# On Greedily Packing Anchored Rectangles


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## Abstract

Consider a set $P$ of points in the unit square $\mathcal{U} = [1, 0)$, one of them being the origin. For each point $p \in P$ you may draw an axis-aligned rectangle in $\mathcal{U}$ with its lower-left corner being $p$. What is the maximum area such rectangles can cover without overlapping each other?

Freedman [18] posed this problem in 1969, asking whether one can always cover at least 50% of $\mathcal{U}$. Over 40 years later, Dumitrescu and Tóth [12] achieved the first constant coverage of 9.1%; since then, no significant progress was made. While 9.1% might seem low, the authors could not find any instance where their algorithm covers less than 50%, nourishing the hope to eventually prove a 50% bound. While we indeed significantly raise the algorithm's coverage to 39%, we extinguish the hope of reaching 50% by giving points for which its coverage stays below 43.3%.

Our analysis studies the algorithm's average and worst-case density of so-called tiles, which represent the staircase polygons in which a point can freely choose its maximum-area rectangle. Our approach is comparatively general and may potentially help in analyzing related algorithms.



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## 1 Introduction

The Lower-Left Anchored Rectangle Packing (LLARP) problem considers a finite set $P \subseteq \mathcal{U} := [0, 1)^2$ of *input points* with $(0, 0) \in P$. The goal is to find a set of non-empty, axis-aligned interior-disjoint rectangles $(r_p)_{p \in P}$ with $p$ being the lower-left corner of $r_p \subseteq \mathcal{U}$ and such that their total area $\sum_{p \in P} |r_p|$ is maximized.

This problem was first introduced by Freedman [18, Unsolved Problem 11, page 345] in 1969. He asked the question whether, for any point set $P$, the rectangles can always be chosen such that they cover at least 50% of $\mathcal{U}$. It is easy to see that this is the best one can hope for, since putting $n$ equally spaced points along the ascending diagonal of $\mathcal{U}$ yields a maximum coverable area of $1/2 + \mathrm{o}(1)$ for $n \to \infty$.

Over the years, the LLARP problem reoccurred in the form of geometric challenges [14] and in miscellaneous books and journals about mathematical puzzles [19, 20, 21]. Still, it took more than 40 years until the first constant lower bound was established: Dumitrescu and Tóth [12] considered a natural greedy algorithm, called GreedyPacking, and proved that it achieves a coverage of 9.1%.

This caused a surge of interest in this old problem, resulting in numerous findings for variants or special cases of the problem (see Section 1.1). Since then, no further significant progress was made towards the original question[1], and even the question whether a maximum area covering can be found in polynomial time remains elusive.

While [12] themselves observed that "*a sizable gap to the conjectured* 50% *remains*", they were unable to find instances where their algorithm *does not* reach 50%. This led them and others to conjecture a much better quality of their algorithm, making it a natural candidate to answer Freedman's question positively, albeit [12] also mentioned that "*obtaining substantial improvements probably requires new ideas*".

Our results indeed attest the greedy algorithm a much better coverage of 39%. However, at the same time, we show that there are instances where the coverage stays below 43.3%.

## 1.1 Related Work

LLARP falls into the class of geometric packing problems, where a typical question is how much of a container can be covered using a set of geometric shapes in two or more dimensions. We concentrate on two-dimensional packing problems with rectangular containers and shapes.

**Complexity of LLARP.** The GreedyPacking algorithm by Dumitrescu and Tóth [12] considers the input points step by step from top-right to bottom-left, always choosing the maximum-area rectangle. They showed that this achieves the same worst-case coverage as an algorithm called TilePacking. The latter partitions the unit square into staircase-shaped *tiles*, one per input point, and chooses a maximal rectangle within each tile (see Section 2 for the formal algorithm description).

While the complexity of LLARP remains unknown, [12] also showed that there is an order of the input points for which the greedy algorithm achieves an optimal packing (albeit of unknown value); how to find that ordering remains unclear. [7] studied the combinatorial structure of optimal solutions, proving that the worst-case number of maximal rectangle packings is exponential in the number of input points.

**LLARP Variants.** After [12], a series of papers studied special cases and variants of LLARP. [6] allowed rectangles to be anchored in any of the four corners and showed that here the worst-case coverage lies in $[7/12, 2/3]$ and in $[5/32, 7/27]$ if the rectangles are restricted to squares. [4] showed that the union of all (possibly overlapping) squares covers at least $1/2$ and proved that finding a maximum corner-anchored square packing is NP-hard. Interestingly, there is only one other LLARP-variant known to be NP-hard, namely if the rectangle's anchors lie in their center [5]. Other results consider specific classes of input points, like points with certain ascending/descending structures [8] or points that lie on the unit square's boundary (for corner-anchored rectangles) [9].

**Further Related Problems and Applications.** Further related problems include the maximum weight independent set of rectangles problem [2, 10, 1] (which was used, e.g., in [5] to derive a PTAS for center-anchored rectangle packings) or geometric knapsack [3, 17] and strip packing problems [15]. In contrast to LLARP and its variants, the size of the objects to be packed is typically part of the input and object placement is less constrained.

[1] A very recent, still unpublished result slightly raised the greedy algorithm's coverage to 10.39% [13].

Note that LLARP-like problems are not of pure theoretical interest, but have applications in, e.g., map labeling. Here, rectangular text labels must be placed under certain constraints (e.g., labels might be scalable but require a fixed ratio and must be placed at a specific anchor) within a given container. We refer to the relatively recent survey [16] for details.

### 1.2 Our Contribution and Techniques

We analyze the greedy algorithm TilePacking from [12] (formally described in Section 2). From a high-level view, TilePacking partitions the unit square into staircase-shaped *tiles*, each anchored at an input point, and chooses an area-maximal rectangle in each tile. A natural way to analyze such an algorithm is to consider the tiles' densities (the ratio between their area-maximal rectangles and their own area) and prove a lower bound on the *average* tile density (which immediately yields the covering guarantee).

Dumitrescu and Tóth [12] follow this approach by defining suitable charging areas $C_t$ for each tile $t$ (trapezoids below/beneath the tile). We also use such a charging scheme, but rely on a much more complex charging area which we refer to as a tile's *crown*. But instead of directly analyzing a tile's charging area, we first extract the critical properties that determine the charging scheme's quality. This general approach (described in Section 3) requires a bound $\xi$ on the tile's *charging ratio* $|C_t|/|t|$ together with some simple properties (basically a form of local convexity characterizing the average tile density).

We derive such a charging ratio bound $\xi_s$ and describe simple, symmetric tiles for which it is tight (Figures 8 and 9). We then take an arbitrary tile and show how to gradually transform it into one of these tiles without increasing its charging ratio. This establishes that $\xi_s$ is indeed a charging ratio bound and allows us to conclude the following theorem.

▶ **Theorem 1.** *Given any set of input points, TilePacking covers at least* $39\%$ *of* $\mathcal{U}$*.*

While the aforementioned transformations to worst-case tiles require some care, we showcase the versatility of our approach by first proving a slightly weaker bound of only 25% (Section 4.2). Its analysis is not only much simpler but, in fact, takes us halfway to Theorem 1, as the used charging ratio bound $\xi_w$ (Proposition 11) is tight for high-density tiles and all that remains is to refine our charging ratio bound for low-density tiles (Proposition 15).

Our second major result constructs an input instance (depicted in Figure 14) for which TilePacking covers significantly less than 50% of the unit square.

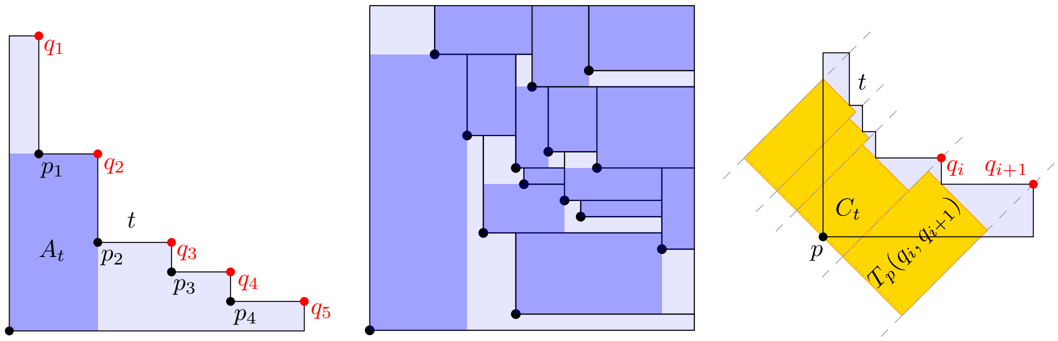
▶ **Theorem 2.** *There is a set* $P$ *of input points for which TilePacking covers at most* $43.3\%$ *of the unit square.*

By the aforementioned worst-case equivalence of TilePacking and GreedyPacking, both Theorems 1 and 2 also hold for the latter.

## 2 Preliminaries and Algorithm Description

Let $\mathcal{U} := [0,1)^2$ denote the unit square. For a point $p \in \mathbb{R}^2$ define $x(p)$ and $y(p)$ as the $x$- and $y$-coordinates of $p$, respectively. For two points $p, p' \in \mathbb{R}^2$ we use the notation $p \preceq p'$ to indicate that $x(p) \leq x(p')$ and $y(p) \leq y(p')$. Similarly, $p \prec p'$ means that $x(p) < x(p')$ and $y(p) < y(p')$. The relations "$\succeq$" and "$\succ$" are defined analogously. For a set $S$ we denote its closure by $\overline{S}$. If $S$ is measurable, we use $|S|$ to denote its area.

To simplify some geometric arguments, we use the following line-notation: We define the line $\ell_q^{\diagup} \subseteq \mathbb{R}^2$ as the line through $q \in \mathbb{R}^2$ of slope $+1$. Similarly, we define the lines $\ell_q^{-}$, $\ell_q^{\diagdown}$, and $\ell_q^{|}$ through $q$ with slope $0$, $-1$, and $\infty$, respectively. For lines of type $R \in \{\diagup, |\}$, we




**(a)** Staircase points $p_i$, $q_j$ and rectangle $A_t$. **(b)** A tile packing produced by TilePacking. **(c)** A tile $t$, its crown $C_t$, and a tower $T_p(q_i, q_{i+1})$.

**Figure 1** Tiles, packings, crowns, and towers. In our figures, tiles are shaded light blue. Upper and lower stair case points are shown in red and black, respectively. A dark blue rectangle represents a (maximal) rectangle of a tile. Crowns are shown in yellow and towers are possibly labeled.

write $\ell_q^R < \ell_{q'}^R$ if $\ell_{q'}^R = \ell_q^R + (x, 0)$ (using element-wise addition) with $x > 0$ and say $\ell_q^R$ *is left of* $\ell_{q'}^R$. Similarly, for lines of type $R \in \{\diagdown, -\}$ we write $\ell_q^R < \ell_{q'}^R$ if $\ell_{q'}^R = \ell_q^R + (0, y)$ with $y > 0$ and say $\ell_q^R$ *is below* $\ell_{q'}^R$. Analogous definitions apply for "$>$", "$\leq$", and "$\geq$".

**Input Sets in General Position.** Remember the problem description from Section 1. We say that the input set $P$ is *in general position* if there are no two (different) points $p, p' \in P$ with $x(p) = x(p')$, $y(p) = y(p')$, or $x(p) + y(p) = x(p') + y(p')$. That is, no two points may share an $x$- or $y$-coordinate and may not lie on the same diagonal of slope $-1$. W.l.o.g., we restrict $P$ to be in general position (see the full version [11] for why this is ok).

