A Decidable Equivalence for a Turing-Complete, Distributed Model of Computation

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───── **Abstract** ─────

Place/Transition Petri nets with inhibitor arcs (PTI nets for short), which are a well-known Turing-complete, distributed model of computation, are equipped with a decidable, behavioral equivalence, called pti-place bisimilarity, that conservatively extends place bisimilarity defined over Place/Transition nets (without inhibitor arcs). We prove that pti-place bisimilarity is sensible, as it respects the causal semantics of PTI nets.

1 Introduction

Place/Transition Petri nets with inhibitor arcs (PTI nets, for short), originally introduced in [2], are a well-known (see, e.g., [7, 19, 26]), Turing-complete (as proved first by Agerwala in [1]), distributed model of computation, largely exploited, e.g., for modeling systems with priorities [17], for performance evaluation of distributed systems [3] and to provide π-calculus [24, 28] with a net semantics [8].

As finite PTI nets constitute a Turing-complete model of computation, essentially all the properties of interest are undecidable, notably the reachability problem, and so even termination: it is undecidable whether a deadlock marking is reachable from the initial one. Also interleaving bisimulation equivalence is undecidable for finite PTI nets, as it is already undecidable [20] on the subclass of finite P/T nets [27]. Similarly, one can prove that also well-known truly-concurrent behavioral equivalences, such as *fully-concurrent* bisimilarity [6], are undecidable [12] for finite PTI nets. Despite this, we show that it is possible to define a *sensible*, behavioral equivalence which is actually *decidable* on finite PTI nets. This equivalence, we call *pti-place bisimilarity*, is a conservative extension of *place bisimilarity on finite P/T nets*, introduced in [4] as an improvement of *strong bisimulation* [25], (a relation proposed by Olderog in [25] on safe nets which fails to induce an equivalence relation), and recently proved decidable in [16].

Place bisimilarity on finite P/T nets is an equivalence over markings, based on relations over the *finite set of net places*, rather than over the (possibly infinite) set of net markings. This equivalence is very natural and intuitive: as a place can be interpreted as a sequential process type (and each token in this place as an instance of a sequential process of that type), a place bisimulation states which kinds of sequential processes (composing the distributed system represented by the finite P/T net) are to be considered as equivalent. Moreover, this equivalence does respect the causal behavior of P/T nets, as van Glabbeek proved in [29] that it is slightly finer than *structure preserving bisimilarity* [29], in turn slightly finer than *fully-concurrent bisimilarity* [6].

We extend this idea in order to be applicable to PTI nets. Informally, a binary relation R over the set S of places is a *pti-place bisimulation* if for all markings m_1 and m_2 which are *bijectively* related via R (denoted by $(m_1, m_2) \in R^\oplus$, where $R^\oplus$ is called the *additive closure* of R), if m_1 can perform transition t_1, reaching marking m_1', then m_2 can perform a transition t_2, reaching m_2', such that

- the pre-sets of t_1 and t_2 are related by $R^\oplus$, the label of t_1 and t_2 is the same, the post-sets of t_1 and t_2 are related by $R^\oplus$, and also $(m_1', m_2') \in R^\oplus$, as required by a place bisimulation [4, 16], but additionally it is required that
- whenever $(s, s') \in R$, s belongs to the inhibiting set of t_1 if and only if s' belongs to the inhibiting set of t_2;

and symmetrically if m_2 moves first. Two markings m_1 and m_2 are pti-place bisimilar, denoted by $m_1 \sim_p m_2$, if a pti-place bisimulation R exists such that $(m_1, m_2) \in R^\oplus$.

We prove that pti-place bisimilarity is an equivalence, but it is not coinductive as the union of pti-place bisimulations may be not a pti-place bisimulation; so, in general, there is not a largest pti-place bisimulation, rather many maximal pti-place bisimulations. In fact, pti-place bisimilarity is the relation on markings given by the union of the additive closure of each maximal pti-place bisimulation. We also prove that $\sim_p$ is sensible, as it respects the causal semantics of PTI nets. As a matter of fact, following the approach in [9, 10], we define a novel, process-oriented, bisimulation-based, behavioral semantics for PTI nets, called *causal-net bisimilarity*, and we prove that this is slightly coarser than pti-place bisimilarity.

The other main contribution of this paper is to show that $\sim_p$ is decidable for finite PTI nets. As a place relation $R \subseteq S \times S$ is finite if the set S of places is finite, there are finitely many place relations for a finite net. We can list all these place relations, say $R_1, R_2, \ldots R_n$. It is possible to decide whether R_i is a pti-place bisimulation by checking two *finite* conditions over a *finite* number of marking pairs: this is a non-obvious observation, as a pti-place bisimulation requires that the pti-place bisimulation conditions hold for the infinitely many pairs (m_1, m_2) belonging to $R_i^\oplus$. Hence, to decide whether $m_1 \sim_p m_2$, it is enough to check, for $i = 1, \ldots n$, whether R_i is a pti-place bisimulation and, in such a case, whether $(m_1, m_2) \in R_i^\oplus$.

The paper is organized as follows. Section 2 recalls the basic definitions about PTI nets, including their causal semantics. Section 3 deals with pti-place bisimilarity, shows that it is an equivalence relation, that it is not coinductive, and that it is slightly finer than causal-net bisimilarity. Section 4 shows that $\sim_p$ is decidable. Finally, Section 5 discusses some related literature and future research.

2   Basic definitions about P/T nets and PTI nets

▶ **Definition 1** (Multiset)**.** *Let $\mathbb{N}$ be the set of natural numbers. Given a finite set S, a* multiset *over S is a function $m : S \to \mathbb{N}$. The* support *set $dom(m)$ of m is $\{s \in S \mid m(s) \neq 0\}$. The set of all multisets over S, denoted by $\mathcal{M}(S)$, is ranged over by m. We write $s \in m$ if $m(s) > 0$. The* multiplicity *of s in m is given by the number $m(s)$. The* size *of m, denoted by $|m|$, is the number $\sum_{s \in S} m(s)$, i.e., the total number of its elements. A multiset m such that $dom(m) = \emptyset$ is called* empty *and is denoted by θ. We write $m \subseteq m'$ if $m(s) \leq m'(s)$ for all $s \in S$. Multiset union $\_ \oplus \_$ is defined as follows: $(m \oplus m')(s) = m(s) + m'(s)$. Multiset difference $\_ \ominus \_$ is defined as follows: $(m_1 \ominus m_2)(s) = max\{m_1(s) - m_2(s), 0\}$. The scalar product of a number j with m is the multiset $j \cdot m$ defined as $(j \cdot m)(s) = j \cdot (m(s))$. By s_i we also denote the multiset with s_i as its only element. Hence, a multiset m over $S = \{s_1, \ldots, s_n\}$ can be represented as $k_1 \cdot s_1 \oplus k_2 \cdot s_2 \oplus \ldots \oplus k_n \cdot s_n$, where $k_j = m(s_j) \geq 0$ for $j = 1, \ldots, n$.*

▶ **Definition 2** (Place/Transition Petri net). *A labeled, finite* Place/Transition *Petri net (P/T net for short) is a tuple $N = (S, A, T)$, where*

- *S is the finite set of* places, *ranged over by s (possibly indexed),*
- *A is the finite set of* labels, *ranged over by ℓ (possibly indexed), and*
- *$T \subseteq (\mathcal{M}(S) \setminus \{\theta\}) \times A \times (\mathcal{M}(S) \setminus \{\theta\})$ is the finite set of* transitions, *ranged over by t (possibly indexed).*

Given a transition $t = (m, \ell, m')$, we use the notation:

- *$^\bullet t$ to denote its* pre-set m *(which cannot be empty) of tokens to be consumed;*
- *$l(t)$ for its* label ℓ, *and*
- *$t^\bullet$ to denote its* post-set m' *(which cannot be an empty multiset) of tokens to be produced.*

Hence, transition t can be also represented as $^\bullet t \xrightarrow{l(t)} t^\bullet$. We also define the flow function *flow : $(S \times T) \cup (T \times S) \to \mathbb{N}$ as follows: for all $s \in S$, for all $t \in T$, $flow(s, t) = {}^\bullet t(s)$ and $flow(t, s) = t^\bullet(s)$. We will use F to denote the* flow relation *$\{(x, y) \in (S \times T) \cup (T \times S) \mid flow(x, y) > 0\}$. Finally, we define pre-sets and post-sets also for places as follows: $^\bullet s = \{t \in T \mid s \in t^\bullet\}$ and $s^\bullet = \{t \in T \mid s \in {}^\bullet t\}$. Note that while the pre-set (post-set) of a transition is, in general, a multiset, the pre-set (post-set) of a place is a set.*

▶ **Definition 3** (Place/Transition net with inhibitor arcs). *A finite Place/Transition net with inhibitor arcs (PTI net for short) is a tuple $N = (S, A, T, I)$, where*

- *(S, A, T) is a finite P/T net;*
- *$I \subseteq S \times T$ is the* inhibiting relation.

Given a transition $t \in T$, we denote by $^\circ t$ its inhibiting set *$\{s \in S \mid (s, t) \in I\}$ of places to be tested for absence of tokens. Hence, a transition t can be also represented as $(^\bullet t, {}^\circ t) \xrightarrow{l(t)} t^\bullet$.*

We use the standard graphical convention for Petri nets. In particular, a pair (s, t) in the inhibiting relation I is graphically represented by an arc from s to t ending with a small circle on the transition side.

