Stackings and One-Relator Products of Groups

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I, Benjamin James Millard, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

The theory of one-relator groups (groups admitting a presentation with a single relator) has a selection of interesting open questions. For example, it is unknown exactly which torsion-free one-relator groups are coherent, or which are hyperbolic. One way in which these questions are currently being studied is by considering immersions of 2-complexes into a group's presentation complex. The specific properties of interest are called non-positive immersions and negative immersions, which put restrictions on the Euler characteristic of the immersing space. In this thesis, we look more closely at these properties, and see how they have been used to study one-relator groups. We also consider one-relator products of groups, a generalisation of one-relator groups where a defining relator is taken over a free product of groups rather than just a free group. We prove that one-relator products admit a stacking, which is a geometric object containing information about the relationship between the defining relator and the underlying free product of groups. We go on to use these stackings to prove that torsion-free one-relator products have the non-positive immersions property. This is a result that has also recently been proved by James Howie and Hamish Short. We discuss how our proof differs and how by using stackings we can find improvements for some of their results. Finally, we discuss the negative immersions property, its conjectural link to hyperbolicity, and how stackings may allow progress in the classification of which one-relator products have negative immersions.

Impact Statement

This thesis provides a way to study properties of one-relator products of groups in a geometric and combinatorial setting, combining ideas from Topology, Group Theory, Geometry and Combinatorics. In particular, it may allow for certain properties known about one-relator groups to be extended to one-relator products. These are groups that are of interest within geometric group theory and are the subject of many open problems for which a new approach may be useful. In Chapter 5 we indicate some of these possible avenues of future research and how the work included in this thesis could be used to make progress.

Acknowledgements

Firstly, I would like to thank my supervisor Lars Louder, not only for suggesting problems for me to work on but also providing support throughout and always being available for discussions. I would also like to thank my whole family, spending time with each of you over the last few years and meeting new family members has made the whole process so much easier. Last but not least I want to thank Sarah for everything, I don't think I would have finished this without you.

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Contents

Chapter 1

Introduction

This thesis was inspired by the use of a simple geometric notion called a stacking, which was introduced in [17] to study properties of one-relator groups (groups admitting a presentation with a single relator). The outline of the thesis will be as follows. In Chapter 2 we provide some background about the history of one-relator groups, introducing some interesting properties these groups have as well as some open questions, such as which one-relator groups are coherent or hyperbolic. We also introduce stackings and describe the results that have been proven using stackings, in particular coherence (for one-relator groups with torsion) and non-positive immersions, both of which stem from proofs of the W-cycles conjecture (proved separately in [17] and [8]).

In Chapter 3 we introduce a generalisation of one-relator groups, one-relator products, seeing some similarities they share with one-relator groups. To do so we will also discuss the relationship between local indicability of groups and having the non-positive immersions property, which will be defined in Chapter 2. We will also discuss relative graphs, which are 2-complexes whose fundamental groups are free products, enabling us to define a relative version of a stacking in Chapter 4.

This will set us up for Chapter 4, which contains the main results of the thesis. Here, we define a generalised version of a stacking for one-relator products, look at the properties of these stackings, and prove in Theorem 4.2.5 that these stackings exist whenever the word defining our relator is not a proper power (or conjugate into a free factor). We then use these new stackings to prove Theorem 4.4.1, which is a version of the W-cycles conjecture for relative graphs whose vertices have non-positive immersions. Using this result we show in Corollary 4.4.2 that one-relator products whose relator is not a proper power and not conjugate into a free factor have the non-positive immersions property. This result has also recently (in 2020) been proved by Howie and Short in [12].

In the final chapter, Chapter 5, we discuss the differences between the results of Chapter 4 and the results of Howie and Short from [12]. In particular, we prove Proposition 5.1.2 and Corollary 5.1.3, showing how the stackings we have defined can be used to improve certain bounds produced in [12] (Theorem 3.3 and Corollary 3.4) by a factor of 5. We also discuss an open problem in the theory of one-relator groups. That is, which one-relator groups are hyperbolic? Here we will discuss some ideas relating this open question to stackings, as well as discussing a way to produce stackings of primitive elements in free groups using Whitehead automorphisms. We also introduce a conjecture about negative immersions for one-relator products, related to a theorem in [15], and talk about some ideas surrounding this conjecture.

Chapter 2

One-Relator Groups

2.1 Introduction to One-Relator Groups

We begin with an introduction to one-relator groups. For a set X denote by $F(X)$ the free group with generating set X.

Definition 2.1.1. A group G is a *one-relator group* if it has a group presentation with exactly one relator, i.e. there exists a set X and an element $w \in F(X)$ such that

$$G \cong \langle X \mid w \rangle \cong \frac{F(X)}{\langle\langle w \rangle\rangle}.$$

Here $\langle\langle w \rangle\rangle$ denotes the normal closure of w in $F(X)$.

The presentation complex of a one-relator group G is constructed by taking a rose with $|X|$ petals each labelled by a distinct element of X and gluing a 2-cell whose boundary reads the word w, see Figure 2.1 for an illustration. The fundamental group of the presentation complex of G is isomorphic to G by construction.

Example 2.1.2. Some initial examples of one-relator groups are:

- *Free groups:* Taking w as the empty word (or w as a primitive word (a member of a basis) in $F(X)$);

- *Surface groups:* $\pi_1(\Sigma_g)$, where Σ_g is a closed orientable surface of genus g.

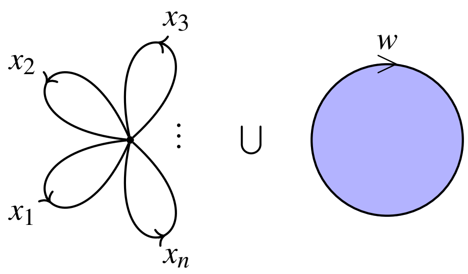


Figure 2.1: Illustration of the presentation complex of the one-relator group $G = \langle X \mid w \rangle$, if $X = \{x_1, \ldots, x_n\}$.

Then

$$\pi_1(\Sigma_g) \cong \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle.$$

- *Finite cyclic groups:* $\langle x \mid x^n \rangle$ for $n \geq 2$.

We discuss some of these and other examples in this section, considering the properties that one-relator groups share.

Given the relatively simple definition of these groups, it would seem safe to assume that it is straightforward to determine their properties. This turns out not to be the case and their study has had a large influence in the development of Combinatorial and Geometric Group Theory. In fact there are still many open questions about one-relator groups, some of which will be discussed here.

The first major progress in the study of one-relator groups was made by Wilhelm Magnus in his thesis, supervised by Max Dehn. He proved a result called the Freiheitssatz (meaning independence theorem) and the techniques used in the proof have been used in many results on one-relator groups since.

Theorem 2.1.3 (Magnus' Freiheitssatz [21])**.** *Let $G = \langle x_1, \ldots, x_n \mid w \rangle$ be a finitely generated one-relator group, where w is a cyclically reduced word. If x_1 appears as a letter in w, then the subgroup of G generated by $x_2, \ldots, x_n$ is free and freely generated by $x_2, \ldots, x_n$.*

Informally this is saying that if w contains a generator as a letter then the only relations in G between the other generators are trivial relations.

The proof of this result led Magnus, in [22] to a solution to the word problem for one-relator groups. Where, for a group G, the word problem asks if there is an algorithm that determines if any given word $w \in G$ is the trivial element in G.

Theorem 2.1.4 (The Word Problem for One-relator Groups, Magnus [22])**.** *The word problem is soluble in all one-relator groups (i.e. such an algorithm as described above exists).*

The word problem was first proven for surface groups only (which are of course one-relator groups) by Dehn in [6]. This proof relied on properties which were later shown to be properties of hyperbolic groups [7]. However, one-relator groups need not be hyperbolic themselves, and Magnus' proof instead used the Freiheitssatz.

2.2 One-relator Groups with Torsion

There is one class of one-relator groups for which many properties are more easily proven and these are the one-relator groups with torsion. The first result about torsion in one-relator groups was a consequence of a result of Lyndon [19] about the cohomological dimension for one-relator groups.

Definition 2.2.1. Let X be a set and let $w \in F(X)$ be a non-trivial element. Then w is a *proper power* if $w = (w')^n$ for some $w' \in F(X)$ and $n > 1$.

Theorem 2.2.2. *[Lyndon [19]] Let $G \cong \langle X \mid w \rangle$ be a one-relator group such that w is not a proper power. Then G is torsion-free.*

This fact together with Newman's Spelling Theorem [23] can be used both to find an algorithm that solves the word problem for one-relator groups with torsion, and proves that one-relator groups with torsion are hyperbolic. Recall the following definition of a group being hyperbolic, introduced by Gromov [7].

Definition 2.2.3. A metric geodesic space X is δ-hyperbolic if geodesic triangles are δ-thin. That is for any geodesic triangle T in X, each side of T is contained in a delta neighbourhood of the others. X is *hyperbolic* if it is δ-hyperbolic for some $\delta > 0$. A finitely generated group G is *hyperbolic* if its Cayley graph with respect to some (or any) finite generating set is hyperbolic.

Some other examples of hyperbolic groups are finite groups, free groups and fundamental groups of surfaces of genus at least 2. These groups are an important area of study as it turns out there are many group theoretical problems that can be solved for hyperbolic groups. For example we mentioned above that hyperbolic groups have soluble word problem. Another example is whether a group is finitely presentable. For full proofs of the results below, which are attributed to Rips, see [1] for example.

Theorem 2.2.4 (Rips)**.** *Hyperbolic groups are finitely presented.*

A group being finitely presented is an algebraic problem, but topologically a group G is finitely presented if there exists a finite simplicial complex X such that $G \cong \pi_1(X)$. One way to find such a complex is by proving the existence of a simply connected simplicial complex Y on which G acts properly discontinuously and cocompactly, since then the quotient Y/G is a compact (and hence finite) simplicial complex with $\pi_1(Y/G) \cong G$ (in fact this is a stronger statement to prove). In the case of hyperbolic groups Rips proved the existence of such a simply connected simplicial complex, now known as the Rips complex.

Theorem 2.2.5 (Rips)**.** *Let G be a hyperbolic group. There exists a simplicial complex Y and an action of G on Y, such that Y is contractible and the action is properly discontinuous and cocompact.*

Other than those with torsion, exactly which one-relator groups are hyperbolic is an open question. A famous question [4] asks whether any one-relator group that doesn't contain a Baumslag-Solitar group is hyperbolic, where Baumslag-Solitar groups are the groups

$$BS(m,n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle.$$

In Chapter 5 we will discuss hyperbolicity and briefly describe why Baumslag Solitar groups are not hyperbolic.