**Tiles and Tile Packings.** A *tile* $t$ is a staircase polygon in $\mathcal{U}$ (see Figure 1a). More formally, $t$ is defined using its *anchor* $p \in \mathcal{U}$ and a set of $k$ *upper staircase points* $\Gamma_t := \{q_1, q_2, \dots, q_k\} \subseteq \overline{\mathcal{U}}$ ordered by increasing $x$-coordinate and such that $q_i \succ p$ for all $q_i$ as well as $q_i \not\preceq q_j$ for all $q_i \neq q_j$. With this we define $t = \{q \in \mathcal{U} \mid q \succeq p \wedge \exists q' \in \Gamma_t \colon q \prec q'\}$. A point $p_i = \big(x(q_{i-1}), y(q_i)\big)$ is called a *lower staircase point*. We define $A_t \subseteq t$ as an (arbitrary) area-maximal rectangle in $t$ and $\rho_t := |A_t|/|t|$ as the tile's *density*. For indexed upper staircase points $q_i$ we often use the shorthands $x_i := x(q_i)$ and $y_i := y(q_i)$.

If $p$ and $\Gamma_t$ do not adhere to $q_i \succ p$ and $q_i \not\preceq q_j$, but only to the weaker requirements $q_i \succeq p$, $q_i \not\preceq q_j$ for all $q_i \neq q_j$), then we say that $t$ is *degenerate*. We show in Lemma 13 that we can transform such tiles into non-degenerate ones without affecting our arguments.

The *hyperbola* of $t$ is $h_t := \{(x + x(p), y + y(p)) \in \mathbb{R}^2_{>0} \mid y = |A_t|/x\}$. Note that all upper staircase points lie between $p$ and $h_t$. Moreover, the points from $\Gamma_t \cap h_t$ span all area-maximal rectangles in $t$. If, $p = (0, 0)$ and $|A_t| = 1$, then $t$ is called *normalized*.

A *tile packing* of the unit square is a partition of $\mathcal{U}$ into tiles. In particular, $\sum_{t \in \mathcal{T}} |t| = |\mathcal{U}| = 1$. We use $A(\mathcal{T}) := \sum_{t \in \mathcal{T}} |A_t|$ to denote the area covered by choosing an area-maximal rectangle $A_t$ for each tile $t$ (the *area covered by* $\mathcal{T}$).

**A Greedy Tile Packing Algorithm.** Let us revisit the algorithm TilePacking by Dumitrescu and Tóth [12]. TilePacking processes the points $P$ from top-right to bottom-left. More formally, it orders $P = \{p_1, p_2, \dots, p_n\}$ such that $\ell_{p_i}^{\diagdown} \geq \ell_{p_{i+1}}^{\diagdown}$. It then defines for each $p_i \in P$ the tile $t_i := \{q \in \mathcal{U} \mid q \succeq p_i\} \setminus \bigcup_{j=1}^{i-1} t_j$, yielding a tile packing $\mathcal{T} = \{t_1, t_2, \dots, t_n\}$.

To build its solution to LLARP, TilePacking picks for each $p \in P$ the rectangle $r_p$ as an (arbitrary) area-maximal rectangle $A_t \subseteq t$ in the tile $t$ containing $p$. Thus, the total area covered by TilePacking is $A(\mathcal{T})$. Figure 1b illustrates the resulting tile packing.

Note that, by this construction, the lower staircase points of each tile $t$ are input points. Moreover, as already mentioned in [12], for each tile we can define a certain *exclusive area* that does not contain an input point.

▶ Observation 3. Consider the tile packing $\mathcal{T}$ produced by TilePacking for a set $P$ of input points. Fix a tile $t \in \mathcal{T}$ and let $p \in P$ denote its anchor point. Then the tile's *exclusive area* $E_t \coloneqq \{\, q \in \mathbb{R}^2 \mid \ell_q^{\searrow} > \ell_p^{\searrow} \wedge \exists q' \in \Gamma_t \colon q \prec q' \,\}$ does not contain any point from $P$.

This observation follows by noting that any such input point $p' \in E_t$ would be processed before $p$ by TilePacking and "shield" at least one upper staircase point $q' \in \Gamma_t$ from $p$, preventing it from becoming an upper staircase point of tile $t$.

## 3 A General Approach for Lower Bounds

Here we present a general approach to derive lower bounds for the area covered by a given tile packing $\mathcal{T}$. Our approach relies on a suitable *charging scheme* $(c_t)_{t\in\mathcal{T}}$ that charges the area of each tile $t \in \mathcal{T}$ to a *charging area* $c_t > 0$.

▶ **Definition 4.** For a given charging scheme, we define $c^* \coloneqq \sum_{t\in\mathcal{T}} c_t$ as the total charged area and $c_t/|t|$ as the *charging ratio* of tile $t$. We call a function $\xi\colon (0,1] \to \mathbb{R}_{\geq 0}$ a *charging ratio bound* with *critical density* $\rho^* \in (0,1]$ if

1. $\xi$ is point-convex at $\rho^*$ with $\xi'(\rho^*) < 0$,
2. $\xi(\rho^*) \geq c^*$ and
3. for any $t \in \mathcal{T}$: $\xi(\rho_t) \leq c_t/|t|$.

Note that a function $f\colon I \to \mathbb{R}, I \subseteq \mathbb{R}$ is said to be *point-convex* at $x \in I$ if $f$ is differentiable at $x$ and the tangent $t$ of $f$ at $x$ satisfies $t(x) \leq f(x)$ for all $x \in I$.

The following lemma uses a charging ratio bound to show that $\rho^*$ is a lower bound on $A(\mathcal{T})$.

▶ **Lemma 5.** *Consider a tile packing $\mathcal{T}$ with a charging scheme $(c_t)_{t\in\mathcal{T}}$ together with a charging ratio bound $\xi$ with critical density $\rho^*$. Then $A(\mathcal{T}) \geq \rho^*$.*

**Proof.** Since $\xi$ is point-convex in $\rho^*$, the tangent $\tau(\rho) \coloneqq \xi(\rho^*) + \xi'(\rho^*) \cdot (\rho - \rho^*)$ of $\xi$ in $\rho^*$ satisfies $\tau(\rho) \leq \xi(\rho)$ for all $\rho \in (0,1]$. Using $A(\mathcal{T}) = \sum_{t\in\mathcal{T}}|A_t| = \sum_{t\in\mathcal{T}}|t| \cdot \rho_t$ we calculate

$$\tau\big(A(\mathcal{T})\big) = \tau\left(\sum_{t\in\mathcal{T}}|t|\cdot\rho_t\right) \leq \sum_{t\in\mathcal{T}}|t|\cdot\tau(\rho_t) \leq \sum_{t\in\mathcal{T}}|t|\cdot\xi(\rho_t) \leq \sum_{t\in\mathcal{T}}|t|\cdot\frac{c_t}{|t|} = c^* \leq \xi(\rho^*), \quad (1)$$

where the second inequality follows from applying Jensen's Inequality to the convex function $\tau$. Combining $\tau\big(A(\mathcal{T})\big) = \xi(\rho^*) + \xi'(\rho^*)\cdot\big(A(\mathcal{T}) - \rho^*\big)$ with Inequality 1 and rearranging gives $\xi'(\rho^*)\cdot A(\mathcal{T}) \leq \xi(\rho^*)\cdot\rho^*$, which yields the desired result after dividing by $\xi(\rho^*) < 0$. ◀

## 4 Charging Scheme and Weak Covering Guarantee

This section introduces the charging scheme we will use to derive our lower bounds for TilePacking's coverage (via the approach presented in Section 3). Then we derive a weak charging ratio bound $\xi_w$, as described in Section 3. While comparatively simple, this already yields that TilePacking covers at least a quarter of the unit square, almost tripling the original guarantee from [12]. Section 5 will refine $\xi_w$ to derive our main result (Theorem 1).

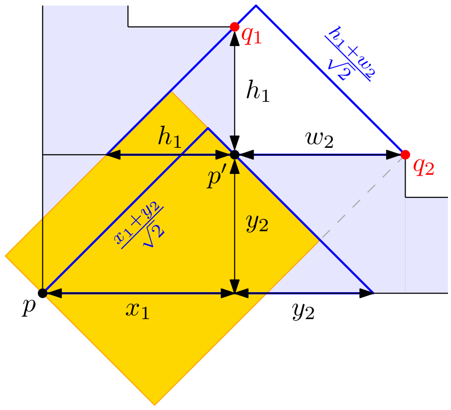



Figure 2 $|T_p(q_1, q_2)|$ is computed via the catheti of the blue triangles.

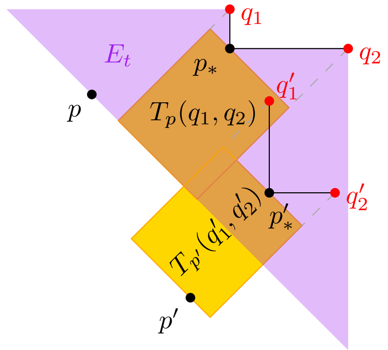



Figure 3 Example for Lemma 8. The shown tower overlap has $p'_* \in E_t$, violating $t$'s exclusive area.

## 4.1 Charging Scheme

Given a tile packing $\mathcal{T}$ constructed by TilePacking, our charging scheme defines an area $C_t$ for each tile $t \in \mathcal{T}$ and charges $t$'s area to $c_t := |C_t|$. We first explain how $C_t$ is constructed from $t$. Afterward, we prove useful properties about these areas and their relation to $\mathcal{T}$.

**Construction of $C_t$.** Consider three points $p, q_1 = (x_1, y_1), q_2 = (x_2, y_2) \in \mathbb{R}^2$ with $q_1, q_2 \succeq p$, $x_1 \leq x_2$, and $y_1 \geq y_2$. The *tower* $T_p(q_1, q_2)$ with *base point* $p$ and *peak* $p_* = (x_1, y_2)$ is the interior of the rectangle enclosed by the lines $\ell_p^{\searrow}$ (the tower's *base*), $\ell_{q_1}^{\nearrow}$ (the tower's *left side*), $\ell_{q_2}^{\nearrow}$ (the tower's *right side*), and $\ell_{p_*}^{\searrow}$ (the tower's *top*). If the subscript $p$ is omitted, the base point is assumed to be the origin $(0, 0)$.

For a tile $t$ with anchor $p$ and $\Gamma_t = \{q_1, q_2, \ldots, q_k\}$ being ordered by increasing $x$-coordinate, we define the charging area as the disjoint union of towers, i.e., $C_t := \bigcup_{i=1}^{k-1} T_p(q_i, q_{i+1})$. We refer to $C_t$ as the *crown* of tile $t$. (See Figure 1c.)

The width and height of a tower $T_p(q_1, q_2)$ correspond to the side lengths of isosceles triangles (see Figure 2), which yields a formula for $|T_p(q_1, q_2)|$. By taking derivatives, we get formulas for the change of the tower's area when moving $q_1$ or $q_2$ horizontally or vertically.

▶ Observation 6. Consider $T_p(q_1, q_2)$ with $q_j - p = (x_j, y_j)$, $j \in \{1, 2\}$. Let $w_2 := x_2 - x_1$, and $h_1 := y_1 - y_2$. Then $|T_p(q_1, q_2)| = (x_1 + y_2) \cdot (w_2 + h_1)/2$.

▶ Observation 7. Consider $T_p(q_1, q_2)$ with $q_j - p = (x_j, y_j)$, $j \in \{1, 2\}$. Let $w_2 := x_2 - x_1$, and $h_1 := y_1 - y_2$. Fix $\alpha \in \mathbb{R}$ and consider the change of $|T_p(q_1, q_2)|$ if either $q_1$ or $q_2$ are moved horizontally or vertically as a linear function of $\varepsilon$:

**(a)** If either $q_1(\varepsilon) := q_1 + (0, \alpha\cdot\varepsilon)$ or $q_2(\varepsilon) := q_2 + (\alpha\cdot\varepsilon, 0)$, then $\partial|T_p(q_1, q_2)|/\partial\varepsilon = \alpha\cdot(x_1 + y_2)/2$ and $\partial^2|T_p(q_1, q_2)|/\partial\varepsilon^2 = 0$.

