Disorders and Permutations

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Abstract

The *additive x-disorder* of a permutation is the sum of the absolute differences of all pairs of consecutive elements. We show that the additive x-disorder of a permutation of $S(n)$, $n \geq 2$, ranges from $n-1$ to $\lfloor n^2/2 \rfloor - 1$, and we give a complete characterization of permutations having extreme such values. Moreover, for any positive integers n and d such that $n \geq 2$ and $n-1 \leq d \leq \lfloor n^2/2 \rfloor - 1$, we propose a linear-time algorithm to compute a permutation $\pi \in S(n)$ with additive x-disorder d.

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1 Introduction

Here we follow a young researcher in computer science who is about to pass an audition for a permanent position in a prestigious university. As she arrived early in the main building of the university, she decides to use one of the elevators to change her mind before reaching the audition room on time. The chosen elevator has n buttons to move to the floor 1, 2, ..., n of the building. To move from a floor a to a floor b, the elevator takes $|b-a|$ seconds, regardless of whether it goes up or down. Our candidate, loving the challenge herself, decides to visit all floors once and only once each. Knowing that she arrived d seconds early, how can she propose a route that takes exactly that long? And for which values d is there at least one solution? It is assumed that the candidate can reach the initial floor of her ballad instantly from the university entry hall and reach the dreaded audition room instantly from the last visited floor. Fig. 1 shows an example.

We tackle this combinatorial problem by studying additive disorders of permutations. Let $\pi \in S(n)$ be a permutation of size $n \geq 2$. The *x-difference sequence* of π is the $(n-1)$-sequence constructed by considering the absolute difference of all pairs of adjacent letters of π, and its *y-difference sequence* is constructed by considering all distances between two consecutive values in π. Moreover, the *additive x-disorder* of π is the sum of the integers in its x-difference sequence and the *additive y-disorder* of π is the sum of the integers in its y-difference sequence. For example, the x-difference sequence of $\pi = 514263 \in S(6)$ is $(4,3,2,4,3)$, its y-difference sequence is $(2,2,3,2,4)$, its additive x-disorder is 16, and its additive y-disorder is 13.

These values associated with permutations are actually statistics: they are maps from combinatorial objects to integers. The literature in algorithmic and combinatorics abounds with examples and studies of similar statistics on permutations. One can cite for instance the major index [5], the inversion number [4], the total displacement [4] (Problem 5.1.1.28), the descent number [2], and the number of cycles [6] of permutations. The present paper is intended to be a first study of these just described disorder statistics.

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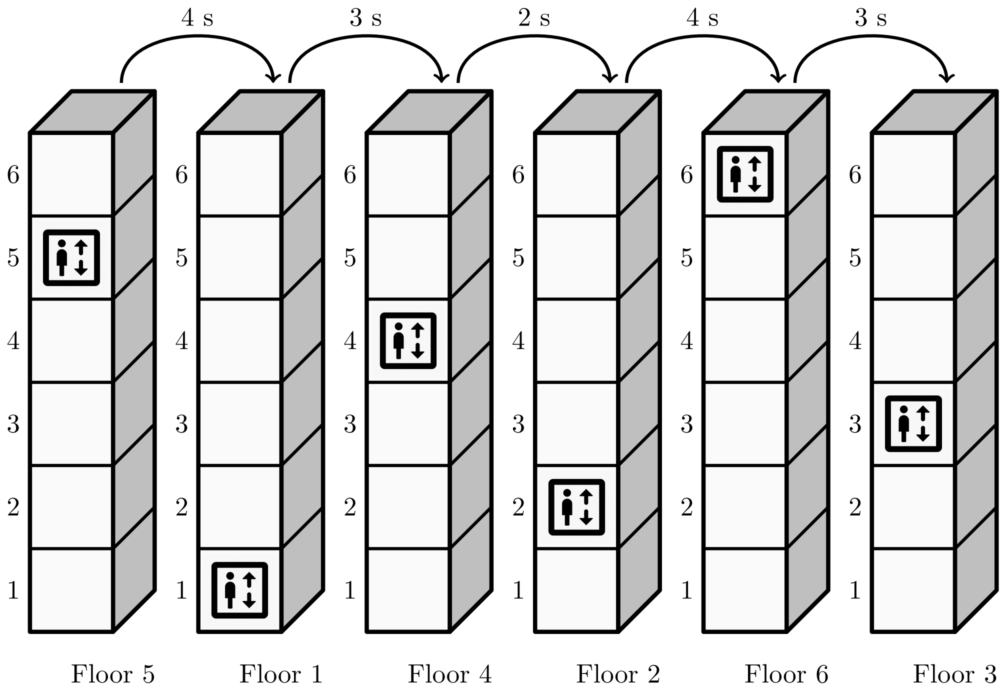


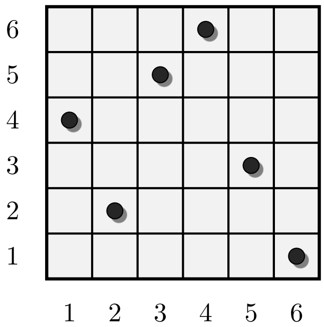
Figure 1 The candidate has to visit exactly each of the six floors of the building. She visits them in following order: 5, 1, 4, 2, 6, and 3. The total duration is 16 s. The candidate would have achieved the maximum duration by visiting floors in the following order: 4, 1, 5, 2, 6, and 3. In this case, the total duration is 17 s (the solution is not unique as the order 3, 5, 1, 6, 2 and 4 provides another solution with total duration 17 s).

The maximum additive x-disorder of a permutation in $S(n)$, $n \geq 2$, is given by Sequence A047838 of the OEIS[1]. More precisely, this sequence is concerned with maximum additive y-disorder, but as we will show soon, the maximum additive x-disorder and the maximum additive y-disorder of permutations in $S(n)$ coincide. It is conjectured[2] that the maximum additive x-disorder of a permutation in $S(n)$ is $\lfloor n^2/2 \rfloor - 1$. We prove that the conjecture is correct.