A possible classification of hyperbolic one-relator groups is suggested in [15] based on the primitivity rank of the group. Let F be a free group, we say that a word $w \in F$ is *primitive* in F if it is a member of a basis of F. In Chapter 5 we will talk about peak reduction [9], which provides a simple algorithm for determining if a word is primitive. This is based on Whitehead automorphisms, which generate the automorphism group of a free group.

Definition 2.2.6. Let F be a free group and $w \in F \setminus \{1\}$. The *primitivity rank* of w is

$$\pi(w) = \min\{rank(K) \,|\, w \in K < F \text{ and w is not primitive in } K\}.$$

By convention, $\pi(w) = \infty$ if w is primitive in F.

The primitivity rank of an element w can in fact be shown to directly relate to another condition described by immersions of 2-complexes.

Let X, Y be 2-complexes (we only need 2-complexes as the presentation complex of any group will be a 2-complex). A map $Y \to X$ is *combinatorial* if its restriction to each cell of Y is a homeomorphism to a cell of X. Furthermore, we say that a combinatorial map $Y \to X$ of 2-complexes is an *immersion* if it is locally injective.

Definition 2.2.7. A compact 2-complex has *negative immersions* (or NI) if for any immersion $Y \looparrowright X$ of a compact, connected 2-complex Y, either $\chi(Y) < 0$ or Y Nielsen reduces to a graph (Nielsen reduction is a stronger version of homotopy equivalence, see 5.5.4 for a full definition). A group G has NI if it has a presentation whose presentation 2-complex has NI.

Example 2.2.8. Free groups have NI since the presentation complex X of a free group is a graph and if $Y \looparrowright X$ is an immersion Y will also be a graph. On the other hand the group $\pi_1(\Sigma_1)$, i.e. the fundamental group of a torus, does not have NI for the obvious reason that $\chi(\Sigma_1) = 0$ and Σ_1 does not Nielsen reduce to a graph since

$w = aba^{-1}b^{-1}$ is not a primitive word in the free group $\langle a,b\rangle$. See Chapter 5 for further information about Nielsen reduction.

The following results were proved in [15] relating the primitivity rank to immersions.

Theorem 2.2.9. *Let F be a free group and $w \in F \setminus \{1\}$.*

- $\pi(w) = 1$ *if and only if w is a proper power.*
- $\pi(w) > 2$ *if and only if the presentation complex of $F/\langle\langle w\rangle\rangle$ has negative immersions.*

In addition to this it is shown that if the one relator group G has negative immersions (so $\pi(w) > 2$) then G contains no Baumslag-Solitar groups, which leads to the conjecture that any one-relator group with negative immersions is in fact hyperbolic. These ideas will be discussed further in Chapter 5.

2.3 Coherence

A question, posed by Baumslag [3], that has recently had attention in the study of one-relator groups is whether all one-relator groups are coherent.

Definition 2.3.1. A group G is *coherent* if every finitely generated subgroup of G is finitely presented.

Some initial examples of coherent groups are free groups (since every finitely generated subgroup is a finitely generated free group) and surface groups (since finitely generated subgroups are also surface groups).

In 2003, Wise claimed in [29] to have a proof that a large class of groups were in fact coherent. These groups are those whose presentation complex has non-positive immersions.

Definition 2.3.2. A compact 2-complex X has *non-positive immersions* (or NPI) if for any immersion $Y \to X$ of a compact, connected 2-complex Y, either $\chi(Y) \leq 0$

or Y Nielsen reduces to a point. A group G has *non-positive immersions* (or NPI) if it has a presentation whose presentation 2-complex has NPI.

Example 2.3.3. Clearly NI $\Rightarrow$ NPI so this provides a set of examples, for example free groups. However, we have extras, and stackings will help us determine which one-relator groups have NPI, for example $\pi_1(\Sigma_1)$ has NPI when it does not have NI, see Example 2.2.8.

For the NPI property, Nielsen reduction can be replaced by the condition that if $\chi(Y) \geq 1$ then Y is homotopy equivalent to a point. For the full definition of Nielsen reduction see 5.5.4.

It turns out that the claimed proof of Wise had a gap (found by Mladen Bestvina) and so remains as a conjecture rather than a true statement:

Conjecture 2.3.4. *If X is a compact 2-complex with NPI, then $\pi_1(X)$ is coherent.*

In order to link this conjecture to one-relator groups we need to know which one-relator groups have NPI. To this end, in 2005, Wise [28] made a related conjecture, known as the W-cycles conjecture. First we need a couple of easy definitions. The first is the notion of a fibre product.

Definition 2.3.5. If $f_1 : \Gamma_1 \looparrowright \Gamma$ and $f_2 : \Gamma_2 \looparrowright \Gamma$ are immersions of graphs, the *fibre product*, $\Gamma_1 \times_\Gamma \Gamma_2$, is defined to be

$$\Gamma_1 \times_\Gamma \Gamma_2 = \{(x,y) \in \Gamma_1 \times \Gamma_2 \,|\, f_1(x) = f_2(y)\}.$$

Remark 2.3.6. *Notice that the fibre product above will be a graph, where vertices of $\Gamma_1 \times_\Gamma \Gamma_2$ are pairs $(x,y) \in V(\Gamma_1) \times V(\Gamma_2)$ such that $f_1(x) = f_2(y)$, and similarly edges are pairs $(x,y) \in E(\Gamma_1) \times E(\Gamma_2)$ such that $f_1(x) = f_2(y)$, with incidence maps being induced by the incidence maps in $\Gamma_1, \Gamma_2, \Gamma$.*

The second definition is what it means for an immersion of a circle to be indivisible.

Definition 2.3.7. If $\Lambda : S^1 \looparrowright \Gamma$ is an immersion of a circle in a graph then Λ is *indivisible* if it does not properly factor through any other immersion $S^1 \looparrowright \Gamma$, i.e. Λ is indivisible if the homotopy class of $\Lambda(S^1)$ in $\pi_1(\Gamma)$ is not a proper power.

We are now able to state Wise's W-cycles Conjecture.

Conjecture 2.3.8 (W-cycles Conjecture [28])**.** *Let $\rho : \Gamma' \to \Gamma$ be an immersion of finite, connected graphs and let $\Lambda : S^1 \to \Gamma$ be an indivisible immersed loop. Let $\mathbb{S}$ be the union of the circular components of $\Gamma' \times_\Gamma S^1$. Then the number of components of $\mathbb{S}$ is at most the rank of Γ'.*

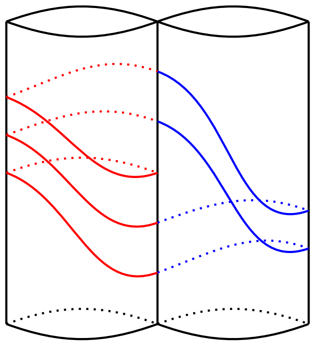
This has been proven separately in [8] and [17]. It can also be used to show that one-relator groups have NPI. Thus, if Conjecture 2.3.4 were to be true it would positively answer Baumslag's question about coherence of one-relator groups. In Chapter 4 we prove a generalisation of this conjecture to one-relator products, which are introduced in Chapter 3. Our proof was inspired by the proof of the W-cycles conjecture in [17] and so we will give an overview of this proof here.

2.4 Stackings and the W-cycle conjecture

This section is an overview of [17]. The idea of a stacking of an immersion $\Lambda : S^1 \looparrowright \Gamma$ is to provide a partial ordering on points in S^1 that is in some sense consistent with incident maps on edges of S^1 and the map Λ. This is done via a 'height function', allowing similar techniques to be used from areas such as Morse Theory. The formal definition is as follows.

Definition 2.4.1. Let Γ be a finite graph, let $\mathbb{S}$ be a disjoint union of circles and let $\Lambda : \mathbb{S} \looparrowright \Gamma$ be continuous. A *stacking* of Λ is an embedding $\hat{\Lambda} : \mathbb{S} \hookrightarrow \Gamma \times \mathbb{R}$ such that $\Lambda = \pi \circ \hat{\Lambda}$, where $\pi : \Gamma \times \mathbb{R} \to \Gamma$ is the trivial $\mathbb{R}$-bundle, $\pi(x,y) = x$.

Example 2.4.2. Consider the rank 2 free group $\langle a,b \rangle$ and let $w : S^1 \to \langle a,b \rangle$ be the map defined by the word $w = ababa$. This is actually a primitive word and so the corresponding one-relator group is free but can also easily be seen to admit a stacking. In fact, as we see in Chapter 5, finding a stacking for primitive elements of $\mathbb{F}_2$ is easy. For example we could use the embedding depicted in Figure 2.2.



$w = ababa$

Figure 2.2: A stacking of the word $w = ababa$ in $\langle a, b \rangle$.

As illustrated by Figure 2.2 the definition of a stacking is very simple and easy to visualise, as well as giving obvious partial orders on the edges and vertices of S^1. In general it is also not hard to compute a stacking of a given word (at least for primitive or relatively short words), as long as it is not a proper power. Given its simplicity it would be easy to dismiss this notion as not providing much information, or to assume it is immediate that words should have stackings. One nice observation about stackings is that they allow you to easily compute the Euler characteristic of the target space Γ just by observing the stacking either from above or below.

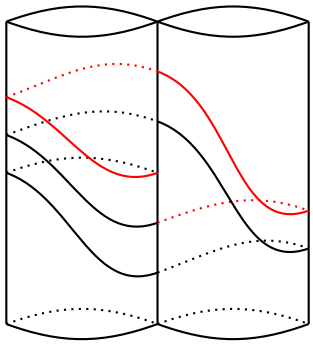
Definition 2.4.3. Let π, ι be the projections of $\Gamma \times \mathbb{R}$ to Γ and $\mathbb{R}$ respectively. Let $\Lambda : \mathbb{S} \looparrowright \Gamma$ be an immersion where $\mathbb{S}$ is a disjoint union of circles, and let $\hat{\Lambda} : \mathbb{S} \hookrightarrow \Gamma \times \mathbb{R}$ be a stacking of Λ. Define the sets $\mathcal{A}_{\hat{\Lambda}}$ and $\mathcal{B}_{\hat{\Lambda}}$ as

$$\mathcal{A}_{\hat{\Lambda}} = \{x \in \mathbb{S} \,|\, \forall y \neq x, \Lambda(x) = \Lambda(y) \Rightarrow \iota(\hat{\Lambda}(x)) > \iota(\hat{\Lambda}(y))\},$$

$$\mathcal{B}_{\hat{\Lambda}} = \{x \in \mathbb{S} \,|\, \forall y \neq x, \Lambda(x) = \Lambda(y) \Rightarrow \iota(\hat{\Lambda}(x)) < \iota(\hat{\Lambda}(y))\}.$$

Loosely, the set $\mathcal{A}_{\hat{\Lambda}}$ is the set of cells in $\mathbb{S}$ that can be seen when looking down from above, and $\mathcal{B}_{\hat{\Lambda}}$ is the set of cells we see from below. For instance, if we return to Example 2.4.2, each of the sets $\mathcal{A}_{\hat{\Lambda}}$ and $\mathcal{B}_{\hat{\Lambda}}$ consist of a single open arc as shown in

Figure 2.3, where an open arc is defined as below.



