



Mathematical Model of Simultaneous Flow between Casson Fluid and Dust Particle over a Vertical Stretching Sheet

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DOI: <https://doi.org/10.30880/ijie.2020.12.03.029>

Received 18 December 2019; Accepted 14 January 2020; Available online 27 February 2020

Abstract: The process involving multiphase flow generally can be found in natural phenomena and many industrial applications. It might be between solid-liquid flow, liquid-liquid flow, gas-solid flow or gas-liquid flow. Their interaction is significant and able to influence the flow characteristics. The experimental work for this topic has been widely performed in order to obtain the best interaction output but it is still incomplete due to the limitation in term of cost and safety issue. Since then, the mathematical model is proposed to counter that constraint. This paper is aims to propose the mathematical model representing the two-phase flow which the interaction of non-Newtonian Casson fluid and solid particles is examined. The flow is investigated moving over a stretching sheet and the combined convection is considered together with the influence of heat generation in fluid phase. The governing equations representing the two-phase model are first transformed into the ordinary differential equations using established similarity transformations where the complexity of the model is reduced. The resulting equations are then solved by employing the Keller-box method with the help of Matlab software. The numerical output in term of velocity and temperature distribution for both phases are illustrated graphically and the value of skin friction and heat transfer coefficients are presented in tabular form for various value of mixed convection parameter, heat generation parameter and Casson parameter. Findings reveal that, the parameter under investigation affects the flow characteristics and present a significant impact to both phases except for mixed convection parameter.

Keywords: Two-phase flow, dusty Casson fluid, heat generation, mixed convection

1. Introduction

Non-Newtonian fluids contaminated with solid particles are termed as dusty Non-Newtonian fluids. It is assumed that these fluids comprise of rigid spheres with similar size, non-interacting dust particles embedded in non-Newtonian fluid, which falls under the category of two-phase flow. This flow case is encountered in all types of non-Newtonian fluid models. Furthermore, the mixture of fluid-particle is of important practical interest in many applications such as petroleum transportation, treatment of waste-water, smoke emission from vehicle, pipping of power plants and corrosive particles in mining. Sandeep et al. [1] reported the theoretical explanations on dusty nanofluid flow through stretching sheet in unsteady state. Meanwhile, in a vertical wavy surface, Siddiqa et al. [2] solved the convective flow of dusty nanofluid by using the primitive variable formulation (PVF). In recent years, Kumar et al. [3] examined the Williamson fluid flow in the presence of solid particles with non-linear thermal radiation effect over a stretching sheet. Under the same impact and geometry, Reddy et al. [4] studied the two-phase model of Old-royd-B and Maxwell fluids with the

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additional influence of chemical reaction. Furthermore, Arifin et al. [5] and Kasim et al. [6] have paid attention to the description of modified magnetic field flow behaviour of dusty Casson and dusty Jeffrey fluids, respectively, in which the fluid is assumed to move by stretching the sheet. Recently, Mackolil and Mahanthesh [7] presented the statistical and exact analysis of a nonlinear convective flow of hybrid nanofluid by considering the influence of particles suspension. The review in the theory of fluid-particle mechanism covering a much range of conditions has also been discussed by several researchers [8-12].

It is worth to mention that, one of the essential aspects in fluid flow problems to be taken into account is the thermal energy transport. Since the fluids are frequently used as heat carriers in the equipment involving cooling and heating systems, an improvement in the thermal energy equations of the fluid model has been suggested. As referred to Kasim [13], the general form of heat generation within the boundary layer region that has a linear relation with temperature is expressed as

$$q''' = \begin{cases} Q_0 (T - T_\infty), & T \geq T_\infty, \\ 0, & T < 0, \end{cases} \quad (1)$$

where Q_0 , T_∞ and T refer to heat generation constant, local and ambient temperatures. Numerous studies of the flow problems set up in Newtonian fluid and different types of non-Newtonian fluids subjected to this condition have been treated by various researchers [14-21]. However, these studies have only focused on the fluid flow without the existence of dust particles.

Keeping all the important aspects as discussed above in mind, the purpose of present study is to apply an energy source of heat generation to dusty Casson fluid, which has a significant impact in the fluid temperature distribution. The heat transfer analysis of this fluid is also accompanied with aligned magnetic field and buoyancy force effects. In addition, the thermal condition of Newtonian heating (NH) is conveniently measured at the boundary of a vertical stretching sheet. Numerical algorithms of this mathematical two-phase model are developed by using Keller-box method and then computed in Matlab software. The implications of the physical parameters on Casson fluid and dust particles which involved throughout the entire flow process are discussed.

