Polymorphic Automorphisms and the Picard Group

Pieter Hofstra
Dept. of Mathematics & Statistics, University of Ottawa, Canada

Jason Parker
Department of Mathematics & Computer Science, Brandon University, Canada

Philip J. Scott[1]
Dept. of Mathematics & Statistics, University of Ottawa, Canada

Abstract

We investigate the concept of definable, or inner, automorphism in the logical setting of partial Horn theories. The central technical result extends a syntactical characterization of the group of such automorphisms (called the covariant isotropy group) associated with an algebraic theory to the wider class of quasi-equational theories. We apply this characterization to prove that the isotropy group of a strict monoidal category is precisely its Picard group of invertible objects. Furthermore, we obtain an explicit description of the covariant isotropy group of a presheaf category.

2012 ACM Subject Classification Theory of computation → Categorical semantics; Theory of computation → Algebraic semantics; Theory of computation → Equational logic and rewriting

Keywords and phrases Partial Horn Theories, Monoidal Categories, Definable Automorphisms, Polymorphism, Indeterminates, Normal Forms

Digital Object Identifier 10.4230/LIPIcs.FSCD.2021.26

Related Version *Previous Version*: `https://arxiv.org/abs/2102.11081`

Funding *Pieter Hofstra*: Research funded by an NSERC Discovery Grant.
Jason Parker: Postdoctoral research funded by NSERC grant of R. Lucyshyn-Wright (Brandon).
Philip J. Scott: Research funded by an NSERC Discovery Grant.

Acknowledgements Pieter Hofstra would like to acknowledge illuminating discussions with Martti Karvonen and Eugenia Cheng. We would also like to thank the three anonymous referees for their insightful comments, corrections, and suggestions.

1 Introduction

In algebra, model theory, and computer science, one encounters the notion of *definable automorphism* (the nomenclature varies by discipline). In first-order logic for example (see e.g. [13]), an automorphism α of a model M is called *definable* (with parameters in M) when there is a formula $\varphi(x, y)$ in the ambient language (possibly containing constants from M) such that for all $a, b \in M$ we have

$$\alpha(a) = b \iff M \models \varphi(a, b).$$

The case of groups is instructive: for a group M, consider the formula $\varphi(x, y)$ given as

$$\varphi(x, y) : \; y = c^{-1}xc$$

for some $c \in M$. This defines an (inner) automorphism of M. Note that in this case the automorphism is also determined by a term $t(x) := c^{-1}xc$ via $a \mapsto t(a)$.

[1] corresponding author

6th International Conference on Formal Structures for Computation and Deduction (FSCD 2021).
Editor: Naoki Kobayashi; Article No. 26; pp. 26:1–26:17
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

These definable automorphisms have various interesting aspects: first of all, they are in some sense *polymorphic* or uniform. This means roughly that the same term t, possibly after replacing constants from M, can also define an automorphism of another model N. Secondly, the definable automorphisms can also provide a generalized notion of *inner automorphism*, even for theories where it does not make sense to speak of group-theoretic conjugation. Indeed, Bergman [1, Theorem 1] shows that in the category of groups, the definable group automorphisms, i.e. the inner automorphisms given by conjugation, can be characterized purely *categorically* by the fact that they extend naturally along any homomorphism. That is: an automorphism $\alpha : G \xrightarrow{\sim} G$ is inner precisely when for any homomorphism $m : G \to H$ there is an extension $\alpha_m : H \xrightarrow{\sim} H$ making diagram (a) commute and also making

$$(a) \quad \begin{tikzcd} G \arrow[r, "m"] \arrow[d, "\alpha"'] & H \arrow[d, "\alpha_m"] \\ G \arrow[r, "m"'] & H \end{tikzcd} \qquad\qquad (b) \quad \begin{tikzcd} H \arrow[r, "n"] \arrow[d, "\alpha_m"'] & K \arrow[d, "\alpha_{nm}"] \\ H \arrow[r, "n"'] & K \end{tikzcd}$$

diagram (b) commute for any further homomorphism $n : H \to K$, so that in particular $\alpha = \alpha_{\mathsf{id}_G}$ by diagram (a). If α is conjugation by $g \in G$, then α_m is conjugation by $m(g) \in H$. Conversely, given any system of group automorphisms $\{\alpha_m : H \xrightarrow{\sim} H \mid m : G \to H\}$ with $\alpha = \alpha_{\mathsf{id}_G}$ that makes diagrams (a) and (b) commute, Bergman shows that there is a unique element $s \in G$ such that α is given by conjugation with s. Bergman therefore refers to such a system $\{\alpha_m \mid m : G \to H\}$ as an *extended inner automorphism* of G.[2]

In categorical logic, we have a canonical method for studying this phenomenon. To any category $\mathbb{C}$, we may associate the functor

$$\mathcal{Z}_{\mathbb{C}} : \mathbb{C} \to \mathsf{Grp} \; ; \qquad \mathcal{Z}_{\mathbb{C}}(C) := \mathsf{Aut}(\pi : C/\mathbb{C} \to \mathbb{C}). \tag{1}$$

Let us unpack this. We have the co-slice category $C/\mathbb{C}$ whose objects are maps $C \to D$ and whose arrows are commutative triangles. The projection functor $\pi : C/\mathbb{C} \to \mathbb{C}$ sends $C \to D$ to D. We then consider the group of natural automorphisms of this projection functor, i.e. the group of *invertible* natural transformations $\alpha : \pi \Rightarrow \pi$. To give such an α is equivalent to giving, for each object $m : C \to D$ of $C/\mathbb{C}$, an automorphism $\alpha_m : D \xrightarrow{\sim} D$, subject to the naturality condition that for any composable pair $m : C \to D, n : D \to E$ in $\mathbb{C}$, we have $\alpha_{nm}n = n\alpha_m$ as in diagram (b) above. Thus, in Bergman's terminology, $\mathcal{Z}_{\mathbb{C}}(C)$ is the group of extended inner automorphisms of C. We call $\mathcal{Z}_{\mathbb{C}}$ the *(covariant) isotropy group (functor)* of $\mathbb{C}$. Another useful way of thinking about this group is by noticing that the assignment $C \mapsto \mathsf{Aut}(C)$ is generally not functorial, unless $\mathbb{C}$ is a groupoid. The isotropy group offers a remedy: the assignment $C \mapsto \mathcal{Z}_{\mathbb{C}}(C)$ *is* functorial, as is straightforward to check, and for each C there is a comparison homomorphism

$$\theta_C : \mathcal{Z}_{\mathbb{C}}(C) \to \mathsf{Aut}(C) \; ; \qquad \alpha \mapsto \alpha_{\mathsf{id}_C} \tag{2}$$

that sends an extended inner automorphism α to its component at the identity of C.[3] We can then turn Bergman's aforementioned result for the category Grp into a *definition* for an arbitrary category $\mathbb{C}$, by defining an automorphism $f : C \xrightarrow{\sim} C$ of an object $C \in \mathbb{C}$ to be *inner* just if f is in the image of $\theta_C : \mathcal{Z}_{\mathbb{C}}(C) \to \mathsf{Aut}(C)$. Less precisely, the automorphism $f : C \xrightarrow{\sim} C$ is inner if it can be coherently extended along any arrow out of C.

---

[2] Earlier versions of this result were also proven by Schupp [12] and Pettet [10].

[3] P. Freyd [2] studied a somewhat similar notion while modelling Reynolds' parametricity for parametric polymorphism. As a special case, his work leads to a *monoid* of natural endomorphisms of the projection functor, whereas in our case, we would obtain the subgroup of invertible elements in this monoid.

(For readers familiar with topos theory and/or earlier papers on the subject of isotropy groups, we point out that in [4, 3] we consider instead the *contravariant* isotropy groups $\mathsf{Aut}(\pi : \mathbb{C}/C \to \mathbb{C})$. Now if $\mathbb{T}$ is a suitable logical theory with classifying topos $\mathcal{B}(\mathbb{T})$, then (a restriction of) the contravariant isotropy group of $\mathcal{B}(\mathbb{T})$ coincides with the covariant isotropy group of the category $\mathsf{fp}\mathbb{T}\mathsf{mod}$ of finitely presented $\mathbb{T}$-models. Moreover, calculation of the latter group generally also yields a description of the covariant isotropy group of the larger category $\mathbb{T}\mathsf{mod}$ of *all* $\mathbb{T}$-models, which is our focus in the present paper.)

In [6], the case where $\mathbb{C}$ is the category of models of an equational theory is analysed. Among other things, a complete syntactic characterization of covariant isotropy for such a $\mathbb{C}$ is obtained, recovering not only Bergman's result for $\mathbb{C} = \mathsf{Grp}$ but also characterizing the definable automorphisms of other common algebraic structures such as monoids and rings. In applying the general characterization in specific instances, one typically needs to analyse the result of adjoining one or more indeterminates to a given model, and this in turn leads one to consider the *word problem* for such models.

The present paper, which is based on the PhD research [9] of the second author, is concerned with the analysis of the notion of isotropy or definable automorphism for (strict) monoidal categories and related structures. It hardly needs arguing that monoidal categories play various important roles in mathematics and theoretical computer science, both as objects of study in their own right, as models of logical theories, and as basic tools for studying other phenomena. However, we should point out here an observation by Richard Garner [5, Proposition 3] to the effect that both Cat and Grpd, the categories of small categories and small groupoids respectively, have *trivial* covariant isotropy, in the sense that for any category/groupoid $\mathbb{C}$ we have $\mathcal{Z}(\mathbb{C}) = 1$, the trivial group. The reason for this is roughly as follows: when considering an inner automorphism α of a category $\mathbb{C}$ in Cat, it must in particular extend to the categories obtained from $\mathbb{C}$ by freely adjoining a new object or arrow; but these latter categories are just obtained from $\mathbb{C}$ via disjoint union, which then (as Garner shows) easily entails that α can only be the identity on $\mathbb{C}$ (and an identical argument works for Grpd). As such, it is perhaps surprising that the category of strict monoidal categories has *non-trivial* isotropy. In fact, and this is the central result of the present paper, the isotropy group of a strict monoidal category is precisely its *Picard group* (its group of $\otimes$-invertible objects).