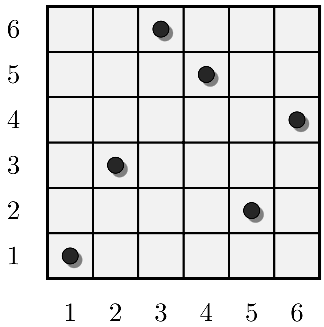
Given an $(n-1)$-sequence of positive integers D, it is shown in [3] that deciding whether there exists some permutation $\pi \in S(n)$ such that D is the x-difference sequence of π is NP-complete. Pursuing this line of research, we complement [3] by showing that for any integer d with $n-1 \leq d \leq \lfloor n^2/2 \rfloor - 1$, there exists a permutation $\pi \in S(n)$ with additive x-disorder d. The proof is constructive. Note that, given an n-sequence of positive integers $D = (d_1, d_2, \ldots, d_n)$, deciding whether there exist two permutations $\pi, \sigma \in S(n)$ such that $d_i = \pi(i) + \sigma(i)$ for $1 \leq i \leq n$ is NP-complete [7].

This paper is organized as follows. Section 2 gives concise background and notation for the disorder setting. We prove that the maximum additive x-disorder of a permutation in $S(n)$ is $\lfloor n^2/2 \rfloor - 1$ in Section 3, and in Section 3.4 that there exists a permutation that achieves any legal additive x-disorder (our approach is constructive).

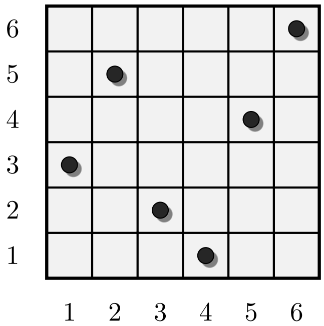
[1] `https://oeis.org/A047838`
[2] More precisely, the upper bound relies on correctness of Sequence A007590 of the OEIS.



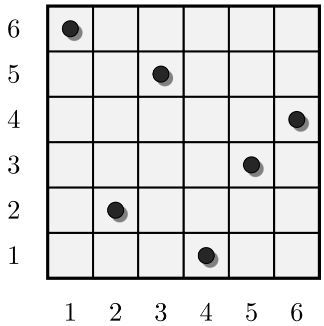
(a) $\pi = 425631$



(b) $\pi^{\mathbf{r}} = 136524$



(c) $\pi^{\mathbf{c}} = 352146$



(d) $\pi^{\mathbf{i}} = 625134$

Figure 2 Trivial bijections of $\pi = 425631$.

2 Definitions

For any non-negative integer n, we let $[n]$ stand for the set $\{1, 2, \dots, n\}$. A *permutation* of size n is a one-to-one mapping $[n] \to [n]$. The set of all permutations of size n is denoted by $S(n)$. For a permutation $\pi \in S(n)$, we write $\pi(i)$ for the integer at position i, $i \in [n]$.

Let $\pi \in S(n)$. The *reverse* of π is the permutation $\pi^{\mathbf{r}}$ defined by $\pi^{\mathbf{r}}(i) = \pi(n - i + 1)$ for every $i \in [n]$. The *complement* of π is the permutation $\pi^{\mathbf{c}}$ defined by $\pi^{\mathbf{c}}(i) = n - \pi(i) + 1$ for every $i \in [n]$. The *inverse* is the regular group theoretical inverse on permutations, $\pi^{\mathbf{i}}$ is defined by $\pi^{\mathbf{i}}(i) = j$ if and only if $\pi(j) = i$ for every $i \in [n]$. See Fig. 2. The reverse, complement, and inverse are called the *trivial bijections* from $S(n)$ to itself [1].

Let $\pi \in S(n)$, $n \geq 2$. The *x-difference sequence* [3] of π, denoted $\Delta_x(\pi)$, is the $(n-1)$-sequence defined by

$$\Delta_x(\pi) = (|\pi(2) - \pi(1)|, |\pi(3) - \pi(2)|, \dots, |\pi(n) - \pi(n-1)|)\,.$$

The *y-difference sequence* of π, denoted $\Delta_y(\pi)$, is the $(n-1)$-sequence defined by $\Delta_y(\pi) = \Delta_x(\pi^{\mathbf{i}})$. See Fig. 3 for an illustration.

The *additive x-disorder* (resp. *additive y-disorder*) of π, denoted by $\delta_x^+(\pi)$ (resp. $\delta_y^+(\pi)$), is defined by $\delta_x^+(\pi) = \sum_{d \in \Delta_x(\pi)} d$ (resp. $\delta_y^+(\pi) = \sum_{d \in \Delta_y(\pi)} d$).

▶ **Example 1.** See Fig. 3 for two examples. Besides, by setting $\pi = 251463$, we have $\Delta_x(\pi) = (3, 4, 3, 2, 3)$, $\Delta_y(\pi) = (2, 5, 2, 2, 3)$, $\delta_x^+(\pi) = 3 + 4 + 3 + 2 + 3 = 15$ and $\delta_y^+(\pi) = 2 + 5 + 2 + 2 + 3 = 14$.