2. Mathematical Formulation

Consider a steady and mixed convection flow of dusty Casson fluid past a vertical stretching sheet in the presence of aligned magnetic field effect under the action of heat generation. A vertical position of stretching sheet is considered as x -axis while y -axis being perpendicular to it, on which the surface sheet is prescribed by thermal boundary condition of NH. In this study, the dusty Casson fluid has been regarded as two component mixtures of Casson fluid and dust particles. Moreover, the dust particles are assumed to be spherical in shape with the same radii and non-interacting. The physical flow model is presented in Fig. 1.

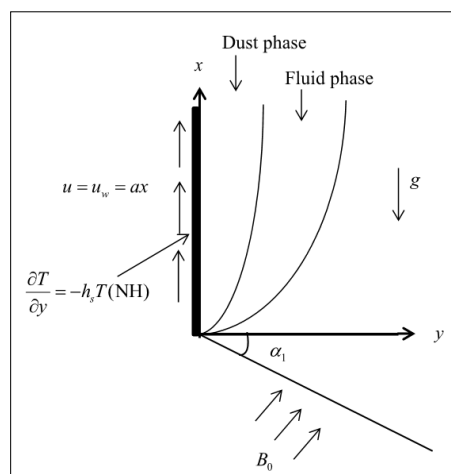


Fig. 1 - Physical model of dusty Casson fluid and coordinate system

As referred to Bhattacharyya et al. [22], the rheological equation for Casson fluid model is

$$\tau_{ij} = \begin{cases} (\mu_B + \rho_y / \sqrt{2\pi}) 2e_{ij}, & \pi > \pi_c \\ (\mu_B + \rho_y / \sqrt{2\pi}) 2e_{ij}, & \pi < \pi_c \end{cases} \quad (2)$$

where μ_B and ρ_y denote plastic dynamic viscosity of non-Newtonian fluid and yield stress, respectively. $\pi = e_{ij}e_{ij}$ corresponds to the product of deformation rate with π_c is its critical value. By considering the above assumptions on dust particles, equation (2) and following [23, 24], the governing equations of Casson fluid (fluid phase) and dust particles (dust phase) under both approximations of Boussinesq and boundary layer can be expressed as

Fluid phase:

$$u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = 0, \quad (3)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \left(1 + \frac{1}{A} \right) \frac{\partial^2 u}{\partial y^2} + \frac{\rho_p}{\rho \tau_v} (u_p - u) - \frac{\sigma}{\rho} B_0^2 \sin^2 \alpha_1 u + \beta^* g (T - T_\infty), \quad (4)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\rho_p c_s}{\tau_T \rho c_p} (T_p - T) + \frac{Q_0}{\rho c_p} (T - T_\infty). \quad (5)$$

Dust phase:

$$\frac{\partial u_p}{\partial x} + \frac{\partial v_p}{\partial y} = 0, \quad (6)$$

$$u_p \frac{\partial u_p}{\partial x} + v_p \frac{\partial v_p}{\partial y} = -\frac{\rho_p}{\rho_p \tau_v} (u_p - u), \quad (7)$$

$$u_p \frac{\partial T_p}{\partial x} + v_p \frac{\partial T_p}{\partial y} = -\frac{\rho_p c_s}{\rho_p \tau_T} (T_p - T) \quad (8)$$

The associated boundary conditions for both phases are

$$u = u_w(x) = \alpha x, \quad v = 0, \quad \frac{\partial T}{\partial y} = -h_s T \text{ at } y = 0, \quad (9)$$

$$u \rightarrow 0, \quad u_p \rightarrow 0, \quad v_p \rightarrow v, \quad T \rightarrow T_\infty, \quad T_p \rightarrow T_\infty \text{ as } y \rightarrow \infty$$

where (u, v) , c_p , T , ρ , α , h_s and Q_0 imply to the velocity components in (x, y) directions, specific heat at constant pressure, temperature, density, thermal diffusivity, viscosity coefficient, heat transfer and heat generation coefficients for fluid phase. In contrast, (u_p, v_p) , c_s , T_p , ρ_p , τ_v and $\tau_T (= 1.5 \gamma \tau_v \text{ Pr})$ represent velocity components in (x, y) directions, specific heat density, temperature, density, velocity and thermal relaxation time of dust particles for dust phase.