**(b)** If either $q_1(\varepsilon) := q_1 + (\alpha \cdot \varepsilon, 0)$ or $q_2(\varepsilon) := q_2 + (0, \alpha \cdot \varepsilon)$, then $\partial|T_p(q_1, q_2)|/\partial\varepsilon = \alpha \cdot \big(w_2 + h_1 - (x_1 + y_2)\big)/2$ and $\partial^2|T_p(q_1, q_2)|/\partial\varepsilon^2 = -\alpha^2$.

**Properties of the Charging Scheme.** The following results capture basic properties of our charging scheme. First, we show that the defined crowns are pairwise disjoint.

▶ **Lemma 8.** *Consider the tile packing $\mathcal{T}$ produced by algorithm TilePacking for a set $P$ of input points. For any two different tiles $t, t' \in \mathcal{T}$, we have $C_t \cap C_{t'} = \emptyset$.*

**Proof.** Fix $t, t' \in \mathcal{T}$ and let $p, p' \in P$ denote their respective anchors. W.l.o.g., assume $\ell^{\diagdown}_{p} > \ell^{\diagdown}_{p'}$, such that TilePacking processes $p$ before $p'$. As crowns consist of towers, it is sufficient to show $T_p(q_1, q_2) \cap T_{p'}(q'_1, q'_2) = \emptyset$ for consecutive $q_1, q_2 \in \Gamma_t$ and $q'_1, q'_2 \in \Gamma_{t'}$. Let $p_*, p'_* \in P$ be the respective peaks of these towers. W.l.o.g., we assume $\ell^{\diagup}_{p_*} < \ell^{\diagup}_{p'_*}$ ($p_*$ lies left of $\ell^{\diagup}_{p'_*}$); the other case follows symmetrically.

If $\ell^{\diagdown}_{p'_*} < \ell^{\diagdown}_{p}$, the towers are separated (the top of $T_{p'}(q'_1, q'_2)$ lies below the base of $T_p(q_1, q_2)$) and cannot intersect. So assume $\ell^{\diagdown}_{p'_*} > \ell^{\diagdown}_{p}$. Then we cannot have $p'_* \prec q_2$, since this would imply that $p'_*$ lies in the exclusive area of $t$, violating Observation 3 (see Figure 3).

Let $\Delta_y \coloneqq q'_1 - p'_*$ and note that $x(\Delta_y) = 0$. Define $\tilde{q}_1 \coloneqq p_* - \Delta_y$ and note that $p'_* \not\succ \tilde{q}_1$, since otherwise $q'_1 = p'_* + \Delta_y \succ \tilde{q}_1 + \Delta_y = p_*$, which (together with $\ell^{\diagdown}_{p_*} > \ell^{\diagdown}_{p} > \ell^{\diagdown}_{p'}$) would mean that $p_*$ lies in the exclusive area of $t'$ (again violating Observation 3).

So $\ell^{\diagup}_{p'_*} > \ell^{\diagup}_{p_*}$, $p'_* \not\prec q_2$, and $p'_* \not\succ \tilde{q}_1$. Together, these imply $x(p'_*) > x(q_2)$ and $y(p'_*) < y(\tilde{q}_1)$, which in turn imply $\ell^{\diagup}_{p'_*} > \ell^{\diagup}_{q_2 - \Delta_y}$. But then, the towers are separated, since $\ell^{\diagup}_{q'_1} = \ell^{\diagup}_{p'_* + \Delta_y} > \ell^{\diagup}_{q_2 - \Delta_y + \Delta_y} = \ell^{\diagup}_{q_2}$ ($T_p(q_1, q_2)$'s right side lies left of $T_{p'}(q'_1, q'_2)$'s left side). ◀

The next lemma's proof shows that all crowns lie inside a pentagon formed by $\mathcal{U}$ and two isosceles triangles left and below of $\mathcal{U}$ (see Figure 4). With Lemma 8 this implies that the total charging area is bounded by the pentagon's area.

▶ **Lemma 9.** *Consider the tile packing $\mathcal{T}$ produced by algorithm TilePacking for a set $P$ of input points. The total charging area of $\mathcal{T}$ is $c^* \leq 3/2$. Moreover, this bound is tight, since for arbitrarily small $\varepsilon > 0$ there are input points $P_\varepsilon$ with $c^* \geq 3/2 - \varepsilon$.*

**Proof.** Define the points $\mathrm{SW} \coloneqq (0, 0)$, $\mathrm{NW} \coloneqq (0, 1)$, and $\mathrm{SE} \coloneqq (1, 0)$. Let $\pentagon$ denote the pentagon enclosed by the lines $\ell^{\diagdown}_{\mathrm{SW}}$, $\ell^{\diagup}_{\mathrm{NW}}$, $\ell^{\diagup}_{\mathrm{SE}}$, $\ell^{-}_{\mathrm{NW}}$, and $\ell^{|}_{\mathrm{SE}}$ (see Figure 4). Since $|\pentagon| = 3/2$ and using Lemma 8, it is sufficient to show that $C_t \subseteq \pentagon$ for any $t \in \mathcal{T}$. For this, in turn, it is sufficient to show that any tower $T_p(q_1, q_2)$ of $C_t$ lies in $\pentagon$.

Fix such a tower $T_p(q_1, q_2)$. Since $p \succeq \mathrm{SW}$, we have $\ell^{\diagdown}_{p} \geq \ell^{\diagdown}_{\mathrm{SW}}$ (the base of $T_p(q_1, q_2)$ lies above $\ell^{\diagdown}_{\mathrm{SW}}$). Similarly, since $q_1, q_2 \in \overline{\mathcal{U}} \subseteq \pentagon$, we have $\ell^{\diagup}_{q_1} \geq \ell^{\diagup}_{\mathrm{NW}}$ (the left side of $T_p(q_1, q_2)$ lies right of the left side of $\pentagon$) and $\ell^{\diagup}_{q_2} \leq \ell^{\diagup}_{\mathrm{SE}}$ (the right side of $T_p(q_1, q_2)$ lies left of the right side of $\pentagon$). Finally, the topmost point $q_1 \in \overline{\mathcal{U}}$ of $T_p(q_1, q_2)$ lies below $\ell^{-}_{NW}$ and the rightmost point $q_2 \in \overline{\mathcal{U}}$ of $T_p(q_1, q_2)$ lies to the left of $\ell^{|}_{SE}$. Together, we get $T_p(q_1, q_2) \subseteq \pentagon$.

For the tightness of the bound, choose $\delta > 0, 1/\delta \in \mathbb{N}$ and define $P^\delta = \{\, \mathrm{SW} \,\} \cup \{\, (k \cdot \delta, 1 - k \cdot \delta^2), (1 - k \cdot \delta^2, k \cdot \delta) \mid k \in \{\, 1, 2, \ldots, 1/\delta - 1 \,\} \,\}$. As illustrated in Figure 4, the crown $C_t$ of tile $t$ with anchor SW converges towards $\pentagon$ as $\delta \to 0$. Therefore, for each $\varepsilon > 0$, we can choose $\delta$ such that, for point set $P_\varepsilon \coloneqq P^\delta$, we have $c^* \geq |C_t| \geq 3/2 - \varepsilon$. ◀

## 4.2 Weak Covering Guarantee for Greedy Tile Packings

This section proves the following, slightly weaker version of Theorem 1:

▶ **Theorem 10.** *For any set of input points, TilePacking covers at least 25% of $\mathcal{U}$.*

Proving this not only serves as a warm-up to illustrate our approach before proving our main result, but – as we will see in Section 5 – brings us halfway towards proving Theorem 1.

So consider a tile packing $\mathcal{T}$ produced by algorithm TilePacking for some set $P$ of input points. To prove Theorem 10, we follow the approach outlined in Section 3, using the charging scheme from Section 4.1. That is, the area of $t \in \mathcal{T}$ is charged to $c_t = |C_t|$, where $C_t$ represents the crown of $t$. To this end, define $\rho^* \coloneqq 1/4$ and the *weak charging ratio bound*

$$\xi_w \colon (0, 1] \to \mathbb{R}_{\geq 0}, \quad \xi_w(\rho) \coloneqq 2 \cdot (1 - \rho).$$

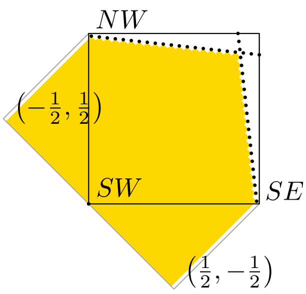



**Figure 4** Pentagon $\pentagon$ and point set $P_\varepsilon$ from Lemma 9.

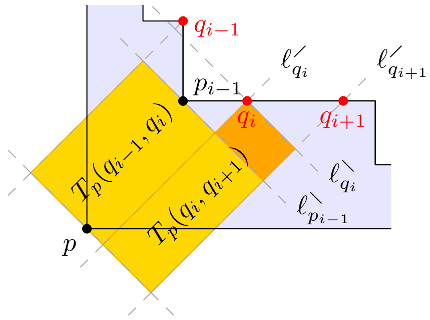



**Figure 5** Removing superfluous points from $\Gamma_t$ reduces $|C_t|$ by the rectangle between $\ell^{\diagup}_{q_i}$, $\ell^{\diagup}_{q_{i+1}}$, $\ell^{\diagdown}_{q_i}$, and $\ell^{\diagdown}_{p_{i-1}}$.

As a linear function, $\xi_w$ is trivially point-convex in $\rho^*$. Moreover, $\xi_w(\rho^*) = 3/2$ and thus, by Lemma 9, $\xi_w(\rho^*) \geq c^*$. In the remainder of this section we prove the following Proposition 11, stating that $\xi_w$ represents a lower bound on the charging ratio of any $t \in \mathcal{T}$. Once this is proven, Theorem 10 follows immediately by applying Lemma 5.

▶ **Proposition 11.** *For any tile $t$ we have $c_t/|t| \geq \xi_w(\rho_t)$.*

**A Lower Bound on the Charging Ratio.** To prove that $\xi_w$ bounds from below the charging ratio $c_t/|t|$ of any tile $t \in \mathcal{T}$, we gradually transform $t$ into a "simpler" tile $\tilde{t}$. Our transformations ensure $\rho_{\tilde{t}} = \rho_t$ and $c_{\tilde{t}}/|\tilde{t}| \leq c_t/|t|$. Eventually, $\tilde{t}$ will be simple enough to directly prove $c_{\tilde{t}}/|\tilde{t}| \geq \xi_w(\rho_{\tilde{t}})$. The following notation expresses progress via such a transformation:

$$\tilde{t} \preceq t \quad :\Leftrightarrow \quad \rho_{\tilde{t}} = \rho_t \text{ and } c_{\tilde{t}}/|\tilde{t}| \leq c_t/|t|.$$

As a simple example, note that both a tile's density and charging-ratio are invariant under translation and concentric scaling w.r.t. its anchor. This gives rise to the following transformation, which allows us to restrict our analysis to normalized tiles.

▶ Observation 12. Translate a tile $t$ such that it is anchored in the origin, then scale it by $1/|A_t|$ around the origin. We call the resulting tile $\tilde{t}$ *normalized*. Then $\tilde{t} \preceq t$.

Consider a tile $t$ with anchor $p$. A transformation may move one of $t$'s upper staircase points to the same $x$- or $y$-coordinate as another point from $\Gamma_t \cup \{p\}$, resulting in a degenerate tile with superfluous points in $\Gamma_t$ (see Section 2). The next lemma states that removing such superfluous points maintains an equal tile with a smaller crown.