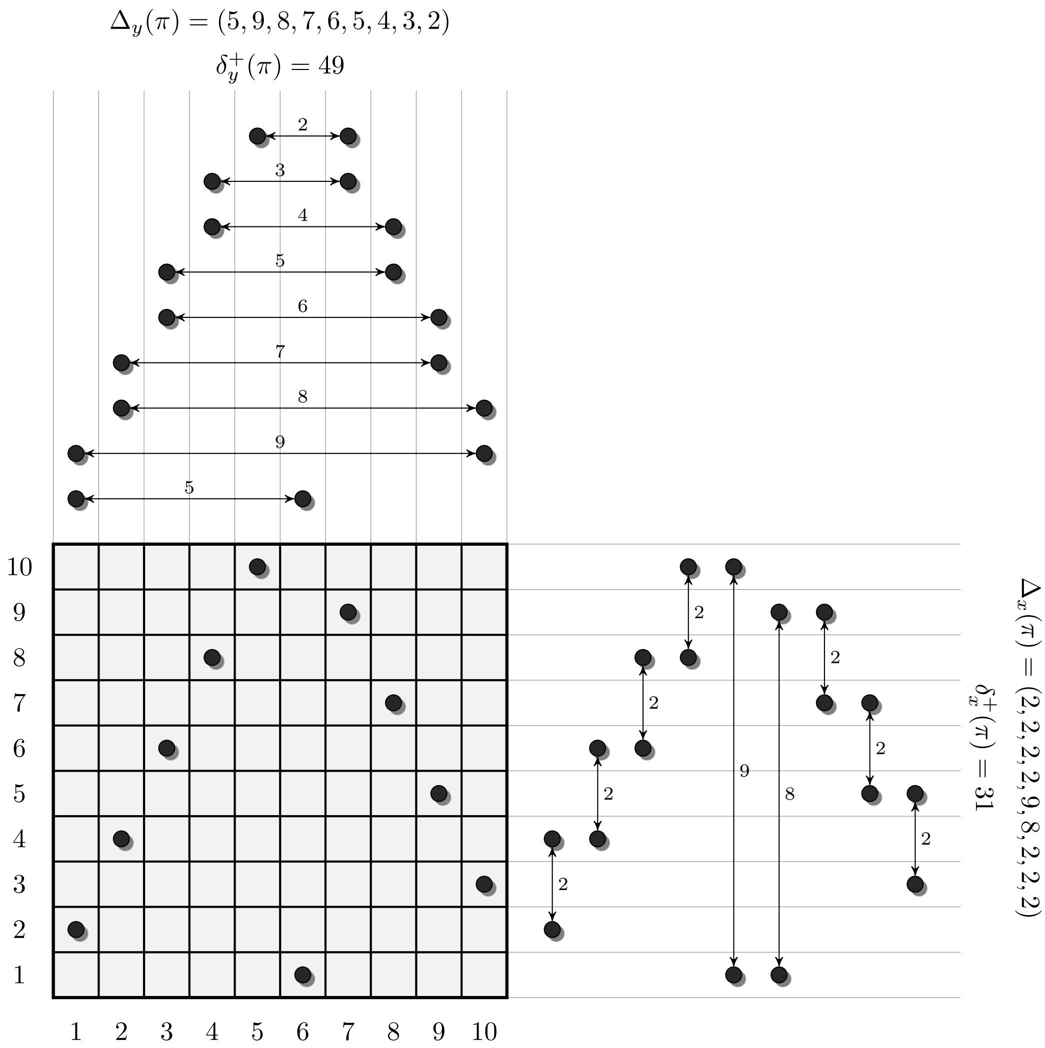


Figure 3 The x-difference sequence, the y-difference sequence, the additive x-disorder, and the additive y-disorder of the permutation $\pi = 2468A19753$ ("A" stands for "10").

3 Bounds on Additive Disorder

In this section, we show that the additive x-disorder and y-disorder of a permutation of $S(n)$ ranges from $n-1$ to $\lfloor n^2/2 \rfloor - 1$. More precisely, we give a complete characterization of permutations having extreme such values (Theorems 3 and 8), and show that every value in this range is the additive x-disorder or y-disorder of some permutation (Theorem 10).

3.1 Basic properties

▶ **Lemma 2.** *For every $\pi \in S(n)$, $n \geq 2$, the four following assertions hold:*

1. $(\Delta_x(\pi))^{\mathbf{r}} = \Delta_x(\pi^{\mathbf{r}})$;
2. $\Delta_x(\pi) = \Delta_x(\pi^{\mathbf{c}})$;
3. $\delta_x^+(\pi) = \delta_y^+(\pi^{\mathbf{i}})$;
4. $\delta_y^+(\pi) = \delta_x^+(\pi^{\mathbf{i}})$.

Proof. The equality $(\Delta_x(\pi))^{\mathbf{r}} = \Delta_x(\pi^{\mathbf{r}})$ is obvious. As for $\Delta_x(\pi) = \Delta_x(\pi^{\mathbf{c}})$, it is enough to observe that, for every $1 \le i < n$,

$$\begin{aligned}|\pi(i+1) - \pi(i)| &= |\pi(i+1) - (n+1) - \pi(i) + (n+1)| \\ &= |(n - \pi(i+1) + 1) - (n - \pi(i) + 1)| \\ &= |\pi^{\mathbf{c}}(i+1) - \pi^{\mathbf{c}}(i)|\,.\end{aligned}$$

The last two assertions are direct consequences of the definition of the y-difference sequence of π as the x-difference sequence of the inverse of π. ◀

The last two assertions of Lemma 2 imply that all results about additive x-disorders of permutations can be rephrased in terms of additive y-disorders and conversely. For this reason, in what follows we shall focus on additive x-disorder and refer to it simply as *additive disorder*.

3.2 Minimum disorder

▶ **Theorem 3.** *The minimum possible additive disorder of a permutation of $S(n)$ is $n-1$. It is attained exactly by the identity permutation and its reverse.*

Proof. In any permutation $\pi \in S(n)$, $|\pi(i+1) - \pi(i)| \ge 1$, so $\delta_x^+(\pi) \ge n-1$. The bound is reached if $\pi(i+1) \in \{\pi(i+1), \pi(i-1)\}$ for all i. In particular, if $\pi(i) \in \{1, n\}$, then $i \in \{1, n\}$ as well (otherwise one of $\pi(i-1), \pi(i+1)$ would not be at distance 1 from $\pi(i)$). Assume $\pi(1) = 1$, then for each j, $\pi(j) = j$ (by induction, $\pi(j+1) \in \{\pi(j)+1, \pi(j)-1\}$, and $\pi(j+1) \ne \pi(j-1) = \pi(j) - 1$). So π is the identity. Similarly if $\pi(1) = n$, then π is the reverse of the identity. ◀

3.3 Maximum disorder

A permutation $\pi \in S(n)$ is *bipartite with threshold k* if $k \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}$ and for every $i \in [n-1]$, either $\pi(i) \le k$ and $\pi(i+1) > k$, or $\pi(i) > k$ and $\pi(i+1) \le k$. Such a permutation has *centered endpoints* if $\{\pi(1), \pi(n)\}$ is either $\{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1\}$ (if $k = \lceil n/2 \rceil$) or $\{\lceil n/2 \rceil, \lceil n/2 \rceil + 1\}$ (if $k = \lfloor n/2 \rfloor$).

▶ **Example 4.** The permutation $\pi = 25371648$ of $S(8)$ is bipartite with threshold 4 and has no centered endpoints. The permutation $\pi = 46172835$ of $S(8)$ is bipartite with threshold 4 and has centered endpoints. The permutation $\pi = 41523$ of $S(5)$ is bipartite with threshold 2 and has centered endpoints. The permutation $\pi = 34152$ of $S(5)$ is bipartite with threshold 3 and has centered endpoints.