The following similarity transformations are adopted to the equations stated above, given as

$$u = \alpha x f'(\eta), \quad v = -(\alpha \nu)^{1/2} f(\eta), \quad u_p = \alpha x F'(\eta), \quad v_p = -(\alpha \nu)^{1/2} F(\eta), \quad (10)$$

$$\eta = \left(\frac{\alpha}{\nu} \right)^{1/2} y, \quad \theta(\eta) = \frac{T - T_\infty}{T_\infty}, \quad \theta_p(\eta) = \frac{T_p - T_\infty}{T_\infty},$$

Substitution of equations (10) into equations (3)-(9) results in the following transformed governing equations

$$\left(1 + \frac{1}{A}\right) f''(\eta) + f(\eta) f''(\eta) - (f'(\eta))^2 + \beta N (F'(\eta) - f'(\eta)) - M \sin^2 \alpha_1 f'(\eta) + \lambda \theta(\eta) = 0, \tag{11}$$

$$\theta''(\eta) + \text{Pr} (f(\eta) \theta'(\eta) + Q \theta(\eta)) + \frac{2}{3} \beta N (\theta_p(\eta) - \theta(\eta)) = 0, \tag{12}$$

$$(F'(\eta))^2 - F(\eta) F'(\eta) + \beta (F'(\eta) - f'(\eta)) = 0, \tag{13}$$

$$\theta_p'(\eta) F(\eta) + \frac{2}{3} \frac{\beta}{\text{Pr} \gamma} (\theta_p(\eta) - \theta(\eta)) = 0, \tag{14}$$

Boundary conditions (9) are written correspondingly as

$$\begin{aligned} f(0) = 0, f'(0) = 1, \theta'(0) = -b(1 + \theta(0)) \quad \text{at } \eta = 0, \\ f'(\eta) \rightarrow 0, F'(\eta) \rightarrow 0, F(\eta) \rightarrow f(\eta), \theta(\eta) \rightarrow 0, \theta_p(\eta) \rightarrow 0 \quad \text{as } \eta \rightarrow \infty \end{aligned} \tag{15}$$

The parameters of Casson, mass concentration of particle phase, magnetic field, fluid-particle interaction, Prandtl number, specific heat ratio of mixture, conjugate parameter for NH, heat generation, mixed convection, Grashoff number and Reynold number appeared in equations (11)-(15) are defined as

$$\begin{aligned} A = \mu_B \frac{\sqrt{2\pi c}}{\rho_y}, N = \frac{\rho_p}{\rho}, M = \frac{\sigma B_0^2}{\rho a}, \beta = \frac{1}{a\tau_v}, \text{Pr} = \frac{\mu c_p}{k}, \gamma = \frac{c_s}{c_p}, \\ b = -h_z \left(\frac{\nu}{a}\right)^{1/2}, Q = \frac{Q_0}{a\rho c_p}, \lambda = \frac{Gr_x}{\text{Re}_x^2}, Gr_x = \frac{g\beta^* T_\infty x^3}{\nu^2}, \text{Re}_x = \frac{u_w(x)x}{\nu}. \end{aligned} \tag{16}$$

The following equation exhibits the exact solution of (11), in which the investigated conditions of magnetic field, mixed convection and dust particles are negligible. It will be used as the validating benchmark of the present results that deserves attention in this study.

$$f(\eta) = \left(1 + \frac{1}{A}\right)^{1/2} \left(1 - \exp\left(-\frac{\eta}{(1+1/A)^{1/2}}\right)\right) \tag{17}$$

The physical quantities of this problem are the skin friction coefficient and Nusselt number, which can be obtained by using the following relations

$$C_f \text{Re}_x^{1/2} = \left(1 + \frac{1}{A}\right) f''(0), \tag{18}$$

$$Nu_x \text{Re}_x^{-1/2} = b \left(\frac{1}{\theta(0)} + 1\right). \tag{19}$$

3. Mathematical Procedure

After performing the transformation on the governing equations (3)-(8), a set of ordinary differential equations of (11)-(15) are solved by utilizing the Keller-box method which has been evinced to be efficient in dealing with the problems involving fluid flow and heat transfer. This is proven by its increased popularity in several investigational studies for analysing the flow problems within this field under a wide variety of situations [25-27]. It is decided in this study to use the finite boundary layer thickness, $\eta_\infty = 8$ with a step size of $\Delta\eta = 0.02$ for the numerical computation of this problem with the aid of Matlab software.