Since the theory of strict monoidal categories is not a purely equational theory, we cannot directly use results from [6]. Instead, we need to work in the setting of *quasi-equational theories*. These are multi-sorted theories in which the operations can be *partial*; equivalently, they are finite-limit theories. These include the theories of categories, groupoids, strict monoidal categories, symmetric/braided/balanced monoidal categories, and crossed modules. They also include what one might call *functor theories*, which are theories describing functors from a small category into a category of models. As a special case, one obtains theories whose categories of models are presheaf categories.[4] Our first main contribution of the paper is then a generalization of the syntactic characterization of isotropy from equational theories to this wider class of quasi-equational theories.

While we have indicated why the non-trivial isotropy of strict monoidal categories is perhaps surprising, there is also a sense in which it is to be expected. Indeed, since strict monoidal categories are monoids internal to Cat, we expect that the isotropy of strict monoidal

[4] Not to be confused with the so-called *theories of presheaf type*, which are theories whose classifying topos happens to be a presheaf topos.

categories is closely related to that of monoids. Since the isotropy of a monoid M is its subgroup of invertible elements, the conjecture that the isotropy of a strict monoidal category is its group of invertible objects is not unreasonable. However, it is not at all immediate that the isotropy of a strict monoidal category should be determined *completely* by its set of objects; the recognition that this *is* the case is the second main contribution of this paper.

A priori, one can try to establish this result in a variety of ways. First of all, it can be approached purely syntactically, by making careful analysis of the word problem for strict monoidal categories. However, several aspects of this analysis can also be cast in more conceptual terms, giving rise to a categorical way of deriving the isotropy of strict monoidal categories from that of monoids. We thus also include a more categorical viewpoint, which applies to several other theories of categorical structures, including crossed modules.

2 Quasi-equational theories

We begin by reviewing the relevant notions from categorical logic. For more details concerning quasi-equational theories and partial Horn logic, we refer to [8]. For a general treatment of categorical logic, see [11].

▶ **Definition 1** (Signatures, Terms, Horn Formulas, Horn Sequents, Quasi-Equational Theories)**.**

- A *signature* Σ is a pair of sets $\Sigma = (\Sigma_{\mathsf{Sort}}, \Sigma_{\mathsf{Fun}})$, where Σ_{Sort} is the set of *sorts* of Σ and Σ_{Fun} is the set of *function/operation symbols* of Σ. Each element $f \in \Sigma_{\mathsf{Fun}}$ comes equipped with a finite tuple of sorts $(A_1, \ldots, A_n, A)$, and we write $f : A_1 \times \ldots \times A_n \to A$.
- Given a signature Σ, we assume that we have a countably infinite set of variables of each sort A. Then one can recursively define the set $\mathsf{Term}(\Sigma)$ of *terms* of Σ in the usual way, so that each term will have a uniquely defined sort. We write $\mathsf{Term}^c(\Sigma)$ for the set of *closed* terms of Σ, i.e. terms containing no variables.
- Given a signature Σ, one can recursively define the set $\mathsf{Horn}(\Sigma)$ of *Horn formulas* of Σ in the usual way, where a Horn formula is a finite conjunction of equations between elements of $\mathsf{Term}(\Sigma)$. We write $\top$ for the empty conjunction.
- A *Horn sequent* over a signature Σ is an expression of the form $\varphi \vdash^{\vec{x}} \psi$, where $\varphi, \psi \in \mathsf{Horn}(\Sigma)$ and have variables among $\vec{x}$.
- A *quasi-equational theory* $\mathbb{T}$ over a signature Σ is a set of Horn sequents over Σ, which we call the *axioms* of $\mathbb{T}$.

One can set up a deduction system of *partial Horn logic* (PHL) for quasi-equational theories, axiomatizing the notion of a *provable sequent* $\varphi \vdash^{\vec{x}} \psi$. Accordingly, for a theory $\mathbb{T}$ we have the notion of a $\mathbb{T}$-provable sequent; moreover, if $\top \vdash^{\vec{x}} \varphi$ is $\mathbb{T}$-provable, then we simply say that $\mathbb{T}$ proves φ, and write $\mathbb{T} \vdash^{\vec{x}} \varphi$.

We refer the reader to [8, Definition 1] for the logical axioms and inference rules of PHL. The distinguishing feature of this deduction system is that equality of terms is *not* assumed to be reflexive, i.e. if $t(\vec{x})$ is a term over a given signature, then $\top \vdash^{\vec{x}} t(\vec{x}) = t(\vec{x})$ is *not* a logical axiom of partial Horn logic, unless t is a variable. In other words, if we abbreviate the equation $t = t$ by $t\downarrow$ (read: *t is defined*), then unless t is a variable, the sequent $\top \vdash^{\vec{x}} t\downarrow$ is *not* a logical axiom of PHL. Furthermore, the logical inference rule of term substitution is then only formulated for *defined* terms.

▶ **Example 2.** We have the following examples of quasi-equational theories:

- Every single-sorted algebraic theory is a quasi-equational theory; this includes the usual algebraic theories of (commutative) monoids, (abelian) groups, (commutative) unital rings, etc.
- The theories of (small) categories, groupoids, categories with a (chosen) terminal object, categories with (chosen) finite products, categories with (chosen) finite limits, locally cartesian closed categories, and elementary toposes, can all be axiomatized as quasi-equational theories over a two-sorted signature (with one sort O for objects and one sort A for arrows). For details see [8, Example 4 and Section 6]. The theory of (small) strict monoidal categories can also be axiomatized as a quasi-equational theory (see Section 4 below).
- If $\mathbb{T}$ is any quasi-equational theory and $\mathcal{J}$ is any small category, then one can axiomatize the functor category $\mathbb{T}\mathsf{mod}^{\mathcal{J}}$ by a quasi-equational theory $\mathbb{T}^{\mathcal{J}}$; see [9, Chapter 5].

In the remainder of the paper, by *theory* we shall mean *quasi-equational theory*, unless explicitly stated otherwise.

We now review the set-theoretic semantics of PHL. This follows the standard pattern of algebraic theories, with the key difference being that function symbols are now only interpreted as *partial* functions. We write $f : A \rightharpoonup B$ for a partial function from A to B, which is by definition a *total* function $f : \mathsf{dom}(f) \to B$ for some subset $\mathsf{dom}(f) \subseteq A$. If Σ is a signature, then a Σ-*structure* M is a family of sets M_C indexed by the sorts C of Σ, together with interpretations of the function symbols $f : A_1 \times \cdots \times A_k \to A$ as partial functions $f^M : M_{A_1} \times \cdots \times M_{A_k} \rightharpoonup M_A$. By induction on the structure of a term t in variable context $x_1 : A_1, \ldots, x_k : A_k$, we obtain its interpretation as a partial function $t^M : M_{A_1} \times \cdots \times M_{A_k} \rightharpoonup M_A$ in a Σ-structure M, while a Horn formula $\varphi(x_1, \ldots, x_k)$ is interpreted as a subset $\varphi(x_1, \ldots, x_k)^M \subseteq M_{A_1} \times \ldots \times M_{A_k}$.

A Σ-structure M *satisfies* a Horn sequent $\varphi \vdash^{\vec{x}} \psi$ if $\varphi(x_1, \ldots, x_k)^M \subseteq \psi(x_1, \ldots, x_k)^M$. When $\mathbb{T}$ is a theory, then a Σ-structure M is a $\mathbb{T}$-*model* when it satisfies all the $\mathbb{T}$-axioms, and hence all the $\mathbb{T}$-provable sequents (by soundness of partial Horn logic).

▶ **Definition 3.** Let Σ be a signature and M, N Σ-structures. A *homomorphism* $h : M \to N$ is a family of total functions $h = (h_A : M_A \to N_A)_{A:\mathsf{Sort}}$ with the property that if $f : A_1 \times \ldots \times A_n \to A$ is any function symbol of Σ and $(a_1, \ldots, a_n) \in \mathsf{dom}\left(f^M\right)$, then $(h_{A_1}(a_1), \ldots, h_{A_n}(a_n)) \in \mathsf{dom}\left(f^N\right)$ and $h_A\left(f^M(a_1, \ldots, a_n)\right) = f^N(h_{A_1}(a_1), \ldots, h_{A_n}(a_n))$. The homomorphism h *reflects definedness* if moreover $(h_{A_1}(a_1), \ldots, h_{A_n}(a_n)) \in \mathsf{dom}\left(f^N\right)$ always implies $(a_1, \ldots, a_n) \in \mathsf{dom}\left(f^M\right)$.

Let us emphasize that the sort components $h_A : M_A \to N_A$ of a homomorphism $h : M \to N$ are *total* functions, rather than partial functions. One could theoretically choose to work with other notions of homomorphism, but for our purposes we have chosen to use the total homomorphisms. When working with homomorphisms we often suppress the sort subscripts. The $\mathbb{T}$-models and their homomorphisms then form a category $\mathbb{T}\mathsf{mod}$, which is complete and cocomplete.

▶ **Definition 4.** A *morphism of theories* $\rho : \mathbb{T} \to \mathbb{S}$ consists of a mapping $A \mapsto \rho(A)$ from the sorts of $\mathbb{T}$ to the sorts of $\mathbb{S}$ and a mapping $f \mapsto \rho(f)$ from the function symbols of $\mathbb{T}$ to the terms of $\mathbb{S}$ that preserves both typing and provability.

When $\rho : \mathbb{T} \to \mathbb{S}$ is a morphism of theories, we have an induced functor $\rho^* : \mathbb{S}\mathsf{mod} \to \mathbb{T}\mathsf{mod}$ by [8, Proposition 28]. This functor ρ^* sends an $\mathbb{S}$-model M to the $\mathbb{T}$-model $\rho^* M$ with $(\rho^* M)_A := M_{\rho(A)}$ for each sort A of $\mathbb{T}$ and $f^{\rho^* M} := \rho(f)^M$ for each function symbol f of $\mathbb{T}$.