We say that permutation π has pattern P_1, P_2, or P_3 if it satisfies the following properties, respectively (see Fig. 4, we then show that as forbidden patterns they characterize permutations with maximal disorder):

- Pattern P_1 (*extreme endpoint*). There is $j \in [n-1]$ such that
  1. $\pi(1) < \pi(j) < \pi(j+1)$,
  2. or $\pi(1) > \pi(j) > \pi(j+1)$,
  3. or $\pi(j) < \pi(j+1) < \pi(n)$,
  4. or $\pi(j) > \pi(j+1) > \pi(n)$.
- Pattern P_2 (*two separated pairs*). There are $i, j \in [n-1]$ and $k \in [n]$ such that
  1. $\pi(i), \pi(i+1) \le k$,
  2. and $\pi(j), \pi(j+1) > k$.

- Pattern P_3 (*three in a row*). There is $j \in [n-2]$ such that
  1. $\pi(j) < \pi(j+1) < \pi(j+2)$,
  2. or $\pi(j) > \pi(j+1) > \pi(j+2)$.

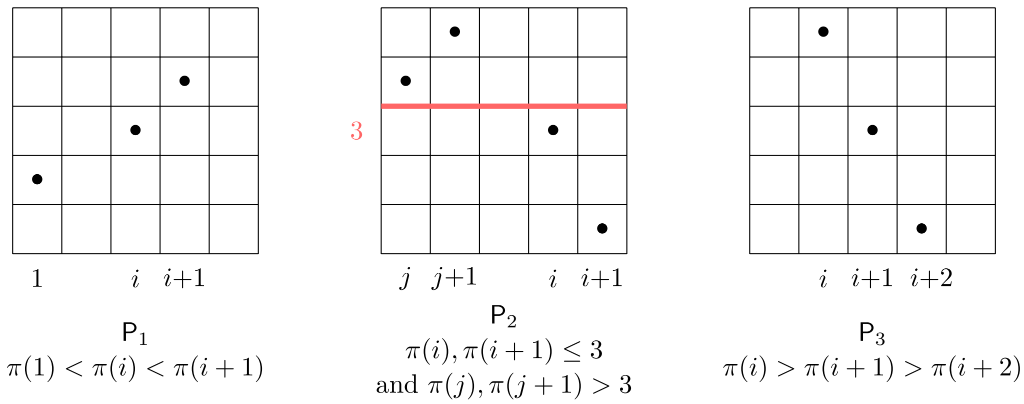


Figure 4 Forbidden patterns in maximal disorder permutations.

▶ **Lemma 5.** *A permutation that does not have patterns P_1, P_2, and P_3 is bipartite and has centered endpoints.*

Proof. Let $a = \max_{i\in[n-1]}(\min\{\pi(i), \pi(i+1)\})$ and $b = \min_{i\in[n-1]}(\max\{\pi(i), \pi(i+1)\})$.

If $b < a$, then for some i, j we have $\pi(i), \pi(i+1) \leq b < a \leq \pi(j), \pi(j+1)$, *i.e.*, π has pattern P_2.

If $a = b$, let j such that $\pi(j) = a = b$. Then one of $\pi(j-1)$, $\pi(j+1)$ must be larger than $\pi(j)$ (by definition of a), and the other must be smaller than $\pi(j)$ (by definition of b). In particular, $j \neq 1, n$ and π has pattern P_3.

If $a > b$. Let $A = \{h \mid \pi(h) \leq a\}$ and $B = \{h \mid \pi(h) \geq b\}$. Then A and B are disjoint, $a = |A|$ and $b = n+1-|B|$. Moreover, each set $\{i, i+1\}$ contains one element in A and one in B, so A, B is a partition of $[n]$ (in other words, $b = a+1$), and $a = |A| \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}$. Overall for every $i \neq n$, $\max\{\pi(i), \pi(i+1)\} > a$ and $\min\{\pi(i), \pi(i+1)\} \leq a$, so π is bipartite with threshold a.

To show that endpoints are centered, first note that if n is even, then $\{1, n\}$ has one element in A, the other in B. If n is odd, either $a = \lceil n/2 \rceil$ and $\{1, n\} \subseteq A$, or $a = \lfloor n/2 \rfloor$ and $\{1, n\} \subseteq B$.

If $\pi(1) < a$ let h be any position such that $\pi(1) < \pi(h) \leq a$. Then $h = n$ (otherwise, by definition of a, $\pi(1) < \pi(h) < \pi(h+1)$ and π has pattern P_1 version 1). So in particular, there can be only one such value of h, so $\pi(1) = a-1$. Furthermore, $\{1, n\} \subseteq A$ so n is odd and $\{\pi(1), \pi(n)\} = \{a, a-1\} = \{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1\}$, so π has centered endpoints.

Similarly if $\pi(1) > b$, we have $\pi(1) = b+1$, $\pi(n) = b$ and n is odd with $\{\pi(1), \pi(n)\} = \{\lceil n/2 \rceil, \lceil n/2 \rceil + 1\}$ and π has centered endpoints.

The same arguments apply if $\pi(n) < a$ or $\pi(n) > b$, so the only case left is $\{\pi(1), \pi(n)\} = \{a, b\}$, which yields that n is even and $\{\pi(1), \pi(n)\} = \{\frac{n}{2}, \frac{n}{2}+1\}$, so π has centered endpoints. ◀

▶ **Lemma 6.** *If π has one of patterns P_1, P_2, or P_3, then there exists π' such that $\delta_x^+(\pi') > \delta_x^+(\pi)$.*

Proof. Pattern P_1 version 1: for some j, $\pi(1) < \pi(j) < \pi(j+1)$. Let π' be the permutation obtained from π by reversing positions 1 to j. Then $\delta_x^+(\pi') = \delta_x^+(\pi) - (\pi(j+1) - \pi(j)) + (\pi(j+1) - \pi(1)) = \delta_x^+(\pi) + \pi(j) - \pi(1) > \delta_x^+(\pi)$. Patterns P_1 versions 2, 3, and 4 are symmetrical.