▶ **Definition 4** (Marking, PTI net system). *A PTI net system $N(m_0)$ is a tuple (S, A, T, I, m_0), where (S, A, T, I) is a PTI net and m_0 is a multiset over S, called the* initial marking. *We also say that $N(m_0)$ is a* marked *net.*

▶ **Definition 5** (Token game). *A transition t is* enabled *at m, denoted $m[t\rangle$, if $^\bullet t \subseteq m$ and $^\circ t \cap dom(m) = \emptyset$. The execution, or* firing, *of t enabled at m produces the marking $m' = (m \ominus {}^\bullet t) \oplus t^\bullet$, written $m[t\rangle m'$.*

▶ **Definition 6** (Firing sequence, reachable marking, safe net). *A* firing sequence *starting at m is defined inductively as follows:*

- *$m[\epsilon\rangle m$ is a firing sequence (where ϵ denotes an empty sequence of transitions) and*
- *if $m[\sigma\rangle m'$ is a firing sequence and $m'[t\rangle m''$, then $m[\sigma t\rangle m''$ is a firing sequence.*

The set of reachable markings *from m is $[m\rangle = \{m' \mid \exists \sigma. m[\sigma\rangle m'\}$. A PTI system $N = (S, A, T, I, m_0)$ is* safe *if for each marking $m \in [m_0\rangle$, we have that $m(s) \leq 1$ for all $s \in S$.*

2.1 Causal semantics for P/T nets and PTI nets

We outline some definitions about the causal semantics of P/T nets, adapted from the literature (cf., e.g., [6, 29, 13, 25]).

▶ **Definition 7** (Acyclic net). *A P/T net $N = (S, A, T)$ is* acyclic *if its flow relation F is acyclic (i.e., $\nexists x$ such that $x F^+ x$, where F^+ is the transitive closure of F).*

The concurrent semantics of a marked P/T net is defined by a class of particular acyclic safe nets, where places are not branched (hence they represent a single run) and all arcs have weight 1. This kind of net is called *causal net*. We use the name C (possibly indexed) to denote a causal net, the set B to denote its places (called *conditions*), the set E to denote its transitions (called *events*), and L to denote its labels.

▶ **Definition 8** (Causal P/T net). *A causal net is a finite marked net $C(\mathsf{m}_0) = (B, L, E, \mathsf{m}_0)$ satisfying the following conditions:*
1. *C is acyclic;*
2. *$\forall b \in B$ $|{}^\bullet b| \leq 1 \wedge |b^\bullet| \leq 1$ (i.e., the places are not branched);*
3. *$\forall b \in B$ $\mathsf{m}_0(b) = \begin{cases} 1 & \text{if } {}^\bullet b = \emptyset \\ 0 & \text{otherwise;} \end{cases}$*
4. *$\forall e \in E$ ${}^\bullet e(b) \leq 1 \wedge e^\bullet(b) \leq 1$ for all $b \in B$ (i.e., all the arcs have weight 1).*
We denote by $Min(C)$ the set m_0, and by $Max(C)$ the set $\{b \in B \mid b^\bullet = \emptyset\}$.

Note that any reachable marking of a causal net is a set, i.e., this net is *safe*; in fact, the initial marking is a set and, assuming by induction that a reachable marking m is a set and enables e, i.e., $\mathsf{m}[e\rangle\mathsf{m}'$, then also $\mathsf{m}' = (\mathsf{m} \ominus {}^\bullet e) \oplus e^\bullet$ is a set, as the net is acyclic and because of the condition on the shape of the post-set of e (weights can only be 1).

As the initial marking of a causal P/T net is fixed by its shape (according to item 3 of Definition 8), in the following, in order to make the notation lighter, we often omit the indication of the initial marking (also in their graphical representation), so that the causal net $C(\mathsf{m}_0)$ is denoted by C.

▶ **Definition 9** (Moves of a causal P/T net). *Given two causal nets $C = (B, L, E, \mathsf{m}_0)$ and $C' = (B', L, E', \mathsf{m}_0)$, we say that C moves in one step to C' through e, denoted by $C[e\rangle C'$, if ${}^\bullet e \subseteq Max(C)$, $E' = E \cup \{e\}$ and $B' = B \cup e^\bullet$.*

▶ **Definition 10** (Folding and Process). *A folding from a causal P/T net $C = (B, L, E, \mathsf{m}_0)$ into a P/T net system $N(m_0) = (S, A, T, m_0)$ is a function $\rho : B \cup E \to S \cup T$, which is type-preserving, i.e., such that $\rho(B) \subseteq S$ and $\rho(E) \subseteq T$, satisfying the following:*
- *$L = A$ and $\mathsf{l}(e) = l(\rho(e))$ for all $e \in E$;*
- *$\rho(\mathsf{m}_0) = m_0$, i.e., $m_0(s) = |\rho^{-1}(s) \cap \mathsf{m}_0|$;*
- *$\forall e \in E, \rho({}^\bullet e) = {}^\bullet\rho(e)$, i.e., $\rho({}^\bullet e)(s) = |\rho^{-1}(s) \cap {}^\bullet e|$ for all $s \in S$;*
- *$\forall e \in E, \rho(e^\bullet) = \rho(e)^\bullet$, i.e., $\rho(e^\bullet)(s) = |\rho^{-1}(s) \cap e^\bullet|$ for all $s \in S$.*
A pair (C, ρ), where C is a causal net and ρ a folding from C to a net system $N(m_0)$, is a process of $N(m_0)$.

▶ **Definition 11** (Moves of a P/T process). *Let $N(m_0) = (S, A, T, m_0)$ be a net system and let (C_i, ρ_i), for $i = 1, 2$, be two processes of $N(m_0)$. We say that (C_1, ρ_1) moves in one step to (C_2, ρ_2) through e, denoted by $(C_1, \rho_1) \xrightarrow{e} (C_2, \rho_2)$, if $C_1[e\rangle C_2$ and $\rho_1 \subseteq \rho_2$.*

Following [9, 10], we define here a possible causal semantics for PTI nets. In order to maintain the pleasant property that a process univocally determines the causal dependencies among its events, it is not enough to just enrich causal P/T nets with inhibitor arcs. Indeed, the reason why a condition is empty may influence the causal relation of events. To solve the problem, in [9, 10] inhibitor arcs are partitioned into two sets: *before* inhibitor arcs and *after* inhibitor arcs. If a condition is connected to an event by a before inhibitor arc, the event fires because the condition has not held yet; if they are connected by an after inhibitor arc, the event fires because the condition does not hold anymore.

▶ **Definition 12** (Causal PTI net). *A causal PTI net is a tuple $C(\mathsf{m}_0) = (B, L, E, Y^{be}, Y^{af}, \mathsf{m}_0)$ satisfying the following conditions, denoting the flow relation of C by F:*

1. *(B, L, E, m_0) is a causal P/T net;*
2. *$(B, L, E, Y^{be} \cup Y^{af}, \mathsf{m}_0)$ is a marked PTI net;*
3. before *and* after *requirements are met, i.e.*
   (a) *If $b\, Y^{be}\, e$, then there exists $e' \in E$ such that $e'\, \mathsf{F}\, b$, and*
   (b) *If $b\, Y^{af}\, e$, then there exists $e' \in E$ such that $b\, \mathsf{F}\, e'$;*
4. *relation $\mathsf{F} \cup \prec_{af} \cup \prec_{be}$ is acyclic, where $\prec_{af} = \mathsf{F}^{-1} \circ Y^{af}$ and $\prec_{be} = (Y^{be})^{-1} \circ \mathsf{F}^{-1}$.*

We denote by $Min(C)$ the set m_0, and by $Max(C)$ the set $\{b \in B \mid b^\bullet = \emptyset\}$.

Relation $\prec_{af} \subseteq E \times E$ states that $e \prec_{af} e'$ if e consumes the token in a place b inhibiting e': this is clearly a causal dependency. Instead, relation $\prec_{be} \subseteq E \times E$ states that $e \prec_{be} e'$ if e' produces a token in a place b inhibiting e: this is clearly a temporal precedence, because the two events can be causally independent, yet they cannot occur in any order, as if e' occurs, then e is disabled.

▶ **Definition 13** (Folding and PTI process). *A folding from a causal PTI net $C = (B, L, E, Y^{be}, Y^{af}, \mathsf{m}_0)$ into a PTI net system $N(m_0) = (S, A, T, I, m_0)$ is a function $\rho : B \cup E \to S \cup T$, which is type-preserving, i.e., such that $\rho(B) \subseteq S$ and $\rho(E) \subseteq T$, satisfying the following:*
- *ρ is a P/T folding from (B, L, E, m_0) into (S, A, T, m_0);*
- *for all $s \in S$ and $e \in E$, if $(s, \rho(e)) \in I$ then for all $b \in B$ such that $\rho(b) = s$, it holds $(b, e) \in Y^{be} \cup Y^{af} \cup \mathsf{F}^{-1}$, and*
  for all $b \in B$ and $e \in E$, if $(b, e) \in Y^{be} \cup Y^{af}$ then $(\rho(b), \rho(e)) \in I$.