▶ **Lemma 13.** *Consider a degenerate tile $t$. The tile $\tilde{t}$ with same anchor $p$ but $\Gamma_{\tilde{t}} := \{ q \in \Gamma_t \mid q \succ p$ and $\nexists q' \in \Gamma_t \colon q \preceq q' \}$ covers the same points, is non-degenerate, and $\tilde{t} \preceq t$.*

**Proof.** Order $\Gamma_t = \{ q_1, q_2, \ldots, q_k \}$ by non-decreasing $x$-coordinate and let $q_0 = q_{k+1} = p$. W.l.o.g. assume there is some $i \in \{1, \ldots, k\}$ with $y(q_i) = y(q_{i+1})$; the case of identical $x$-coordinates follows analogously. Let $\tilde{t}$ denote the (possibly still degenerate) tile with anchor $p$ and $\Gamma_{\tilde{t}} = \Gamma_t \setminus \{q_i\}$. Note that $\{ q \in \mathcal{U} \mid q \succeq p \wedge q \prec q_i \} \subseteq \{ q \in \mathcal{U} \mid q \succeq p \wedge q \prec q_{i+1} \}$, which implies $\tilde{t} = t$ and, thus, $\rho_t = \rho_{\tilde{t}}$. Removing $q_i$ affects the towers $T_p(q_i, q_{i+1})$ with peak $q_i$ (only if $i < k$) and $T_p(q_{i-1}, q_i)$ with peak $p_{i-1}$. Figure 5 illustrates the situation.

We now show that $c_{\tilde{t}} \leq c_t$, such that $\tilde{t} \preceq t$; the lemma's statement then follows by iteration. If $i = k$, then $c_{\tilde{t}} = c_t - |T(q_{i-1}, q_i)| \leq c_t$. So assume $i < k$. Then $c_{\tilde{t}} = c_t - |\Box| \leq c_t$, where $\Box$ is the rectangle enclosed by the lines $\ell^{\diagup}_{q_i}$, $\ell^{\diagup}_{q_{i+1}}$, $\ell^{\diagdown}_{q_i}$, and $\ell^{\diagdown}_{p_{i-1}}$ (see Figure 5). ◀

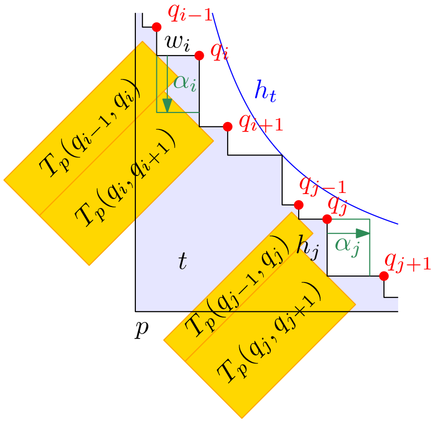



**Figure 6** Moving inner points in Lemma 14. Note that $\alpha_j > 0$ and $\alpha_i < 0$ in this case.

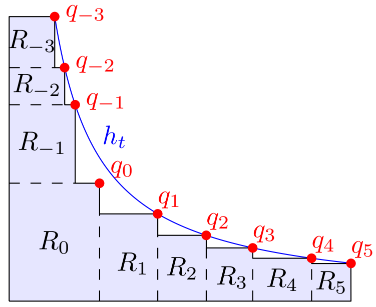



**Figure 7** Notation for Proposition 11 with $l = 3$ and $m = 5$.

Note that $h_t \cap \Gamma_t$ consists of the upper staircase points that form maximum rectangles in the tile. We now prove that we can transform $t$ such that at most one $q \in \Gamma_t$ lies not on $h_t$.

▶ **Lemma 14.** *A normalized tile $t$ can be transformed into a tile $\tilde{t} \preceq t$ with $|\Gamma_{\tilde{t}} \setminus h_{\tilde{t}}| \leq 1$.*

**Proof.** Assume $|\Gamma_t \setminus h_t| > 1$ and order $\Gamma_t = \{ q_1, q_2, \dots, q_k \}$ by increasing $x$-coordinate. To simplify border cases, define $q_0 = q_1$ and $q_{k+1} = q_k$. Choose $q_i, q_j \in \Gamma_t \setminus h_t$ with $i < j$. Consider the transformation $q_i(\varepsilon) := q_i + (0, \alpha_i \cdot \varepsilon)$ and $q_j(\varepsilon) := q_j + (\alpha_j \cdot \varepsilon, 0)$ with $\alpha_i, \alpha_j \in \mathbb{R}$. Then the tile and crown areas become functions $t(\varepsilon)$ and $c_t(\varepsilon)$ of $\varepsilon$. We show that there are non-zero $\alpha_i, \alpha_j$ such that $t(\varepsilon)$ and, thus, $\rho_{t(\varepsilon)}$ remain constant and $c_t(\varepsilon)$ does not increase. Eventually, this results in a tile $\tilde{t} \preceq t$ that has an additional point on $h_t$ or that has a degenerate staircase point which we can remove by Lemma 13. In both cases $|\Gamma_t \setminus h_t|$ is decreased and the lemma follows by iteration. Figure 6 illustrates the transformation.

The transformation changes only the towers $T(q_{i-1}, q_i)$, $T(q_i, q_{i+1})$, $T(q_{j-1}, q_j)$, and $T(q_j, q_{j+1})$. ($i < j$ ensures that these changes do not interfere with each other.) For $l \in \{ 0, 1, \dots, k+1 \}$ let $q_l = (x_l, y_l)$ and define $w_l := x_l - x_{l-1}$ for $l \neq 0$ and $h_l := y_l - y_{l+1}$ for $l \neq k+1$. Then the transformation changes the area of the tile according to $\partial t(\varepsilon)/\partial\varepsilon = \alpha_i \cdot w_i + \alpha_j \cdot h_j$. To keep $|t|$ constant, we make this zero by setting $\alpha_j = -\alpha_i \cdot w_i / h_j$.

It remains to find a non-zero $\alpha_i$ such that $c_t(\varepsilon)$ is non-increasing in $\varepsilon$. For this let $T_i(\varepsilon) := |T(q_{i-1}, q_i)| + |T(q_i, q_{i+1})|$ and define $T_j(\varepsilon)$ analogously. By Observation 7, $\partial^2 T_l(\varepsilon)/\partial\varepsilon^2 = -\alpha_l^2$ for $l \in \{ i, j \}$. This yields $\partial^2 c_t(\varepsilon)/\partial\varepsilon^2 = -\alpha_i^2 - \alpha_j^2 < -\alpha_i^2$, which is a negative constant. This allows us to choose $\alpha_i \in \{ -1, +1 \}$ such that $c_t(\varepsilon)$ is non-increasing. ◀

With these results, we are ready to prove our first covering guarantee for TilePacking.

**Proof of Proposition 11.** Let $t$ be a tile. By Observation 12 and Lemma 14, we can assume that $t$ is normalized and $|\Gamma_t \setminus h_t| \leq 1$. If $|\Gamma_t \setminus h_t| = 1$, let $q_0 \in \Gamma_t \setminus h_t$, otherwise let $q_0 \in \Gamma_t$ arbitrary. Relabel the points $\Gamma_t = \{ q_{-l}, \dots, q_m \}$ in increasing order of their $x$-coordinates. To simplify border cases, let $q_{-l-1} = q_{-l}$ and $q_{m+1} = q_m$. Let further $w_i := x_i - x_{i-1}$ for $i \neq l-1$ and $h_i := y_i - y_{i+1}$ for $i \neq m+1$ (using $q_i = (x_i, y_i)$). For $i = -l, \dots, m$ inductively define the rectangles $R_i := \{ q \in t \mid q \prec q_i \} \setminus \bigcup_{|j|<|i|} R_j$. Finally, for $i = 1, \dots, m$, let $T_i := T(q_{i-1}, q_i)$; for $i = -l, \dots, -1$ let $T_i := T(q_i, q_{i+1})$. See Figure 7 for an illustration.

Note that $c_t = \sum_{i=-l}^{-1}|T_i| + \sum_{i=1}^{m}|T_i|$ and $|t| = \sum_{j=-l}^{m}|R_j|$. We will first show that for $i \in \{-l, \dots, m\} \setminus \{-1, 0, 1\}$ we have $|T_i| \geq 2|R_i|$. Afterward, we show $|T_{-1}| + |T_1| \geq 2(|R_{-1}| + |R_0| + |R_1| - 1)$. With these inequalities and since $\rho_t = |A_t|/|t| = 1/|t|$ due to the normalization, the desired statement follows via

$$c_t = \sum_{i=-l}^{-1}|T_i| + \sum_{i=1}^{m}|T_i| \geq \sum_{i=-l}^{m} 2|R_i| - 2 = 2|t| - 2 = 2|t| \cdot (1 - \rho_t) = |t| \cdot \xi_w(\rho_t).$$

We now show the above bounds, starting with $|T_i| \geq 2|R_i|$ for $|i| \geq 2$. W.l.o.g. we assume $i \geq 2$; the case $i \leq -2$ follows by symmetry. Note that $i \geq 2$ implies $q_i, q_{i-1} \in h_t$ and thus (since $t$ is normalized) $y_j = 1/x_j$ for $j \in \{i-1, i\}$. This yields $x_{i-1}/y_i = x_{i-1} \cdot x_i$ as well as $w_i/h_{i-1} = (x_i - x_{i-1})/(y_{i-1} - y_i) = x_{i-1} \cdot x_i$. We use these identities together with $|R_i| = w_i \cdot y_i$ to bound the formula for $|T_i|$ from Observation 6:

$$\begin{aligned}|T_i| &= 1/2 \cdot (x_{i-1} + y_i)(w_i + h_{i-1}) = w_i \cdot y_i \cdot 1/2 \cdot (1 + x_{i-1}/y_i)(1 + h_{i-1}/w_i)\\ &= |R_i| \cdot \frac{1}{2} \cdot (1 + x_{i-1} \cdot x_i) \cdot \left(1 + \frac{1}{x_{i-1} \cdot x_i}\right) = |R_i| \cdot \frac{1}{2} \cdot \frac{(1 + x_{i-1} \cdot x_i)^2}{x_{i-1} \cdot x_i} \geq 2|R_i|,\end{aligned}$$

using that the function $x \mapsto (1+x)^2/x$ over $[0, \infty)$ has a minimum value of 4 at $x = 1$.

It remains to show that $|T_{-1}| + |T_1| \geq 2(|R_{-1}| + |R_{-1}| + |R_0| - 1)$. Note that, if $m = 0$, we have $q_{m+1} = q_m$ by definition and $|R_1| = 0$ and $|T_1| = 0$ hold. Similarly, if $l = 0$ then $|R_{-1}| = 0$ and $|T_{-1}| = 0$. We assume that $l > 0$ or $m > 0$, as otherwise $\xi_w(\rho_t) = \xi_w(1) = 0$ and the proposition becomes trivial. W.l.o.g. let $m > 0$; the other case follows symmetrically.

For $\alpha \in \{-1, +1\}$ (which we fix later) and $\varepsilon \geq 0$ define the transformation $y_0(\varepsilon) \coloneqq y_0 + \alpha \cdot \varepsilon$, where $\varepsilon$ is chosen such that $y_0(\varepsilon) \in [y_1, 1/x_0]$ [2], which moves $q_0$ either up- or downward, depending on $\alpha$. Thus, with $f(\varepsilon) \coloneqq |T_{-1}| + |T_1| - 2(|R_{-1}| + |R_1| + |R_0| - 1)$, where the $T_i$ and $R_i$ depend on $y_0$ and thus $\varepsilon$, our goal becomes to prove $f(0) \geq 0$. To this end, consider how $f(\varepsilon)$ changes with $\varepsilon$. The rectangles $|R_j|$ ($j \in \{-1, 0, 1\}$) change linearly or remain constant. By Observation 7, $\partial^2|T_1|/\partial\varepsilon^2 = 0$. Similarly, if $l > 0$ we have $\partial^2|T_{-1}|/\partial\varepsilon^2 = -\alpha^2 = -1$ by Observation 7, and if $l = 0$ we have $\partial^2|T_{-1}|/\partial\varepsilon^2 = 0$ (because $|T_{-1}|$ remains zero). Thus, in all cases $\partial^2 f(\varepsilon)/\partial\varepsilon^2 \leq 0$, meaning its minimum $f_{\min}$ lies at one of the borders, where either $y_0 = y_1$ or $y_0 = 1/x_0$. We consider both possibilities and show that each time $f_{\min} \geq 0$ (which finishes the proof, since $f(0) \geq f_{\min} \geq 0$).