Pattern P_2: Depending on the relative order of i and j, of $\pi(i)$ and $\pi(i+1)$, and of $\pi(j)$ and $\pi(j+1)$ we have a total of eight cases to check. Assuming $i < j$ leaves the four following alternatives, which correspond to two distinct patterns, up to symmetry:

- Pattern P_2' (*monotonous pairs*). There are $i < j \in [n-1]$ such that
  1. $\pi(i) < \pi(i+1) < \pi(j) < \pi(j+1)$
  2. or $\pi(i+1) < \pi(i) < \pi(j+1) < \pi(j)$.
- Pattern P_2'' (*non-monotonous pairs*).There are $i < j \in [n-1]$ such that
  1. $\pi(i) < \pi(i+1) < \pi(j+1) < \pi(j)$
  2. or $\pi(i+1) < \pi(i) < \pi(j) < \pi(j+1)$.

Pattern P_2' version 1: for some $i < j$, $\pi(i) < \pi(i+1) < \pi(j) < \pi(j+1)$. Let π' be the permutation obtained from π by reversing positions $i+1$ to j. Then $\delta_x^+(\pi') = \delta_x^+(\pi) - (\pi(i+1) - \pi(i)) - (\pi(j+1) - \pi(j)) + (\pi(j) - \pi(i)) + (\pi(j+1) - \pi(i+1)) = \delta_x^+(\pi) + 2(\pi(j) - \pi(i+1)) > \delta_x^+(\pi)$.

Pattern P_2'' version 1: for some $i < j$, $\pi(i) < \pi(i+1) < \pi(j+1) < \pi(j)$. Let π' be the permutation obtained from π by reversing positions $i+1$ to j. Then $\delta_x^+(\pi') = \delta_x^+(\pi) - (\pi(i+1) - \pi(i)) - (\pi(j) - \pi(j+1)) + (\pi(j) - \pi(i)) + (\pi(j+1) - \pi(i+1)) = \delta_x^+(\pi) + 2(\pi(j+1) - \pi(i)) > \delta_x^+(\pi)$.

This completes the proof of P_2.

Pattern P_3 version 1: for some j, $\pi(j) < \pi(j+1) < \pi(j+2)$. Let π' be the permutation obtained from π by moving $\pi(j+1)$ to position 1. Then $\delta_x^+(\pi') = \delta_x^+(\pi) - (\pi(j+1) - \pi(j)) - (\pi(j+2) - \pi(j+1)) + (\pi(j+2) - \pi(j)) + |\pi(j+1) - \pi(1)| = \delta_x^+(\pi) + |\pi(j+1) - \pi(1)| > \delta_x^+(\pi)$. Pattern P_3 version 2 is symmetrical. ◀

▶ **Lemma 7.** *The additive disorder of a bipartite permutation $\pi \in S(n)$ is*

$$\delta_x^+(\pi) = \lfloor n^2/2 \rfloor - |\pi(1) - \lceil n/2 \rceil| - |\pi(n) - \lceil n/2 \rceil| .$$

Proof. Let $m = \lfloor n/2 \rfloor$, and k be a threshold for which π is bipartite. If n is even then $\lfloor n^2/2 \rfloor - 1 = 2m^2 - 1$, and for $n = 2m+1$, $\lfloor n^2/2 \rfloor - 1 = \frac{1}{2}((2m+1)^2 - 1) - 1 = 2m^2 + 2m - 1$

By the definition of bipartite, for any i, $|\pi(i+1) - \pi(i)| = |\pi(i) - k| + |\pi(i+1) - k|$. Thus,

$$\begin{aligned} \delta_x^+(\pi) + |\pi(1) - k| + |\pi(n) - k| &= |\pi(1) - k| + \left(\sum_{i=1}^{n-1} |\pi(i+1) - \pi(i)| \right) + |\pi(n) - k| \\ &= 2 \sum_{i=1}^{n} |\pi(i) - k|. \end{aligned}$$

We introduce the partition $H \cup L$ of $[1, n]$ as $L = \{i \mid \pi(i) \leq k\}$ and $H = \{i \mid \pi(i) > k\}$ (in particular, $|L| = k$ and $|H| = n - k$). Note that $i \to \pi(i) - k$ is a bijection between H and $[1, n-k]$, and that $i \to k - \pi(i)$ is a bijection between L and $[0, k-1]$. We have

$$\sum_{i \in H} |\pi(i) - k| = \sum_{i \in H} (\pi(i) - k) = \sum_{j=1}^{n-k} j = (n-k)(n-k+1)$$

and

$$\sum_{i \in L} |\pi(i) - k| = \sum_{i \in L} (k - \pi(i)) = \sum_{j=0}^{k-1} j = (k-1)k.$$

So overall

$$\begin{aligned} 2\sum_{i=1}^{n} |\pi(i) - k| &= (n-k)(n-k+1) + (k-1)k \\ &= (n-k)^2 + k^2 + n - 2k. \end{aligned}$$

If n is even ($n = 2m$), then $k = n - k = m$ and $n - 2k = 0$.

$$2\sum_{i=1}^{n} |\pi(i) - k| = 2m^2 \qquad = \frac{n^2}{2}.$$

Also, $|\pi(i) - k| = |\pi(i) - \lceil n/2 \rceil|$ for $i = 1$ and $i = n$, so this concludes the proof when n is even.

If n is odd ($n = 2m + 1$), k can be m or $m + 1$. If $k = m$, then $n - k = m + 1$ and $n - 2k = 1$.

$$2\sum_{i=1}^{n} |\pi(i) - k| = (m+1)^2 + m^2 + 1.$$

Also, $\pi(1)$ and $\pi(n)$ are both greater than k and $k = \lceil n/2 \rceil - 1$, so

$$|\pi(1) - k| + |\pi(n) - k| = |\pi(1) - \lceil n/2 \rceil| + |\pi(n) - \lceil n/2 \rceil| + 2.$$

This gives the following disorder:

$$\delta_x^+(\pi) = m^2 + (m+1)^2 - |\pi(1) - \lceil n/2 \rceil| - |\pi(n) - \lceil n/2 \rceil| - 1.$$

Otherwise, if $k = m + 1$, then $n - k = m$ and $n - 2k = -1$.