4. Results and Discussion

The primary focus of this study is the response of flow behaviour and temperature distribution, as well as the drag between fluid and solid surface towards the variations in mixed convection, λ , Casson, A and heat generation, Q parameters. Numerical results of those parameters can only be acknowledged prior to the exposure of current solutions with exact solution (17). For this, Table 1 has been presented for the purpose of comparing the acquired numerical solution of the limiting case arising in this problem with analytical solution. It can be noticed that, the displayed results are in a close agreement which ascertains the authenticity in the numerical method used.

Further, Table 2 shows the skin friction coefficient, $-C_f \text{Re}_x^{1/2}$ and Nusselt number, $Nu_x \text{Re}_x^{-1/2}$ for the three parameters considered in this study. The variation of both physical quantities with Q is similar to that with A , besides that the values of $Nu_x \text{Re}_x^{-1/2}$ corresponds to Q are greater while the magnitude values of $-C_f \text{Re}_x^{1/2}$ are slightly higher for almost values of A . Meanwhile, the presence of parameter λ resulting in decreasing in magnitude value of $-C_f \text{Re}_x^{1/2}$ and increasing in value of $Nu_x \text{Re}_x^{-1/2}$. Note that, the salient feature of these results is that they represent the drag and temperature between the fluid flowing and sheet surface from the mathematical point of view.

Moreover, it can also be observed that, the parameters of λ and A have significantly influenced the wall friction, but the opposite behaviour is noticed for Q . Hence, the velocity profile with regard to λ and A , together with temperature profile for various values of Q are evaluated and presented graphically, as shown in Figs. 2-4. From Fig. 2, the fluid velocity increases with the increase in λ that resulted from the growing effect of buoyancy force. Meanwhile, a reverse trend is perceived when A is enhanced, as shown in Fig. 3. This can be attributed to the high fluid viscosity which then increases the resistance in the fluid motion. Further, it can be observed that in Fig. 4, higher values of Q motivate fluid temperature as more energy is developed and consequently improving the heat supplied into fluid. It is prominent to mention here that, the flow and temperature distributions of dust particles have been decisively affected by the three considered parameters. More interestingly, the behaviour displayed by dust particles showed the similar propensity with fluid phase under consideration of those parameters.

Table 1 - Comparison of $-f''(0)$ for various values of A

A	$-f''(0)$	
	Exact expression (17)	Present
1	0.70711	0.70711
3	0.86603	0.86603
5	0.91287	0.91287
7	0.93541	0.93542
10	0.95346	0.95346

Table 2 - Variation of skin friction coefficient and Nusselt number for various values of λ , Q and A

λ	Q	A	$-C_f \text{Re}_x^{1/2}$	$Nu_x \text{Re}_x^{-1/2}$
0.1	0.5	2	1.51007	1.08527
0.2			1.49954	1.08819
0.3			1.47871	1.09335
0.5			1.45902	1.10489
0.6	0	2	1.50446	2.33569
	0.1		1.50194	2.12917
	0.2		1.49844	1.90605
	0.3		1.49325	1.66286
0.5	0.5	0.5	2.10681	1.24756
		1	1.70814	1.17044
		1.5	1.55230	1.12269
		2	1.46843	1.09583