If at $f_{\min}$ we have $y_0 = 1/x_0$, let $t_{\text{high}}$ denote the corresponding tile. Note that $q_0$ lies on the hyperbola $h_t$. But then $|R_0| = 1$ and, thus, $f_{\min} = |T_{-1}| + |T_1| - 2(|R_{-1}| + |R_1|)$. Moreover, with $q_0 \in h_t$ we can apply the calculations for $|i| > 1$ to get $|T_{-1}| \geq 2|R_{-1}|$ and $|T_1| \geq 2|R_1|$, such that $f_{\min} \geq 0$.

So assume that at $f_{\min}$ we have $y_0 = y_1$ and let $t_{\text{low}}$ denote the corresponding tile. Note that $R_0$ and $R_1$ form a rectangle from the origin to the point $q_1$ on $h_t$, such that $|R_0| + |R_1| = 1$. Thus, $f_{\min} = |T_{-1}| + |T_1| - 2|R_{-1}|$. Define the (degenerate) tile $t'$ with $\Gamma_{t'} = \{q_{-1}, q_0, q_1\}$ and anchor $p$, such that its crown area is $c_{t'} = |T_1| + |T_2|$. By Lemma 13, for the (non-degenerate) tile $\tilde{t}'$ with $\Gamma_{\tilde{t}'} = \Gamma_{t'} \setminus \{q_0\}$ we have $c_{\tilde{t}'} \leq c_{t'}$. The crown $c_{\tilde{t}'}$ consists of the single tower $T(q_{-1}, q_1)$. Since $q_{-1}, q_1 \in h_t$, we can apply the calculations for $|i| > 1$ to get $c_{\tilde{t}'} = |T(q_{-1}, q_1)| \geq 2|R_{-1}|$. Putting everything together we get

$$f_{\min} = |T_{-1}| + |T_1| - 2|R_{-1}| = c_{t'} - 2|R_{-1}| \geq c_{\tilde{t}'} - 2|R_{-1}| \geq 0. \qquad \blacktriangleleft$$

[2] These boundaries ensure that the tile remains valid and normalized. Note that if $l = 0$, moving $q_0$ upward also causes the dummy point $q_{-1}$ to move upward, such that $|R_{-1}|$ and $|T_{-1}|$ remain zero.

# 5 Strong Covering Guarantee for Greedy Tile Packings

This section proves our strong covering guarantee for TilePacking, namely Theorem 1. We use the same approach as for our weak covering guarantee from Section 4.2 but derive a stronger charging ratio bound. More exactly, instead of $\xi_w$ we use

$$\xi_s\colon (0,1] \to \mathbb{R}_{\geq 0}, \quad \xi_s(\rho) \coloneqq \begin{cases} 1-\rho\cdot\bigl(1+\sinh(1-1/\rho)\bigr) & \text{if } \rho \leq 1/2 \\ \xi_w(\rho) = 2\cdot(1-\rho) & \text{if } \rho > 1/2. \end{cases} \tag{2}$$

Most properties required for our approach from Definition 4 are easily verified for $\xi_s$ (whose function graph can be seen in Figure 12). Indeed, for $\rho^* \coloneqq \xi_s^{-1}(3/2) \approx 0.3901$, we have $\partial\xi_s(\rho)/\partial\rho|_{\rho=\rho^*} \approx -5.1 < 0$. Moreover, $\xi_s$ is point-convex at $\rho^*$, since it is convex on $(0,1/2]$ and on $(1/2,1]$ its tangent $t_\xi$ at $\rho^*$ lies below $\xi_s$ ($t_\xi$ is steeper and $t_\xi(1/2) \approx 0.94 < 1 = \xi_w(1/2)$). Also, by choice of $\rho^*$ and by Lemma 9, we have $\xi_s(\rho^*) = 3/2 \geq c^*$ for the total charged area $c^*$ of a tile packing $\mathcal{T}$ produced by algorithm TilePacking.

The following proposition states the remaining required property of Definition 4.

▶ **Proposition 15.** *For any tile $t$ we have $c_t/|t| \geq \xi_s(\rho_t)$ and this bound is tight.*

With this, Theorem 1 follows by applying Lemma 5. The remainder of this section outlines the analysis of this proposition.

**Transformation to Worst-case Tiles.** For tiles $t$ of density $\rho_t$ larger than $1/2$, Proposition 15 follows from Proposition 11, since in this regime $\xi_s(\rho_t) = \xi_w(\rho_t)$. The tightness for such high densities follows since for any $\rho_t \in (1/2,1]$ there is a (symmetric) *step tile* $t = t_l(\rho_t)$ of density $\rho_t$ (depicted in Figure 9) with $c_t/|t| = \xi_s(\rho_t)$. Thus, we restrict our further study to tiles of density at most $1/2$. We will show how to gradually transform any such tile $t$ into a (symmetric) *hyperbola tile* $t_h(\rho_t) \preceq t$ (depicted in Figure 8). Again, the tightness follows from the existence of a tile $t = t_h(\rho_t)$ with $c_t/|t| = \xi_s(\rho_t)$.

Before we outline the transformation process into such worst-case low-density tiles, we need to cope with the fact that $t_h(\rho_t)$ is not a staircase polygon and, thus, not captured by our tile definition. However, one can see $t_h(\rho_t)$ as the result of defining $\Gamma_t$ as $k$ equally spaced points from the hyperbola $\{\,(x,y) \in [0,s) \mid y = 1/x\,\}$ and taking the limit $k \to \infty$. The next paragraph formalizes this intuition by introducing *generalized tiles* and some related notions.

**Generalized Tiles and Crown Contribution.** A *generalized tile* $t$ is defined equivalently to "normal" tiles, with the only difference that $\Gamma_t$ may be infinite. All other tile definitions (e.g., point set definition of $t$, maximum-area rectangle $A_t$, or density $\rho_t$) stay intact.

From now on the term *tile* always refers to a normalized and non-degenerate generalized tile. (Note that the points which cause a generalized tile to be degenerate, cannot reside in a slide. As such Observation 12 and Lemma 13 easily transfer to generalized tiles.) We require that the $x$-coordinates of $\Gamma_t$ can be partitioned into $k$ inclusion-wise maximal, closed intervals $I_1, I_2, \ldots, I_k$, ordered by increasing $x$-coordinates. For $i \in \{\,1,2,\ldots,k\,\}$ let $q_i^-, q_i^+ \in \Gamma_t$ denote the points realizing the left- and rightmost $x$-coordinate of $I_i$, respectively. Note that $I_i$ may be a point interval, such that $q_i^- = q_i^+$. A *section* of $\Gamma_t$ is a tuple as follows:

- a *step* $(q_i^+, q_{i+1}^-)$, if $q_i^+, q_{i+1}^- \in h_t$;
- a *slide* $(q_i^-, q_i^+)$, if $q_i^- \neq q_i^+$ and $\{\,q \in \Gamma_t \mid x(q) \in I_i\,\} \subseteq h_t$;
- a *double step* $(q_{i-1}^+, q_i^-, q_{i+1}^-)$, if $q_{i-1}^+, q_{i+1}^- \in h_t$ and $q_i^- = q_i^+ \notin h_t$; or
- the *corners* $(q_1, q_2)$, if $q_1 \notin h_t$ and $q_2 \in h_t$ as well as $(q_{k-1}, q_k)$ if $q_k \notin h_t$ and $q_{k-1} \in h_t$.

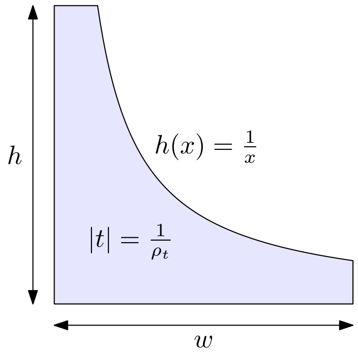



**Figure 8** Symmetric ($w = h$) low-density tile $t = t_h(\rho_t)$ for which $\xi_s$ is tight.

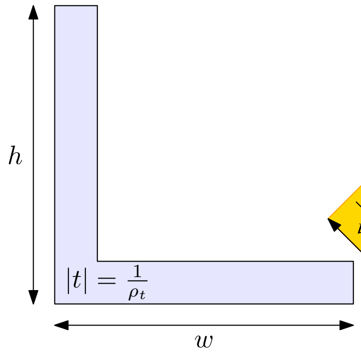



**Figure 9** Symmetric ($w = h$) high-density tile $t = t_l(\rho_t)$ for which $\xi_s$ is tight.

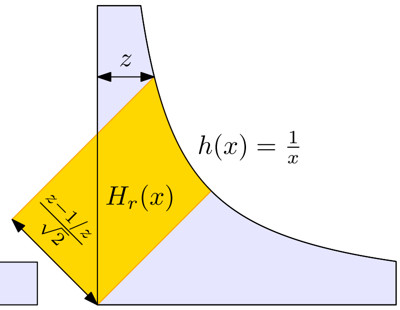



**Figure 10** Crown contribution of a slide.

We further define $t(x_L, x_R) = \{ (x, y) \in t \mid x_L \le x \le x_R \}$. After applying Lemma 14, all tiles resulting from our transformations can be described as a sequence of such sections. Figure 13 illustrates generalized tiles and the different sections.

Note that a slide can be understood as the limit case of $k \to \infty$ equally spaced upper staircase points. As the slide is the only part of a generalized tile which differs from normal tiles, our charging scheme from Section 4.1 naturally extends to generalized tiles. This yields the following tower complement for slides:

▶ **Definition 16.** For a tile $t$ with a slide $(q_1, q_2)$, rotate the hyperbola by $\pi/4$ around the anchor of $t$, to obtain the rotated hyperbola $h_r(x) = \sqrt{x^2 + 2}$. The area under $h_r$ between $q_1$ and $q_2$ will be denoted as $H(q_1, q_2)$.

▶ Observation 17. For a tile $t$ with slide $(q_1, q_2)$ we get $|H(q_1, q_2)| := \left[\frac{1}{4} \cdot (z^2 - z^{-2}) + \ln z\right]_{x_1}^{x_2}$.

**Proof.** $|H(q_1, q_2)|$ can be calculated via integration: The indefinite integral under $h_r(x) = \sqrt{x^2 + 2}$ is $H_r(x) := \int h_r(x)\,\mathrm{d}x = x/2 \cdot \sqrt{2 + x^2} + \operatorname{arsinh}(x/\sqrt{2})$, and for $x = (z - 1/z)/\sqrt{2}$, we get $H_r((z - 1/z)/\sqrt{2}) = 1/4 \cdot (z^2 - z^{-2}) + \ln z$, where $z = x_1$ or $z = x_2$ (see Figure 10). ◀

**Overview of the Transformation Process.**[3] Figure 11 gives an overview of how we gradually transform an arbitrary tile $t$ with density $\rho_t \le 1/2$ into a worst-case hyperbola tile $t_h(\rho_t)$. Starting with an arbitrary tile $t$ (I), Lemma 18 either enforces $\Gamma_t \subseteq h_t$ or $\Gamma_t \setminus h_t = \{ q \}$.