$$2\sum_{i=1}^{n} |\pi(i) - k| = m^2 + (m+1)^2 - 1.$$

Also, $k = \lceil n/2 \rceil$, so

$$|\pi(1) - k| + |\pi(n) - k| = |\pi(1) - \lceil n/2 \rceil| + |\pi(n) - \lceil n/2 \rceil|.$$

This gives the same formula for the additive disorder:

$$\delta_x^+(\pi) = m^2 + (m+1)^2 - |\pi(1) - \lceil n/2 \rceil| - |\pi(n) - \lceil n/2 \rceil| - 1.$$

Note that $m^2 + (m+1)^2 - 1 = 2m^2 + 2m = \frac{1}{2}((2m+1)^2 - 1) = \lfloor n^2/2 \rfloor$, so this completes the proof when n is odd. ◀

▶ **Theorem 8.** *The maximum possible additive disorder of a permutation of $S(n)$ is $\lfloor n^2/2 \rfloor - 1$. It is attained exactly by bipartite permutations with centered endpoints.*

Proof. First, note that according to Lemma 7, any bipartite permutation with centered endpoints has disorder $\lfloor n^2/2 \rfloor - 1$.

Conversely, let π be a permutation with maximal disorder. It may not have any of the patterns P_1, P_2, or P_3 by Lemma 6, hence it is bipartite with centered endpoints by Lemma 5. ◀

Algorithm 1 Given positive integers n and d, the algorithm returns a permutation $\pi \in S(n)$ such that $\delta_x^+(\pi) = d$.

```
Realization(n, d)
if d < n − 1 or d > ⌊n²/2⌋ − 1 then
    return error
else if d = n − 1 then
    return 12...n
else if d ≤ ⌊(n − 1)²/2⌋ + 1 then
    π ← Realization(n − 1, d − 2)
    i ← max(2, Position(π, n − 1))
    return Insertion(π, i, n)
else
    d′ ← ⌊n²/2⌋ − d
    π ← 12...⌈n/2⌉
    σ ← n(n − 1)...(⌈n/2⌉ + 1)
    if d′ ≤ ⌊n/2⌋ then
        i ← ⌊n/2⌋ + 1
        j ← i − d′
    else
        i ← ⌊n/2⌋ + 1 + (−1)^(n mod 2)(d′ − ⌊n/2⌋)
        j ← 1
    π ← PutFirst(π, j)
    if i ∈ π then
        π ← PutLast(π, i)
    else
        σ ← PutLast(σ, i)
    return Interleave(π, σ)
```

3.4 All disorders in the range can be achieved

To state the upcoming algorithm, let us set some definitions. For any word u of length n, any word v of length m, any $i \in [n]$, and any letter a, let

- `Position(u, a)` be the position of a in u when a occurs in u;
- `Insertion(u, i, a)` be the word $u(1) \dots u(i-1)au(i) \dots u(n)$;
- `PutFirst(u, a)` (resp. `PutLast(u, a)`), where a is at position i in u, be the word $au(1) \dots u(i-1)u(i+1) \dots u(n)$ (resp. $u(1) \dots u(i-1)u(i+1) \dots u(n)a$);
- `Interleave(u, v)` be the word $u(1)v(1)u(2)v(2) \dots u(k)v(k)w$ where $k = \min\{n, m\}$ and w is the suffix of u of length $n - m$ if $n - m \geq 0$ or the suffix of v of length $m - n$ otherwise.

Let us now consider the algorithm `Realization`, taking as inputs a value $n \geq 2$ and an integer d, and outputting when this is possible a permutation of $S(n)$ having d as additive disorder (see Algorithm 1).

▶ **Example 9.** Table 1 shows some permutations built by `Realization(n, d)`.

Table 1 The permutations built by `Realization`$(n,\ d)$, where $2 \leq n \leq 7$ and $1 \leq d \leq 23$.

	n						
d	1	2	3	4	5	6	7
1	–	12	–	–	–	–	–
2	–	–	123	–	–	–	–
3	–	–	132	1234	–	–	–
4	–	–	–	1243	12345	–	–
5	–	–	–	1432	12354	123456	–
6	–	–	–	1423	12543	123465	1234567
7	–	–	–	2413	15432	123654	1234576
8	–	–	–	–	15423	126543	1234765
9	–	–	–	–	25413	165432	1237654
10	–	–	–	–	15243	165423	1276543
11	–	–	–	–	25143	265413	1765432
12	–	–	–	–	–	165243	1765423
13	–	–	–	–	–	265143	2765413
14	–	–	–	–	–	162435	1765243
15	–	–	–	–	–	162534	2765143
16	–	–	–	–	–	261534	1762435
17	–	–	–	–	–	361524	1762534
18	–	–	–	–	–	–	2761534
19	–	–	–	–	–	–	3761524
20	–	–	–	–	–	–	1726453
21	–	–	–	–	–	–	1726354
22	–	–	–	–	–	–	2716354
23	–	–	–	–	–	–	3716254

We note that Algorithm 1 runs in polynomial time in n and d. In fact, provided the data structure used for permutations allows constant-time insertions of elements before and after n, then it is actually linear. To this end, double-ended queues with a pointer to the highest value are a solution.

▶ **Theorem 10.** *For any n and any $n-1 \leq d \leq \lfloor n^2/2 \rfloor - 1$, Algorithm 1 yields a permutation $\pi \in S(n)$ with $\delta_x^+(\pi) = d$ in linear-time w.r.t. n.*

Proof. The proof is by induction on n, assume that the theorem is true for $n-1$. We write π^* for the permutation returned by `Realization`(n, d). We distinguish three cases depending on the value of d.

If $d = n-1$, then π^* is the identity permutation and $\delta_x^+(\pi^*) = d$.

