q_{|Q|-1}} = \ell\overrightarrow{P_0P_{|P|-1}}$ for some $\ell \in \mathbb{Z}$. Similarly $\overrightarrow{P_0P_{|P|-1}} = \ell\overrightarrow{Q_0Q_{|Q|-1}}$ for some $\ell \in \mathbb{Z}$ and then either $\overrightarrow{P_0P_{P_{|P|-1}}} = \overrightarrow{Q_0Q_{Q_{|Q|-1}}}$ or $\overrightarrow{P_0P_{P_{|P|-1}}} = -\overrightarrow{Q_0Q_{Q_{|Q|-1}}}$. ◀

B Omitted Figures

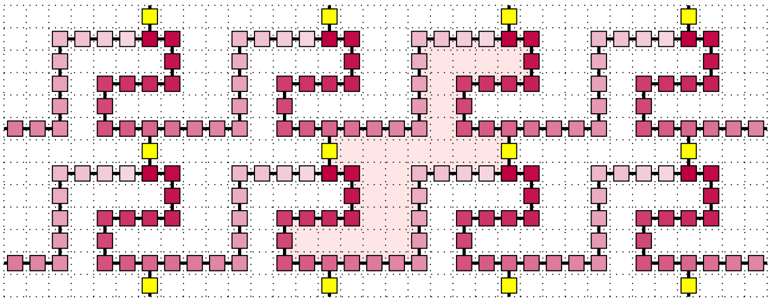


Figure B.1 Illustration of a bi-pumpable terminal assembly. We consider two bi-pumpable paths P and Q such that $\overrightarrow{P_0P_{|P|-1}}$ is not colinear with $\overrightarrow{Q_0Q_{|Q|-1}}$. The $2D$ plane is filled by these paths and α can be characterised by its restriction to the red area and the vectors $\overrightarrow{P_0P_{|P|-1}}$ and $\overrightarrow{Q_0Q_{|Q|-1}}$.

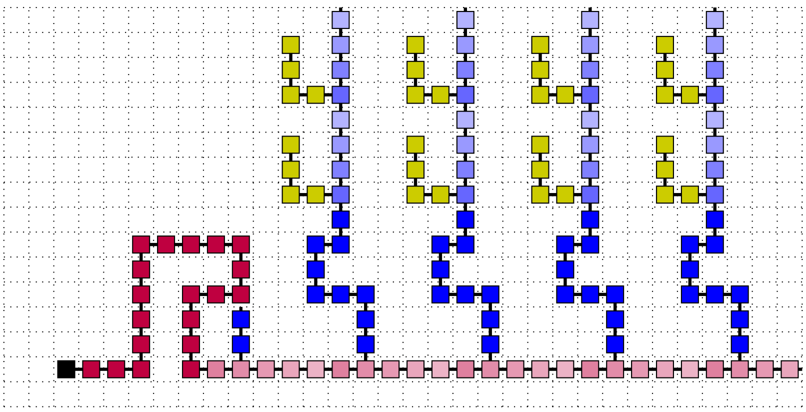


Figure B.2 An aperiodic terminal assembly α: the seed is in black, a comb and its backbone are in light red, a finite path linking the seed to the comb is in dark red, a comb and its translation are in blue (dark blue for the transient part and light blue for the periodic part) and a finite path growing on the periodic part of the tooth is in yellow. Note that the first comb cannot fully grow.

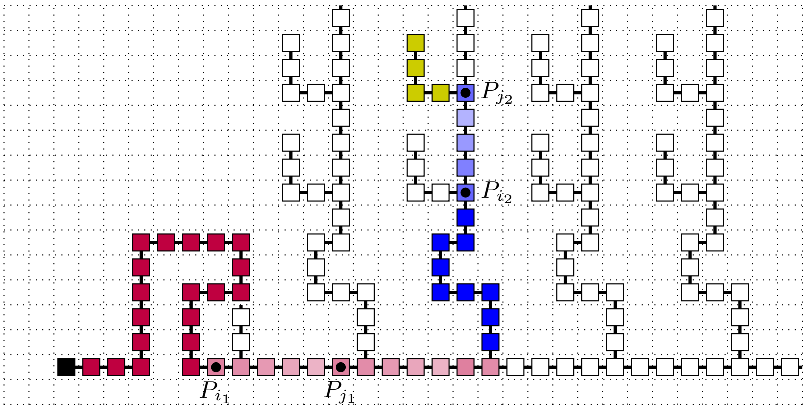


Figure B.3 The producible path P (the tiles which does not belong to P are in white).