▶ **Lemma 18.** *Let $t$ be a tile. Then either there exists a tile $\tilde{t} \preceq t$ with $\Gamma_{\tilde{t}} \subset h_{\tilde{t}}$, or $t$ contains a double step $(q_1, q_2)$ with $x_1 \le 1 \le x_2$ or a corner $(q_1, q_2)$ with $x_1 < 1$ (if $q_2 \notin h_t$, $x_2 > 1$ if $q_1 \notin h_t$).*

This lemma yields two different cases: In the first case (II), $t$ contains a double-step $(q_1, q, q_2)$ and we can enforce $x(q_1) \le 1 \le x(q_2)$. In the second case (III), $q$ is part of a corner where $x(q') \ge 1$ or $x(q') \le 1$ can be enforced for corners $(q, q')$ or $(q', q)$, respectively. (The case where $\Gamma_{\tilde{t}} \subset h_{\tilde{t}}$ will be dealt with later, in case (VIII).)

The next step from cases (II)/(III) to cases (IV)/(V) is based on smaller transformation/property statements about adjacent sections:

[3] Due to space limitations, the proofs for the following Lemmas 18–29 can only be found in the arXiv version of this paper (see [11]).

▶ **Lemma 19.** *Let $t$ be a tile, $q_1, q_2, q_3 \in \Gamma_t$ the leftmost three points (in order) such that $q_1 = q_2$ or $(q_1, q_2)$ is a step; and $q_2 = q_3$ or $(q_2, q_3)$ is a slide. Then these sections can be replaced by up to one step $s = (q_1, q_2)$ and up to one slide $h = (q_2, q_3)$ such that*

1. *If $s$ exists, then $x_1 x_2 \geq 1/\sqrt{2}$*
2. *If $h$ exists, then $x_1 x_2 \leq 1/\sqrt{2}$*

*giving us a tile $\tilde{t} \preceq t$.*

Intuitively Lemma 19 states that the leftmost sections will only be a step and hyperbola when they satisfy precise properties and otherwise the tile can be transformed such that one of the sections vanished.

▶ **Lemma 20.** *Let $t$ be a tile with a step $(q_1, q_2)$ and a slide $(q_2, q_3)$. If $x_1 \geq 1/x_3$, then the two sections can be replaced by a slide $(q_1, q_4)$ and a step $(q_4, q_2)$, resulting in a tile $\tilde{t} \preceq t$.*

Lemma 20 describes when “swapping” steps with an adjacent slide lowers $c_t$.

▶ **Lemma 21.** *Let $t$ be a tile and $(q_1, q_2)$,$(q_2, q_3)$ be steps with $x_3 \leq 1$. Then the two steps can be replaced with a step $(q_1, q_4)$ and a slide $(q_4, q_3)$, resulting in a tile $\tilde{t} \preceq t$.*

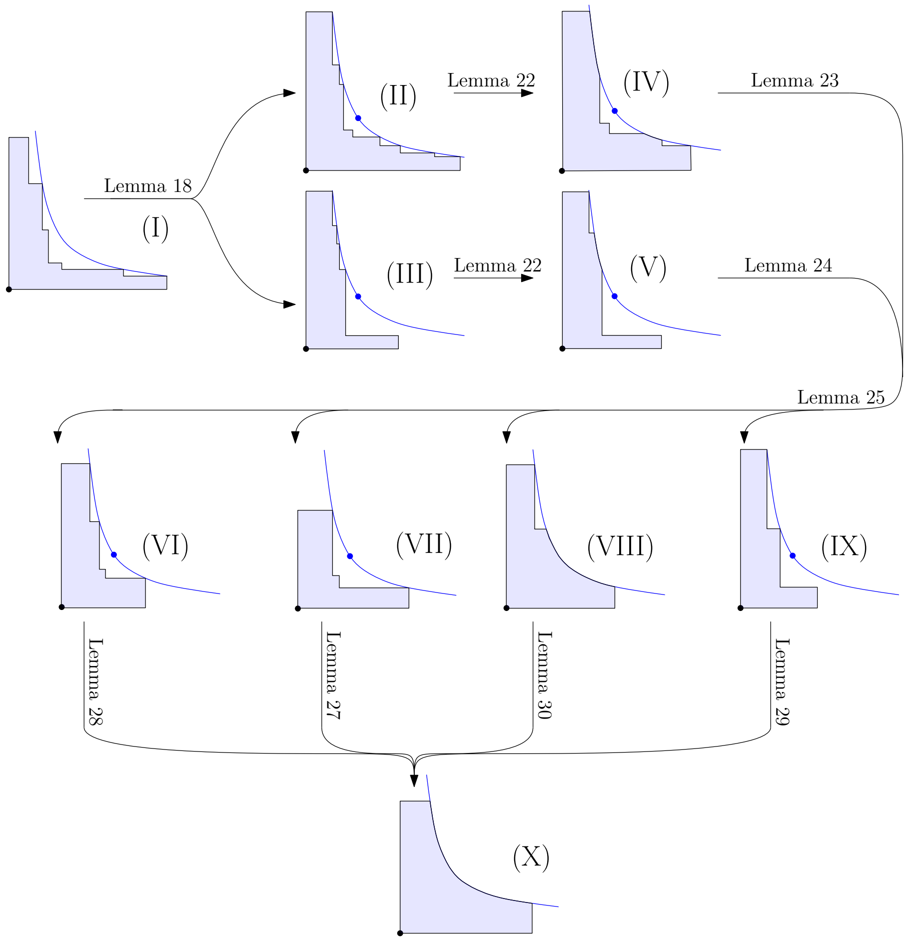



**Figure 11** Transforming low-density tiles $t$ with $\rho_t \leq 1/2$ to a corresponding worst-case hyperbola tile $t_h(s) \preceq t$. For the normalized tiles in Cases (II) and later, the blue dot marks the point $(1, 1)$.

Lemma 21 describes when "merging" two adjacent steps into a step and adjacent slide lowers $c_t$. Note that each of these lemmas hold up to reflection on the x=y axis, due to symmetry. Using the transformations/properties in these lemmas we can show the following Lemma 22, which allows us to only consider tiles containing a single step.

▶ **Lemma 22.** *Consider a tile $t$ with $\rho_t \leq 1/2$. Then there exists a tile $\tilde{t} \preceq t$, containing at most one step. Furthermore, if $\Gamma_t \subseteq h_t$ then $\Gamma_{\tilde{t}} \subseteq h_{\tilde{t}}$.*

Now down to only one step, the following two lemmas (Lemmas 23 and 24) show that we can transform the tiles from (IV) and (V), respectively, such that either $\Gamma_t \subseteq h_t$ or $t$ contains no slides:

▶ **Lemma 23.** *Let $t$ be a tile with a double step $(q_1, q_2, q_3)$ and a slide $(q_3, q_4)$. We can replace both sections by a double step $(q_1, q_4)$ or a sequence of steps and slides between $q_1$ and $q_4$, obtaining a tile $t' \preceq t$.*

▶ **Lemma 24.** *Let $t$ be a tile with a slide $(q_1, q_2)$ and a corner $(q_2, q_3)$. We can replace both sections by a corner $(q_1, q_2')$ or a sequence of steps and slides between $q_1$ and $q_2'$, obtaining a tile $t' \preceq t$.*

These lemmas, together with the following Lemma 25, can then be used to reduce the remaining cases to those illustrated in (VI) to (IX):

▶ **Lemma 25.** *Let $t$ be a tile with $\rho_t \leq 1/2$, then there exists a tile $\tilde{t} \preceq t$ with $\Gamma_{\tilde{t}} \subseteq h_{\tilde{t}}$ or $\tilde{t}$ consists of a step and a corner or a double step and possibly a step.*

The following Lemma 26 allows us to restrict ourselves to tiles $t$ with $\rho_t = 1/2$ or where all points in $\Gamma_t$ are on $t$'s hyperbola, and is essential to proving the subsequent lemmas.

▶ **Lemma 26.** *Let $t$ be a tile with $\rho_t \leq 1/2$. Then there exists a tile $\tilde{t}$ with $\Gamma_{\tilde{t}} \subseteq h_{\tilde{t}}$ or $\rho_{\tilde{t}} = 1/2$ such that, if $|C_{\tilde{t}}|/|\tilde{t}| \geq \xi_s(\rho_{\tilde{t}})$ then also $|C_t|/|t| \geq \xi_s(\rho_t)$.*

For each of the four cases we then separately show $t_h(\rho) \preceq t$ (Lemmas 27–30).

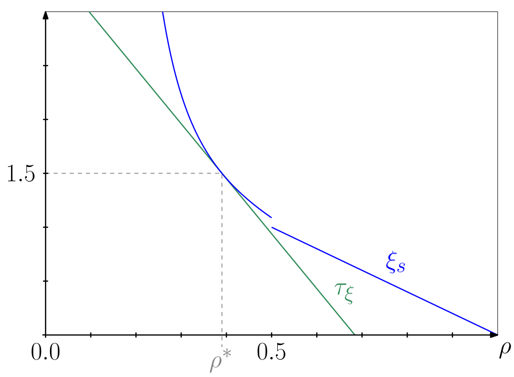



**Figure 12** The charging ratio bound $\xi_s$ from Section 5 and its tangent $\tau_\xi$ at $\rho^* = \xi_s^{-1}(3/2)$. This illustrates that $\xi_s$ is point-convex at $\rho^*$.

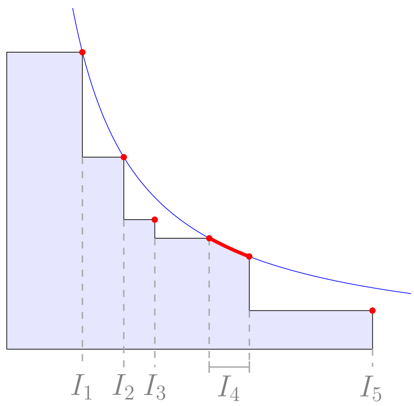



**Figure 13** A generalized tile with four sections formed by the five intervals $I_1$ to $I_5$ (of which only $I_4$ is a proper interval). They form a step (between $I_1$ and $I_2$), a double-step (between $I_2$ and $I_4$), a slide ($I_4$), and a corner ($I_4, I_5$).

▶ **Lemma 27.** *Let $t$ be a tile with $\rho_t \leq 1/2$ consisting only of a double-step $(q_1, q_2, q_3)$. Then $|C_t|/|t| \geq \xi_s(\rho_t)$.*

▶ **Lemma 28.** *Let $t$ be a tile with $\rho_t \leq 1/2$ consisting only of a double step $(q_1, q_2, q_3)$ and a step $(q_3, q_4)$. Then $|C_t|/|t| \geq \xi_s(\rho_t)$*

▶ **Lemma 29.** *Let $t$ be a tile with $\rho_t \leq 1/2$ consisting only of a step $(q_1, q_2)$ where $x_1x_2 \geq 1/\sqrt{2}$ and a corner $(q_2, q_3)$. Then there exists a tile $t' \preceq t$ only consisting of two steps.*

▶ **Lemma 30.** *Let $t$ be a tile with $\rho_t \leq 1/2$ with $\Gamma_t \subseteq h_t$. Then $|C_t|/|t| \geq \xi_s(\rho_t)$.*

**Proof.** By Lemma 22 we can assume that $t$ contains at most one step. Since $\Gamma_t \subseteq h_t$, $t$ can only consist of steps and slides. Then $t$ must contain at least one slide, as otherwise $t$ consists of exactly one step, contradicting $\rho_t \leq 1/2$. Using Lemma 20, we can then ensure that $t$ has at most one slide: Assuming this is not the case, $t$ has a slide $(q_1, q_2)$, a step $(q_2, q_3)$ and another slide $(q_3, q_4)$. Using the constraints of Lemma 20 we get $x_1 > 1/x_3 > 1/x_4 > x_2 > x_1$, a contradiction, so $t$ has exactly one slide and possibly one step.