A pair (C, ρ), where C is a causal PTI net and ρ a folding from C to a PTI net system $N(m_0)$, is a PTI process *of $N(m_0)$.*

Each inhibitor arc in the causal net has a corresponding inhibitor arc in the net system. The only case wherea condition b is not connected by an inhibitor arc to an event e is when b is in the post-set of e: as b starts to hold only after e occurs, the only possibility is to put a before arc. This would make the relation $\prec_{be}$ reflexive, invalidating item 4 of Definition 12. However, since b is in the post-set of e, we are sure that e happens before b is fulfilled, hence making useless the presence of a before inhibitor arc. For this reason, with the requirement $(b, e) \in Y^{be} \cup Y^{af} \cup \mathsf{F}^{-1}$, we ask for the presence of an inhibitor arc only if there exists no flow from e to b.

▶ **Definition 14** (Moves of a PTI process). *Let $N(m_0) = (S, A, T, I, m_0)$ be a PTI net system and let (C_i, ρ_i), for $i = 1, 2$, be two PTI processes of $N(m_0)$, where $C_i = (B_i, L, E_i, Y_i^{be}, Y_i^{af}, \mathsf{m}_0)$. We say that (C_1, ρ_1) moves in one step to (C_2, ρ_2) through e, denoted by $(C_1, \rho_1) \overset{e}{\longrightarrow} (C_2, \rho_2)$, if the following hold:*
- *${}^\bullet e \subseteq Max(C_1)$, $E_2 = E_1 \cup \{e\}$, $B_2 = B_1 \cup e^\bullet$, $\rho_1 \subseteq \rho_2$, i.e. the P/T process of (C_1, ρ_1) moves in one step through e to the P/T process of (C_2, ρ_2).*
- *Given two relations $\mathcal{B}$ and $\mathcal{A}$, defined as*
  - *$\forall b \in e^\bullet$, $\forall e' \in E_1$ we have $b\, \mathcal{B}\, e'$ if and only if $(\rho_2(b), \rho_2(e')) \in I$,*
  - *$\forall b \in B_2$ such that $b^\bullet \neq \emptyset$, we have $b\, \mathcal{A}\, e$ if and only if $(\rho_2(b), \rho_2(e)) \in I$,*
  we have $\{b \in B_2 \mid b\, \mathcal{A}\, e\} \cap Max(C_1) = \emptyset$.
- *Finally, $Y_2^{be} = Y_1^{be} \cup \mathcal{B}$ and $Y_2^{af} = Y_1^{af} \cup \mathcal{A}$.*

The item $\{b \in B_2 \mid b\, \mathcal{A}\, e\} \cap Max(C_1) = \emptyset$ models the fact that a transition can fire only if all its inhibiting places are free. Indeed, an event can fire only if its (so far known) inhibiting conditions are not maximal. Note that, by construction, before arcs can connect
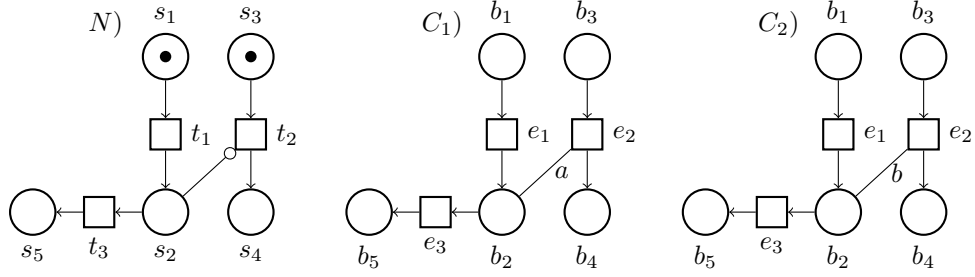
Figure 1 A marked PTI net and two PTI causal nets corresponding to its two maximal processes.

only new inhibiting conditions to past events and in particular we do not allow before arcs connecting a condition in the post-set of a newly added event e with the event e itself. Moreover, after arcs can only connect old inhibiting conditions to the new event e and since $\{b \in B_2 \mid b \mathcal{A} e\} \cap Max(C_1) = \emptyset$, the old inhibiting conditions cannot be in the pre-set of the newly added event e. Therefore, both relations $\prec_2^{be}$ and $\prec_2^{af}$ are acyclic, and since F_2 is acyclic too, (C_2, ρ_2) is truly a process of $N(m_0)$.

▶ **Example 15.** Consider the three nets in Figure 1, where we use the graphical convention that *before* inhibitor arcs and *after* inhibitor arcs are represented by lines between a condition and an event: the former labeled by b, the latter labeled by a. The initial marking of N is $m_0 = s_1 \oplus s_3$. The shape of a process generated by $N(m_0)$ may depend on the order of transitions in a given transition sequence. As a matter of fact, transition sequences containing the same transitions but in a different order may generate different processes, e.g. C_1 and C_2. Indeed, C_1 represents the transition sequence $t_1 \, t_3 \, t_2$, while C_2 represents the transition sequence $t_2 \, t_1 \, t_3$. In the former case, t_1 fires first, so that t_2 can only fire after the inhibiting token in s_2 has been cleaned up by transition t_3: therefore the causal net has an after arc between b_2 and e_2. In the latter case, t_2 is the first transition to fire and there are no tokens in the inhibiting place b_2, therefore the causal net has a before arc between b_2 and e_2. Note that the underlying causal P/T net of these two processes is the same, but before and after inhibitor arcs are different.

We are now ready to introduce a novel behavioral relation for PTI nets, namely causal-net bisimulation, which is an interesting relation in its own right, as the induced equivalence, namely *causal-net bisimilarity*, on P/T nets coincides with *structure-preserving bisimilarity* [29], and so it is slightly finer than *fully-concurrent bisimilarity* [6]. However, since we conjecture that causal-net bisimilarity is undecidable (already on finite P/T nets), we will use this behavioral relation only for comparison with pti-place bisimilarity, showing the the latter is a finer, but decidable, approximation of the former.

▶ **Definition 16** (Causal-net bisimulation)**.** *Let* $N = (S, A, T, I)$ *be a PTI net. A* causal-net bisimulation *is a relation R, composed of triples of the form (ρ_1, C, ρ_2), where, for $i = 1, 2$, (C, ρ_i) is a process of $N(m_{0i})$ for some m_{0i}, such that if $(\rho_1, C, \rho_2) \in R$ then*

i) $\forall t_1, C', \rho_1'$ *such that* $(C, \rho_1) \xrightarrow{e} (C', \rho_1')$, *where* $\rho_1'(e) = t_1$, $\exists t_2, \rho_2'$ *such that* $(C, \rho_2) \xrightarrow{e} (C', \rho_2')$, *where* $\rho_2'(e) = t_2$, *and* $(\rho_1', C', \rho_2') \in R$;

ii) *symmetrically,* $\forall t_2, C', \rho_2'$ *such that* $(C, \rho_2) \xrightarrow{e} (C', \rho_2')$, *where* $\rho_2'(e) = t_2$, $\exists t_1, \rho_1'$ *such that* $(C, \rho_1) \xrightarrow{e} (C', \rho_1')$, *where* $\rho_1'(e) = t_1$, *and* $(\rho_1', C', \rho_2') \in R$.

Two markings m_1 and m_2 of N are cn-bisimilar, denoted by $m_1 \sim_{cn} m_2$, if there exists a causal-net bisimulation R containing a triple $(\rho_1^0, C^0, \rho_2^0)$, where C^0 contains no events and $\rho_i^0(Min(C^0)) = \rho_i^0(Max(C^0)) = m_i$ for $i = 1, 2$.

If $m_1 \sim_{cn} m_2$, then these two markings have the same causal PTI nets, so that the executions originating from the two markings have the same causal dependencies (determined by F and $\prec_{af}$) and the same temporal dependencies (determined by $\prec_{be}$). Causal-net bisimilarity $\sim_{cn}$ is an equivalence relation (see the preliminary version of this paper [11]).

3 Pti-place bisimilarity

We now present pti-place bisimilarity, which conservatively extends *place bisimilarity* [4, 16] to the case of PTI nets. First, an auxiliary definition.

3.1 Additive closure and its properties

▶ **Definition 17** (Additive closure). *Given a PTI net $N = (S, A, T, I)$ and a* place relation *$R \subseteq S \times S$, we define a* marking relation *$R^\oplus \subseteq \mathcal{M}(S) \times \mathcal{M}(S)$, called the* additive closure of R, *as the least relation induced by the following axiom and rule.*

$$\frac{}{(\theta, \theta) \in R^\oplus} \qquad \frac{(s_1, s_2) \in R \ (m_1, m_2) \in R^\oplus}{(s_1 \oplus m_1, s_2 \oplus m_2) \in R^\oplus}$$

Note that two markings are related by $R^\oplus$ only if they have the same size; in fact, the axiom states that the empty marking is related to itself, while the rule, assuming by induction that m_1 and m_2 have the same size, ensures that $s_1 \oplus m_1$ and $s_2 \oplus m_2$ have the same size.

▶ **Proposition 18.** *For each relation $R \subseteq S \times S$, if $(m_1, m_2) \in R^\oplus$, then $|m_1| = |m_2|$.*

Note also that the membership $(m_1, m_2) \in R^\oplus$ may be proved in several different ways, depending on the chosen order of the elements of the two markings and on the definition of R. For instance, if $R = \{(s_1, s_3), (s_1, s_4), (s_2, s_3), (s_2, s_4)\}$, then $(s_1 \oplus s_2, s_3 \oplus s_4) \in R^\oplus$ can be proved by means of the pairs (s_1, s_3) and (s_2, s_4), as well as by means of $(s_1, s_4), (s_2, s_3)$. An alternative way to define that two markings m_1 and m_2 are related by $R^\oplus$ is to state that m_1 can be represented as $s_1 \oplus s_2 \oplus \ldots \oplus s_k$, m_2 can be represented as $s_1' \oplus s_2' \oplus \ldots \oplus s_k'$ and $(s_i, s_i') \in R$ for $i = 1, \ldots, k$.