$w = ababa$

Figure 2.3: The red arc is exactly the set $\mathcal{A}_{\hat{\Lambda}}$, where the endpoints of this arc are not included.

Definition 2.4.4. An *open arc* of $\mathbb{S}$ is a connected, simply connected, open subset of $\mathbb{S}$, which is a union of edges and vertices of $\mathbb{S}$. A stacking shall be called *good* if $\mathcal{A}_{\hat{\Lambda}}$ and $\mathcal{B}_{\hat{\Lambda}}$ intersect each component of $\mathbb{S}$.

It is easy to check that if Λ is an immersion, the connected components of $\mathcal{A}_{\hat{\Lambda}}$ and $\mathcal{B}_{\hat{\Lambda}}$ are open arcs or components of $\mathbb{S}$ ([17] Lemma 8). Furthermore, if we define reducibility in the following manner.

Definition 2.4.5. Λ is *reducible* if there is an edge of Γ that is traversed exactly once by Λ.

Then it is also easy to see that Λ is reducible if and only if $\mathcal{A}_{\hat{\Lambda}} \cap \mathcal{B}_{\hat{\Lambda}}$ contains the interior of an edge e ([17] Lemma 9) since there can then be no (interior of an) edge either above or below e and e must be traversed exactly once by Λ.

The main use of these sets is the obvious if not rather surprising link to Euler characteristic.

Lemma 2.4.6 ([17] Lemma 10)**.** *Let $\hat{\Lambda} : \mathbb{S} \hookrightarrow \Gamma \times \mathbb{R}$ be a stacking of a surjective immersion $\Lambda : \mathbb{S} \looparrowright \Gamma$. The number of open arcs of $\mathcal{A}_{\hat{\Lambda}}$ (or $\mathcal{B}_{\hat{\Lambda}}$) is $-\chi(\Gamma)$.*

The proof follows immediately from the observation that, due to Λ being surjective, for any vertex x of Γ, if $\nu(x)$ is the valence of x, there are exactly $\nu(x) - 2$ ends of open arcs of $\mathcal{A}_{\hat{\Lambda}}$ for which the image of this endpoint in Γ is x. Therefore, the number of open arcs is

$$\frac{1}{2}\sum_{x\in V(\Gamma)}(\nu(x)-2),$$

and we know by the handshaking lemma that the sum of all valences of vertices in a graph is twice the number of edges.

We now have an understanding of stackings, but the motivation for these was the proof of the W-cycles conjecture, the link is through the proof of the following stronger statement from [17].

Theorem 2.4.7 ([17] Theorem 2)**.** *Let $\rho : \Gamma' \looparrowright \Gamma$ be an immersion of finite, connected graphs and let $\Lambda : S^1 \looparrowright \Gamma$ be an indivisible immersion. Suppose that $\mathbb{S}$, the union of circular components of $\Gamma' \times_\Gamma S^1$ is non-empty, so we have a natural covering map $\sigma : \mathbb{S} \to S^1$. Then either*

$$deg(\sigma) \leq -\chi(\Gamma'),$$

or the pullback immersion $\Lambda' : \mathbb{S} \looparrowright \Gamma'$ is reducible.

There are two main parts to the proof of this theorem. One is that for indivisible immersions, stackings do actually exist and the second is to use the sets $\mathcal{A}_{\hat{\Lambda}}$ and $\mathcal{B}_{\hat{\Lambda}}$ and their relation to Euler characteristic to provide the inequality given the existence of stackings. The following result provides this second part of the proof, assuming Λ has a stacking $\hat{\Lambda}$.

Lemma 2.4.8 ([17] Lemma 12)**.** *If $\hat{\Lambda}$ is a good stacking of Λ then either Λ' is reducible or*

$$deg(\sigma) \leq -\chi(\Lambda'(\mathbb{S})).$$

Recall that a good stacking was defined in Definition 2.4.4. For a proof of this statement see [17]. In Chapter 4 we will prove a version of this result for one-relator products using an adjusted version of a stacking, as we will no longer be working with graphs and circles but 2-complexes instead. The final stage of the proof for W-cycles is to prove the existence of stackings, for this objects called towers are required. These are useful in the study of one-relator groups and one-relator products so we will introduce these here and make further use of them throughout the thesis.

2.4.1 Towers and the Existence of Stackings

Definition 2.4.9. A connected covering map is *infinite cyclic* if its deck group is isomorphic to $\mathbb{Z}$. Equivalently an infinite cyclic cover has fibre $\mathbb{Z}$ (the fibre of a path connected cover $p: Y \to X$ is $p^{-1}(x)$ for any choice $x \in X$).

The following is an example of an easy infinite cyclic covering map.

Example 2.4.10. Suppose we have a continuous map $f: X \to S^1$ for a CW complex X. Let $p: \mathbb{R} \to S^1$ be the covering map $t \mapsto e^{2\pi it}$. Consider the pullback $X \times_{S^1} \mathbb{R}$. Let $q: X \times_{S^1} \mathbb{R} \to X$ be projection onto the first coordinate, then q is a cover and for any $x \in X$,

$$q^{-1}(x) = \{(x,y) \mid y \in \mathbb{R},\, p(y) = f(x)\} = \{(x,y) \mid y \in p^{-1}(f(x))\}.$$

Therefore, the map $q^{-1}(x) \to p^{-1}(f(x))$ given by $(x,y) \mapsto y$ is a bijection and $p^{-1}(f(x)) = \mathbb{Z}$. Thus q is a cover with fibre $\mathbb{Z}$ and so it is an infinite cyclic cover.

In general, an infinite cyclic cover is determined by a homomorphism of the fundamental group of your space onto the infinite cyclic group $\mathbb{Z}$. The following is standard covering map theory. Let X be a CW complex and suppose that $f: \pi_1(X) \twoheadrightarrow \mathbb{Z}$ is an epimorphism. Let $N = Ker(f) \leq \pi_1(X)$, then N is a normal subgroup and $\pi_1(X)/N \cong \mathbb{Z}$. Furthermore, the subgroup N corresponds to a covering map $p: Y \to X$ with deck group $\pi_1(X)/N \cong \mathbb{Z}$, i.e. p is an infinite cyclic cover.

Definition 2.4.11. Let X be a CW complex. A (cyclic) tower is a map $X_0 \to X$ that decomposes as a composition of a finite sequence of maps

$$X_0 \looparrowright \cdots \looparrowright X_n = X$$

such that each map is either an inclusion of a subcomplex or an (infinite cyclic) covering map.

Mostly, we will be using cyclic towers but it is useful to note the more general definition of a tower, where we allow for any connected covering rather than just infinite cyclic covers.

Definition 2.4.12. Let $f : Y \to X$ be a cellular map of compact CW complexes. A *(cyclic) tower lifting* of f is a map $f' : Y \to X'$ such that there is a (cyclic) tower $g : X' \to X$ and $gf' = f$. A (cyclic) tower lifting f' is *maximal* if the only (cyclic) tower lifting of f' is the trivial one (i.e. X'=X, see Figure 2.4).

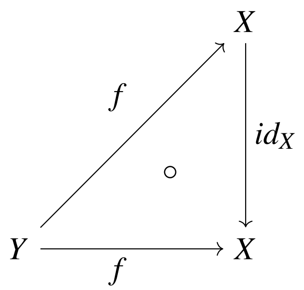


Figure 2.4: Commutative diagram showing the trivial tower lifting of a map $f : Y \to X$.

It is not obvious that maximal tower liftings should even exist, its quite conceivable that we could keep taking covers and inclusions of subcomplexes indefinitely. However, it turns out they do indeed exist and allow us to use inductive arguments via towers.

Lemma 2.4.13 ([11] Lemma 3.1)**.** *Let $f : Y \to X$ be a cellular map of compact CW complexes. Then there exists a maximal (cyclic) tower lifting $Y \to X'$ of f.*

This result was proven by Howie in [11] using induction on $V(Y) - V(Im(f))$, where for a CW complex C, $V(C)$ denotes the number of 0-cells of C. For if

$V(Y) - V(Im(f)) = 0$ then Y and $Im(f)$ have the same number of 0-cells (so no 0-cells are identified under f). Suppose that there exists a non-trivial loop γ in $Im(f)$ then since $f : Y \to Im(f)$ is surjective and no 0-cells are identified, γ pulls back to a loop γ' in Y. Then γ' cannot be null-homotopic otherwise γ is null-homotopic. Hence the induced map $\pi_1(Y) \to \pi_1(Im(f))$ on fundamental groups must be an epimorphism. Since this map is an epimorphism we cannot lift over an infinite cyclic cover, which gives the base case for induction. For the inductive step it is straightforward to show that the quantity $V(Y) - V(Im(f))$ will decrease if we can find another proper tower lifting.

Remark 2.4.14. *Note that in the general tower case, if $V(Y) - V(Im(f)) \neq 0$ it is possible to lift via a covering map so the maximal tower lifting $f' : Y \to X'$ will be π_1-surjective. This is not necessarily true for cyclic towers.*

It turns out that these towers can be used to prove the existence of stackings, leading to the following result.

Lemma 2.4.15. *Any indivisible immersion $\Lambda : S^1 \looparrowright \Gamma$ of a circle into a graph has a stacking $\hat{\Lambda} : S^1 \hookrightarrow \Gamma \times \mathbb{R}$.*

In Chapter 4 we prove a generalised result for one-relator products of groups with NPI whose defining word is neither a proper power nor conjugate into a vertex, but for now the proof for graphs is as follows. By Lemma 2.4.13 we can take a maximal tower lifting of Λ. It can be deduced that at the top of the tower we have a circle and the map from S^1 is a finite-to-one cover. This means it is in fact the identity since Λ is not a proper power and hence we can find a stacking for the maximal tower lifting. Using this an inductive proof can be achieved on the length of a 'stackable' tower lifting.

2.4.2 One-relator groups have NPI

In Chapter 2, we stated Conjecture 2.3.4, a conjecture of Wise, that a group whose presentation complex has NPI will be coherent and we stated that one-relator groups have NPI without a reason as to why. This is in fact a neat consequence of the work on W-cycles and was proved separately in [17] and [8]. The proof we include here is the method from [17] using the stackings result, we have included the full proof as we will use similar ideas in Chapter 4 to prove that torsion-free one-relator products have NPI and it is useful to know what we are aiming for. The proof of the following theorem is from [17].