It is enough to show $\Delta := |t|\xi_s(\rho_t) - |C_t| < 0$, since the statement follows by rearranging. First assume that $t$'s only section is a slide $(q_1, q_2)$. Then $|t| = |t(0, x_1)| + |t(x_1, x_2)| = 1 + \ln(x_2/x_1)$, and we get:

$$\begin{aligned}
\Delta &= |t|\xi_s(\rho_t) - |H(q_1, q_2)| = |t| - 1 + \sinh(|t| - 1) - \left[1/4 \cdot (z^2 - z^{-2}) + \ln z\right]_{x_1}^{x_2} \\
&= \left(x_2^{-2} - x_2^2 + x_1^2 - x_1^{-2}\right)/4 + \sinh(|t| - 1) + |t| - (1 + \ln(x_2/x_1)) \\
&= \left(x_2^{-2} - x_2^2 + x_1^2 - x_1^{-2}\right)/4 + \sinh(\ln(x_2/x_1)) \\
&= 2\sinh((\ln(x_1) + \ln(x_2))/2)^2 \sinh(\ln(x_1/x_2)) < 0
\end{aligned}$$

where the last inequality directly follows from $x_1 < x_2$.

Now assume that $t$ consists of a step $(q_1, q_2)$ and a slide $(q_2, q_3)$ (w.l.o.g. ordered in this way). In such a case we have $|t| = |t(x_0, x_1)| + |t(x_1, x_2)| + |t(x_2, x_3)| = 1 + (x_2 - x_1)/x_2 + \ln(x_3/x_2)$, or rearranged, $x_3 = x_2 e^{x_1/x_2 + |t| - 2}$. Again we calculate

$$\begin{aligned}
\Delta &= |t|\xi_s(\rho_t) - (|T(q_1, q_2)| + |H(q_2, q_3)|) \\
&= |t| - 1 + \sinh(|t| - 1) - \frac{1}{2}\left(\frac{1}{x_1} - \frac{1}{x_2} + x_2 - x_1\right)\left(x_1 + \frac{1}{x_2}\right) - \left[\frac{z^2 - z^{-2}}{4} + \ln z\right]_{x_2}^{x_3}.
\end{aligned}$$

Taking the derivative of $\Delta$ w.r.t. $|t|$ (after inserting $x_3 = x_2 e^{x_1/x_2 + |t| - 2}$), we obtain $\partial\Delta/\partial|t| = -2\cosh(x_1/x_2 + |t| - 2 + \ln x_2)^2 < 0$. This indicates that $\Delta$ is maximized for smallest $|t|$. So assume $|t| = 2$ now, or equivalently, $x_3 = x_2 e^{x_1/x_2}$. By Lemma 19 we can further assume that $x_2 = 1/(\sqrt{2}x_1)$. Using the substitution $x_1 = 2^{-3/4}\sqrt{u}$ we get

$$\begin{aligned}
\Delta &= \frac{1}{2\sqrt{2}}\left(-3 - \frac{\sqrt{2}}{e} + \sqrt{2}e + \frac{1 - e^u}{u} + u + \frac{u}{2e^u}\right) \quad \text{with the derivative} \\
\frac{\partial\Delta}{\partial u} &= \frac{(1 - u)(u^2 + 2e^{2u} - 2e^u(1 + u))}{4\sqrt{2}u^2 e^u}.
\end{aligned}$$

The derivative above has only one zero, namely $u = 1$: From $0 < x_1^2 \leq x_1x_2 = 1/\sqrt{2}$, we can deduce $u \in (0, 2]$ by the substitution. For the right factor of the derivative's numerator we then get $u^2 + 2e^{2u} - 2e^u(1 + u) > 2e^u(e^u - (1 + u)) > 0$ from the Taylor series of $e^u$. Hence checking $\Delta$ at $u$'s boundaries and $u = 1$ is sufficient, where we get $\lim_{u \to 0} \Delta \approx -0.24 < 0$ (apply L'Hospital's rule on $(1 - e^u)/u$), $\Delta|_{u \to 1} \approx -0.07 < 0$ and $\Delta|_{u \to 2} \approx -0.26 < 0$. ◀

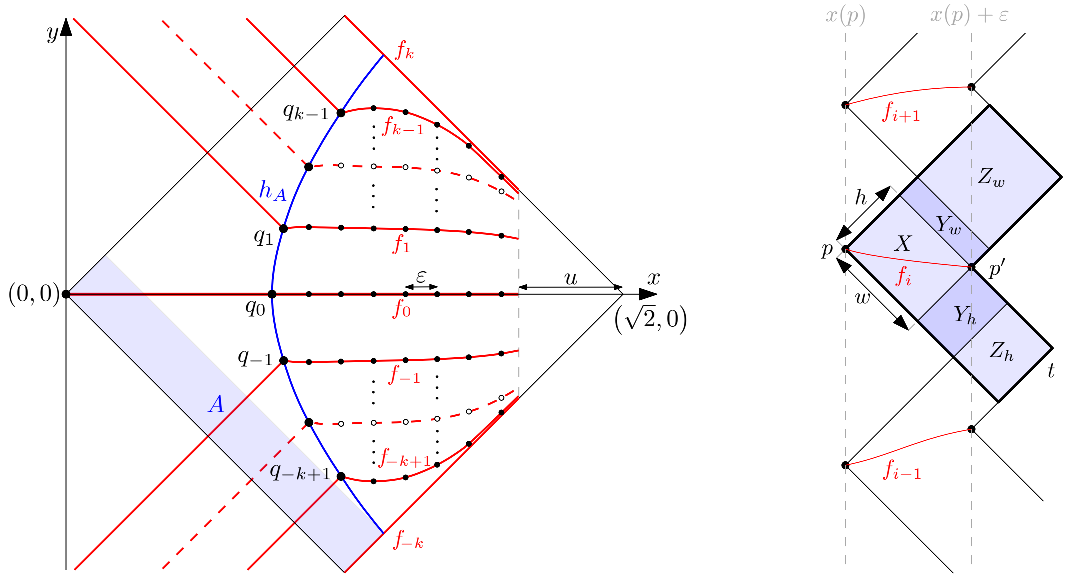



**Figure 14** Left: Our construction. Right: A tile anchored at $p \in P_{\varepsilon,k} \setminus \{ (0,0) \}$. For demonstration purposes, the areas are not shown to scale; in truth $X, Z_w, Z_h$ are significantly smaller than $Y_w, Y_h$.

With all previous lemmas combined, we are now ready to give the proof for Proposition 15.

**Proof of Proposition 15.** By Proposition 11 we already get the result for $\rho_t > 1/2$. It remains to show the result for tiles with $\rho_t \leq 1/2$. Lemma 25 allows us to limit ourselves to certain tiles $t$: Tiles with $\Gamma_t \subseteq h_t$ (Lemma 30 yields the result), tiles with only a double step (use Lemma 27), tiles with a step and a double step (use Lemma 28) and tiles that consist of a step and a corner (use Lemma 29). As such the bound follows.

It remains to show the tightness. First note that for $\rho_t = 1$, the tightness is trivial (choose an arbitrary tile $t$ with $|\Gamma_t| = 1$). For $\rho < 1$, we show that $\xi_s$ exactly corresponds to the tiles $t_l(\rho)$ and $t_h(\rho)$ shown in Figures 8 and 9.

Let $t = t_l(\rho)$ with $\Gamma_t = \{ q_1, q_2 \} \subset h_t$, $x_1 = y_2$. We have $|t| = |t(0,x_1)| + |t(x_1,x_2)| = 1 + (1 - x_1^2)$, or rearranged $x_1 = \sqrt{2 - 1/\rho}$ and we get

$$|C_t|/|t| = |T(q_1,q_2)|/|t| = 1/2 \cdot (1/x_1 - x_1 + 1/x_1 - x_1)(x_1 + x_1)/\big(2 - x_1^2\big) = 2 - 2\rho.$$

Now let $t = t_h(\rho)$. $t$ consists of a slide $(q_1, q_2) = ((x_1, 1/x_1), (1/x_1, x_1))$, hence $|t| = |t(0,x_1)| + |t(x_1,x_2)| = 1 + \ln((1/x_1)/x_1) = 1 - 2\ln x_1$, or rearranged, $x_1 = e^{(1-1/\rho)/2}$. So

$$|C_t|/|t| = |H(q_1,q_2)|\rho = \rho\left[\frac{1}{4}\big(z^2 - z^{-2}\big) + \ln z\right]_{e^{(1-1/\rho)/2}}^{e^{(1/\rho-1)/2}} = 1 - \rho(1 + \sinh(1 - \frac{1}{\rho})).$$

Note that $t$ is only valid in the sense of generalized tiles. However, $t$ can be arbitrarily well approximated by a non-generalized tile with area and crown size arbitrarily close to $|t|$ and $|C_t|$, respectively, by densely placing an increasing number of points on the hyperbola. ◀

## 6 Upper Bound

To show Theorem 2, we construct a point set where TilePacking covers at most roughly $(1 - e^{-2})/2$. Our goal is to construct a tile $\hat{t}$ at the origin where each maximal rectangle has the same size $A$. We therefore place $2k + 1$ points $q_i$ into $\mathcal{U}$ densely on a hyperbola $h_A$ centered at the origin. The remaining tiles will have a density close to 1/2.

Such tiles can be realized when they have two nearly equal-sized maximal rectangles with minimum overlap. Hence, we add for each point $q_i$ on $h_A$ a set of (almost) evenly spaced points $p_{i,j}$ with distance roughly $\varepsilon$ between each other. To get two maximal rectangles of roughly same area for each such tile, the exact coordinates of the points $p_{i,j}$ must be chosen carefully, as the placement of such points influences the size of maximal rectangles for surrounding points. That is why we place the points on arcs of functions $f_i$ described by differential equations, where each $f_i$ depends on the two neighboring curves $f_{i-1}$ and $f_{i+1}$.

We may only use finitely many points $q_i \in h_A$ and $p_{i,j} \in f_i$. Both discretizations introduce an error term. Exploiting our choice of the functions $f_i$ and how they relate to each other, we can show that both error terms vanish as $k$ goes to infinity and $\varepsilon$ goes to zero.

To aid the analysis, we rotate $\mathcal{U}$ by $\pi/4$ around the origin, obtaining a *rotated unit square* $\mathcal{U}_r$ (see Figure 14). Here, TilePacking processes the points from right to left.

We start by formally defining the point set $P_{\varepsilon,k}$ and the functions $f_i$. Let $h_A = \{ (x, y) \in \mathcal{U}_r \mid x^2 - y^2 = 2A \}$ be the right branch of a hyperbola centered at the origin that lives in $\mathcal{U}_r$. First we define the upper part of the construction with non-negative $y$ coordinates. For $i = 0, \dots, k-1$, densely choose $k$ points $q_i \in h_A$ such that $y(q_i) \geq 0$ and $\sqrt{2A} = x(q_0) < x(q_1) < \cdots < x(q_{k-1})$. Define further $f_0(x) = 0$ and $f_k(x) = \sqrt{2} - x$. For $0 < i < k$, define $f_i : [0, \sqrt{2}) \to \mathbb{R}$ using

$$f_i(x(q_i)) = y(q_i) \text{ and } \tag{3}$$

$$f_i'(x) = \begin{cases} -1 & \text{for } x \leq x(q_i) \\ 1 - 2\dfrac{f_i(x) - f_{i-1}(x)}{f_{i+1}(x) - f_{i-1}(x)} & \text{for } x > x(q_i). \end{cases} \tag{4}$$

This means, each $f_i$ with $0 < i < k$ has slope $-1$ in $[0, x(q_i))$, then it intersects $h_A$ at $q_i$ according to Equation (3), and then it has a slope depending on the current values of $f_{i-1}, f_i$ and $f_{i+1}$ according to Equation (4). For the symmetric part with negative $y$ coordinates we define $q_{-i} = (x(q_i), -y(q_i))$ and $f_{-i}(x) = -f_i(x)$ for $0 < i \leq k$. Observe that, for $0 \leq i < k$, we get $q_{-i} \in h_A$ and the $f_{-i}$ adhere to Equation (4). We are now ready to define the point set $P_{\varepsilon,k}$ for $\varepsilon > 0$ and $k \in \mathbb{N}$ as

$$P_{\varepsilon,k} = \{ (0,0) \} \cup \bigcup_{i,j \in \mathbb{Z}} \left\{ (j\delta,\ f_i(j\delta)) \;\middle|\; -k < i < k, x(q_i) \leq j\delta < \sqrt{2} \right\}.$$

We require that $q_i \in P_{\varepsilon,k}$ for all $-k < i < k$, so we choose $\varepsilon$ such that it divides all $x(q_i)$.