▶ **Proposition 19** ([14]). *For each place relation $R \subseteq S \times S$, the following hold:*
1. *If R is an equivalence relation, then $R^\oplus$ is an equivalence relation.*
2. *If $R_1 \subseteq R_2$, then $R_1^\oplus \subseteq R_2^\oplus$, i.e., the additive closure is monotone.*
3. *If $(m_1, m_2) \in R^\oplus$ and $(m_1', m_2') \in R^\oplus$, then $(m_1 \oplus m_1', m_2 \oplus m_2') \in R^\oplus$, i.e., the additive closure is additive.*
4. *If R is an equivalence relation and, moreover, $(m_1 \oplus m_1', m_2 \oplus m_2') \in R^\oplus$ and $(m_1, m_2) \in R^\oplus$, then $(m_1', m_2') \in R^\oplus$, i.e., the additive closure of an equivalence place relation is subtractive.*

When R is an equivalence relation, it is rather easy to check whether two markings are related by $R^\oplus$. An algorithm, described in [14], establishes whether an R-preserving bijection exists between the two markings m_1 and m_2 of equal size k in $O(k^2)$ time. Another algorithm, described in [22, 15], checks whether $(m_1, m_2) \in R^\oplus$ in $O(n)$ time, where n is the size of S. However, these performant algorithms heavily rely on the fact that R is an equivalence relation, hence also subtractive (case 4 of Proposition 19). If R is not an equivalence relation, which is typical for place bisimulations, the naive algorithm for checking whether $(m_1, m_2) \in R^\oplus$ would simply consider m_1 represented as $s_1 \oplus s_2 \oplus \ldots \oplus s_k$, and then would scan all the possible permutations of m_2, each represented as $s_1' \oplus s_2' \oplus \ldots \oplus s_k'$, to check that $(s_i, s_i') \in R$ for $i = 1, \ldots, k$. Of course, this naive algorithm has worst-case complexity $O(k!)$.

▶ **Example 20.** Consider $R = \{(s_1, s_3), (s_1, s_4), (s_2, s_4)\}$, which is not an equivalence relation. Suppose we want to check that $(s_1 \oplus s_2, s_4 \oplus s_3) \in R^\oplus$. If we start by matching $(s_1, s_4) \in R$, then we fail because the residual (s_2, s_3) is not in R. However, if we permute the second marking to $s_3 \oplus s_4$, then we succeed because the required pairs (s_1, s_3) and (s_2, s_4) are both in R.

Nonetheless, the problem of checking whether $(m_1, m_2) \in R^\oplus$ has polynomial time complexity because it can be considered as an instance of the problem of finding a perfect matching in a bipartite graph, where the nodes of the two partitions are the tokens in the two markings, and the edges are defined by the relation R. In fact, the definition of the bipartite graph takes $O(k^2)$ time (where $k = |m_1| = |m_2|$) and, then, the Hopcroft-Karp-Karzanov algorithm [18] for computing the maximum matching has worst-case time complexity $O(h\sqrt{k})$, where h is the number of the edges in the bipartire graph ($h \leq k^2$) and to check whether the maximum matching is perfect can be done simply by checking that the size of the matching equals the number of nodes in each partition, i.e., k. Hence, in evaluating the complexity of the algorithm in Section 4, we assume that the complexity of checking whether $(m_1, m_2) \in R^\oplus$ is in $O(k^2\sqrt{k})$.

A related problem is that of computing, given a marking m_1 of size k, the set of all the markings m_2 such that $(m_1, m_2) \in R^\oplus$. This problem can be solved with a worst-case time complexity of $O(n^k)$ because each of the k tokens in m_1 can be related via R to n places at most.

Now we list some necessary, and less obvious, properties of additively closed place relations that will be useful in the following.

▶ **Proposition 21** ([14]). *For each family of place relations $R_i \subseteq S \times S$, the following hold:*
1. $\emptyset^\oplus = \{(\theta, \theta)\}$, *i.e., the additive closure of the empty place relation yields a singleton marking relation, relating the empty marking to itself.*
2. $(\mathcal{I}_S)^\oplus = \mathcal{I}_M$, *i.e., the additive closure of the identity on places $\mathcal{I}_S = \{(s, s) \mid s \in S\}$ yields the identity relation on markings $\mathcal{I}_M = \{(m, m) \mid m \in \mathcal{M}(S)\}$.*
3. $(R^\oplus)^{-1} = (R^{-1})^\oplus$, *i.e., the inverse of an additively closed relation R equals the additive closure of its inverse R^{-1}.*
4. $(R_1 \circ R_2)^\oplus = (R_1^\oplus) \circ (R_2^\oplus)$, *i.e., the additive closure of the composition of two place relations equals the compositions of their additive closures.*

3.2 Pti-place bisimulation and its properties

We are now ready to introduce pti-place bisimulation, which is a non-interleaving behavioral relation defined over the net places. Note that for P/T nets, place bisimulation [4, 16] and pti-place bisimulation coincide because $I = \emptyset$.

▶ **Definition 22** (Pti-place bisimulation). *Let $N = (S, A, T, I)$ be a PTI net. A* pti-place bisimulation *is a relation $R \subseteq S \times S$ such that if $(m_1, m_2) \in R^\oplus$ then*
1. $\forall t_1$ *such that $m_1[t_1\rangle m_1'$, $\exists t_2$ such that $m_2[t_2\rangle m_2'$ and*
   (a) $({}^\bullet t_1, {}^\bullet t_2) \in R^\oplus$, $(t_1^\bullet, t_2^\bullet) \in R^\oplus$, $l(t_1) = l(t_2)$, and $(m_1 \ominus {}^\bullet t_1, m_1 \ominus {}^\bullet t_2) \in R^\oplus$,
   (b) $\forall s, s' \in S.(s, s') \in R \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$.
2. $\forall t_2$ *such that $m_2[t_2\rangle m_2'$, $\exists t_1$ such that $m_1[t_1\rangle m_1'$ and*
   (a) $({}^\bullet t_1, {}^\bullet t_2) \in R^\oplus$, $(t_1^\bullet, t_2^\bullet) \in R^\oplus$, $l(t_1) = l(t_2)$, and $(m_1 \ominus {}^\bullet t_1, m_1 \ominus {}^\bullet t_2) \in R^\oplus$,
   (b) $\forall s, s' \in S.(s, s') \in R \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$.

Two markings m_1 and m_2 are pti-place bisimilar*, denoted by $m_1 \sim_p m_2$, if there exists a pti-place bisimulation R such that $(m_1, m_2) \in R^\oplus$.*

Note that, by additivity of $R^{\oplus}$ (cf. Proposition 19), from $(m_1 \ominus {}^{\bullet}t_1, m_2 \ominus {}^{\bullet}t_2) \in R^{\oplus}$ and $(t_1^{\bullet}, t_2^{\bullet}) \in R^{\oplus}$ we derive $(m_1', m_2') \in R^{\oplus}$, which is the condition required in the original definition of place bisimulation in [4]. Our slightly stronger formulation is more adequate for the proof of Theorem 27.

Conditions 1(b) and 2(b) make sure that the relation R respects the inhibiting behavior of places. Indeed, an inhibiting place for one of the two transitions cannot be related via R to a non-inhibiting place for the other transition. These conditions might appear rather restrictive, and one may wonder whether they can be weakened or omitted altogether. However, their presence is strictly necessary in the crucial step of the proof of Lemma 29. Moreover, these conditions are also essential for proving Theorem 27.

▶ **Example 23.** Consider the PTI net N_1 in Figure 2. Not only the loop labeled by b on the left is unwound on the right, but also the a-labeled transition on the left is replicated three times on the right. The relation

$$R = \{(s_0, s_5), (s_1, s_{11}), (s_2, s_4), (s_2, s_7), (s_3, s_6), (s_3, s_8), (s_3, s_9), (s_3, s_{10})\}$$

is a pti-place bisimulation and so, e.g., $2 \cdot s_2 \oplus s_3 \sim_p s_4 \oplus s_7 \oplus s_9$.

Now consider the PTI net N_2 in Figure 2. In this case, the b-labeled transition on the left can be matched by the b-labeled transition on the right, even if their inhibiting set differ in size, because both (s_1, s_1') and (s_3, s_1') are in the following bisimulation. Indeed, the relation

$$R' = \{(s_1, s_1', (s_2, s_2'), (s_3, s_1'), (s_4, s_4'), (s_5, s_4'), (s_6, s_4')\}$$

is a pti-place bisimulation and so, e.g., $s_1 \oplus s_3 \oplus 2 \cdot s_2 \oplus s_5 \sim_p 2 \cdot s_1' \oplus 2 \cdot s_2' \oplus s_4'$.

▶ **Example 24.** Consider the PTI net in Figure 3, depicting two models of unbounded producer-consumer with priority.