In particular, for every sort A of $\mathbb{T}$ there is a forgetful functor $U_A : \mathbb{T}\mathsf{mod} \to \mathsf{Set}$ sending a model M to the carrier set M_A (induced by the theory morphism from the single-sorted empty theory to $\mathbb{T}$ that sends the unique sort of the former theory to the sort A). Each such functor also has a left adjoint F_A (see e.g. [8, Theorem 29]), giving for a set X the free $\mathbb{T}$-model $F_A(X)$ generated by X: $F_A \dashv U_A : \mathsf{Set} \rightleftarrows \mathbb{T}\mathsf{mod}$.

▶ **Definition 5.** For a $\mathbb{T}$-model M, we can form the extension $\mathbb{T}(M)$, the *diagram theory of* M, adapted from ordinary model theory [13]. It is the extension of $\mathbb{T}$ by

- A constant $\overline{a} : A$ and an axiom $\top \vdash \overline{a} \downarrow$ for every element $a \in M_A$ (for every sort A).
- An axiom $\top \vdash \overline{f(a_1, \ldots, a_k)} = f(\overline{a_1}, \ldots, \overline{a_k})$ for every function symbol $f : A_1 \times \cdots \times A_k \to A$ and tuple $(a_1, \ldots, a_k) \in \mathsf{dom}\left(f^M\right)$.

For better readability, we will generally omit the bar notation on constants of M. Clearly M is a model of $\mathbb{T}(M)$, and in fact it is the *initial* model: $\mathbb{T}(M)\mathsf{mod} \simeq M/\mathbb{T}\mathsf{mod}$ (see [9, Lemma 2.2.4] for a proof). The obvious theory morphism $\mathbb{T} \to \mathbb{T}(M)$ corresponds to the forgetful functor $M/\mathbb{T}\mathsf{mod} \to \mathbb{T}\mathsf{mod}$.

One of the central constructions in the present paper is that of *adjoining an indeterminate* to a model. Given a $\mathbb{T}$-model M and a sort A of $\mathbb{T}$, we form a new model $M\langle \mathsf{x}_A \rangle$ which is the result of freely adjoining a new element x_A of sort A to M. Formally, one can define $M\langle \mathsf{x}_A \rangle$ as $M + F_A(1)$, where $F_A(1)$ is the free $\mathbb{T}$-model on one generator of sort A. Consequently, homomorphisms $M\langle \mathsf{x}_A \rangle \to N$ are in natural bijective correspondence with pairs (h, n) consisting of a homomorphism $h : M \to N$ and an element $n \in N_A$. We will write $\mathbb{T}(M, \mathsf{x}_A)$ for the theory extending the diagram theory $\mathbb{T}(M)$ by a new constant $\mathsf{x}_A : A$ and a new axiom $\top \vdash \mathsf{x}_A \downarrow$. One can then equivalently define the $\mathbb{T}$-model $M\langle \mathsf{x}_A \rangle$ as the initial model of $\mathbb{T}(M, \mathsf{x}_A)$. For a sequence of (not necessarily distinct) sorts $A_1, \ldots, A_k$, we will also write $\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_k)$ for the theory extending $\mathbb{T}(M)$ by new, pairwise distinct constants $\mathsf{x}_i : A_i$ and axioms $\top \vdash \mathsf{x}_i \downarrow$ for each $1 \leq i \leq k$.

Finally, we note that for a $\mathbb{T}$-model M, an indeterminate x_A of sort A, and an arbitrary sort B, we have

$$M\langle \mathsf{x}_A \rangle_B = \{t \in \mathsf{Term}^c\left(\mathbb{T}(M), \mathsf{x}_A\right) \mid t : B \text{ and } \mathbb{T}(M, \mathsf{x}_A) \vdash t \downarrow\} /{=}, \tag{3}$$

i.e. the carrier set $M\langle \mathsf{x}_A \rangle_B$ of the $\mathbb{T}$-model $M\langle \mathsf{x}_A \rangle$ at the sort B is the quotient of the set of provably defined closed terms of sort B, possibly containing x_A and constants from M, modulo the partial congruence relation of $\mathbb{T}(M, \mathsf{x}_A)$-provable equality. For more details, see [9, Remark 2.2.7].

3 Isotropy

We now embark on the syntactic description of the covariant isotropy group of a theory. First, let us briefly review the simpler situation of a single-sorted equational theory $\mathbb{T}$. That is, we describe the isotropy group of a $\mathbb{T}$-model M (details are in [6]). The elements of the model $M\langle \mathsf{x} \rangle$ (for x an indeterminate) can be described explicitly as congruence classes of terms $t(\mathsf{x})$, built from the indeterminate x, constants from M, and the operation symbols of $\mathbb{T}$. Two such terms are congruent if they are $\mathbb{T}(M, \mathsf{x})$-provably equal. For example, if $\mathbb{T}$ is the theory of monoids and M is a monoid with $m_1, m_2, m_3 \in M$, unit e, and $m_1 m_2 = m_3$, then the terms $t = \mathsf{x} m_1 \mathsf{x} m_1 m_2 \mathsf{x}$ and $\mathsf{x} e m_1 \mathsf{x} e m_3 \mathsf{x}$ are congruent.

For a set-theoretic $\mathbb{T}$-model M, each congruence class $[t] \in M\langle \mathsf{x} \rangle$ can be interpreted as a function $t^M : M \to M$, via substitution into the indeterminate x. We thus have a mapping

$$M\langle \mathsf{x} \rangle \to [M, M]\,; \qquad [t] \mapsto t^M$$

where $[M, M]$ is the set of functions from M to itself (well-definedness follows from soundness of the set-theoretic semantics of equational logic). Moreover, this mapping is a homomorphism of monoids, where the monoid structure on $M\langle \mathsf{x} \rangle$ is given by substitution: $[t] \cdot [s] := [t[s/\mathsf{x}]]$, the unit being $[\mathsf{x}]$. We then restrict on both sides to the invertible elements, obtaining a group homomorphism $\mathsf{Inv}(M\langle \mathsf{x} \rangle) \to \mathsf{Perm}(M)$ from the group of substitutionally invertible (congruence classes of) terms to the permutation group of the set M. However, we do not wish to just consider arbitrary permutations of the set M, but rather *automorphisms* of the $\mathbb{T}$-model M; in fact, we want to consider *inner* automorphisms, i.e. automorphisms that extend naturally along any homomorphism $M \to N$. On the level of terms $[t] \in M\langle \mathsf{x} \rangle$, this is achieved by the following definition: $[t]$ is said to *commute generically with* a function symbol $f : A^n \to A$ (A being the unique sort of $\mathbb{T}$) if

$$\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n) \vdash \; t[f(\mathsf{x}_1, \ldots, \mathsf{x}_n)/\mathsf{x}] = f(t[\mathsf{x}_1/\mathsf{x}], \ldots, t[\mathsf{x}_n/\mathsf{x}]).$$

We then form the subgroup $\mathsf{DefInn}(M)$ of $\mathsf{Inv}(M\langle \mathsf{x} \rangle)$ on those $[t]$ that commute generically with all function symbols of the theory. This ensures that such a $[t]$ induces an *automorphism* of the $\mathbb{T}$-model M and not merely a permutation of its underlying set, thus yielding a mapping $(-)^M : \mathsf{DefInn}(M) \to \mathsf{Aut}(M)$. However, it turns out that such an automorphism induced by an element of $\mathsf{DefInn}(M)$ is also *inner*. Indeed, given $h : M \to N$, we obtain a homomorphism $h\langle \mathsf{x} \rangle : M\langle \mathsf{x} \rangle \to N\langle \mathsf{x} \rangle$ of the substitution monoids, which restricts to a group homomorphism $\mathsf{DefInn}(M) \to \mathsf{DefInn}(N)$. It can then be shown that the subgroup $\mathsf{DefInn}(M)$ is isomorphic to the covariant isotropy group of M, where $\theta_M : \mathcal{Z}(M) \to \mathsf{Aut}(M)$ is the comparison homomorphism (2):

$$\begin{tikzcd}
 & \mathsf{DefInn}(M) \arrow[r, "\subseteq"'] \arrow[d, "(-)^M"] & \mathsf{Inv}(M\langle \mathsf{x} \rangle) \arrow[d, "(-)^M"] \\
\mathcal{Z}(M) \arrow[ur, dashed, "\cong"] \arrow[r, "\theta_M"'] & \mathsf{Aut}(M) \arrow[r, "\subseteq"'] & \mathsf{Perm}(M)
\end{tikzcd}$$

We now explain how to extend this result to a (multi-sorted) *quasi-equational theory* $\mathbb{T}$. The main technical difficulties in this extension involve accommodating multi-sortedness and the possibility of certain terms not being provably defined. To handle multi-sortedness, we need to consider, for a $\mathbb{T}$-model M, the model $M\langle \mathsf{x}_A \rangle$ obtained by adjoining an indeterminate x_A of sort A for any sort A of $\mathbb{T}$. Since substitution corresponds to composition under the interpretation mapping $t \mapsto t^M$, it follows that $M\langle \mathsf{x}_A \rangle_A$ carries a monoid structure (recall (3) for the definition of this set), defined as before in terms of substitution into the indeterminate x_A. We now write

$$M\langle \bar{\mathsf{x}} \rangle := \prod_{A:\mathsf{Sort}} M\langle \mathsf{x}_A \rangle_A$$

for the sort-indexed product monoid of these substitution monoids. An element of $M\langle \bar{\mathsf{x}} \rangle$ is therefore a sort-indexed family of congruence classes of terms $[s_A]_A$, where $s_A \in \mathsf{Term}^c(\mathbb{T}(M), \mathsf{x}_A)$ is of sort A and $\mathbb{T}(M, \mathsf{x}_A) \vdash s_A \downarrow$. Given such a tuple $[s_A]_A$, its interpretation gives us, at each sort A, a *total* function $s_A^M : M_A \to M_A$ (because s_A is provably defined in $\mathbb{T}(M, \mathsf{x}_A)$), defined via substitution into the indeterminate x_A (cf. [9, Remark 2.2.12]). The central definitions towards characterizing those $[s_A]_A \in M\langle \bar{\mathsf{x}} \rangle$ that induce elements of isotropy for M are then as follows:

▶ **Definition 6.** Let M be a $\mathbb{T}$-model and $[s_C]_C \in M\langle \bar{\mathsf{x}} \rangle$.