Theorem 2.4.16 ([17], [8])**.** *Let X be a compact, connected 2-complex with one 2-cell α. If the attaching map of α is an indivisible immersion, then X has non-positive immersions.*

Proof. Take an immersion $Y \looparrowright X$ where Y is a connected, compact 2-complex. Firstly, if Y has no 2-cells then it is a graph and $\chi(Y) \geq 1$ if and only if $\chi(Y) = 1$ and Y is a tree, so the conditions for NPI are easily satisfied. Proceed by induction on the number of 2-cells of Y. Let Γ be the 1-skeleton of X and Γ' the 1-skeleton of Y. Then since Y has 2-cells, the union $\mathbb{S}$ of circular components of $\Gamma' \times_\Gamma S^1$ is non-empty, where $\alpha : S^1 \looparrowright \Gamma$ is the indivisible immersion corresponding to α. Notice further that $\Gamma' \looparrowright \Gamma$ is an immersion since $Y \looparrowright X$ is. Therefore if $\sigma : \mathbb{S} \to S^1$ is the natural covering map, $deg(\sigma)$ is exactly the number of 2-cells of Y. By Theorem 2.4.7 either $deg(\sigma) \leq -\chi(\Gamma')$, in which case $\chi(Y) = \chi(\Gamma') + |\{\text{2-cells of } Y| \leq 0$ as we desired, or the pullback immersion $\Lambda' : \mathbb{S} \looparrowright \Gamma'$ is reducible. In this case there is an edge of Γ' that is traversed exactly once by $\mathbb{S}$, i.e. Y has a 2-cell with a free face. Thus, we can perform Nielsen reduction to Y (in this case we deformation retract to the space obtained by removing the free face and the interior of the 2-cell) to reduce the number of 2-cells of Y completing the induction. □

Another interesting consequence of the W-cycles conjecture is a coherence result. In particular it is possible to prove the following.

Theorem 2.4.17 ([16], [30])**.** *If G is a one-relator group with torsion, then G is coherent.*

This follows from the W-cycles conjecture in the case of torsion. There are two steps to the proof. The first is that if we have a one-relator group with torsion $F/\langle\langle w^n \rangle\rangle$, then the orbicomplex defined by taking the rose and gluing in a single 2-cell with a degree n cone point has a finite-sheeted cover that is a compact 2-complex so we can always unwrap our orbicomplex to a 2-complex. Secondly, in the torsion case Theorem 2.4.7 can be reformulated using the fact that $deg(\sigma) = n|\{\text{2-cells of } Y\}|$ to say either Y is irreducible or

$$(n-1)|\{\text{2-cells of } Y\}| \leq -\chi(Y).$$

In particular, since $n > 1$ (which is where the proof fails without torsion), we have for example

$$|\{\text{2-cells of } Y\}| \leq rank(\pi_1(Y)),$$

which is an upper bound on the number of two cells. The remainder of the proof is to tie these ideas together, since we aim to prove coherence we would start with a subgroup of G and show using folding techniques that it can be represented as the limit of a sequence of immersions to the unwrapped cover of the original orbicomplex. Then use the above to show each of these immersions has a bound on the number of 2-cells and after passing to a subsequence of these (where maps are homeomorphisms) it can be shown that the original choice of subgroup of G must also be finitely presented.

Remark 2.4.18. *The above proof cannot be directly applied to one-relator products for the reason that it is unknown whether in this case the orbicomplex will have a finite-sheeted cover where the covering space is a* 2*-complex.*

Very recently, in [18] Louder and Wilton have provided a proof that one-relator groups with negative immersions are also coherent. In fact they prove this result for a larger class that also encompasses one-relator groups with torsion, but we will not state this here as the definitions will not be used later in this thesis.

There are many other interesting properties and results known or conjectured about one-relator groups for which we have only given a taste here. However, our main area of interest for the next couple of chapters is one-relator products and the results described above are the ones which we would like to be able to generalise to one-relator products.

Chapter 3

One-relator Products

3.1 Locally Indicable Groups

Locally indicable groups will be used later in our proof for the existence of a stacking for torsion-free one-relator products, but they are also very useful in proving other generalisations of results of free groups.

Definition 3.1.1. A group G is said to be *indicable* if there exists an epimorphism (surjective homomorphism) $G \twoheadrightarrow \mathbb{Z}$. A group G is said to be *locally indicable* if every non-trivial, finitely generated subgroup is indicable.

Example 3.1.2. Trivial examples of locally indicable groups are free abelian groups and free groups, since their finitely generated subgroups are free abelian and free respectively and so have obvious epimorphisms to $\mathbb{Z}$.

The following lemma from [16] is a generalisation of Stallings' folding to 2-complexes. It will allow us to construct immersions of 2-complexes using finitely presented subgroups. The following lemma can also be proved by taking a maximal tower lifting, see Remark 2.4.14.

Lemma 3.1.3 ([16])**.** *Any combinatorial map of 2-complexes $Y \to X$ factors as*

$$Y \to Z \looparrowright X,$$

where $Y \to Z$ is surjective and π_1-surjective and $Z \looparrowright X$ is an immersion.

Such a factorisation can be found by first folding the one-skeleton to an immersion and then identifying any 2-cells that map to the same 2-cell.

A non-trivial class of locally indicable groups that is relevant to our discussion is the class of groups with NPI. The fact that groups with NPI are locally indicable is a standard result for which we have included the proof. First, we explain how to construct immersions of 2-complexes using finitely presented subgroups. This result is contained in the proof of Lemma 6.12 of [15], but it is useful to understand the construction so we will explain it here as well.

Lemma 3.1.4 (Lemma 6.12 [15])**.** *Let X be a connected, compact 2-complex. If $H \to \pi_1(X)$ is a homomorphism from a finitely presented group H then there exists a connected, compact 2-complex Y and a combinatorial map $f : Y \to X$ such that $\pi_1(Y) \cong H$.*

Proof. Construct a combinatorial map $Y \to X$ representing the homomorphism $H \to \pi_1(X)$ as follows. H is finitely presented so it is isomorphic to a presentation of the form $H \cong \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$. Let $G = \pi_1(X)$. Let R be a rose with m petals and let $R \to X$ be the combinatorial map (after necessary subdivision) sending the petals to choices of loops in X whose homotopy classes represent the images of the elements x_i in G. Now each r_j is the boundary of a singular disk diagram $D_j \to X$ so construct Y by gluing D_j to R along the corresponding boundaries for each j. If the disk diagrams identify disjoint cells of R then there is a path between these cells that has trivial image in X, i.e. there is folding occurring, so we can adjust the disk diagram as necessary to allow for this. We now have a combinatorial map $Y \to X$ such that $\pi_1(Y) \cong H$. See Figure 3.1 for an illustration of this construction. □

Lemma 3.1.5. *Suppose X is a 2-complex with non-positive immersions, then $\pi_1(X)$ is locally indicable.*

This proof is from [29], although in that case it is in fact proved for a weaker version

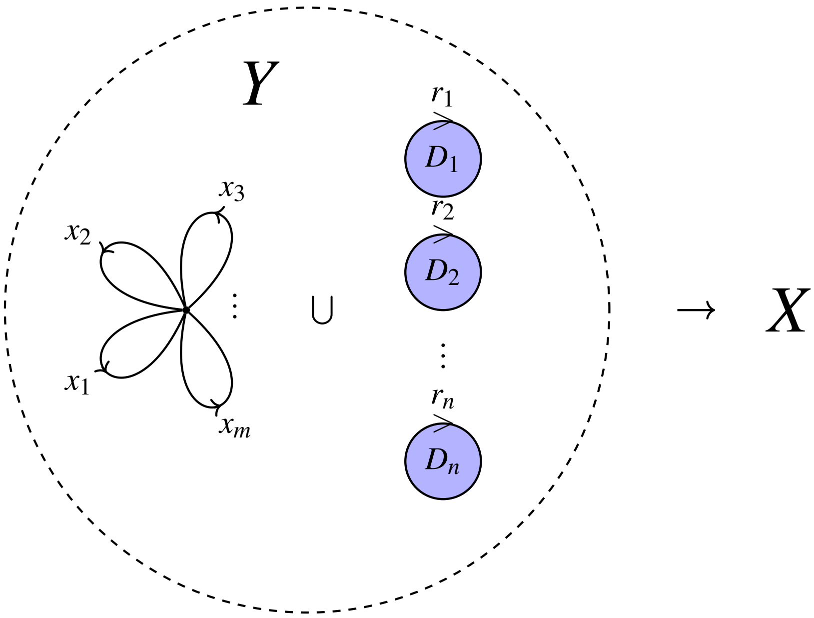


Figure 3.1: Diagram showing how to construct a combinatorial map of 2-complexes using a finitely presented subgroup, as in Lemma 3.1.4.

of NPI where $\chi(Y) > 0$ only implies $\pi_1(Y) = 1$ rather than contractibility. We will include the same proof here without explicitly using towers, which were used by Wise, although as noted above the factorisation from Lemma 3.1.3 can also be found using a maximal tower lifting.

Proof. Suppose $H \leq \pi_1(X)$ is finitely generated and not indicable (i.e. it does not admit an epimorphism to $\mathbb{Z}$). Let $H = \langle a_1, \ldots, a_n \rangle$, then since H is not indicable there exists m_i such that $a_i^{m_i} \in [H,H]$ because H is finitely so the abelianisation must be finite. Let $a_i^{m_i} = w_i$ where w_i is a product of commutators of the a_i and their inverses, so $w_i \in [H,H]$. Define a group

$$K = \langle k_1, \ldots, k_n \,|\, k_1^{m_1} = w_1(k), \ldots, k_n^{m_n} = w_n(k) \rangle,$$

where $w_i(k)$ is the word w_i after the switching from the alphabet $\{h_1, \ldots, h_n\}$ to the alphabet $\{k_1, \ldots, k_n\}$. The map $K \to H$ defined by $k_i \mapsto h_i$ defines an epimorphism by construction, and K is finitely presented and not indicable. By Lemma 3.1.4 since there exists a homomorphism $K \to H \to X$, there exists a combinatorial map $Y \to X$ from a connected, compact 2-complex Y such that $\pi_1(Y) \cong K$. By Lemma 3.1.3,

$Y \to X$ factors through an immersion $Z \looparrowright X$ such that the induced map $\pi_1(Y) \to \pi_1(Z)$ is surjective.