In the left part, denoted as $UPAC$ for readability, the producer p can generate two products of type a and b and stock them in w_a and w_b (for "warehouse") respectively. Transitions o_a and o_b model the order of a client c from the warehouse, which may then be shipped (place s) and delivered (transition d). Product a has priority both in the production and ordering phases, and this is modelled by two inhibiting arcs between w_a and b and o_b. Roughly speaking, if there is an a in the warehouse, then no b can be produced or ordered. Moreover, only one product of type a can be stored in either w_a or w_a', as the inhibiting arcs between a warehouse for a and the other a-transition do not allow to perform the latter until the former has been freed by the execution of transition o_a.

In the right part, denoted as $UPBC$, we duplicate the production and ordering phases of product b, and remove one of the two lines of product a. The behavior of the system remains the same, and this is proved by the pti-place bisimulation

$$R = \{(p, \overline{p}), (w_a, \overline{w}_a), (w_a', \overline{w}_a), (w_b, \overline{w}_b), (w_b, \overline{w}_b'), (s, \overline{s}), (c, \overline{c})\}.$$
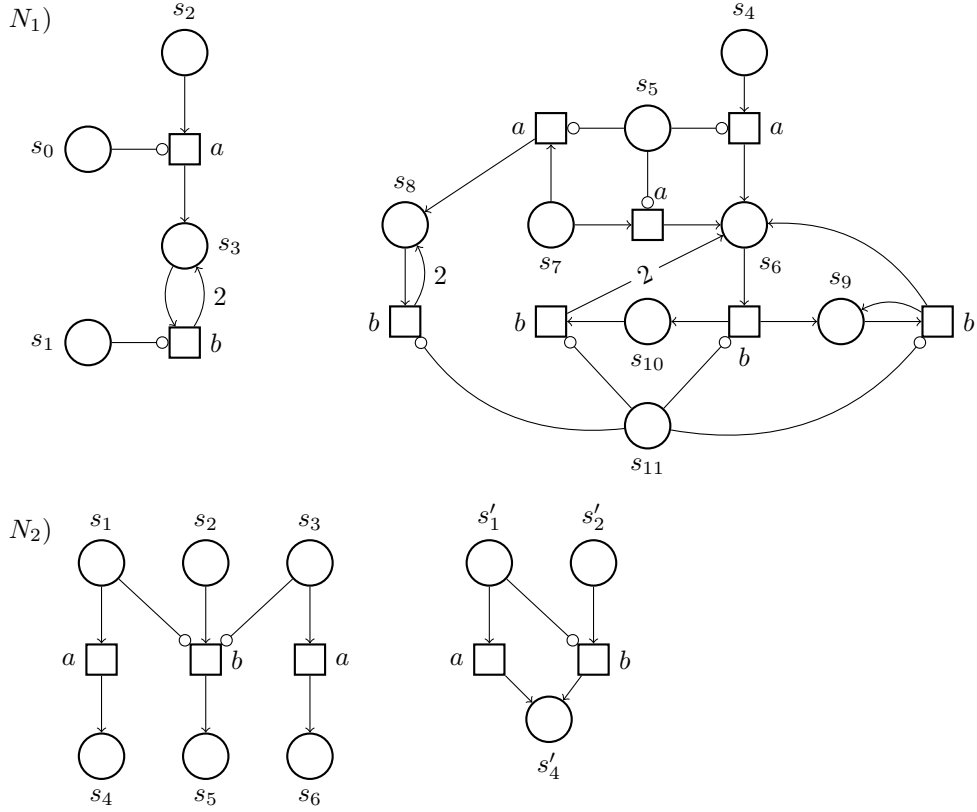
We now prove that $\sim_p$ is an equivalence relation.

▶ **Proposition 25.** *For each PTI net $N = (S, A, T, I)$, relation $\sim_p \subseteq \mathcal{M}(S) \times \mathcal{M}(S)$ is an equivalence relation.*

Proof. Direct consequence of the fact that for each PTI net $N = (S, A, T, I)$, the following hold:
1. The identity relation $\mathcal{I}_S = \{(s, s) \mid s \in S\}$ is a pti-place bisimulation;
2. the inverse relation $R^{-1} = \{(s', s) \mid (s, s') \in R\}$ of a pti-place bisimulation R is a pti-place bisimulation;

Figure 2 Two PTI nets, whose transitions are labeled either by a or by b.

3. the relational composition $R_1 \circ R_2 = \{(s, s'') \mid \exists s'.(s, s') \in R_1 \wedge (s', s'') \in R_2\}$ of two pti-place bisimulations R_1 and R_2 is a pti-place bisimulation.

See Appendix A.1 for details. ◀

By Definition 22, pti-place bisimilarity can be defined in the following way:

$\sim_p = \bigcup \{R^\oplus \mid R \text{ is a pti-place bisimulation}\}$.

By monotonicity of the additive closure (Proposition 19(2)), if $R_1 \subseteq R_2$, then $R_1^\oplus \subseteq R_2^\oplus$.

Hence, we can restrict our attention to maximal pti-place bisimulations only:

$\sim_p = \bigcup \{R^\oplus \mid R \text{ is a } maximal \text{ pti-place bisimulation}\}$.

However, it is not true that

$\sim_p = (\bigcup \{R \mid R \text{ is a } maximal \text{ pti-place bisimulation}\})^\oplus$

because the union of pti-place bisimulations may be not a pti-place bisimulation (as already observed for place bisimulation in [4, 16]), so that its definition is not coinductive.

▶ **Example 26.** Consider the net in Figure 4, whose transitions are $t_1 = (s_2, s_3) \xrightarrow{a} s_1$, $t_2 = (s_2 \oplus s_3, \theta) \xrightarrow{a} s_5$ and $t_3 = (s_3, s_2) \xrightarrow{a} s_4$. Clearly, R_1 and R_2, defined as follows, are both maximal pti-place bisimulations.

$$R_1 = \{(s_2, s_2), (s_3, s_3)\} \cup (\{s_1, s_4, s_5\} \times \{s_1, s_4, s_5\})$$
$$R_2 = \{(s_2, s_3), (s_3, s_2)\} \cup (\{s_1, s_4, s_5\} \times \{s_1, s_4, s_5\})$$
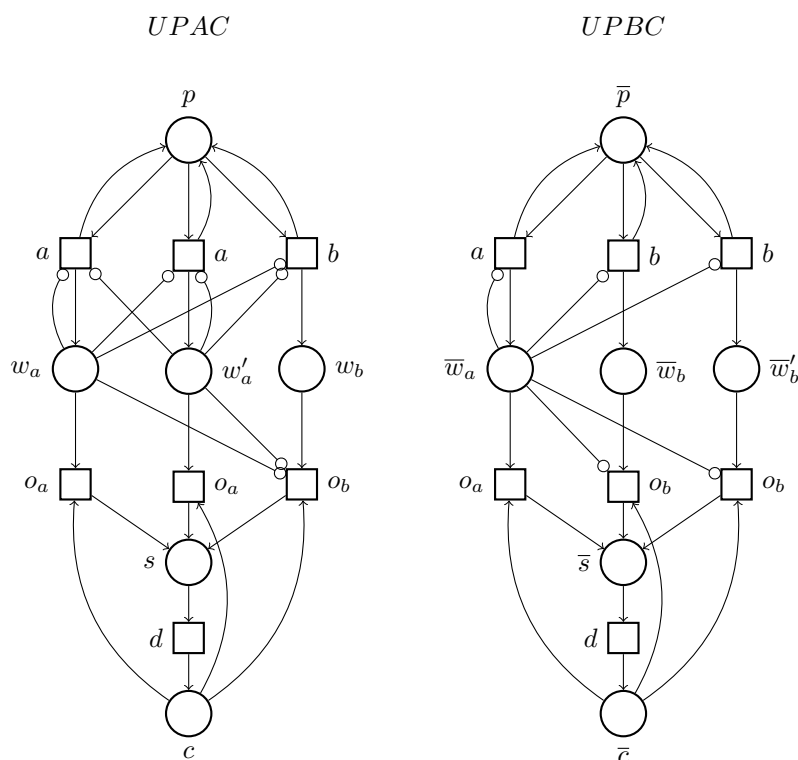
Figure 3 A PTI net representing two unbounded producers/consumers with priority. For simplicity, we display the labels of transitions instead of their names.

Note that the union $R = R_1 \cup R_2$ is not a pti-place bisimulation as, for example, $(2 \cdot s_2, s_2 \oplus s_3) \in R^\oplus$, but the pti-place bisimulation conditions are not satisfied. Indeed, if $2 \cdot s_2$ moves first by $2 \cdot s_2[t_1\rangle s_1 \oplus s_2$, then $s_2 \oplus s_3$ can only try to respond with $s_2 \oplus s_3[t_2\rangle s_5$ since t_1 and t_3 are inhibited. However, this is not possible because we have that $(^\bullet t_1, {}^\bullet t_2) \notin R^\oplus$, and, even worse, $(s_2, \theta) \notin R^\oplus$.

## 3.3 Pti-place bisimilarity is finer than causal-net bisimilarity

▶ **Theorem 27** (Pti-place bisimilarity implies causal-net bisimilarity). *Let $N = (S, A, T, I)$ be a PTI net and m_1, m_2 two of its markings. If $m_1 \sim_p m_2$, then $m_1 \sim_{cn} m_2$.*

**Proof.** See Appendix A.2.   ◀

There are at least the following three important technical differences between causal-net bisimilarity and pti-place bisimilarity.