In order to be able to choose $P_{\varepsilon,k}$ as shown above, we need that the $f_i$ are well-behaved: They must be defined in $[0, \sqrt{2})$, and should only intersect $h_A$ at $q_i$. Intuitively, this is true, since the differential equation drives each function $f_i$ to the midpoint of the functions $f_{i-1}$ and $f_{i+1}$. The proof is given in the following lemma.

▶ **Lemma 31.** *Each function $f_i$ intersects $h_A$ exactly once, namely at $q_i$. Furthermore, $f_i(x)$ is differentiable for all $i = -k, \dots, k$, and $f_{-k}(x) < \cdots < f_k(x)$ holds for all $x \in [0, \sqrt{2})$.*[4]

We are now able to show Lemma 32: All tiles $t \neq \hat{t}$ have a density of close to $1/2$, unless they are too close to the right corner of $\mathcal{U}_r$, in which case their area is negligible. Afterwards, it only remains to optimize the parameter $A$, which is done in Theorem 33.

▶ **Lemma 32.** *Let $u, k > 0$ and $\hat{\mathcal{U}} = (\bigcup_{p \in P_{\varepsilon,k}, x(p) \leq \sqrt{2}-u} t_p) \setminus \hat{t}$. Then TilePacking covers $|\hat{\mathcal{U}}|/2 + c_k(\varepsilon)$ area in $\hat{\mathcal{U}}$ for $P_{\varepsilon,k}$ where $\lim_{\varepsilon \to 0} c_k(\varepsilon) = 0$.*

[4] Due to space limitations, the proof can only be found in the arXiv version of this paper (see [11]).

**Proof.** Consider a point $p \neq (0,0)$, $x(p) \leq \sqrt{2} - u$ that lies on some curve $f_i$ and creates the tile $t$. Assuming $\varepsilon < u$, there exists another point $p' = (x(p) + \varepsilon, f_j(x(p) + \varepsilon)) \in P_{\varepsilon,k}$. By Lemma 31, we can assume that $f_{i+1}(x) - f_i(x) > \varepsilon$ for all $i = -k, \ldots, k-1$, $x \leq \sqrt{2} - u$. It follows from $|f_i'| \leq 1$ that $p'$ is a lower staircase point of $t$. (For the same reason there cannot be points further to the right that are lower staircase points.) The tile $t$ is therefore only restricted by the tiles from points with $x$-coordinate $x(p')$. Therefore, $t$ has exactly two maximal rectangles which TilePacking can choose from (see Figure 14).

TilePacking will choose the larger one of the two maximal rectangles (call them $R_1$ and $R_2$). Since there are multiple points with the same $x$-coordinate as $p$, any of them can be processed first by TilePacking. This gives rise to areas $Z_w$,$Z_h$ that may be covered by the tile $t$ or by tiles directly above or below it (see Figure 14). W.l.o.g. we assume that TilePacking covers both $Z_w$ and $Z_h$ when choosing the maximal rectangle for $t$ (since assuming this for all such tiles may only increase the covered area).

Since all $f_i$ are differentiable in $[0, \sqrt{2})$, Taylor's Theorem provides a function $g(x)$ with $\lim_{x \to 0} g(x) = 0$ such that $f_i(x+\varepsilon) = f_i(x) + f_i'(x) \cdot \varepsilon + g(\varepsilon) \cdot \varepsilon$. Denote by $w, h$ the dimensions of the rectangle $X = R_1 \cap R_2$. Then $w = (\varepsilon + f_i(x(p) + \varepsilon) - f_i(x(p)))/\sqrt{2} = (\varepsilon + f_i'(x(p))\varepsilon + g(\varepsilon)\varepsilon)/\sqrt{2} = (1 + f_i'(x(p)) + g(\varepsilon))\varepsilon/\sqrt{2}$ and similarly $h = (1 - f_i'(x(p)) - g(\varepsilon))\varepsilon/\sqrt{2}$.

From $|f_i'| \leq 1$ for all $f_i$, one can easily see that the rectangles $X, Z_w$ and $Z_h$ have widths and heights in $\mathrm{O}(\varepsilon)$, giving them a total area of $W := X + Z_w + Z_h = \mathrm{O}(\varepsilon^2)$.

The tile $t$ also contains two additional rectangles with an area of $Y_w = w((f_i(x(p)) - f_{i-1}(x(p)))/\sqrt{2} - h)$ and $Y_h = h((f_{i+1}(x(p)) - f_i(x(p)))/\sqrt{2} - w)$. Note that this also holds if $x(q_{i\pm1}) > x(p)$, as we extended $f_{i\mp1}$ with lines of slope $\pm 1$. In this case the two rectangles are restricted by $q_{i\mp1}$'s tile, respectively. Hence, when evaluating the functions at $x(p)$:

$$\begin{aligned}
|Y_w - Y_h| &= |w(f_i - f_{i-1})/\sqrt{2} - h(f_{i+1} - f_i)/\sqrt{2}| \\
&= |(1 + f_i' + g(\varepsilon))(\varepsilon(f_i - f_{i-1}))/2 - (1 - f_i' - g(\varepsilon))(\varepsilon(f_{i+1} - f_i))/2| \\
&= |(f_{i+1}(g(\varepsilon) + f_i' - 1) - f_{i-1}(g(\varepsilon) + f_i' + 1)) + 2f_i| \cdot \varepsilon/2 \\
&= |(g(\varepsilon)(f_{i+1} - f_{i-1}) + f_i'(f_{i+1} - f_{i-1}) - f_{i-1} - f_{i+1} + 2f_i)| \cdot \varepsilon/2 \\
&= |g(\varepsilon)||f_{i+1} - f_{i-1}| \cdot \varepsilon/2 \leq |g(\varepsilon)| \cdot \varepsilon\sqrt{2},
\end{aligned}$$

where the last inequality holds by Lemma 31, which gives us $f_{i+1}(x) - f_{i-1}(x) \leq f_k(x) - f_{-k}(x) \leq 2\sqrt{2}$ for $x \in [0, \sqrt{2})$.

W.l.o.g. assume $Y_w > Y_h$. Then for the tile $t$ with area $|t| = W + Y_w + Y_h$, TilePacking covers at most $W + Y_w \leq W + Y_w/2 + (Y_h + |g(\varepsilon)| \cdot \varepsilon\sqrt{2})/2 = |t|/2 + \mathrm{O}(\varepsilon(|g(\varepsilon)| + \varepsilon))$. As $\hat{\mathcal{U}}$ is the union of such tiles and $|P_{\varepsilon,k}| = \mathrm{O}(k/\varepsilon)$, we have a total coverage of $|\hat{\mathcal{U}}|/2 + \mathrm{O}(k(|g(\varepsilon)| + \varepsilon))$. This immediately gives us the function $c_k(\varepsilon) = \mathrm{O}(k(|g(\varepsilon)| + \varepsilon))$ with $\lim_{\varepsilon \to 0} c_k(\varepsilon) = 0$. ◀

▶ **Theorem 33.** *TilePacking has no better lower bound than* $(1 - e^{-2})/2$.

**Proof.** We analyze the area $\rho$ covered by TilePacking on $P_{\varepsilon,k}$ for some fixed $k$ and $u$ as $\varepsilon$ approaches 0. The bound then follows from letting $k$ go to $\infty$ and $u$ go to 0.

By Lemma 32, TilePacking covers half of $\hat{\mathcal{U}} = (\bigcup_{p \in P_{\varepsilon,k}, x(p) \leq \sqrt{2}-u} t_p) \setminus \hat{t}$ (plus $c_k(\varepsilon)$ that approaches 0 for $\varepsilon \to 0$) for each $u > 0$. Additionally, at most $u^2$ area is covered from all tiles at points $p$ with $x(p) > \sqrt{2} - u$. TilePacking covers $A + Q$ area in $\hat{t}$, where an error term $Q$ is introduced because the $q_i$ points only provide an approximation of $h_A$. $Q$ can easily be bounded by, e.g., $Q \leq \max_i(x(q_i) - x(q_{i-1})) + \max_i(y(q_i) - y(q_{i-1}))$ (the biggest rectangular strip that fits between two consecutive $q_i$ points, in $\mathcal{U}$). (Note that all $q_i$ lie in $P_{\varepsilon,k}$, so no additional error is introduced.) In total, using $E = Q + c_k(\varepsilon) + u^2$,

$$\rho \le A + |\hat{\mathcal{U}}|/2 + E \le A + (1 - |\hat{t}|)/2 + E \le A + (1 - (A + \int_A^1 \frac{A}{x}\,\mathrm{d}x))/2 + E$$
$$\le (1 + A + A\ln A)/2 + E.$$

Minimizing the last term leads to $\rho \le (1 - e^{-2})/2 + E$ at $A = e^{-2}$. $Q$ approaches 0 when $k \to \infty$ since the $q_i$ lie densely on $h_A$. Hence $E$ approaches 0 for large $k$ and small $u, \varepsilon$. ◀

## 7 Conclusion

We have shown that TilePacking's worst-case coverage lies between 39% and 43.3%. Our lower bound substantially improves over the previous best lower bound of roughly 9.1% [12], while our upper bound is the first non-trivial upper bound for TilePacking. Note that both bounds easily transfer to the (arguably more natural) GreedyPacking algorithm [12].

Our analysis crucially relies on a novel charging scheme and on a new analysis framework. The latter reduces the task of proving good coverage to finding good lower bounds on the tiles' charging ratios (see Section 3). The versatility of this approach shows in the fact that already a comparatively simple and short analysis yields a lower bound of 25%. Moreover, our approach provides structural insights: e.g., it allows us to characterize the exact shape of worst-case tiles as a function of their density (see Figures 8 and 9). We believe that our framework might help to analyze similar algorithms for (variants of) LLARP.

Concerning the remaining gap of size roughly 4 percentage points between our bounds, we believe that both bounds can be improved. For the lower bound, one shortcoming of our analysis is that tiles are analyzed individually, ignoring their local relationships in the unit square. Our lower bound basically predicts that the worst-case instance of TilePacking should consist solely of tiles whose shapes resemble Figure 8, which seems impossible.

Regarding the upper bound, there is still a noticeable gap between the maximal area that is coverable by the crowns (see Figure 4) and the area into which the crowns fall in our upper bound construction (see Figure 14). In particular, our construction uses only one tile (in the origin) with a large charging ratio, while all other tiles can be shown to have a charging ratio of roughly 1. Also note that our upper bound construction might be of interest with respect to the approximation variant of LLARP: an optimal solution should be able to fill most of the unit square, such that our results would imply a corresponding bound on the approximation ratio of TilePacking.

We leave as a major open question to find new algorithms for LLARP that might tackle the 50% conjecture. Note that our upper bound is tailored towards a specific greedy algorithm, so there is reasonable hope that other (possibly also greedy) algorithms might still achieve a coverage of 50%.

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