Since X has NPI either $\chi(Z) \leq 0$ or Z is contractible. Since Y is not contractible (as K is non-trivial), $Y \to X$ cannot factor through a contractible complex since the induced map on fundamental groups is non-trivial by construction, so Z is not contractible. Therefore, $\chi(Z) \leq 0$ and Z is a 2-complex so

$$\chi(Z) = \beta_0(Z) - \beta_1(Z) + \beta_2(Z),$$

where $\beta_i(Z)$ is the ith Betti number of Z. Then, $\beta_0(Z) = 1$ and $\beta_2(Z) \geq 0$ so $\beta_1(Z) \geq 1$ and we have epimorphisms $K \cong \pi_1(Y) \twoheadrightarrow \pi_1(Z) \twoheadrightarrow H_1(Z) \twoheadrightarrow \mathbb{Z}$, contradicting the fact that K is not indicable. Thus all finitely generated subgroups of $\pi_1(X)$ are indicable so $\pi_1(X)$ is locally indicable. □

A consequence of Lemma 3.1.5 and Theorem 2.4.16 is that all torsion-free one-relator groups are locally indicable. This result was proved in [5] without the use of NPI by Brodskii and was originally posed as a question by Baumslag in [3].

When we talk about one-relator products of groups they will always be products of locally indicable groups as these give the closest generalisation of one-relator groups and have many useful properties. In fact our results will restrict to the case that the groups have NPI which are of course locally indicable by Lemma 3.1.5. Recall the definition of a free product of groups:

Definition 3.1.6. If A, B are groups with presentations $A = \langle X_A \,|\, R_A \rangle$ and $B = \langle X_B \,|\, R_B \rangle$, then the free product of A and B, $A * B$, has the presentation

$$A * B \cong \langle X_A \cup X_B \,|\, R_A \cup R_B \rangle.$$

Free products can then be used to define one-relator products in the following manner.

Definition 3.1.7. A group G is a one-relator product of locally indicable groups (or just a *one-relator product* if the context is clear) if

$$G \cong \frac{A*B}{N},$$

where A and B are locally indicable groups and N is the normal closure of a single element w in $A*B$ such that w is not conjugate into A or B.

The first result we mentioned in Chapter 2 was Magnus' Freiheitssatz and it turns out that a similar result of Howie, [11], also holds for one-relator products.

Theorem 3.1.8 ([11] Theorem 4.3)**.** *Let $G \cong \frac{A*B}{N}$ be a one-relator product of locally indicable groups. Then the canonical maps $A \to G$ and $B \to G$ are injective.*

Notice that in the case of G being a one-relator group and A and B being $\mathbb{Z}$ for example, the fact w is not conjugate into A or B means that each generator appears in the word w so we get the original conclusion of Magnus' Freiheitssatz. Howie's proof of this result, in [11], is an inductive proof making use of towers, as introduced in Chapter 2. This leads on to a similar result, also of Howie, again making use of towers.

Theorem 3.1.9 ([10] Theorem 4.2)**.** *Let $G \cong \frac{A*B}{N}$ be a one-relator product of locally indicable groups. The following are equivalent.*

1. *G is locally indicable;*

2. *G is torsion-free;*

3. *w is not a proper power in $A*B$.*

Compare this result to Theorem 2.2.2 for one-relator groups, again we see that the only time we get torsion in a one-relator product is if the word w is a proper power exactly as before.

3.2 Relative Graphs

In order to apply ideas related to stackings to one-relator products, we first need to have a more geometric/topological understanding of free products. To this end we will use objects known as relative graphs.

Recall the Grushko Decomposition Theorem, which says that any finitely generated group G can be decomposed into a free product $G \cong G_1 * \cdots * G_p * F_q$, where each G_i is freely indecomposable (i.e. it is not isomorphic to a free product of non-trivial groups) and not $\mathbb{Z}$, and F_q is a free group of rank q. This decomposition is unique in the sense that the numbers p, q are unique and in the sense that the freely indecomposable factors in any two decompositions are conjugate in pairs. For a topological description of such a free decomposition we use relative graphs, which are CW complexes whose fundamental group is a free product of groups, using notation from [14].

Definition 3.2.1. A *relative graph* is a CW-complex Γ equipped with a pair of collections of disjoint subcomplexes

$$\text{Ell}(\Gamma) = \{Y_1, \ldots, Y_p\} \subset \text{Verts}(\Gamma) = \{V_1, \ldots, V_b\}$$

such that all cells not contained in some $Y \in \text{Verts}(\Gamma)$ are one-dimensional, and if $\pi_1(Y) \neq 1$ for $Y \in \text{Verts}(\Gamma)$, then $Y \in \text{Ell}(\Gamma)$.

The *vertices* of Γ are the elements of $\text{Verts}(\Gamma)$ and the *edges*, denoted by $\text{Edges}(\Gamma)$, of Γ are the one-cells not contained in a vertex of Γ. Vertices in $\text{Ell}(\Gamma)$ are called *elliptic vertices*.

The fundamental group of a relative graph is easily computed using Seifert Van Kampen. If Γ is a (connected) relative graph and $\Gamma/\text{Verts}(\Gamma)$ is the graph obtained from Γ by collapsing each $Y \in \text{Verts}(\Gamma)$ to a point, then

$$\pi_1(\Gamma) = \left(\mathop{\ast}_{V \in \text{Verts}(\Gamma)} \pi_1(V) \right) * \pi_1(\Gamma/\text{Verts}(\Gamma)),$$

where only the vertices in $\mathrm{Ell}(\Gamma)$ have non-trivial fundamental group.

An initial easy result about relative graphs and the NPI property is the following. Note that this result is in fact an if and only if, since inclusions of vertices are immersions so any space immersing into a vertex also immerses into the entire relative graph.

Lemma 3.2.2. *Let Γ be a relative graph such that each vertex of Γ has NPI. Then Γ has NPI.*

Proof. Consider an immersion $f : Y \looparrowright \Gamma$. Then Y is a relative graph with $\mathrm{Verts}(Y)$ being the components of pre-images of vertices of Γ and $\mathrm{Edges}(Y)$ the pre-images of edges. Then

$$\chi(Y) = \sum_{V \in \mathrm{Verts}(Y)} \chi(V) - |\mathrm{Edges}(Y)|.$$

For any vertex $V \in \mathrm{Verts}(Y)$ the restriction of f to V is an immersion into a vertex of Γ, which has NPI, so either $\chi(V) \leq 0$ or V is contractible. Since Y is connected there are at least $|\mathrm{Verts}(Y)| - 1$ edges, with exactly this number if and only if Y is a relative tree. Thus,

$$\chi(Y) \leq 1 + \sum_{V \in \mathrm{Verts}(Y)} (\chi(V) - 1),$$

with equality if and only if Y is a relative tree. But $\chi(V) - 1 \leq 0$ with equality if and only if V is contractible, so

$$\chi(Y) \leq 1,$$

with equality if and only if Y is a relative tree and each vertex of Y is contractible, i.e. Y is contractible. Thus Γ has NPI. □

For the results in the following chapter we will need an adjusted version of the Euler characteristic, which is based on the Scott complexity of a relative graph from [14] (our characteristic is essentially minus the Scott complexity).

Definition 3.2.3. Let Γ be a relative graph. Define the *relative characteristic of* Γ,

$\chi_r(\Gamma)$ to be

$$\chi_r(\Gamma) = \chi(\Gamma/\text{Verts}(\Gamma)) - |\text{Ell}(\Gamma)|.$$

Notice that in terms of the Grushko decomposition, if our vertex spaces are freely indecomposable, then $\chi_r(\Gamma) = 1 - p - q$, where p is the number of Elliptic vertices (whose fundamental group is not isomorphic to $\mathbb{Z}$) and q is the rank of the free part.

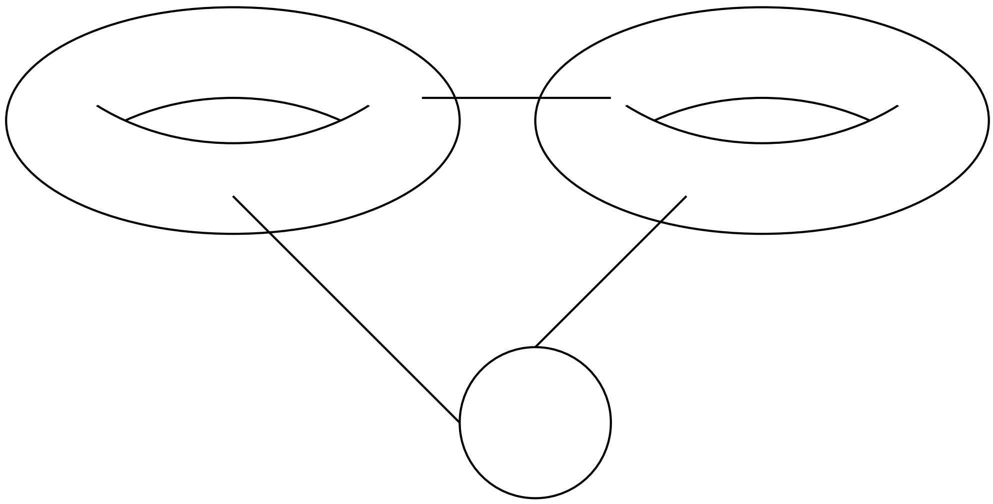


Figure 3.2: Here Γ is a relative graph with $\pi_1(\Gamma) \cong \mathbb{Z}^2 * \mathbb{Z}^2 * \mathbb{F}_2$, and $\chi_r(\Gamma) = -3$.

An initial easy result about the relative characteristic and NPI is as follows.

Lemma 3.2.4. *Let Γ be a relative graph with NPI. Then $\chi_r(\Gamma) \geq \chi(\Gamma)$.*

Proof. Consider a vertex $V \in \text{Verts}(\Gamma)$, since inclusions are immersions either V Nielsen reduces to a point, so $\chi(V) = 1$ and $V \notin \text{Ell}(\Gamma)$ or $\chi(V) \leq 0$. Thus,

$$\chi(\Gamma) = \chi(\Gamma/\text{Verts}(\Gamma)) + \sum_{V \in Verts(\Gamma)} (\chi(V) - 1),$$

so

$$\chi(\Gamma) \leq \chi(\Gamma/\mathrm{Verts}(\Gamma)) + \sum_{V \in Ell(\Gamma)} (0-1) = \chi_r(\Gamma).$$

□

Example 3.2.5. To illustrate that it is possible to have a relative graph with $\chi_r(\Gamma) < \chi(\Gamma)$, we could for instance take a relative graph where some vertices are spheres. See Figure 3.3 as an example. This shows us that we really need vertices to have NPI for Lemma 3.2.4 to be true and not just vertices with locally indicable fundamental group.