1. A causal-net bisimulation is a very complex relation – composed of cumbersome triples of the form (ρ_1, C, ρ_2) – that must contain infinitely many triples if the net system offers a never-ending behavior. On the contrary, a pti-place bisimulation is always a very simple finite relation over the finite set S of places.

2. A causal net bisimulation proving that $m_1 \sim_{cn} m_2$ is a relation specifically designed for showing that m_1 and m_2 generate the same causal nets, step by step. If we want to prove that, e.g., $n \cdot m_1$ and $n \cdot m_2$ are causal-net bisimilar (which may not hold!), we have to construct a new causal-net bisimulation to this aim. Instead, a pti-place bisimulation
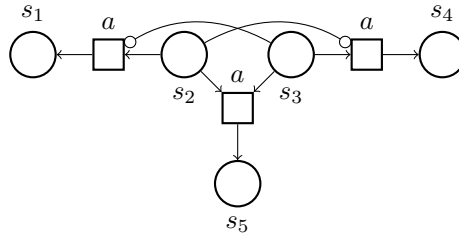
Figure 4 A PTI net.
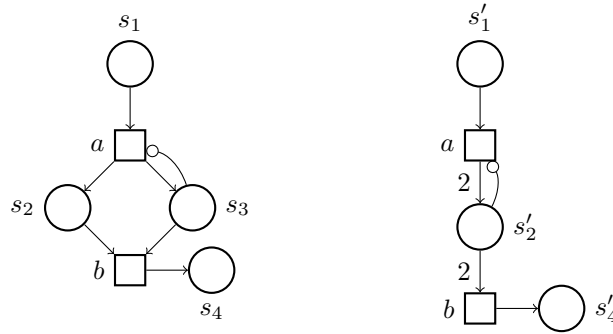
Figure 5 Two PTI nets.

R relates those places which are considered equivalent under all the possible R-related contexts. Hence, if R justifies that $m_1 \sim_p m_2$ as $(m_1, m_2) \in R^\oplus$, then for sure the same R justifies that $n \cdot m_1$ and $n \cdot m_2$ are pti-place bisimilar, as also $(n \cdot m_1, n \cdot m_2) \in R^\oplus$.

3. Finally, while pti-place bisimilarity is decidable (see the next section), it is not known whether causal-net bisimilarity is decidable on finite PTI nets.[1]

However, these technical advantages of pti-place bisimilarity over causal-net bisimilarity are balanced by an increased discriminating power of the former over the latter, that, in some cases, might appear even excessive, as the following intriguing example shows.

▶ **Example 28.** Consider the net in Figure 5. First of all, note that $s_2 \sim_{cn} s_2'$, because both are stuck markings. However, we have that $2 \cdot s_2 \not\sim_{cn} 2 \cdot s_2'$ because $2 \cdot s_2$ is stuck, while $2 \cdot s_2'$ can perform b. This observation is enough to conclude that $s_2 \not\sim_p s_2'$, because a pti-place bisimulation R relates places that are equivalent under any R-related context: if $(s_2, s_2') \in R$ then $(2 \cdot s_2, 2 \cdot s_2') \in R^\oplus$, but these two markings do not satisfy the pti-place bisimulation conditions, so R is not a pti-place bisimulation.

Nonetheless, it is interesting to observe that $s_1 \sim_{cn} s_1'$, because they generate the same causal PTI nets, step by step; moreover, even for any $n \geq 1$ we have $n \cdot s_1 \sim_{cn} n \cdot s_1'$. However, $s_1 \not\sim_p s_1'$ because it is not possible to build a pti-place bisimulation R containing the pair (s_1, s_1'). The problem is that it would be necessary to include, into the candidate pti-place relation R, also the pair (s_2, s_2'), which is not a pti-place bisimulation pair, as discussed above. Therefore, no pti-place bisimulation R can relate s_1 and s_1'.

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[1] Esparza observed [12] that, for finite P/T nets with at least two unbounded places, all the behavioral relations ranging from interleaving bisimilarity to fully-concurrent bisimilarity [6] are undecidable. Even if his proof does not apply to causal-net bisimilarity, we conjecture that this equivalence is undecidable as well.

4 Pti-place bisimilarity is decidable

In order to prove that $\sim_p$ is decidable, we first need a technical lemma which states that it is decidable to check whether a place relation $R \subseteq S \times S$ is a pti-place bisimulation.

▶ **Lemma 29.** *Given a finite PTI net $N = (S, A, T, I)$ and a place relation $R \subseteq S \times S$, it is decidable whether R is a pti-place bisimulation.*

Proof. It is enough to check two finite conditions on transitions and places of the net; full detail in Appendix A.3. ◀

▶ **Theorem 30** (Pti-place bisimilarity is decidable). *Given a PTI net $N = (S, A, T, I)$, for each pair of markings m_1 and m_2, it is decidable whether $m_1 \sim_p m_2$.*

Proof. If $|m_1| \neq |m_2|$, then $m_1 \nsim_p m_2$ by Proposition 18. Otherwise, we can assume that $|m_1| = k = |m_2|$. As $|S| = n$, the set of all the place relations over S is of size 2^n. Let us list such relations as: $R_1, R_2, \ldots, R_{2^n}$. Hence, for $i = 1, \ldots, 2^n$, by Lemma 29 we can decide whether the place relation R_i is a pti-place bisimulation and, in such a case, we can check whether $(m_1, m_2) \in R_i^{\oplus}$ in $O(k^2\sqrt{k})$ time. As soon as we have found a pti-place bisimulation R_i such that $(m_1, m_2) \in R_i^{\oplus}$, we stop concluding that $m_1 \sim_p m_2$. If none of the R_i is a pti-place bisimulation such that $(m_1, m_2) \in R_i^{\oplus}$, then we can conclude that $m_1 \nsim_p m_2$. Since this procedure might scan all place relations, the worst-case complexity of the algorithm is exponential in the number of places n. ◀

5 Conclusion

Pti-place bisimilarity is the only decidable behavioral equivalence for finite PTI nets, which constitute a powerful, Turing-complete distributed model of computation, widely used in theory and applications of concurrency (e.g., [1, 3, 9, 7, 8, 17, 19, 26]). Thus, it is the only equivalence for which it is possible (at least, in principle) to verify algorithmically the (causality-preserving) correctness of an implementation by exhibiting a pti-place bisimulation between its specification and implementation. It is also sensible, because it respects the causal behavior of PTI nets, since it is finer than causal-net bisimilarity. Of course, pti-place bisimilarity is a rather discriminating behavioral equivalence, as illustrated in Example 28, and a proper evaluation of its usefulness on real case studies is left for future research.

In our interpretation, (pti-)place bisimilarity is an attempt of giving semantics to *unmarked*, rather than marked, nets, shifting the focus from the usually undecidable question *When are two markings equivalent?* to the decidable (but more restrictive) question *When are two places equivalent?* A possible answer to the latter question may be: two places are equivalent if, whenever the same number of tokens are put on these two places, the behavior of the marked nets is the same. If we reinterpret Example 28 in this perspective, we clearly see that place s_2 and place s_2' cannot be considered as equivalent because, even if the marking s_2 and s_2' are equivalent (as they are both stuck), the marking $2 \cdot s_2$ is not equivalent to the marking $2 \cdot s_2'$ (as only the latter can move). More specifically, a (pti-)place bisimulation R considers two places s_1 and s_2 as equivalent if $(s_1, s_2) \in R$, as, by definition of (pti-)place bisimulation, they must behave the same in any R-related context.

The decidability result for pti-place bisimilarity is based on the fact that the net model is finite, even if the associated reachability graph may be unboundedly large or even infinite: indeed, one can decide pti-place bisimilarity simply checking a large, but finite, number of conditions on the shape of the finite net, rather than inspecting its (possibly, infinitely many) reachable markings.

Turing completeness is achieved in PTI nets by means of their ability to test for zero. Other Turing-complete models of computation may exploit different mechanisms to this aim. For instance, in the π-calculus [24, 28] Turing completeness is achieved by means of the ability to generate unboundedly new names (by means of the interplay between recursion and the restriction operator), but this feature is not describable by means of a finite net model [8, 23]. For this reason, we think it is hard to find a sensible, decidable behavioral equivalence for the whole π-calculus.

To the best of our knowledge, this is the second paper proving the decidability of a behavioral equivalence for a Turing-complete formalism. In fact, in [21] it is proved that (interleaving) bisimilarity is decidable for a small process calculus, called HOcore, with higher-order communication (but without restriction), that is, nonetheless, Turing-complete.

Future work will be devoted to see whether the pti-place bisimulation idea can be extended to other, possibly even larger classes of nets, such as *lending* Petri nets [5], where transitions are allowed to consume tokens from a place even if it does not contain enough tokens, thus enabling negative-valued markings.

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A   Properties of pti-place bisimilarity

A.1   Pti-place bisimilarity is an equivalence

▶ **Proposition 31.** *For each PTI net $N = (S, A, T, I)$, the following hold:*

1. *The identity relation $\mathcal{I}_S = \{(s, s) \mid s \in S\}$ is a pti-place bisimulation;*
2. *the inverse relation $R^{-1} = \{(s', s) \mid (s, s') \in R\}$ of a pti-place bisimulation R is a pti-place bisimulation;*
3. *the relational composition $R_1 \circ R_2 = \{(s, s'') \mid \exists s'.(s, s') \in R_1 \wedge (s', s'') \in R_2\}$ of two pti-place bisimulations R_1 and R_2 is a pti-place bisimulation.*

Proof. The proof is almost standard, due to Proposition 21.