- If $f : A_1 \times \ldots \times A_n \to A$ is a function symbol of Σ, then we say that $([s_C])_C$ *commutes generically with* f if the Horn sequent

  $$f(\mathsf{x}_1, \ldots, \mathsf{x}_n) \downarrow \vdash \; s_A[f(\mathsf{x}_1, \ldots, \mathsf{x}_n)/\mathsf{x}_A] = f\left(s_{A_1}[\mathsf{x}_1/\mathsf{x}_{A_1}], \ldots, s_{A_n}[\mathsf{x}_n/\mathsf{x}_{A_n}]\right)$$

  is provable in $\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n)$.

- We say that $([s_C])_C$ is *invertible* if for each sort A there is some $\left[s_A^{-1}\right] \in M\langle \mathsf{x}_A \rangle_A$ with

$$\mathbb{T}(M, \mathsf{x}_A) \vdash s_A\left[s_A^{-1}/\mathsf{x}_A\right] = \mathsf{x}_A = s_A^{-1}[s_A/\mathsf{x}_A].$$

- We say that $([s_C])_C$ *reflects definedness* if for every function symbol $f : A_1 \times \ldots \times A_n \to A$ in Σ with $n \geq 1$, the sequent

$$f\left(s_{A_1}[\mathsf{x}_1/\mathsf{x}_{A_1}], \ldots, s_{A_n}[\mathsf{x}_n/\mathsf{x}_{A_n}]\right) \downarrow \; \vdash f(\mathsf{x}_1, \ldots, \mathsf{x}_n) \downarrow$$

  is provable in $\mathbb{T}(M, \mathsf{x}_1, \ldots, \mathsf{x}_n)$.

The condition that $[s_C]_C$ commutes generically with the function symbols of $\mathbb{T}$ then ensures that $[s_C]_C$ induces not just an endofunction of each carrier set M_C but in fact an endo*morphism* of the $\mathbb{T}$-model M. Invertibility of $[s_C]_C$ then ensures that these endomorphisms are bijective. However, due to the fact that function symbols are interpreted as partial maps, a (sortwise) bijective homomorphism is not in general an isomorphism in $\mathbb{T}\mathsf{mod}$: a bijective homomorphism is an isomorphism precisely when it also reflects definedness (cf. [9, Lemma 2.2.33]). Thus, the third condition ensures that the inverses $\left[s_A^{-1}\right]$ also induce endomorphisms.

Let us write $\mathsf{DefInn}(M)$ again for the subgroup of the product monoid $M\langle \bar{\mathsf{x}} \rangle$ consisting of those elements satisfying the three conditions above. We then have the following characterization, of which detailed proofs can be found in [9, Theorems 2.2.41, 2.2.53]:

▶ **Theorem 7.** *Let $\mathbb{T}$ be a quasi-equational theory. Then for any $M \in \mathbb{T}\mathsf{mod}$ we have*

$$\mathcal{Z}(M) \cong \mathsf{DefInn}(M) = \left\{ [s_C]_C \in M\langle \bar{\mathsf{x}} \rangle \;\middle|\; [s_C]_C \begin{array}{l} \textit{is invertible, commutes generically with} \\ \textit{all operations, and reflects definedness.} \end{array} \right\}.$$

4 Monoidal categories and the Picard group

With this description of the isotropy group of an arbitrary quasi-equational theory, we now turn to the specific example of strict monoidal categories. We can axiomatize these using the following signature Σ (where the first two ingredients comprise the signature for *categories*):

- two sorts O and A (for objects and arrows);
- function symbols $\mathsf{dom}, \mathsf{cod} : A \to O$, $\mathsf{id} : O \to A$, and $\circ : A \times A \to A$;
- function symbols $\otimes_O : O \times O \to O$, $\otimes_A : A \times A \to A$;
- constant symbols $I_O : O$ and $I_A : A$.

Whenever reasonable, we omit the subscripts on $\otimes$ and I. As axioms, we take those for categories and add (omitting the hypothesis $\top$):

- $x \otimes y \downarrow, \qquad I \downarrow,$
- $x \otimes (y \otimes z) = (x \otimes y) \otimes z, \qquad x \otimes I = x = I \otimes x,$
- $\mathsf{dom}(f \otimes g) = \mathsf{dom}(f) \otimes \mathsf{dom}(g), \qquad \mathsf{cod}(f \otimes g) = \mathsf{cod}(f) \otimes \mathsf{cod}(g),$
- $f \circ h \downarrow \wedge\, g \circ k \downarrow \vdash (f \otimes g) \circ (h \otimes k) = (f \circ h) \otimes (g \circ k),$
- $\mathsf{id}(x \otimes y) = \mathsf{id}(x) \otimes \mathsf{id}(y), \qquad \mathsf{id}(I_O) = I_A.$

Note that in this fragment of logic, we need to include axioms forcing the tensor products and unit object to be *total* operations. Because of strict associativity, we may omit brackets when dealing with nested expressions involving tensor products. We shall henceforth denote this theory by $\mathbb{T}$, and write $\mathsf{StrMonCat}$ for its category of models, whose objects are small strict monoidal categories and whose morphisms are strict monoidal functors. Our goal is now to prove the following:

▶ **Theorem 8.** *The covariant isotropy group* $\mathcal{Z} : \mathsf{StrMonCat} \to \mathsf{Grp}$ *is naturally isomorphic to the functor* $\mathsf{Pic} : \mathsf{StrMonCat} \to \mathsf{Grp}$ *that sends a strict monoidal category* $\mathbb{C}$ *to its Picard group* $\mathsf{Pic}(\mathbb{C})$*, i.e. the group of* $\otimes$*-invertible elements in the monoid of objects of* $\mathbb{C}$*.*

Because a strict monoidal category is a monoid object in Cat, we have two functors

$$\mathrm{Ob}, \mathrm{Arr} : \mathsf{Cat}(\mathsf{Mon}) = \mathsf{StrMonCat} \rightrightarrows \mathsf{Mon}.$$

We shall ultimately prove that the diagram

$$\begin{tikzcd} \mathsf{StrMonCat} \arrow[rr, "\mathrm{Ob}"] \arrow[dr, "\mathcal{Z}"'] & & \mathsf{Mon} \arrow[dl, "\mathcal{Z}_{\mathsf{Mon}}"] \\ & \mathsf{Grp} & \end{tikzcd} \tag{4}$$

commutes up to natural isomorphism, showing that the covariant isotropy group of $\mathsf{StrMonCat}$ is completely determined by the covariant isotropy group of Mon. Since we have proved in [6, Example 4.3] that the latter sends a monoid M to its subgroup of invertible elements, Theorem 8 then follows.[5]

4.1 Monoidal categories and indeterminates

In this section we analyse the process of adjoining an indeterminate to a strict monoidal category. Let us first describe explicitly the result of adjoining an indeterminate to a *monoid*.

▶ **Definition 9.** Let M be a monoid, and X a set of symbols disjoint from M.

- A *word* over $M\langle X\rangle$ is formal string of symbols from the alphabet $M \cup X$.
- A word w is in *(expanded) normal form* when it has the form $w \equiv m_0x_0m_1x_1 \cdots x_{n-1}m_n$ for $m_i \in M$ and $x_j \in X$. In other words, w is in expanded normal form if it contains no two consecutive elements of M, and if every occurrence of some $x \in X$ in w is flanked on both sides by an element of M.

We then have (by taking an arbitrary word, multiplying adjacent elements from M and inserting the unit of M wherever necessary):

▶ **Lemma 10.** *When* $M = (M, \cdot, e)$ *is a monoid, every element* w *of the monoid* $M\langle \mathsf{x}\rangle$ *has a canonical representative* $w = m_0\mathsf{x}m_1\mathsf{x}\cdots\mathsf{x}m_n$ *in expanded normal form.*

Moreover, the unit of $M\langle\mathsf{x}\rangle$ *is represented as the word* e *and multiplication is given by* $(m_0\mathsf{x}m_1\mathsf{x}\cdots\mathsf{x}m_j) \cdot (m'_0\mathsf{x}m'_1\mathsf{x}\cdots\mathsf{x}m'_k) = m_0\mathsf{x}m_1\mathsf{x}\cdots\mathsf{x}(m_j \cdot m'_0)\mathsf{x}m'_1\cdots\mathsf{x}m'_k$.