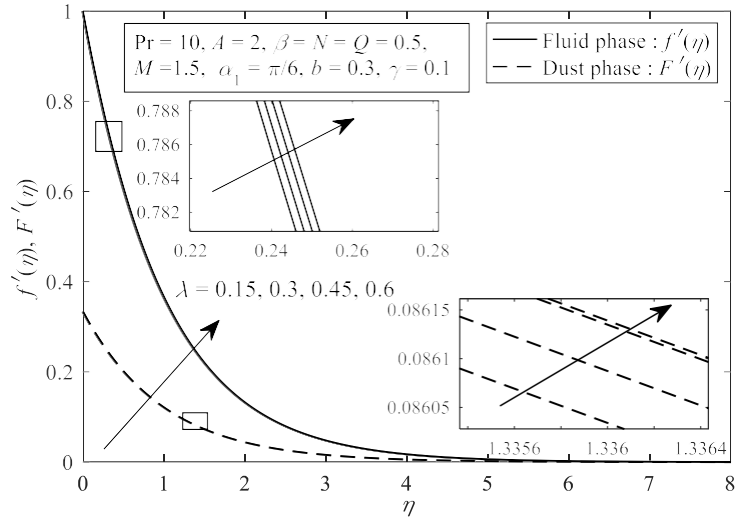


Fig. 2 - Variation of $f'(\eta)$ and $F'(\eta)$ for various values of λ

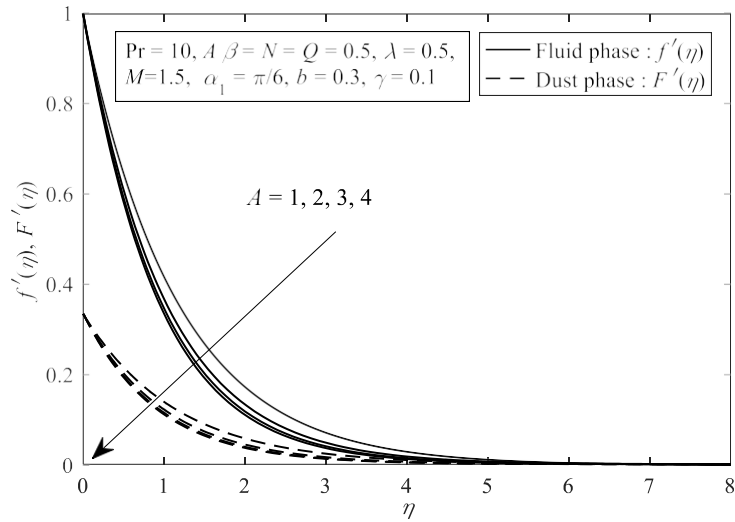


Fig. 3 - Variation of $f'(\eta)$ and $F'(\eta)$ for various values of A

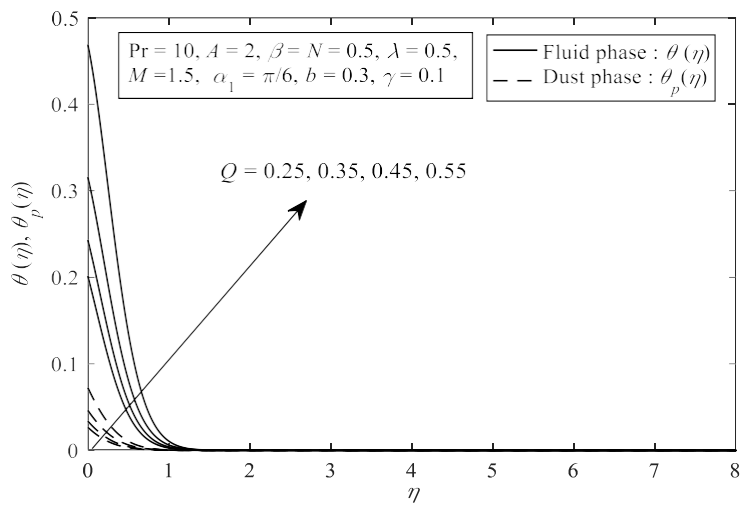


Fig. 4 - Variation of $\theta(\eta)$ and $\theta_p(\eta)$ for various values of Q

5. Conclusion

The present investigational is concerned with the theoretical study of two-phase mixed convection flow of dusty Casson fluid over a vertical stretching sheet. It is worth to mention that, the present mathematical model gives interpretation of two-phase flow behavior, where the existing single phase flow models are unable to cater the solid-fluid system as it is concerned on fluid flow only. Therefore, the numerical results acquired in this study have been clarified by treating fluid phase as Casson fluid while dust particles constitutes dust phase. The interplay between aligned magnetic field, heat generation and thermal condition of NH has also been examined. To solve this problem, it is convenient to introduce the similarity transformation into the governing equations of the respective model so that the numerical simulation can be performed. It is concluded that no noticeable difference is observed in the flow field of both fluid and dust phases when large value of mixed convection parameter, λ is imposed in comparison to that of Casson parameter, A . In addition, temperature profile is found to attain its free stream promptly.

Acknowledgement

The authors gratefully acknowledge the financial support received from Universiti Malaysia Pahang for (RDU 182306, RDU182307, RDU 190303 & RDU 192602).

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