(1) $\mathcal{I}_S$ is a pti-place bisimulation as for each $(m, m) \in \mathcal{I}_S^\oplus$ whatever transition t the left (or right) marking m performs a transition (say, $m[t\rangle m'$), the right (or left) instance of m in the pair does exactly the same transition $m[t\rangle m'$ and, of course, $({}^\bullet t, {}^\bullet t) \in \mathcal{I}_S^\oplus$, $(t^\bullet, t^\bullet) \in \mathcal{I}_S^\oplus$, $l(t) = l(t)$, $(m \ominus {}^\bullet t, m \ominus {}^\bullet t) \in \mathcal{I}_S^\oplus$, by Proposition 21(2), and, also, $\forall s \in S.(s, s) \in \mathcal{I}_S \Rightarrow (s \in {}^\circ t \Leftrightarrow s \in {}^\circ t)$, as required by the pti-place bisimulation definition.

(2) Suppose $(m_2, m_1) \in (R^{-1})^\oplus$ and $m_2[t_2\rangle m_2'$. By Proposition 21(3) $(m_2, m_1) \in (R^\oplus)^{-1}$ and so $(m_1, m_2) \in R^\oplus$. Since R is a pti-place bisimulation, item 2 of the bisimulation game ensures that there exist t_1 and m_1' such that $m_1[t_1\rangle m_1'$, with $({}^\bullet t_1, {}^\bullet t_2) \in R^\oplus$, $l(t_1) = l(t_2)$, $(t_1^\bullet, t_2^\bullet) \in R^\oplus$ and $(m_1 \ominus {}^\bullet t_1, m_2 \ominus {}^\bullet t_2) \in R^\oplus$; moreover, $\forall s, s' \in S.(s, s') \in R \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$. Summing up, if $(m_2, m_1) \in (R^{-1})^\oplus$, to the move $m_2[t_2\rangle m_2'$, m_1 replies with the move $m_1[t_1\rangle m_1'$, such that (by Proposition 21(3)) $({}^\bullet t_2, {}^\bullet t_1) \in (R^{-1})^\oplus$, $l(t_2) = l(t_1)$, $(t_2^\bullet, t_1^\bullet) \in (R^{-1})^\oplus$, $(m_2 \ominus {}^\bullet t_2, m_1 \ominus {}^\bullet t_1) \in (R^{-1})^\oplus$ and, moreover, $\forall s, s' \in S.(s', s) \in R^{-1} \Rightarrow (s' \in {}^\circ t_2 \Leftrightarrow s \in {}^\circ t_1)$, as required. The case when m_1 moves first is symmetric and thus omitted.

(3) Suppose $(m, m'') \in (R_1 \circ R_2)^\oplus$ and $m[t_1\rangle m_1$. By Proposition 21(4), we have that $(m, m'') \in R_1^\oplus \circ R_2^\oplus$, and so there exists m' such that $(m, m') \in R_1^\oplus$ and $(m', m'') \in R_2^\oplus$. As $(m, m') \in R_1^\oplus$ and R_1 is a pti-place bisimulation, if $m[t_1\rangle m_1$, then there exist t_2 and m_2 such that $m'[t_2\rangle m_2$ with $({}^\bullet t_1, {}^\bullet t_2) \in R_1^\oplus$, $l(t_1) = l(t_2)$, $(t_1^\bullet, t_2^\bullet) \in R_1^\oplus$ and $(m \ominus {}^\bullet t_1, m' \ominus {}^\bullet t_2) \in R_1^\oplus$; moreover, $\forall s, s' \in S.(s, s') \in R_1 \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$. But as $(m', m'') \in R_2^\oplus$ and R_2 is a pti-place bisimulation, we have also that there exist t_3 and m_3 such that $m''[t_3\rangle m_3$ with $({}^\bullet t_2, {}^\bullet t_3) \in R_2^\oplus$, $l(t_2) = l(t_3)$, $(t_2^\bullet, t_3^\bullet) \in R_2^\oplus$ and $(m' \ominus {}^\bullet t_2, m'' \ominus {}^\bullet t_3) \in R_2^\oplus$; moreover, $\forall s', s'' \in S.(s', s'') \in R_2 \Rightarrow (s' \in {}^\circ t_2 \Leftrightarrow s'' \in {}^\circ t_3)$. Summing up, for $(m, m'') \in (R_1 \circ R_2)^\oplus$, if $m[t_1\rangle m_1$, then there exist t_3 and m_3 such that $m''[t_3\rangle m_3$ and (by Proposition 21(4)) $({}^\bullet t_1, {}^\bullet t_3) \in (R_1 \circ R_2)^\oplus$, $l(t_1) = l(t_3)$, $(t_1^\bullet, t_3^\bullet) \in (R_1 \circ R_2)^\oplus$ and $(m \ominus {}^\bullet t_1, m'' \ominus {}^\bullet t_3) \in (R_1 \circ R_2)^\oplus$; moreover, $\forall s, s'' \in S.(s, s'') \in R_1 \circ R_2 \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s'' \in {}^\circ t_3)$, as required. The case when m'' moves first is symmetric and so omitted. ◀

▶ **Proposition 32.** *For each PTI net* $N = (S, A, T, I)$, *relation* $\sim_p \subseteq \mathcal{M}(S) \times \mathcal{M}(S)$ *is an equivalence relation.*

Proof. Direct consequence of Proposition 31. ◀

A.2   Pti-place bisimilarity is finer than causal-net bisimilarity

▶ **Theorem 33** (Pti-place bisimilarity implies causal-net bisimilarity). *Let* $N = (S, A, T, I)$ *be a PTI net and* m_1, m_2 *two of its markings. If* $m_1 \sim_p m_2$, *then* $m_1 \sim_{cn} m_2$.

Proof. If $m_1 \sim_p m_2$, then there exists a pti-bisimulation R_1 such that $(m_1, m_2) \in R_1^\oplus$. Let us consider

$$R_2 \overset{def}{=} \{(\rho_1, C, \rho_2) | (C, \rho_1) \text{ is a PTI process of } N(m_1) \text{ and}$$
$$(C, \rho_2) \text{ is a PTI process of } N(m_2) \text{ and}$$
$$\forall b \in B \ (\rho_1(b), \rho_2(b)) \in R_1\}.$$

We want to prove that R_2 is a causal-net bisimulation. First of all, consider a triple of the form $(\rho_1^0, C^0, \rho_2^0)$, where C^0 is the causal PTI net without events and ρ_1^0, ρ_2^0 are such that $\rho_i^0(Min(C^0)) = \rho_i^0(Max(C^0)) = \rho_i^0(B^0) = m_i$ for $i = 1, 2$, and $(\rho_1^0(b), \rho_2^0(b)) \in R_1$ for all $b \in B^0$. Then $(\rho_1^0, C^0, \rho_2^0)$ must belong to R_2, because (C^0, ρ_i^0) is a process of $N(m_i)$, for $i = 1, 2$ and, by hypothesis, $(m_1, m_2) \in R_1^\oplus$. Hence, if R_2 is a causal-net bisimulation, then the triple $(\rho_1^0, C^0, \rho_2^0) \in R_2$ ensures that $m_1 \sim_{cn} m_2$.

Assume $(\rho_1, C, \rho_2) \in R_2$. In order for R_2 to be a cn-bisimulation, we must prove that

(i) $\forall t_1, C', \rho_1'$ such that $(C, \rho_1) \xrightarrow{e} (C', \rho_1')$, where $\rho_1'(e) = t_1$, $\exists t_2, \rho_2'$ such that $(C, \rho_2) \xrightarrow{e} (C', \rho_2')$, where $\rho_2'(e) = t_2$, and $(\rho_1', C', \rho_2') \in R_2$;

(ii) symmetrical, if (C, ρ_2) moves first.

Assume $(C, \rho_1) \xrightarrow{e} (C', \rho_1')$ with $\rho_1'(e) = t_1$. Since $(\rho_1, C, \rho_2) \in R_2$, for all $b \in Max(C)$ we have $(\rho_1(b), \rho_2(b)) \in R_1$ and therefore $(\rho_1(Max(C)), \rho_2(Max(C))) \in R_1^\oplus$. Since $\rho_1(Max(C))$ $[t_1\rangle \rho_1'(Max(C'))$ and R_1 is a pti-place bisimulation, there exist t_2, m_2 such that $\rho_2(Max(C))$ $[t_2\rangle m_2$ with $(^\bullet t_1, ^\bullet t_2) \in R_1^\oplus$, $l(t_1) = l(t_2)$, $(t_1^\bullet, t_2^\bullet) \in R_1^\oplus$, $(\rho_1(Max(C)) \ominus ^\bullet\rho_1'(e), \rho_2(Max(C)) \ominus ^\bullet\rho_2'(e)) \in R_1^\oplus$ and, moreover, $\forall s, s' \in S.(s, s') \in R_1 \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$. Note that, since $(t_1^\bullet, t_2^\bullet) \in R_1^\oplus$ and $(\rho_1(Max(C)) \ominus ^\bullet\rho_1'(e), \rho_2(Max(C)) \ominus {}^\bullet\rho_2'(e)) \in R_1^\oplus$, by additivity of additive closure (cf. Proposition 19), $(\rho_1(Max(C)) \ominus ^\bullet\rho_1'(e) \oplus t_1^\bullet, \rho_2(Max(C)) \ominus ^\bullet\rho_2'(e) \oplus t_2^\bullet) \in R_1^\oplus$, i.e. $(\rho_1'(Max(C')), m_2) \in R^\oplus$.