We now turn to the process of adjoining an indeterminate *object* x_O, i.e. an indeterminate of sort O, to a strict monoidal category $\mathbb{C}$. In order to determine the objects of $\mathbb{C}\langle\mathsf{x}_O\rangle$, we note that the functor $\mathrm{Ob} : \mathsf{StrMonCat} \to \mathsf{Mon}$ has both adjoints:

$$\begin{tikzcd} \mathsf{StrMonCat} \arrow[r, "\mathrm{Ob}"] & \mathsf{Mon} \arrow[l, bend right, "\Delta"'] \arrow[l, bend left, "\nabla"] \arrow[l, phantom, shift right=2, "\perp"] \arrow[l, phantom, shift left=2, "\perp"] \end{tikzcd}$$

Here Δ sends a monoid M to the discrete strict monoidal category on M and ∇ sends M to the indiscrete strict monoidal category on M. In fact, if $\mathcal{E}$ is *any* category with finite limits,

[5] For a general functor $F : \mathcal{E} \to \mathcal{F}$ it is *not* the case that $\mathcal{Z}_{\mathcal{E}} \cong \mathcal{Z}_F \circ F$. In fact, in [3] it is explained that in general the relationship between $\mathcal{Z}_{\mathcal{E}}$ and $\mathcal{Z}_{\mathcal{F}} \circ F$ takes the form of a *span*. The commutativity of (4) may thus be expressed by saying that both legs of the span associated with Ob are isomorphisms.

then the forgetful functor $\mathrm{Ob} : \mathsf{Cat}(\mathcal{E}) \to \mathcal{E}$ has both adjoints (the proof is a completely straightforward analogue of the argument for $\mathcal{E} = \mathsf{Set}$). As such, $\mathrm{Ob} : \mathsf{StrMonCat} \to \mathsf{Mon}$ preserves all limits and colimits. Now by definition $\mathbb{C}\langle \mathsf{x}_O \rangle \cong \mathbb{C} + F\mathbf{1}$, where $F\mathbf{1}$ is the free strict monoidal category on a single object; moreover, the latter is easily seen to be isomorphic to $\Delta(F1)$, the discrete strict monoidal category on the free monoid $F1$ on one generator. We thus have

$$\mathrm{Ob}(\mathbb{C}\langle \mathsf{x}_O \rangle) \cong \mathrm{Ob}(\mathbb{C} + F\mathbf{1}) \cong \mathrm{Ob}(\mathbb{C}) + \mathrm{Ob}(F\mathbf{1}) = \mathrm{Ob}(\mathbb{C}) + F1 \cong \mathrm{Ob}(\mathbb{C})\langle \mathsf{x} \rangle.$$

This shows that the object forgetful functor preserves the process of adjoining an indeterminate of sort O.[6]

We now describe the monoid of arrows of $\mathbb{C}\langle \mathsf{x}_O \rangle$. It is not true that $\mathrm{Arr} : \mathsf{StrMonCat} \to \mathsf{Mon}$ preserves arbitrary binary coproducts, but it *does* preserve the specific binary coproduct $\mathbb{C} + F\mathbf{1}$:

► **Lemma 11.** *If* $\mathbb{C} \in \mathsf{StrMonCat}$*, then* $\mathrm{Arr}(\mathbb{C}\langle \mathsf{x}_O \rangle) \cong \mathrm{Arr}(\mathbb{C})\langle \mathsf{x} \rangle$*.*

Proof. We sketch a syntactic proof, noting that the result can also be deduced categorically from the fact that the endofunctor $- + F1 : \mathsf{Mon} \to \mathsf{Mon}$ preserves pullbacks.

An element of $\mathrm{Arr}(\mathbb{C}\langle \mathsf{x}_O \rangle)$ is a congruence class of terms t built up from the operations of $\mathbb{T}$, arrows of $\mathbb{C}$, and the term $\mathsf{id}(\mathsf{x}_O)$. One can show by induction that every such term t is congruent to one of the form $t = f_1 \otimes \mathsf{id}(\mathsf{x}_O) \otimes f_2 \otimes \mathsf{id}(\mathsf{x}_O) \otimes \cdots \otimes \mathsf{id}(\mathsf{x}_O) \otimes f_n$ where each f_i is an arrow of $\mathbb{C}$. Thus, the monoid $\mathrm{Arr}(\mathbb{C}\langle \mathsf{x}_O \rangle)$ is isomorphic, by Lemma 10, to $\mathrm{Arr}(\mathbb{C})\langle \mathsf{x} \rangle$. ◄

In fact, we may describe the relationship between the functor $(-) + F\mathbf{1}$ adjoining an indeterminate object to a strict monoidal category and the functor $(-) + F1$ adjoining an indeterminate element to a monoid as follows.

► **Proposition 12.** *The functor* $(-) + F\mathbf{1} : \mathsf{Cat}(\mathsf{Mon}) \to \mathsf{Cat}(\mathsf{Mon})$ *is naturally isomorphic to* $\mathsf{Cat}(- + F1)$*.*

We thus obtain the following explicit description of the strict monoidal category $\mathbb{C}\langle \mathsf{x}_O \rangle$:

Objects: Words $a_1\mathsf{x}a_2\mathsf{x}\cdots\mathsf{x}a_n$ where each a_i is an object of $\mathbb{C}$.
Morphisms: Words $f_1\mathsf{x}f_2\mathsf{x}\cdots\mathsf{x}f_n$ where each f_i is an arrow of $\mathbb{C}$.
Domain: $\mathsf{dom}(f_1\mathsf{x}\cdots\mathsf{x}f_n) = \mathsf{dom}(f_1)\mathsf{x}\cdots\mathsf{x}\mathsf{dom}(f_n)$.
Codomain: $\mathsf{cod}(f_1\mathsf{x}\cdots\mathsf{x}f_n) = \mathsf{cod}(f_1)\mathsf{x}\cdots\mathsf{x}\mathsf{cod}(f_n)$.
Identities: $\mathsf{id}(a_1\mathsf{x}\cdots\mathsf{x}a_n) = \mathsf{id}(a_1)\mathsf{x}\cdots\mathsf{x}\mathsf{id}(a_n)$.
Composition: $(f_1\mathsf{x}\cdots\mathsf{x}f_n) \circ (g_1\mathsf{x}\cdots\mathsf{x}g_n) = f_1g_1\mathsf{x}\cdots\mathsf{x}f_ng_n$.
Tensors: $(a_1\mathsf{x}\cdots\mathsf{x}a_n) \otimes (b_1\mathsf{x}\cdots\mathsf{x}b_m) = a_1\mathsf{x}\cdots\mathsf{x}(a_n \otimes b_1)\mathsf{x}\cdots\mathsf{x}b_m$.
Tensor units: I_O, I_A (tensor units of $\mathbb{C}$ regarded as one-letter words).

Next, we address the issue of adjoining an indeterminate *arrow* x_A to $\mathbb{C}$. Here we cannot invoke a simple categorical fact about coproducts, because $\mathrm{Arr} : \mathsf{StrMonCat} \to \mathsf{Mon}$ does not preserve coproducts of the relevant kind (which, to be explicit, is coproducts with the free strict monoidal category $F\mathbf{2}$, where $\mathbf{2}$ is the free-living arrow). We are thus forced to carry out a direct *syntactic* analysis of the objects and arrows of $\mathbb{C}\langle \mathsf{x}_A \rangle$. Note that these are generated, under the operations of domain, codomain, identities, composition, and tensor

[6] Note that for a functor $\rho^* : \mathbb{S}\mathsf{mod} \to \mathbb{T}\mathsf{mod}$ induced by a theory morphism $\rho : \mathbb{T} \to \mathbb{S}$ it is *not* in general the case that $\rho^*(M\langle \mathsf{x} \rangle) \cong (\rho^* M)\langle \mathsf{x} \rangle$.

product, from the objects and arrows of $\mathbb{C}$, together with the new arrow x_A. In particular, there will be two new objects $\mathsf{dom}(\mathsf{x}_A)$ and $\mathsf{cod}(\mathsf{x}_A)$, and corresponding identity arrows $\mathsf{id}(\mathsf{dom}(\mathsf{x}_A))$, $\mathsf{id}(\mathsf{cod}(\mathsf{x}_A))$.

▶ **Definition 13.** Let $\mathbb{C} \in \mathsf{StrMonCat}$. A closed term $t \in \mathsf{Term}^c(\mathbb{C}, \mathsf{x}_A)$ of sort O is in *normal form* when it is of the form $t = a_1 \otimes \mathsf{x}_1 \otimes \cdots \otimes \mathsf{x}_{k-1} \otimes a_k$, where each a_i is an object of $\mathbb{C}$ and each $\mathsf{x}_i \in \{\mathsf{dom}(\mathsf{x}_A), \mathsf{cod}(\mathsf{x}_A)\}$. A closed term $t \in \mathsf{Term}^c(\mathbb{C}, \mathsf{x}_A)$ of sort A is in *normal form* when it is of the form $t = f_1 \otimes \mathsf{x}_1 \otimes \cdots \otimes \mathsf{x}_{k-1} \otimes f_k$, where each f_i is an arrow of $\mathbb{C}$ and each $\mathsf{x}_i \in \{\mathsf{x}_A, \mathsf{id}(\mathsf{dom}(\mathsf{x}_A)), \mathsf{id}(\mathsf{cod}(\mathsf{x}_A))\}$.

We may now describe $\mathbb{C}\langle \mathsf{x}_A \rangle$ in terms of normal forms. It is straightforward to prove, by directly verifying the universal property, that the category described below is indeed isomorphic to $\mathbb{C}\langle \mathsf{x}_A \rangle$. Alternatively, one can endow the set $\{t \in \mathsf{Term}^c(\mathbb{C}, \mathsf{x}_A) \mid \mathbb{T}(\mathbb{C}, \mathsf{x}_A) \vdash t \downarrow\}$ with a rewriting system and show that each term has a unique normal form.