If $n \leq d \leq \lfloor (n-1)^2/2 \rfloor + 1$, then by induction π (line 7) is a permutation of $[n-1]$ with $\delta_x^+(\pi') = d-2$ (since $n-2 \leq d-2 \leq \lfloor (n-1)^2/2 \rfloor - 1$). By the choice of i, we have $2 \leq i \leq n$ and $\{\pi_{i-1}, \pi_i\} = \{n-1, x\}$ for some $1 \leq x < n-1$. Then $\pi^* = (\pi_1, \ldots, \pi_{i-1}, n, \pi_i, \ldots, \pi_n)$, so

$$\begin{aligned}
\delta_x^+(\pi^*) &= \delta_x^+(\pi) - |\pi_{i-1} - \pi_i| + |n - \pi_i| + |n - \pi_{i-1}| \\
&= \delta_x^+(\pi) - (n-1-x) + (n-(n-1)) + (n-x) \\
&= (d-2) + 2 = d.
\end{aligned}$$

Finally, if $\lfloor (n-1)^2/2 \rfloor + 2 \le d \le \lfloor n^2/2 \rfloor - 1$. We have $d' = \lfloor n^2/2 \rfloor - d$ (line 11). The value of d' is bounded as follows

$$1 \le d' \le \lfloor n^2/2 \rfloor - \lfloor (n-1)^2/2 \rfloor - 2 = \lfloor n^2/2 \rfloor - \lfloor (n^2+1)/2 \rfloor + n - 2 \le n - 2.$$

Note that π^* is built as the interleaving of π (containing $\{1, \dots, \lceil n/2 \rceil\}$) and σ (containing $\{\lceil n/2 \rceil + 1, \dots, n\}$), so it is a bipartite permutation. By Lemma 7, it suffices to verify that $|\pi^*(1) - \lceil n/2 \rceil| + |\pi^*(n) - \lceil n/2 \rceil| = d'$. Values i and j are defined lines 14 to 19. First remark, using $d' \le n-2$, that $1 \le j \le n$, $1 \le i \le n$, and $i \ne j$. We show that (i) $\pi^*(1) = j$, (ii) $\pi^*(n) = i$, and (iii) $|j - \lceil n/2 \rceil| + |i - \lceil n/2 \rceil| = d'$. Property (i) is clear since $1 \le j \le \lfloor n/2 \rfloor$ by construction, so $\pi(1) = j$ after line 20, and finally $\pi^*(1) = j$. Towards (ii), note that i is the last element of either π or σ after line 24, so it suffices to show that $i \le \lceil n/2 \rceil$ if and only if n is odd. We now discuss specific cases depending on the value of d' and the parity of n.

If $d' \le \lfloor n/2 \rfloor$, we have $i = \lfloor n/2 \rfloor + 1$, so $i \le \lceil n/2 \rceil$ iff n is odd (so $\pi^*(n) = i$). Furthermore, $j \le \lceil n/2 \rceil \le i$, so $|j - \lceil n/2 \rceil| + |i - \lceil n/2 \rceil| = i - j = d'$.

If $d' > \lfloor n/2 \rfloor$, we have $j = 1$ and $i = \lfloor n/2 \rfloor + 1 + (-1)^{n \bmod 2}(d' - \lfloor n/2 \rfloor)$. So $|j - \lceil n/2 \rceil| = \lceil n/2 \rceil - 1$. If n is odd, $i < \lceil n/2 \rceil$ and $|i - \lceil n/2 \rceil| = d' - \lceil n/2 \rceil + 1$ and $|i - \lceil n/2 \rceil| + |j - \lceil n/2 \rceil| = d'$. If n is even, $i = d' + 1 \ge \lceil n/2 \rceil$, and $|i - \lceil n/2 \rceil| + |j - \lceil n/2 \rceil| = d'$.

The linearity of the algorithm w.r.t. n is clear. Indeed, the only trick consists, in the case starting at line 7, in having a constant-time insertion of the letter n in the permutation π returned by the recursive call. Since n is always inserted adjacent to the letter $n-1$, it is enough to store the position of the last letter to achieve the claimed complexity. ◀

4 Concluding remarks

There are many questions left open in this paper. Below we briefly discuss three directions for further research.

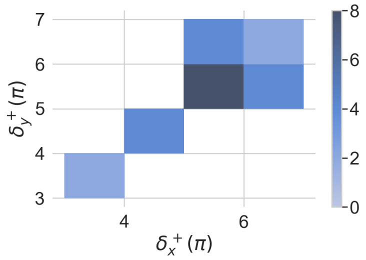
1. Sure enough, our candidate that arrived d seconds early has to start at some given floor i to reach the audition at some another floor j. How can she propose a route that starts at floor i, ends at floor j and takes exactly that long? And for which values d is there at least one solution? Note that for $n = 4$, if one focus on permutations that start with 1 and end with 4, we have $\delta_x^+(1234) = 3$, $\delta_x^+(1324) = 5$ but no permutation $\pi \in S(4)$ starting with 1 and ending with 4 achieves $\delta_x^+(\pi) = 4$.
2. Given an $(n-1)$-sequence D_x, it is NP-complete to decide whether there exists a permutation $\pi \in S(n)$ such that $\Delta_x(\pi) = D_x$. This was proved by M. De Biasi [3]. It is natural to ask for the following extension: Given two $(n-1)$-sequences D_x and D_y, how hard is the problem to decide whether there exists a permutation $\pi \in S(n)$ such that $\Delta_x(\pi) = D_x$ and $\Delta_y(\pi) = D_y$? What about the case $D_x = D_y$? See Table 2 for the landscape of $S(4)$.
3. We have shown that for any positive integer d_x, $n - 1 \le d_x \le \lfloor n^2/2 \rfloor - 1$, one can construct in linear-time a permutation $\pi \in S(n)$ such that $d_x = \delta_x^+(\pi)$. The most natural question to ask is: Given two positive integers d_x and d_y, how hard is the problem to decide whether there exists a permutation π such that $d_x = \delta_x^+(\pi)$ and $d_y = \delta_y^+(\pi)$? Again, what about the case $d_x = d_y$? See Table 3 for the landscape of $S(4)$ and refer to Fig. 5 for visualizing the distribution of points $(\delta_x^+(\pi), \delta_y^+(\pi))$ for all permutations $\pi \in S(n)$, $4 \le n \le 11$. More generally, towards a better understanding of the important aspects of differences in large permutations, the study of the distribution of the points $(\delta_x^+(\pi), \delta_y^+(\pi))$ for $\pi \in S(n)$ is likely to be a promising direction (see Fig. 6 and Fig. 5).

Table 2 Permutations of $S(4)$ with given difference sequences.