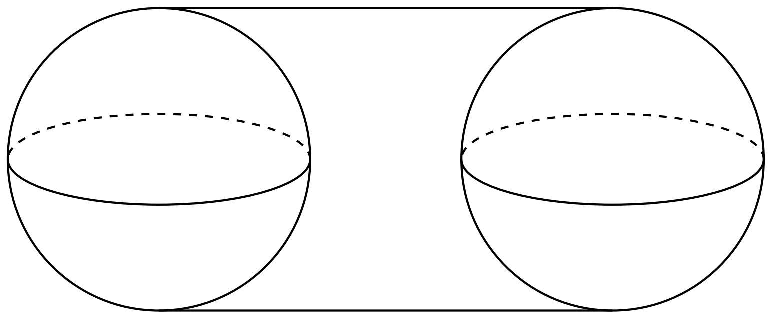


Figure 3.3: Γ is the relative graph with two spheres as vertices, connected by two edges. So $\pi_1(\Gamma) \cong \mathbb{Z}$, $\chi_r(\Gamma) = 0$ and $\chi(\Gamma) = 2$. However, Γ clearly does not have NPI as it admits immersions from a sphere which is not contractible.

Definition 3.2.6. Let Γ, Γ' be relative graphs. A continuous map $\varphi : \Gamma \to \Gamma'$ is a *morphism of relative graphs* if elements of $\mathrm{Verts}(\Gamma)$ are mapped combinatorially to elements of $\mathrm{Verts}(\Gamma')$ and interiors of edges of Γ are mapped homeomorphically to interiors of edges of Γ'.

In the case of free groups we can identify subgroups of free groups with immersions into graphs, using a technique developed by Stallings in [25], now known as Stallings' folding. Similarly, subgroups of free products can be identified with immersions of relative graphs, defined below, using an adjusted Stallings' folding technique.

Definition 3.2.7. Let Γ be a relative graph and $V \in \mathrm{Verts}(\Gamma)$. The *star* of V, $st(V)$, is the space obtained by taking a copy of V and for each oriented $e \in \mathrm{Edges}(\Gamma)$ with terminal endpoint in v, attach a copy of e to v at the corresponding point.

Definition 3.2.8. Let $\varphi : \Gamma' \to \Gamma$ be a morphism of relative graphs. Say that φ is an *immersion of relative graphs* if for each $Y \in \text{Verts}(\Gamma)$ and each connected component X of $\varphi^{-1}(Y)$,

$$\pi_1(\varphi_X) : \pi_1(st(X), \partial st(X)) \to \pi_1(st(Y), \partial st(Y))$$

is an injective map.

If $\varphi : \Gamma \to \Gamma'$ is a morphism of graphs, then φ is called a *Stallings fold* if φ only identifies a pair of edges with a common endpoint. Stallings' theorem [25] tells us that any morphism of graphs $\Gamma \to \Gamma'$ factors as $\Gamma \to \bar{\Gamma} \to \Gamma'$ such that $\bar{\Gamma} \to \Gamma'$ is an immersion of graphs and $\Gamma \to \bar{\Gamma}$ is a composition of Stallings folds. An analogue can be proved in the case of relative graphs, these moves are from [14].

Define a family of folding moves for a relative graph Γ:

1. *Enlarging a subcomplex*: Replace $Y \in Verts(\Gamma)$ by a larger subcomplex of Γ.

2. *Collapse a vertex*: Replace $Y \in Verts(\Gamma)$ with a point.

3. *Attaching 2-cells*: Attach 2-cells to elements of $Y \in Ell(\Gamma)$.

4. *Replace a vertex with a tree*: If $Y \in Verts(\Gamma)$ has $\pi_1(Y) = 1$, let T be a tree whose boundary points (valence one vertices) are exactly the set of 0-cells v of Y for which there exists $e \in Edges(\Gamma)$ with $\alpha(e) = v$ for some $\alpha \in \{\iota, \tau\}$. Replace Y with T by gluing the boundary of T to the corresponding 0-cells in Γ.

5. *Folding*: Let $Y \in Verts(\Gamma)$ and $e, f \in Edges(\Gamma)$ distinct relative edges such that $\alpha(e), \alpha(f) \in Y$ for some $\alpha \in \{\iota, \tau\}$. Let p be a path in Y between $\alpha(e)$ and $\alpha(f)$. Let D be a square and attach D to Γ by attaching three sides of D along $ep\bar{f}$. Collapse D by collapsing the face of D attached to f to the remaining three faces (here faces of D are 1-dimensional, since D is a square).

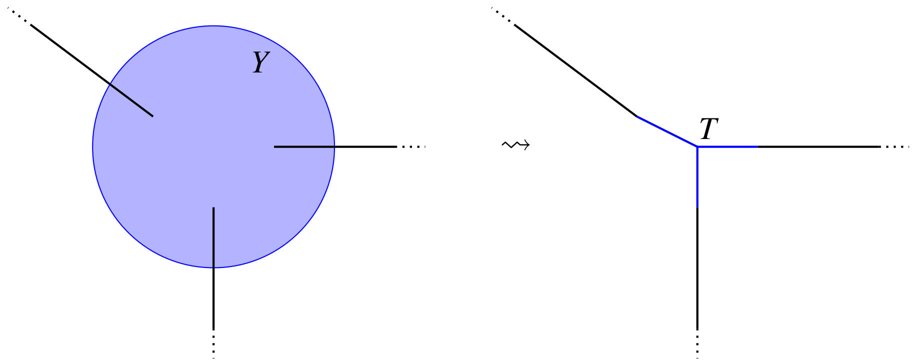


Figure 3.4: (Replace a vertex by a tree): If a vertex Y has $\pi_1(Y) = 1$ we can replace it with a subtree T.

If the fourth side h of D connects two 0-cells of Γ collapse it to a point.

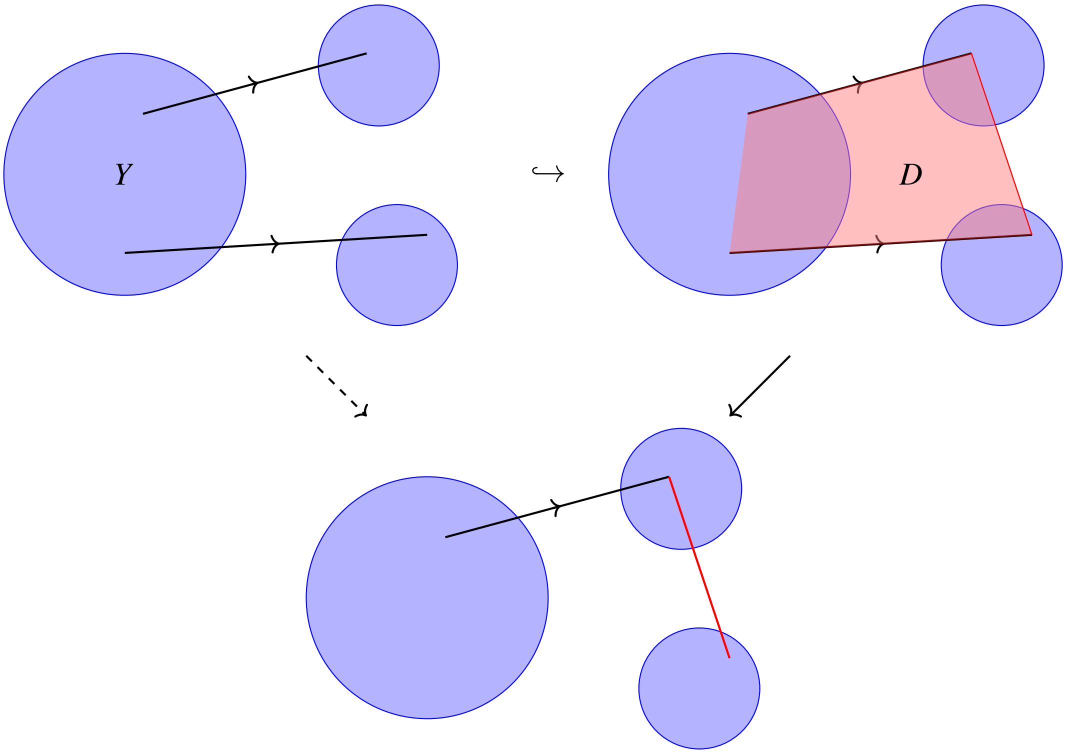


Figure 3.5: This diagram represents a folding move. The final step is to collapse the red edge if it joins distinct vertices, i.e. if it is not a path in Γ.

Lemma 3.2.9 ([14])**.** *Let Γ, Γ' be relative graphs. If $\varphi : \Gamma \to \Gamma'$ is a combinatorial map, then there is a relative graph $\bar{\Gamma}$ with $\chi_r(\bar{\Gamma}) \geq \chi_r(\Gamma)$, a π_1-onto map $F : \Gamma \to \bar{\Gamma}$ and an immersion of relative graphs $\bar{\varphi} : \bar{\Gamma} \to \Gamma'$, such that $\bar{\varphi} \circ F$ is homotopic to φ and F is a composition of folds. Immersions of relative graphs are injective on fundamental groups.*

The following proof is from [14] and was proved for Scott complexity. We include the proof here to confirm it goes through for relative characteristic as well.

Proof. If φ maps a vertex, v, that is not a point to a point, then collapse it. If $v \in \mathrm{Ell}(\Gamma)$, $\chi_r(\Gamma)$ increases, otherwise it leaves it unchanged. If φ is not injective on $\pi_1(Y)$ for some $Y \in \mathrm{Ell}(\Gamma)$, attach enough 2-cells so that it is injective, if this causes the fundamental group to become trivial replace Y with a tree (and remove from $\mathrm{Ell}(\Gamma)$), again this can only increase the relative characteristic. If φ is not an immersion of relative graphs then there exists a $Y' \in \mathrm{Verts}(\Gamma')$ and a connected component Y of $\varphi^{-1}(Y')$ such that $\pi_1(\varphi_Y)$ is not injective. Therefore we have $e, f \in \mathrm{Edges}(\Gamma)$ that are both incident to Y with the same image in Γ'. Let D be a disk (thought of as a rectangle) and attach to Γ along e, f and a path between e and f in Y, let h be the fourth side of D. Fold Γ by collapsing f to e and add h to the vertices it connects. If h attaches distinct vertices, $|\mathrm{Ell}(\Gamma)|$ decreases (or stays the same if one of the vertices has trivial fundamental group) and $\chi(\Gamma/\mathrm{Verts}(\Gamma))$ is unchanged, so χ_r increases. If h connects a vertex to itself, $|\mathrm{Ell}(\Gamma)|$ can increase by at most 1 and $\chi(\Gamma/\mathrm{Verts}(\Gamma))$ increases by at least 1. □

Chapter 4

The W-cycles Conjecture for One-relator Products

In Chapter 2 we defined what it meant for a 2-complex (or a group) to have non-positive immersions (NPI). We mentioned that in [29] Wise had conjectured that for any compact 2-complex X with NPI, $\pi_1(X)$ is coherent, i.e. every finitely generated subgroup of G is finitely presentable. This is motivated by a question of Baumslag [3] asking whether every one-relator group is coherent. We also summarised the proof that all torsion-free one-relator groups have NPI using the w-cycles conjecture, providing a link to coherence.