Objects: closed terms of sort O in normal form.
Arrows: closed terms of sort A in normal form.
Domain: $\mathsf{dom}(f_1 \otimes \mathsf{x}_1 \otimes \cdots \otimes \mathsf{x}_{k-1} \otimes f_k) = \mathsf{dom}(f_1) \otimes \mathsf{y}_1 \otimes \cdots \otimes \mathsf{y}_{k-1} \otimes \mathsf{dom}(f_k)$ where $\mathsf{y}_i = \mathsf{dom}(\mathsf{x}_A)$ when $\mathsf{x}_i = \mathsf{x}_A$ or $\mathsf{x}_i = \mathsf{id}(\mathsf{dom}(\mathsf{x}_A))$, and $\mathsf{y}_i = \mathsf{cod}(\mathsf{x}_A)$ otherwise.
Codomain: $\mathsf{cod}(f_1 \otimes \mathsf{x}_1 \otimes \cdots \otimes \mathsf{x}_{k-1} \otimes f_k) = \mathsf{cod}(f_1) \otimes \mathsf{y}_1 \otimes \cdots \otimes \mathsf{y}_{k-1} \otimes \mathsf{cod}(f_k)$ where $\mathsf{y}_i = \mathsf{cod}(\mathsf{x}_A)$ when $\mathsf{x}_i = \mathsf{x}_A$ or $\mathsf{x}_i = \mathsf{id}(\mathsf{cod}(\mathsf{x}_A))$, and $\mathsf{y}_i = \mathsf{dom}(\mathsf{x}_A)$ otherwise.
Identities: $\mathsf{id}(a_1 \otimes \mathsf{x}_1 \otimes \cdots \otimes \mathsf{x}_{k-1} \otimes a_k) = \mathsf{id}(a_1) \otimes \mathsf{id}(\mathsf{x}_1) \otimes \cdots \otimes \mathsf{id}(\mathsf{x}_{k-1}) \otimes \mathsf{id}(a_k)$.
Composition: For $t = f_1 \otimes \mathsf{x}_1 \otimes \cdots \otimes \mathsf{x}_{k-1} \otimes f_k$ and $s = g_1 \otimes \mathsf{x}'_1 \otimes \cdots \otimes \mathsf{x}'_{k-1} \otimes g_k$ with $\mathsf{cod}(t) = \mathsf{dom}(s)$, define $s \circ t = (g_1 f_1) \otimes \mathsf{z}_1 \otimes \cdots \otimes \cdots \otimes \mathsf{z}_{k-1} \otimes (g_k f_k)$, where z_i is defined from x_i and x'_i in the evident way.
Tensors: $(a_1 \otimes \mathsf{x}_1 \otimes \cdots \otimes \mathsf{x}_{n-1} \otimes a_n) \otimes (b_1 \otimes \mathsf{y}_1 \otimes \cdots \otimes \mathsf{y}_{m-1} \otimes b_m) = a_1 \otimes \mathsf{x}_1 \otimes \cdots \otimes \mathsf{x}_{n-1} \otimes (a_n \otimes b_1) \otimes \mathsf{y}_1 \otimes \cdots \otimes \mathsf{y}_{m-1} \otimes b_m$.
Tensor units: I_O, I_A (tensor units of $\mathbb{C}$ regarded as one-letter words).

4.2 Isotropy group

We are now in a position to analyse the isotropy group of a strict monoidal category. By the results of the previous section, we know that an element of isotropy of a strict monoidal category $\mathbb{C}$ may be taken to be of the form (s_O, s_A), where s_O and s_A are closed terms in normal form of sort O and A respectively.

The first observation is that elements of isotropy of the monoid $\mathrm{Ob}(\mathbb{C})$ induce elements of isotropy of $\mathbb{C}$ (as we shall see in the next section, this is not specific to strict monoidal categories.) In what follows, we write $\mathcal{Z}(\mathbb{C})$ for the isotropy group of a strict monoidal category $\mathbb{C}$, and $\mathcal{Z}_{\mathsf{Mon}}(M)$ for the isotropy group of a monoid M (which is the group of invertible elements of M by [6, Example 4.3]).

▶ **Lemma 14.** *Let $\mathbb{C} \in \mathsf{StrMonCat}$. When a is an invertible object in the monoid $\mathrm{Ob}(\mathbb{C})$ with inverse b, the pair $(a \otimes \mathsf{x}_O \otimes b, \mathsf{id}(a) \otimes \mathsf{x}_A \otimes \mathsf{id}(b))$ is an element of $\mathcal{Z}(\mathbb{C})$.*

Proof. To show that $(a \otimes \mathsf{x}_O \otimes b, \mathsf{id}(a) \otimes \mathsf{x}_A \otimes \mathsf{id}(b))$ is an element of isotropy, one can straightforwardly verify that it is invertible, commutes generically with all operations of $\mathbb{T}$, and reflects definedness (for details, see [9, Proposition 3.9.35]). However, it is less work to show directly that given a strict monoidal functor $F : \mathbb{C} \to \mathbb{D}$, we obtain an automorphism α_F of $\mathbb{D}$ as follows. On objects we set $\alpha_F(d) = Fa \otimes d \otimes Fb$, while on arrows we set $\alpha_F(f) = \mathsf{id}(Fa) \otimes f \otimes \mathsf{id}(Fb)$. It is routine to check that this defines an automorphism and that the family α_F is natural in F. ◀

The above lemma gives us a mapping $\theta_{\mathbb{C}} : \mathcal{Z}_{\mathsf{Mon}}(\mathrm{Ob}(\mathbb{C})) \to \mathcal{Z}(\mathbb{C})$. It is easily verified that this is in fact a group homomorphism, natural in $\mathbb{C}$.

Next, we define a retraction σ of θ. This is done categorically using the right adjoint ∇ to Ob. Concretely, given an element of isotropy $\alpha \in \mathcal{Z}(\mathbb{C})$, we define an element $\sigma_{\mathbb{C}}(\alpha) \in \mathcal{Z}_{\mathsf{Mon}}(\mathrm{Ob}(\mathbb{C}))$ as follows: consider a monoid homomorphism $h : \mathrm{Ob}(\mathbb{C}) \to N$. This corresponds by the adjunction $\mathrm{Ob} \dashv \nabla$ to a strict monoidal functor $\tilde{h} : \mathbb{C} \to \nabla(N)$; the component of α at $\tilde{h}$ is an automorphism of $\nabla(N)$, whence $\mathrm{Ob}\,(\alpha_{\tilde{h}})$ is an automorphism of N (using the fact that $\mathrm{Ob} \circ \nabla = 1$). This leads to:

▶ **Lemma 15.** *If $\mathbb{C} \in \mathsf{StrMonCat}$, the map $\sigma_{\mathbb{C}} : \mathcal{Z}(\mathbb{C}) \to \mathcal{Z}_{\mathsf{Mon}}(\mathrm{Ob}(\mathbb{C}))$ is a group homomorphism.*

Interpreting this syntactically, we find that if $(s_O, s_A) \in \mathcal{Z}(\mathbb{C})$, then $s_O \in \mathcal{Z}_{\mathsf{Mon}}(\mathrm{Ob}(\mathbb{C}))$, and hence $s_O = a \otimes \mathsf{x}_O \otimes b$ for an invertible object a with inverse b. We also see that $\sigma_{\mathbb{C}}$ is a retraction of $\theta_{\mathbb{C}}$, i.e. that $\sigma_{\mathbb{C}} \circ \theta_{\mathbb{C}} = 1$.

Since $\theta_{\mathbb{C}}$ is a section, it now remains to show that $\theta_{\mathbb{C}}$ is an epimorphism of groups, i.e. is surjective. So we must show for any element of isotropy $(s_O, s_A) = (a \otimes \mathsf{x}_O \otimes b, s_A) \in \mathcal{Z}(\mathbb{C})$ (with invertible object a and inverse b) that we have $s_A = \mathsf{id}(a) \otimes \mathsf{x}_A \otimes \mathsf{id}(b)$. To this end, we first note that since (s_O, s_A) commutes generically with the operations dom and cod we get

$$a \otimes \mathsf{dom}(\mathsf{x}_A) \otimes b = s_O[\mathsf{dom}(\mathsf{x}_A)/\mathsf{x}_O] = \mathsf{dom}(s_A)$$

and likewise

$$a \otimes \mathsf{cod}(\mathsf{x}_A) \otimes b = s_O[\mathsf{cod}(\mathsf{x}_A)/\mathsf{x}_O] = \mathsf{cod}(s_A).$$

Thus, by uniqueness of normal forms, s_A must have the form $f \otimes \mathsf{x}_A \otimes g$ for some morphisms $f : a \to a$ and $g : b \to b$ of $\mathbb{C}$. So we must now show that $f = \mathsf{id}(a)$ and $g = \mathsf{id}(b)$, and for that we use the fact that (s_O, s_A) commutes generically with id, giving

$$f \otimes \mathsf{id}(\mathsf{x}_O) \otimes g = s_A[\mathsf{id}(\mathsf{x}_O)/\mathsf{x}_A] = \mathsf{id}(s_O) = \mathsf{id}(a \otimes \mathsf{x}_O \otimes b) = \mathsf{id}(a) \otimes \mathsf{id}(\mathsf{x}_O) \otimes \mathsf{id}(b).$$

We now get the desired equalities $f = \mathsf{id}(a)$ and $g = \mathsf{id}(b)$ by appealing to the uniqueness of normal forms. This concludes the proof of Theorem 8.

5 Further examples and applications

In this section we briefly explore some further theories of interest, and indicate the extent to which the analysis of the case of strict monoidal categories can be generalized.

5.1 Internal categories

The analysis of strict monoidal categories reveals that it is profitable, at least for the purposes of understanding isotropy, to regard strict monoidal categories as internal categories in the category Mon of monoids. This naturally raises the following question: are there other algebraic theories $\mathbb{T}$ for which the forgetful functor $\mathrm{Ob} : \mathsf{Cat}(\mathbb{T}\mathsf{mod}) \to \mathbb{T}\mathsf{mod}$ induces an isomorphism on the level of isotropy groups?