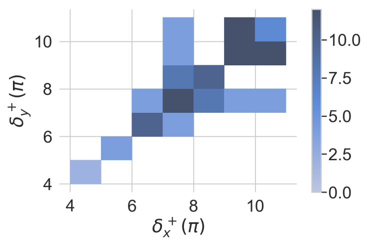
D_x	D_y	$\pi \in S(4)$ with $\Delta_x(\pi) = D_x$ and $\Delta_y(\pi) = D_y$
$(1,1,1)$	$(1,1,1)$	1234, 4321
$(1,1,3)$	$(1,1,3)$	3214
$(1,1,3)$	$(3,1,1)$	2341
$(1,2,1)$	$(1,2,1)$	1243, 2134, 3421, 4312
$(1,2,3)$	$(2,1,2)$	2314, 3241
$(1,3,1)$	$(1,3,1)$	2143, 3412
$(2,1,2)$	$(1,2,3)$	3124, 4213
$(2,1,2)$	$(2,1,2)$	1324, 4231
$(2,1,2)$	$(3,2,1)$	1342, 2431
$(2,3,2)$	$(2,3,2)$	2413, 3142
$(3,1,1)$	$(1,1,3)$	4123
$(3,1,1)$	$(3,1,1)$	1432
$(3,2,1)$	$(2,1,2)$	1423, 4132

Table 3 Permutations of $S(4)$ with given disorders.

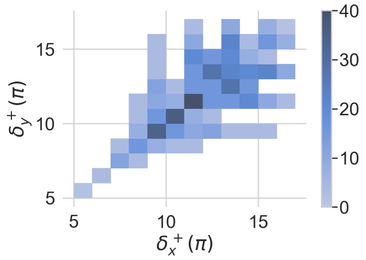
d_x	d_y	$\pi \in S(4)$ with $\delta_x^+(\pi) = d_x$ and $\delta_x^+(\pi) = d_y$
3	3	1234, 4321
4	4	1243, 2134, 3421, 4312
5	5	1324, 1432, 2143, 2341, 3214, 3412, 4123, 4231
5	6	1342, 2431, 3124, 4213
6	5	1423, 2314, 3241, 4132
7	7	2413, 3142



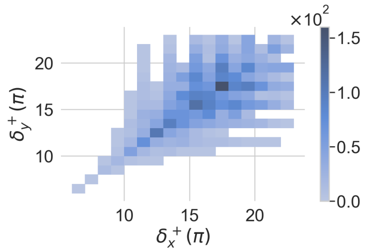
(a) $\pi \in S(4)$



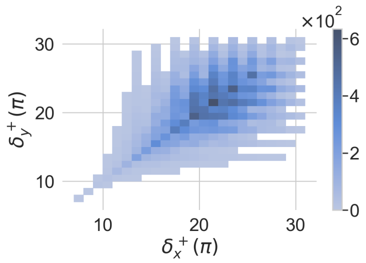
(b) $\pi \in S(5)$



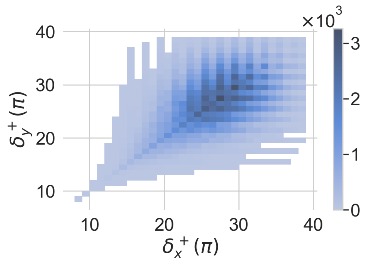
(c) $\pi \in S(6)$



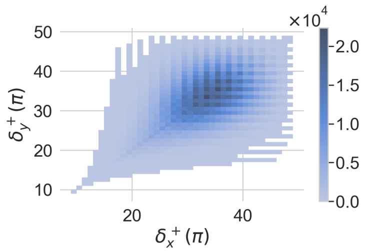
(d) $\pi \in S(7)$



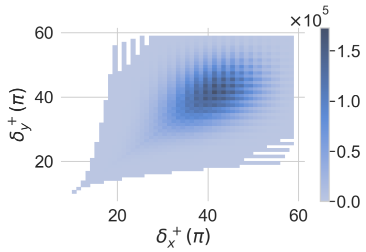
(e) $\pi \in S(8)$



(f) $\pi \in S(9)$

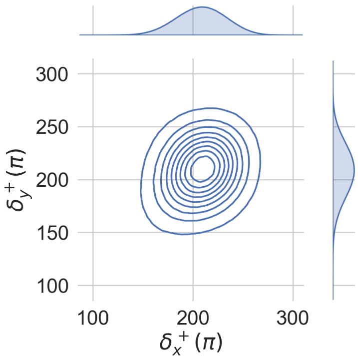


(g) $\pi \in S(10)$

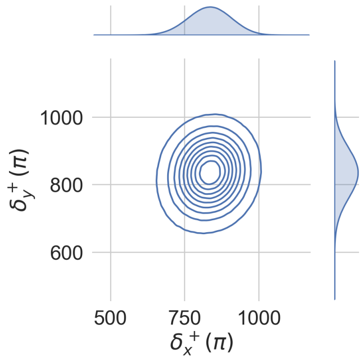


(h) $\pi \in S(11)$

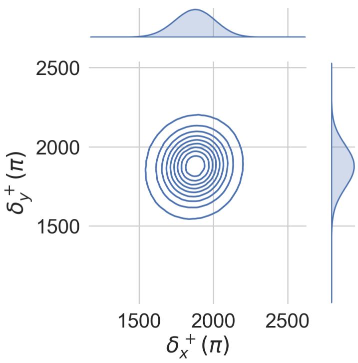
Figure 5 Bivariate histograms of pairs $(\delta_x^+(\pi), \delta_y^+(\pi))$ for all permutations $\pi \in S(n)$, $4 \leq n \leq 11$.



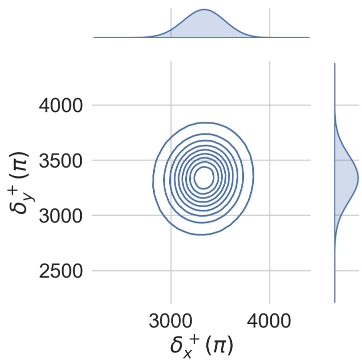
(a) $\pi \in S(25)$



(b) $\pi \in S(50)$



(c) $\pi \in S(75)$



(d) $\pi \in S(100)$

Figure 6 Kernel Density Estimate (KDE) of pairs $(\delta_x^+(\pi), \delta_y^+(\pi))$ for 10^7 random permutations $\pi \in S(n)$, $n \in \{25, 50, 75, 100\}$.

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