Therefore, since t_1 and t_2 have the same pre-sets/post-sets up to R_1, it is possible to derive $(C, \rho_2) \xrightarrow{e} (C'', \rho_2')$, where ρ_2' is such that $\rho_2'(e) = t_2$ and $(\rho_1'(b), \rho_2'(b)) \in R_1$ for each $b \in e^\bullet$ (which is really possible because $(t_1^\bullet, t_2^\bullet) \in R_1^\oplus$). Now we prove that $C' = C''$. The underlying P/T parts of C' and C'' are obviously the same (so C' and C'' have the same events, the same conditions and the same flow relation), therefore we have to check that also the newly added (after/before) inhibitor arcs are the same, i.e.,

- $\forall b \in B'$ such that $b^\bullet \neq \emptyset$ we have $b \, \mathcal{A}_1 \, e \iff b \, \mathcal{A}_2 \, e$, and
- $\forall b \in e^\bullet \, \forall e' \in E$ we have $b \, \mathcal{B}_1 \, e' \iff b \, \mathcal{B}_2 \, e'$,

where we denote $\mathcal{A}_1$ (resp. $\mathcal{B}_1$) the after (before) inhibitor arcs obtained by extending C to C' and $\mathcal{A}_2$ (resp. $\mathcal{B}_2$) the after (before) inhibitor arcs obtained by extending C to C''. However, these additional requests are trivially satisfied because we know that $\forall s, s' \in S.(s, s') \in R_1 \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$. In fact, if $b \, \mathcal{A}_1 \, e$, then, by Definition 13, there is an inhibitor arc from $\rho_1(b)$ to t_1, i.e., $\rho_1(b) \in {}^\circ t_1$. Since $(\rho_1(b), \rho_2(b)) \in R_1$, this implies that $\rho_2(b) \in {}^\circ t_2$ and so $b \, \mathcal{A}_2 \, e$. The implication on the other side is symmetrical, and therefore omitted. The argument for relations $\mathcal{B}_1, \mathcal{B}_2$ is the same, and therefore omitted.

To conclude, we have $C' = C''$. Thus, $(C, \rho_2) \xrightarrow{e} (C', \rho_2')$ with $\rho_2'(e) = t_2$ and $(\rho_1'(b), \rho_2'(b)) \in R_1$ for each $b \in e^\bullet$. Hence, for all $b' \in B'$ it holds that $(\rho_1'(b'), \rho_2'(b')) \in R_1$, because for all $b' \in B$ this holds by hypothesis and for all $b' \in e^\bullet$ this follows by construction (thanks to the fact that $(t_1^\bullet, t_2^\bullet) \in R_1^\oplus$). As a consequence $(\rho_1', C', \rho_2') \in R_2$.

The case where (C, ρ_2) moves first is symmetrical and therefore omitted. Thus, R_2 is a causal-net bisimulation and, since $(\rho_1^0, C^0, \rho_2^0) \in R_2$, we have $m_1 \sim_{cn} m_2$. ◄

A.3 It is decidable whether a place relation is a pti-place bisimulation

▶ **Lemma 34.** *Given a finite PTI net $N = (S, A, T, I)$ and a place relation $R \subseteq S \times S$, it is decidable whether R is a pti-place bisimulation.*

Proof. We want to prove that R is a pti-place bisimulation if and only if the following two finite conditions are satisfied:

1. $\forall t_1$ such that $^\bullet t_1[t_1\rangle$, $\forall m$ such that $(^\bullet t_1, m) \in R^\oplus$, $\exists t_2$ such that $^\bullet t_2[t_2\rangle$ and
   (a) $^\bullet t_2 = m$,
   (b) $(t_1^\bullet, t_2^\bullet) \in R^\oplus$, $l(t_1) = l(t_2)$,
   (c) $\forall s, s' \in S.(s, s') \in R \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$.
2. $\forall t_2$ such that $^\bullet t_2[t_2\rangle$, $\forall m$ such that $(m, ^\bullet t_2) \in R^\oplus$, $\exists t_1$ such that $^\bullet t_1[t_1\rangle$ and
   (a) $^\bullet t_1 = m$,
   (b) $(t_1^\bullet, t_2^\bullet) \in R^\oplus$, $l(t_1) = l(t_2)$,
   (c) $\forall s, s' \in S.(s, s') \in R \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$.

First we prove the implication from left to right, only for condition 1, as the other is symmetrical. If R is a pti-place bisimulation and $(^\bullet t_1, m) \in R^\oplus$, then from $^\bullet t_1[t_1\rangle t_1^\bullet$ it follows that there exists t_2 such that $^\bullet t_2[t_2\rangle t_2^\bullet$ with $^\bullet t_2 = m$, $(t_1^\bullet, t_2^\bullet) \in R^\oplus$, $l(t_1) = l(t_2)$ and, moreover, $\forall s, s' \in S.(s, s') \in R \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$. Therefore, conditions (a), (b) and (c) are trivially satisfied.

Now we prove the implication from right to left, i.e., if conditions 1 and 2 hold for R, then R is a pti-place bisimulation. Suppose $(m_1, m_2) \in R^\oplus$ and $m_1[t_1\rangle m_1'$. Let $q = \{(s_1, s_1'), (s_2, s_2'), \ldots, (s_k, s_k')\}$ be any multiset of associations that can be used to prove that $(m_1, m_2) \in R^\oplus$. So this means that $m_1 = s_1 \oplus s_2 \oplus \ldots \oplus s_k$, $m_2 = s_1' \oplus s_2' \oplus \ldots \oplus s_k'$ and that $(s_i, s_i') \in R$ for $i = 1, \ldots, k$. If $m_1[t_1\rangle m_1'$, then $m_1' = m_1 \ominus {}^\bullet t_1 \oplus t_1^\bullet$. Consider the multiset of associations $p = \{(\overline{s}_1, \overline{s}_1'), \ldots, (\overline{s}_h, \overline{s}_h')\} \subseteq q$, with $\overline{s}_1 \oplus \ldots \oplus \overline{s}_h = {}^\bullet t_1$.

Note that $({}^\bullet t_1, \overline{s}_1' \oplus \ldots \oplus \overline{s}_h') \in R^\oplus$ and that ${}^\bullet t_1[t_1\rangle$. Hence, by condition 1, there exists a transition t_2 such that ${}^\bullet t_2[t_2\rangle$, ${}^\bullet t_2 = \overline{s}_1' \oplus \ldots \oplus \overline{s}_h'$, $(t_1^\bullet, t_2^\bullet) \in R^\oplus$, $l(t_1) = l(t_2)$, and $\forall s, s' \in S.(s, s') \in R \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$. By hypothesis, ${}^\circ t_1 \cap dom(m_1) = \emptyset$, so since $(m_1, m_2) \in R^\oplus$ and condition (c) holds, we have that ${}^\circ t_2 \cap dom(m_2) = \emptyset$. Therefore, since ${}^\bullet t_2 \subseteq m_2$, also $m_2[t_2\rangle m_2'$ is firable, where $m_2' = m_2 \ominus {}^\bullet t_2 \oplus t_2^\bullet$, and we have that $({}^\bullet t_1, {}^\bullet t_2) \in R^\oplus$, $(t_1^\bullet, t_2^\bullet) \in R^\oplus$, $l(t_1) = l(t_2)$, $(m_1 \ominus {}^\bullet t_1, m_2 \ominus {}^\bullet t_2) \in R^\oplus$ and, moreover, $\forall s, s' \in S.(s, s') \in R \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$, as required, where $(m_1 \ominus {}^\bullet t_1, m_2 \ominus {}^\bullet t_2) \in R^\oplus$ holds as, from the set q of matching pairs for m_1 and m_2, we have removed those in p.

If $m_2[t_2\rangle m_2'$, then we have to use an argument symmetric to the above, where condition 2 is used instead. Hence, we have proved that conditions 1 and 2 are enough to prove that R is a pti-place bisimulation.

Finally, the complexity of this procedure is as follows. For condition 1, we have to consider all the net transitions, and for each t_1 we have to consider all the markings m such that $({}^\bullet t_1, m) \in R_i^\oplus$, and for each pair (t_1, m) we have to check whether there exists a transition t_2 such that $m = {}^\bullet t_2$, $l(t_1) = l(t_2)$, $(t_1^\bullet, t_2^\bullet) \in R_i^\oplus$ and, moreover, that $\forall s, s' \in S.(s, s') \in R \Rightarrow (s \in {}^\circ t_1 \Leftrightarrow s' \in {}^\circ t_2)$. And the same for condition 2. Hence, this procedure has worst-case time complexity $O(q \cdot n^p \cdot q \cdot (p^2 \sqrt{p} + n^2 \cdot p))$, where $q = |T|$, $n = |S|$ and p is the least number such that $|{}^\bullet t| \leq p$, $|{}^\circ t| \leq p$, and $|t^\bullet| \leq p$ for all $t \in T$, as the number of markings m related via R_i to ${}^\bullet t_1$ is n^p at most, checking whether $(t_1^\bullet, t_2^\bullet) \in R_i^\oplus$ takes $O(p^2 \sqrt{p})$ in the worst case and, finally, checking the conditions on the inhibiting sets is $n^2 \cdot p$ at most. ◀