Let us first state which of the ideas from the case of monoids carry over to a general algebraic theory $\mathbb{T}$. First of all, we still have a string of adjunctions

$$\begin{tikzcd} \mathsf{Cat}(\mathbb{T}\mathsf{mod}) \arrow[rr, "\mathrm{Ob}"] & & \mathbb{T}\mathsf{mod} \arrow[ll, bend right, "\Delta"', "\perp"] \arrow[ll, bend left, "\nabla", "\perp"'] \end{tikzcd}$$

with $\mathrm{Ob} \circ \nabla \cong 1 \cong \mathrm{Ob} \circ \Delta$, since $\mathbb{T}\mathsf{mod}$ has finite limits. This allows us to deduce the existence of a pair of natural comparison homomorphisms

$$\theta_{\mathbb{C}} : \mathcal{Z}_{\mathbb{T}}(\mathrm{Ob}(\mathbb{C})) \to \mathcal{Z}(\mathbb{C}) \ ; \qquad \sigma_{\mathbb{C}} : \mathcal{Z}(\mathbb{C}) \to \mathcal{Z}_{\mathbb{T}}(\mathrm{Ob}(\mathbb{C}))$$

with $\sigma \circ \theta = 1$ (here $\mathcal{Z}$ denotes the isotropy of $\mathsf{Cat}(\mathbb{T}\mathsf{mod})$ and $\mathcal{Z}_{\mathbb{T}}$ that of $\mathbb{T}\mathsf{mod}$). We thus have:

▶ **Lemma 16.** *Let $\mathbb{T}$ be any algebraic theory and $\mathbb{C}$ any internal category in $\mathbb{T}\mathsf{mod}$. Then $\mathcal{Z}_{\mathbb{T}}(\mathrm{Ob}(\mathbb{C}))$ is a retract of $\mathcal{Z}(\mathbb{C})$, naturally in $\mathbb{C}$.*

In the case of strict monoidal categories, we were able to prove syntactically that the embedding-retraction pair (θ, σ) is an isomorphism. The same proof can also be applied in at least two other cases of interest. Recall that a *crossed module* (A, G, δ, a) consists of a pair of groups A, G, a group homomorphism $\delta : A \to G$, and a group homomorphism $a : G \to \mathsf{Aut}(A)$ from G to the automorphism group of A, making certain diagrams commute. If XMod denotes the category of crossed modules and their morphisms, then it is also true that XMod is equivalent to the category $\mathsf{Cat}(\mathsf{Grp})$ of internal categories in Grp (cf. e.g. [7, XII.8]).

▶ **Proposition 17.** *The isotropy group of a crossed module (A, G, δ, a) is isomorphic to G.*

Proof. When composing the functor $\mathrm{Ob} : \mathsf{Cat}(\mathsf{Grp}) \to \mathsf{Grp}$ with the equivalence $\mathsf{XMod} \xrightarrow{\sim} \mathsf{Cat}(\mathsf{Grp})$, one obtains the forgetful functor which sends a crossed module (A, G, δ, a) to G. Moreover, the isotropy group of a group G is G itself by [6, Example 4.1]. ◀

▶ **Proposition 18.** *The covariant isotropy group $\mathcal{Z} : \mathsf{StrSymMonCat} \to \mathsf{Grp}$ of strict **symmetric** monoidal categories is constant, with value the trivial group.*

Proof. The isotropy group of commutative monoids is trivial by [6, Example 4.4]. ◀

We want to emphasize that the preceding proposition is *not* inconsistent with Theorem 8: while Theorem 8 asserts that the covariant isotropy group of a strict symmetric monoidal category $\mathbb{C}$ *in the category* $\mathsf{StrMonCat}$ is isomorphic to its Picard group (which may be non-trivial), the preceding proposition asserts that the covariant isotropy group of $\mathbb{C}$ *in the full subcategory* $\mathsf{StrSymMonCat}$ is trivial. In other words, if $\mathcal{A}$ is a full subcategory of $\mathcal{B}$, with covariant isotropy groups $\mathcal{Z}_{\mathcal{A}} : \mathcal{A} \to \mathsf{Grp}$ and $\mathcal{Z}_{\mathcal{B}} : \mathcal{B} \to \mathsf{Grp}$, then $\mathcal{Z}_{\mathcal{A}}(A)$ may differ from $\mathcal{Z}_{\mathcal{B}}(A)$ for an object A in the full subcategory $\mathcal{A}$.

5.2 Presheaf categories

Using Theorem 7, we can also compute the covariant isotropy of any presheaf category $\mathsf{Set}^{\mathcal{J}}$ for a small category $\mathcal{J}$. We first axiomatize $\mathsf{Set}^{\mathcal{J}}$ as the category of models of a quasi-equational theory.

▶ **Definition 19 (Presheaf Theory).** Let $\mathcal{J}$ be a small category. We define the signature $\Sigma^{\mathcal{J}}$ to have one sort X_i for each $i \in \mathrm{Ob}(\mathcal{J})$ and one function symbol $\alpha_f : X_i \to X_j$ for each arrow $f : i \to j$ in $\mathcal{J}$.

We define the *presheaf theory* $\mathbb{T}^{\mathcal{J}}$ to be the quasi-equational theory over the signature $\Sigma^{\mathcal{J}}$ with the following axioms:

- $\top \vdash^{x:X_i} \alpha_f(x) \downarrow$ *for any* $f : i \to j$ *in* $\mathcal{J}$ (i.e. each α_f is *total*).
- $\top \vdash^{x:X_i} \alpha_{\mathsf{id}_i}(x) = x$ *for every* $i \in \mathrm{Ob}(\mathcal{J})$ (i.e. each α_{id_i} acts as an identity).
- $\top \vdash^{x:X_i} \alpha_g(\alpha_f(x)) = \alpha_{g\circ f}(x)$ for any composable pair $i \xrightarrow{f} j \xrightarrow{g} k$ in $\mathcal{J}$.

We will lighten notation and write i instead of X_i and f instead of α_f. We write x_i for an indeterminate of sort i. It is completely straightforward to verify that we have an isomorphism of categories $\mathbb{T}^{\mathcal{J}}\mathsf{mod} \cong \mathsf{Set}^{\mathcal{J}}$ (for details, see [9, Proposition 5.1.8]). So to compute the covariant isotropy group $\mathcal{Z}_{\mathsf{Set}^{\mathcal{J}}} : \mathsf{Set}^{\mathcal{J}} \to \mathsf{Grp}$ of the category $\mathsf{Set}^{\mathcal{J}}$, it is equivalent to compute the covariant isotropy group $\mathcal{Z}_{\mathbb{T}^{\mathcal{J}}} : \mathbb{T}^{\mathcal{J}}\mathsf{mod} \to \mathsf{Grp}$ of the theory $\mathbb{T}^{\mathcal{J}}$.

According to Theorem 7, we have for a $\mathbb{T}^{\mathcal{J}}$-model (i.e. a functor) $F : \mathcal{J} \to \mathsf{Set}$ that

$$\mathcal{Z}(F) \cong \left\{ [s_i]_i \in \prod_{i\in\mathcal{J}} F\langle \mathsf{x}_i\rangle_i \mid [s_i]_i \text{ is invertible and commutes gen. with all } f : i \to j \right\}.$$

Note that since all terms are provably defined in $\mathbb{T}^{\mathcal{J}}$, we can omit the condition that $[s_i]_i$ reflects definedness. We now require the following preparatory lemma.

▶ **Lemma 20.** *Let* $M \in \mathbb{T}^{\mathcal{J}}\mathsf{mod}$*. If* $f, f' : i \to j$ *are parallel arrows in* $\mathcal{J}$ *and* $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_i) \vdash f(\mathsf{x}_i) = f'(\mathsf{x}_i)$*, then* $f = f'$*.*

Proof. The assumption $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_i) \vdash f(\mathsf{x}_i) = f'(\mathsf{x}_i)$ implies that for any homomorphism (i.e. natural transformation) $\eta : M \to N$ in $\mathsf{Set}^{\mathcal{J}}$ we have $N(f) = N(f')$, since given any $a \in N_i$ there is a homomorphism $[\eta, a] : M\langle \mathsf{x}_i\rangle \to N$ sending x_i to a (cf. also [9, Lemma 3.1.2]). We now take $N : \mathcal{J} \to \mathsf{Set}$ to be $N := M + \mathcal{J}(i, -)$ and η to be the coproduct inclusion. Then $f = f \circ \mathsf{id}(i) = N(f)(\mathsf{id}(i)) = N(f')(\mathsf{id}(i)) = f' \circ \mathsf{id}(i) = f'$, as required. ◀

As a consequence of this lemma, we find that any term congruence class $[t] \in M\langle \mathsf{x}_i\rangle$ has a *unique* representation as $t \equiv a$ for some $a \in M_j$ or $t \equiv f(\mathsf{x}_i)$ for some f with domain i, depending on whether the indeterminate x_i occurs in t.

Let $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$ be the group of natural automorphisms of the identity functor $\mathsf{Id}_{\mathcal{J}} : \mathcal{J} \to \mathcal{J}$ of a small category $\mathcal{J}$, which is sometimes called the *center* of $\mathcal{J}$. We now have:

▶ **Proposition 21.** *Let* $\mathcal{J}$ *be a small category. For any* $M \in \mathbb{T}^{\mathcal{J}}\mathsf{mod}$ *we have*

$$\mathcal{Z}(M) = \left\{ \left([\psi_i(\mathsf{x}_i)]\right)_i \in \prod_{i\in\mathcal{J}} M\langle \mathsf{x}_i\rangle_i : \psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}}) \right\}.$$

Proof. It is straightforward to prove the right-to-left inclusion using the assumption that ψ is a natural automorphism of $\mathsf{Id}_{\mathcal{J}}$, so let us turn to the less obvious converse inclusion. So suppose that $([s_i])_{i\in\mathcal{J}} \in \mathcal{Z}(M) \subseteq \prod_i M\langle \mathsf{x}_i\rangle_i$. By the lemma, as well as the fact that invertible terms must contain the indeterminate, we may represent $s_i = \psi_i(\mathsf{x}_i)$, where $\psi_i : i \to i$ is a map in $\mathcal{J}$. We show that $\psi := (\psi_i)_{i\in\mathcal{J}}$ is a natural automorphism of $\mathsf{Id}_{\mathcal{J}}$. First, each $\psi_i : i \to i$ is an isomorphism: take the inverse $([t_i])_i$ of $([s_i])_i$, and represent this inverse as $\chi_i(\mathsf{x}_i)$ for $\chi_i : i \to i$. Since $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_i)$ then proves the equations $(\psi_i \circ \chi_i)(\mathsf{x}_i) = \psi_i(\chi_i(\mathsf{x}_i)) = \mathsf{x}_i = \mathsf{id}_i(\mathsf{x}_i)$ and $(\chi_i \circ \psi_i)(\mathsf{x}_i) = \mathsf{id}_i(\mathsf{x}_i)$, it follows by Lemma 20 that ψ_i is the inverse of χ_i.