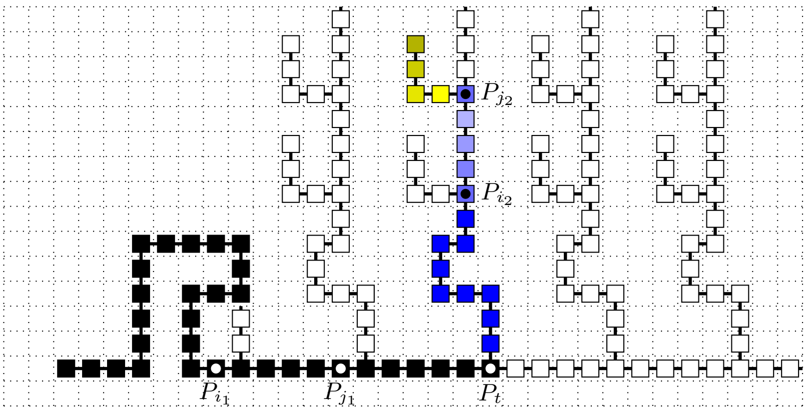
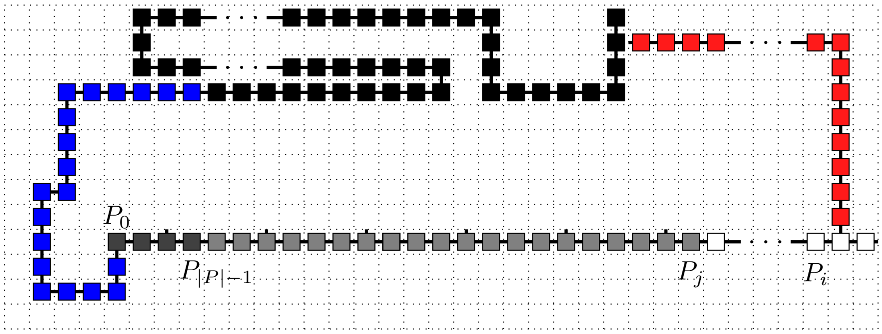
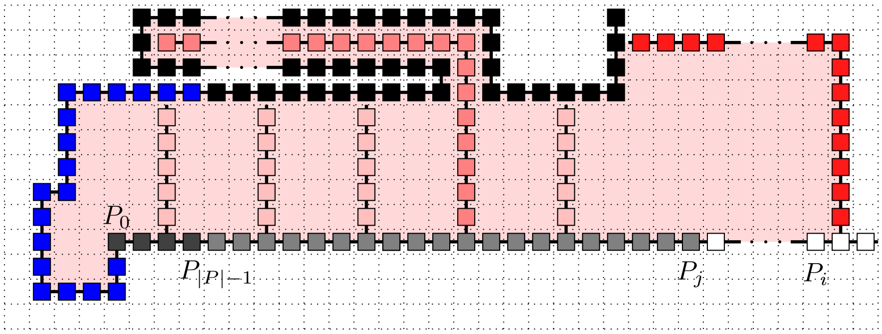


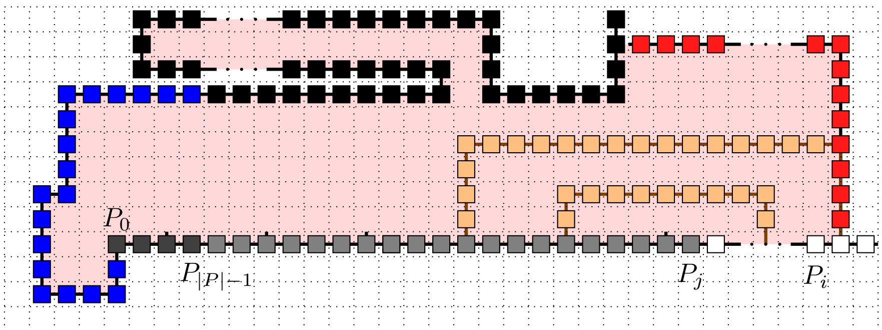
Figure B.4 The path $P_{[t+1,\dots,|P|-1]}$ is producible by $(T, \sigma \cup P_{[0,\dots,t]})$. The terminal assembly is still α, the new seed is in black.



a) The seed σ is in black, P (in dark grey) is simply pumpable and the pumping of P is in light grey up to index j, and in white afterwards. Assembly β is the union of the seed and the blue path. γ is the union of β and the gray paths. Path $Q_{0,\ldots,m}$ is in red and intersects the seed.



b) All the translations of Q growing on the light gray part of $(P)^*$ start in the red area of the grid. If β blocks them all, some of the translations of Q must intersect with each other before intersecting β. Thus, one of the translations must fully grow.



c) The path S is ultimately periodic and grows on $(P)^*_{|P|,\ldots,j-|P|-1}$. There is two examples of S in orange, one leaves the red area by intersecting $Q_{0,\ldots,m}$ and the other one by intersecting $P_{j+1,\ldots,k}$. In both cases, we can find an arc of $(P)^*$ of width at least $|P|$.

Figure B.5 Illustration of Proof 15.