As stated in Chapter 2, for one-relator groups with torsion there has been recent progress (see [16]) and the W-cycles conjecture can again be applied to prove that all one-relator groups with torsion are coherent. In this chapter we provide a generalisation of the W-cycles conjecture to relative graphs rather than just graphs. Note that recent work of Howie and Short in [12] has shown that such one relator products with torsion (i.e. the attaching map of the additional 2-cell is a proper power) are coherent. They also provide a separate proof that torsion-free one-relator products of groups with NPI have NPI, which is the initial consequence of our version of the relative W-cycles conjecture, although they use a different method not using stackings. In Chapter 5 we discuss the differences between our proof and theirs.

4.1 Existence of Stackings

In the original definition stackings are defined for circles, where the circle represents a word in a free group. If we were to take the same definition but replace the graph (whose fundamental group is the free group) with a relative graph, we would lose the 2-dimensional structure of the relative graph and would only ever be able to prove results about the one skeleton. For example, if two subpaths in the circle have homotopic images in the relative graph this information is lost in the one-skeleton. To fix this issue we instead want to find a stacking of the cover of our relative graph corresponding to the subgroup $\langle w \rangle$. In the original case this more general stacking replaces the circle with a circle that has branches protruding from it (or trees attached if we don't assume vertices have freely indecomposable fundamental group), each branch being a copy of $\mathbb{R}$ and the central circle is precisely the circle we stacked in the usual sense.

Definition 4.1.1. Let Γ be a relative graph and take an immersion $\lambda : S^1 \looparrowright \Gamma$ such that the homotopy class $[\lambda(S^1)]$ is not conjugate into a vertex of Γ. Let $\Lambda' : \mathbb{S}'_\lambda \to \Gamma$ be the covering map associated to the subgroup $\langle \lambda(S^1) \rangle \leq \pi_1(\Gamma)$. Then λ factors through this cover via an embedding $e : S^1 \to \mathbb{S}'_\lambda$. Let $\mathbb{S}_\lambda$ be the component of $\mathbb{S}'_\lambda \setminus (\text{Edges}(\mathbb{S}'_\lambda) \setminus e(S^1))$ with $\pi_1(\mathbb{S}_\lambda) \cong \mathbb{Z}$. The *word cover* of λ, $\Lambda : \mathbb{S}_\lambda \to \Gamma$ is the restriction of Λ' to $\mathbb{S}_\lambda$. If the homotopy class $[\lambda(S^1)] \in \pi_1(\Gamma)$ is not a proper power say Λ is an *indivisible word cover*. See Figure 4.1 for an illustration of this definition.

Remark 4.1.2. *The space* $\mathbb{S}_\lambda$ *defined above can be viewed as a relative graph by letting vertices be pre-images of vertices in* Γ *and edges being pre-images of edges in* Γ.

The fact that the word is not conjugate into a vertex means that edges of the relative graph are crossed and the cover has the structure as described in the definition. These facts will be used implicitly in later proofs.

Throughout this section the vertices of relative graphs will have NPI, and hence

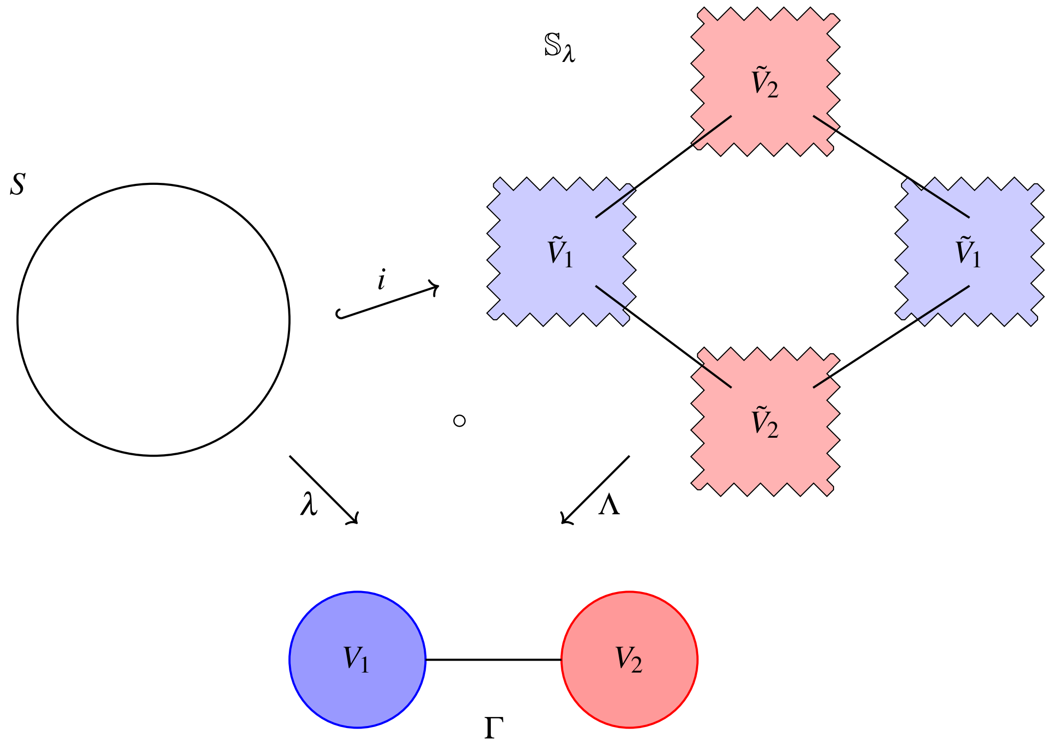


Figure 4.1: This diagram shows a word cover with the map i just being the inclusion of the circle in $\mathbb{S}_\lambda$. Each $\tilde{V}_i$ is the universal cover of V_i and the relative edges of $\mathbb{S}_\lambda$ map to the single edge in Γ.

also locally indicable fundamental groups, as the existence of a stacking will rely on this.

We have the object we wish to stack, so our next definition extends stackings over graphs to stackings over relative graphs by embedding word covers in $\Gamma \times \mathbb{R}$ rather than embedding the word itself. A stacking will give us a partial ordering on cells in an indivisible word cover. We will go on to prove that stackings exist for a certain class of relative graphs (those whose vertices have NPI), assuming the word is not a proper power.

Definition 4.1.3. Let Γ be a relative graph and take an immersion $\lambda : S \looparrowright \Gamma$, where S is a disjoint union of circles. A *stacking* of the corresponding disjoint union of word covers $\Lambda : \mathbb{S}_\lambda \to \Gamma$, is an embedding $\hat{\Lambda} : \mathbb{S}_\lambda \hookrightarrow \Gamma \times \mathbb{R}$ such that $\pi\hat{\Lambda} = \Lambda$, where $\pi : \Gamma \times \mathbb{R} \to \Gamma$ is the trivial $\mathbb{R}$-bundle. See Figure 4.2 for an illustration of a stacking.

Mention of the map λ will largely be omitted to avoid excessive notation and we

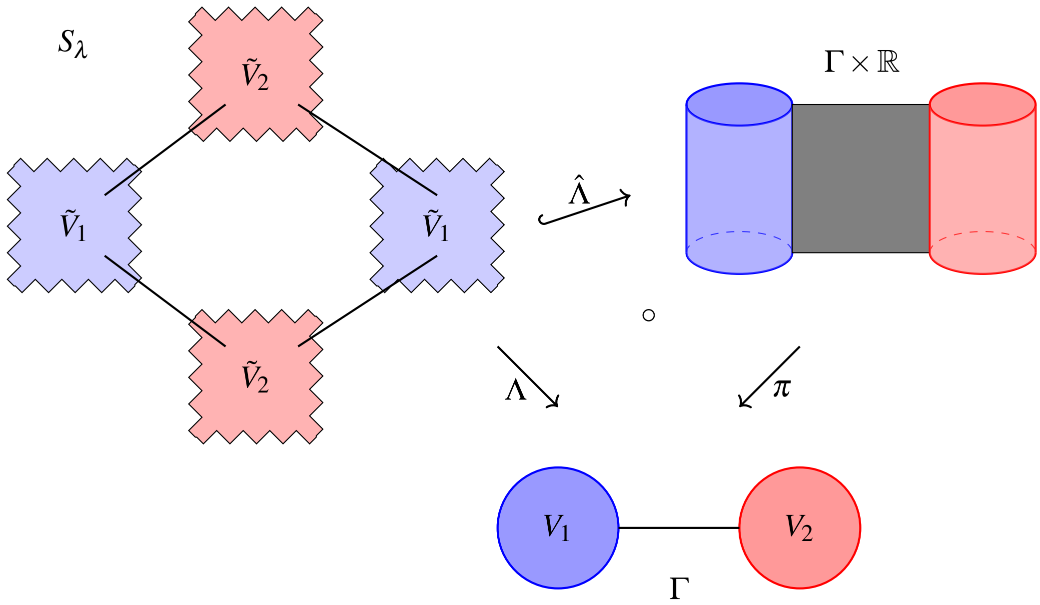


Figure 4.2: A stacking of the word cover S_λ is an embedding in the product $\Gamma \times \mathbb{R}$ such that the diagram shown commutes.

will use $\mathbb{S}_\lambda = \mathbb{S}$. The underlying map of a disjoint union of circles will always be assumed. On top of this we will talk about $\mathbb{S}$ as a relative graph, so edges of $\mathbb{S}$ are the pre-images of edges in Γ and vertices of $\mathbb{S}$ are the pre-images of vertices in Γ.

In order to prove that stackings indeed exist, we will use a tower argument similar to the ones used in [11] and [17]. Recall that a covering map is infinite cyclic if its deck group is isomorphic to $\mathbb{Z}$.

Recall Lemma 2.4.13: If $f : Y \to X$ be a cellular map of compact CW complexes. Then there exists a maximal cyclic tower lifting $Y \to X'$ of f. This is an essential component for inductive proofs using towers. In order to make our inductive proofs work we will not only need this but also a result about pulling our embeddings (stackings) back to an infinite cyclic cover. The following lemma was proved in [17] for Γ being a graph, we quickly check here that the same proof also goes through for infinite cyclic covers of other spaces (for example we are mainly interested in relative graphs).

Lemma 4.1.4. *Consider an infinite cyclic cover $\tilde{\Gamma} \to \Gamma$ of CW complexes. Then there is an embedding $\tilde{\Gamma} \times \mathbb{R} \hookrightarrow \Gamma \times \mathbb{R}$ such that the diagram commutes:*

$$\begin{array}{ccc} \tilde{\Gamma} \times \mathbb{R} & \longrightarrow & \tilde{\Gamma} \\ \big\downarrow & & \big\downarrow \\ \Gamma \times \mathbb{R} & \longrightarrow & \Gamma \end{array}$$

where the top and bottom maps and coordinate projections.