To show that ψ is natural, let $f : j \to k$ be any arrow in $\mathcal{J}$, and let us show that $\psi_k \circ f = f \circ \psi_j$. We know that $\left([\psi_i(\mathsf{x}_i)]\right)_i = [s_i]_i$ commutes generically with the function symbol $f : X_j \to X_k$ of $\Sigma^{\mathcal{J}}$, which implies that $\mathbb{T}^{\mathcal{J}}(M, \mathsf{x}_j) \vdash (\psi_k \circ f)(\mathsf{x}_j) = (f \circ \psi_j)(\mathsf{x}_j)$, from which we obtain the required $\psi_k \circ f = f \circ \psi_j$ again by Lemma 20. Thus $\psi : \mathsf{Id}_{\mathcal{J}} \xrightarrow{\sim} \mathsf{Id}_{\mathcal{J}}$ is indeed a natural automorphism with $([s_i])_i = \left([\psi_i(\mathsf{x}_i)]\right)_i$. ◀

▶ **Corollary 22.** *Let $\mathcal{J}$ be a small category. For any functor $F : \mathcal{J} \to \mathsf{Set}$ we have $\mathcal{Z}(F) \cong \mathsf{Aut}\,(\mathsf{Id}_{\mathcal{J}})$, and hence the covariant isotropy group functor of $\mathsf{Set}^{\mathcal{J}}$ is constant on the automorphism group of $\mathsf{Id}_{\mathcal{J}}$.*

Proof. Given $([s_i])_{i\in\mathcal{J}} \in \mathcal{Z}(F)$, we know by Proposition 21 that there is some $\psi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$ with $[s_i]_i = [\psi_i(\mathsf{x}_i)]_i$. We now show that this assignment $([s_i])_i \mapsto \psi$ is a well-defined group isomorphism $\mathcal{Z}(F) \xrightarrow{\sim} \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$. It is well-defined, because if there is also some $\chi \in \mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$ with $[s_i]_i = [\psi_i(\mathsf{x}_i)]_i = [\chi_i(\mathsf{x}_i)]_i$, then from Lemma 20 we obtain $\psi = \chi$. It is clearly injective, it is surjective by Proposition 21, and it is readily seen to preserve group multiplication, so that it is indeed a group isomorphism. ◀

We can now use Corollary 22 to characterize the covariant isotropy groups of certain presheaf categories of interest.

▶ **Proposition 23.** *If M is a monoid, then the covariant isotropy group $\mathcal{Z} : \mathsf{Set}^M \to \mathsf{Grp}$ of the category of M-sets and M-equivariant maps is constant on $\mathsf{Inv}(Z(M))$, the subgroup of invertible elements of the center of M. In particular, if G is a group, then the covariant isotropy group $\mathcal{Z} : \mathsf{Set}^G \to \mathsf{Grp}$ is constant on $Z(G)$.*

Proof. The result follows immediately from Corollary 22 and the observation that the automorphism group of the identity functor on the monoid M, regarded as a one-object category, is isomorphic to $\mathsf{Inv}(Z(M))$. ◀

▶ **Proposition 24.** *Let $\mathcal{J}$ be a **rigid** category, i.e. a category whose objects have no non-identity automorphisms (e.g. $\mathcal{J}$ could be a preorder or poset). Then the covariant isotropy group $\mathcal{Z} : \mathsf{Set}^{\mathcal{J}} \to \mathsf{Grp}$ is trivial.*

We point out that Corollary 22 illustrates an important difference between covariant isotropy $\mathsf{Set}^{\mathcal{J}} \to \mathsf{Grp}$ and contravariant isotropy $\left(\mathsf{Set}^{\mathcal{J}}\right)^{\mathrm{op}} \to \mathsf{Grp}$. Indeed, the latter is generally *not* constant, but is a representable functor $F \mapsto \mathsf{Set}^{\mathcal{J}}[F, Z]$ for a suitable presheaf of groups Z, that is, an internal group object in $\mathsf{Set}^{\mathcal{J}}$. The connection between covariant and contravariant isotropy is then as follows: the group of global sections of Z is isomorphic to the group $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$:

$$\Gamma(Z) = \mathsf{Set}^{\mathcal{J}}(1, Z) \cong \mathcal{Z}(F) \text{ for } F : \mathcal{J} \to \mathsf{Set}.$$

6 Conclusions and future work

We have shown how a syntactic description of polymorphic automorphisms can be fruitfully applied to characterize the covariant isotropy of several kinds of structures of relevance in logic, algebra, and computer science. Most notably, we have shown that the covariant isotropy group of a strict monoidal category coincides with its Picard group of $\otimes$-invertible objects. We have also shown that the covariant isotropy group of a presheaf category $\mathsf{Set}^{\mathcal{J}}$ behaves quite differently from the contravariant one, in that it is the *constant* group with value $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$.

There are several open questions and possible lines for further inquiry:

1. The generalization from algebraic to quasi-equational theories presented in this paper is the first step on a path upwards through the various fragments of logic. In particular, we hope to generalize some of the techniques to determine the isotropy groups of some *geometric* theories of interest.

2. We have shown how to determine the covariant isotropy groups of presheaf categories, but we have left open the question of how to determine the isotropy of *sheaf* toposes. In particular, it would be of interest to determine the covariant isotropy of the topos of nominal sets (also known as the Schanuel topos).
3. For a theory $\mathbb{T}$ and small category $\mathcal{J}$, there is a theory $\mathbb{S} = \mathbb{S}(\mathbb{T}, \mathcal{J})$ with $\mathbb{S}\mathsf{mod} \cong \mathbb{T}\mathsf{mod}^{\mathcal{J}}$ (in Section 5.2 we considered the special case where $\mathbb{T}$ is the theory of sets). In [9, Chapter 5] the second author has obtained, under mild assumptions on $\mathbb{T}$, a description of the covariant isotropy group of $\mathbb{T}^{\mathcal{J}}\mathsf{mod}$ in terms of $\mathsf{Aut}(\mathsf{Id}_{\mathcal{J}})$ and the isotropy group of $\mathbb{T}$.
4. We have not yet investigated in detail how isotropy behaves with respect to morphisms of theories $\rho : \mathbb{T} \to \mathbb{S}$. We have seen a rather special case in Section 4 with $\mathrm{Ob} : \mathsf{StrMonCat} \to \mathsf{Mon}$, but the general case is more involved.
5. One possible perspective on the theory of strict monoidal categories is that it is a *tensor product* of the theory of categories with that of monoids. This leads to the question of whether, under suitable hypotheses on the theories $\mathbb{T}$ and $\mathbb{S}$, we can describe the isotropy of the tensor product theory $\mathbb{T} \otimes \mathbb{S}$ in terms of that of $\mathbb{T}$ and $\mathbb{S}$.
6. One can define, for a 2-category $\mathcal{E}$ and object $X \in \mathcal{E}$, the 2-group of pseudo-natural auto-equivalences of $X/\mathcal{E} \to \mathcal{E}$. This leads to a 2-dimensional version of isotropy, taking values in 2-groups. It is then possible to show that the 2-isotropy group of a (non-strict) monoidal category (regarded as an object of the 2-category of monoidal categories and strong monoidal functors) is the Picard 2-group. This will be presented in forthcoming work.

References

1 George M. Bergman. An inner automorphism is only an inner automorphism, but an inner endomorphism can be something strange. *Publicacions Matematiques*, 56(1):91–126, 2012.

2 Peter J. Freyd. Core algebra revisited. *Theor. Comput. Sci.*, 375(1-3):193–200, 2007. `doi:10.1016/j.tcs.2006.12.033`.

3 Jonathon Funk, Pieter Hofstra, and Sakif Khan. Higher isotropy. *Theory and Applications of Categories*, 33(20):537–582, 2018.

4 Jonathon Funk, Pieter Hofstra, and Benjamin Steinberg. Isotropy and crossed toposes. *Theory and Applications of Categories*, 26(24):660–709, 2012.

5 Richard Garner. Inner automorphisms of groupoids. Preprint, available at `arXiv:1907.10378`, 2019.

6 Pieter J. W. Hofstra, Jason Parker, and Philip J. Scott. Isotropy of algebraic theories. In Sam Staton, editor, *Proceedings of the Thirty-Fourth Conference on the Mathematical Foundations of Programming Semantics, MFPS 2018, Dalhousie University, Halifax, Canada, June 6-9, 2018*, volume 341 of *Electronic Notes in Theoretical Computer Science*, pages 201–217. Elsevier, 2018. `doi:10.1016/j.entcs.2018.11.010`.

7 Saunders Mac Lane. *Categories for the working mathematician.* Springer, second edition, 1997.

8 Erik Palmgren and Steven J. Vickers. Partial horn logic and cartesian categories. *Ann. Pure Appl. Log.*, 145(3):314–353, 2007. `doi:10.1016/j.apal.2006.10.001`.

9 Jason Parker. *Isotropy groups of quasi-equational theories.* PhD thesis, Université d'Ottawa/University of Ottawa, 2020. URL: `https://ruor.uottawa.ca/handle/10393/41032`.

10 Martin R. Pettet. On inner automorphisms of finite groups. *Proceedings of the American Mathematical Society*, 106(1):87–90, May 1989.

11 Andrew M. Pitts. Categorical logic. In *Handbook of Logic in Computer Science: Volume 5. Algebraic and Logical Structures*, pages 40–128. Oxford University Press, 2000.

12 Paul E. Schupp. A characterization of inner automorphisms. *Proceedings of the American Mathematical Society*, 101(2):226–228, October 1987.

13 Joseph R. Shoenfield. *Mathematical logic*. CRC Press, 2018.