Proof. Let $g \in \pi_1(\Gamma)$, then g acts by deck transformations on the cover $\tilde{\Gamma}$. The map $\tilde{\Gamma} \to \Gamma$ is an infinite cyclic cover by assumption, so it has deck group $\mathbb{Z}$. Thus, we have a homomorphism $\pi_1(\Gamma) \to \mathbb{Z}$ that allows elements of $\pi_1(\Gamma)$ to act by integer translation on $\mathbb{R}$. Consider the diagonal action of $\pi_1(\Gamma)$ on $\tilde{\Gamma} \times \mathbb{R}$, i.e. $\pi_1(\Gamma)$ acts diagonally by deck transformations on $\tilde{\Gamma}$ and by translation on the $\mathbb{R}$.

By construction, the quotient of $\tilde{\Gamma} \times \mathbb{R}$ by this action is homeomorphic to $\Gamma \times \mathbb{R}$. Let $X = \tilde{\Gamma} \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ for example, then translates of X by an integer are disjoint so by passing to the quotient the embedding $X \hookrightarrow \tilde{\Gamma} \times \mathbb{R}$ produces an embedding $X \hookrightarrow \Gamma \times \mathbb{R}$. Now take any homeomorphism $\mathbb{R} \to \left(-\frac{1}{2}, \frac{1}{2}\right)$ to provide an embedding $e : \tilde{\Gamma} \times \mathbb{R} \hookrightarrow \Gamma \times \mathbb{R}$ factoring through X via this homeomorphism.

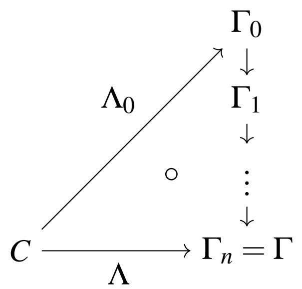
Notice that if $\iota : \tilde{\Gamma} \to \Gamma$ is the given infinite cyclic cover, then for any $(x, y) \in \tilde{\Gamma} \times \mathbb{R}$, $e(x, y) = (\iota(x), e_{\mathbb{R}}(y))$ by construction, where $e_{\mathbb{R}}$ is dependent on the choice of homeomorphism $\mathbb{R} \to \left(-\frac{1}{2}, \frac{1}{2}\right)$. Therefore, the diagram commutes since $\pi \circ e(x, y) = \iota(x) = \iota \circ \tilde{\pi}(x, y)$. $\square$

Now that we are all set for our inductive arguments we can begin the proof of existence of a stacking in the case when Λ is a single indivisible word cover. To this end, we will first prove that a stacking exists if we restrict $\mathbb{S}$ to a compact subset. Afterwards, we will show that we can find a stacking of the entirety of $\mathbb{S}$ by exhausting $\mathbb{S}$ with a nested sequence of compact subsets.

The following proposition relies on vertices of Γ having NPI rather than just locally indicable fundamental groups. We only prove the weaker case as it is exactly what we need for our later results, however this proposition should also be true for vertices with locally indicable fundamental groups.

Proposition 4.1.5. *Let* Γ *be a relative graph whose vertices have NPI, and let* $\Lambda : \mathbb{S} \to \Gamma$ *be an indivisible word cover. Let* $C \subset \mathbb{S}$ *be a connected, compact subset of* $\mathbb{S}$ *containing each edge of* $\mathbb{S}$ *with* $\pi_1(C) \cong \mathbb{Z}$*. There exists an embedding* $\hat{\Lambda} : C \hookrightarrow \Gamma \times \mathbb{R}$ *such that* $\pi\hat{\Lambda} = \Lambda$ *where* $\pi : \Gamma \times \mathbb{R} \to \Gamma$ *is the projection to* Γ*.*

Proof. For the base case of our induction. By Lemma 2.4.13 there exists a maximal cyclic tower lifting $\Lambda_0 : C \to \Gamma_0$ of Λ:



Let $\Lambda_i : C \to \Gamma_i$ be the lift of Λ to Γ_i. By maximality of the tower lifting, Λ_0 is a surjective map and so Γ_0 is compact since C is compact. By definition of a tower lifting the map $\Gamma_0 \to \Gamma$ is an immersion since it is a composition of inclusions and covering maps. By Lemma 3.2.2, Γ, and hence also Γ_0, has NPI since the vertices of Γ have NPI. Therefore, $\pi_1(\Gamma_0)$ is locally indicable by Lemma 3.1.5. Consider any vertex V of C (where C is a relative graph), then V is a simply connected subcomplex of C and the restriction of Λ_0 to V is a maximal cyclic tower lifting of $\Lambda|_V$, otherwise Λ_0 could not be maximal. Now $\pi_1(\Gamma_0)$ is locally indicable so $\pi_1(\Lambda_0(V))$ is indicable (must admit an epimorphism to $\mathbb{Z}$ if non-trivial) as it is a finitely generated subgroup. If $H_1(\Lambda_0(V))$ contains an infinite cyclic factor then there exists an infinite cyclic cover that lifts $\Lambda_0|_V$ since $\pi_1(V) = 1$, by standard covering map theory. This contradicts maximality of the tower lifting. Thus, $H_1(\Lambda_0(V))$ is finite since it is a finitely generated abelian group containing no $\mathbb{Z}$ factor. Therefore, $\pi_1(\Lambda_0(V))$ is an indicable group with finite first homology (abelianisation) and hence it must be trivial as it cannot admit an epimorphism to $\mathbb{Z}$. So $\pi_1(\Lambda_0(V)) = \{1\}$ and the map

$\Lambda_0|_V$ must be an embedding. Since this is true for each vertex it follows that $\pi_1(\Gamma_0)$ is free, but $\pi_1(C) \cong \mathbb{Z}$, so $\pi_1(\Gamma_0) \cong \mathbb{Z}$ by maximality of the tower (otherwise we could lift over an infinite cyclic cover). It follows that C and Γ_0 are homeomorphic as CW complexes and the map Λ_0 is a finite-to-one covering map. However, $\Lambda : \mathbb{S} \to \Gamma$ is an indivisible word cover (i.e. the word is not a proper power), so the cover is one-to-one and thus $C \to \Lambda_0$ is a homeomorphism. Therefore, there is a trivial stacking $\hat{\Lambda}_0 : C \hookrightarrow \Lambda_0 \times \mathbb{R}$.

We will now proceed by induction on the length of the tower n. Suppose that the statement of the theorem holds true whenever $C \to \Gamma$ has a maximal tower lifting of length $n-1$, $n \geq 1$ (we have proved above the statement holds when the maximal tower lift has length $n = 0$).

Take any $C \to \Gamma$ as in the statement of the theorem and suppose $C \to \Gamma$ has a maximal tower lifting of length n. Then $C \to \Gamma_{n-1}$ has a maximal tower lifting of length $n-1$ so there exists a stacking $\hat{\Lambda}_{n-1} : C \hookrightarrow \Gamma_{n-1} \times \mathbb{R}$.

If the map $\iota : \Gamma_{n-1} \to \Gamma_n$ given by the tower is an inclusion of subcomplexes then $\hat{\Lambda}_n := (\iota \times id_{\mathbb{R}}) \circ \hat{\Lambda}_{n-1} : C \to \Gamma_n \times \mathbb{R}$ is an embedding, where $(i \times id_{\mathbb{R}}) : \Gamma_{n-1} \times \mathbb{R} \to \Gamma_n \times \mathbb{R}$ sends $(x, y) \mapsto (i(x), y)$. The following square is trivially commutative,

$$\begin{array}{ccc} \Gamma_{n-1} \times \mathbb{R} & \xrightarrow{\pi_{n-1}} & \Gamma_{n-1} \\ {\scriptstyle (\iota \times id_{\mathbb{R}})} \downarrow & \circ & \downarrow {\scriptstyle \iota} \\ \Gamma_n \times \mathbb{R} & \xrightarrow[\pi_n]{} & \Gamma \end{array}$$

so $\pi_n \circ \hat{\Lambda}_n = \pi_n \circ (\iota \times id_{\mathbb{R}}) \circ \hat{\Lambda}_{n-1} = \iota \circ \pi_{n-1} \circ \hat{\Lambda}_{n-1} = \iota \circ \Lambda_{n-1} = \Lambda_n$ by definition of a tower lifting. Thus, $\hat{\Lambda}_n$ is a stacking of $\Lambda_n = \Lambda$.

Otherwise ι is an infinite cyclic cover and Lemma 4.1.4 provides the embedding $e : \Gamma_{n-1} \times \mathbb{R} \to \Gamma_n \times \mathbb{R}$ such that the following diagram commutes.

$$\begin{array}{ccc} \Gamma_{n-1} \times \mathbb{R} & \xrightarrow{\pi_{n-1}} & \Gamma_{n-1} \\ e \Big\downarrow & \circ & \Big\downarrow \iota \\ \Gamma_n \times \mathbb{R} & \xrightarrow[\pi_n]{} & \Gamma \end{array}$$

In this case we define $\hat{\Lambda}_n = e \circ \hat{\Lambda}_{n-1} : C \to \Gamma_n \times \mathbb{R}$. This is clearly an embedding and as before we have

$$\pi_n \circ \hat{\Lambda}_n = \pi_n \circ e \circ \hat{\Lambda}_{n-1} = \iota \circ \pi_{n-1} \circ \hat{\Lambda}_{n-1} = \iota \circ \Lambda_{n-1} = \Lambda_n,$$

so $\hat{\Lambda}_n$ defines a stacking of $\Lambda = \Lambda_n$ completing our inductive proof. □

The next result is our first main result showing that we can in fact find a stacking of the entirety of $\mathbb{S}$, rather than just a compact subcomplex. First we define certain compact subcomplexes of $\mathbb{S}$, providing an example of a nested sequence of compact subsets whose union is the entire cover. For the existence proof explicit sets are not required but we will use these sets later and it is useful to have a more visual idea for the proof anyway.

Definition 4.1.6. Let X be a CW-complex and let $x \in X^0$ (where X^0 is the 0-skeleton of X). Inductively define sets $C(x,m)$ for each natural number m: $C(x,0) = \{x\}$, and

$$C(x,m+1) = C(x,m) \cup \overline{\{c \subset X \mid \text{c is an open cell of } X \text{ with } \partial c \cap C(x,m) \neq \emptyset\}}.$$

Remark 4.1.7. *If X is the universal cover of a finite CW-complex then X is locally finite so $C(x,m)$ is finite and hence compact for every $m \in \mathbb{N}$. Notice however that these sets need not be simply connected, we will address this later.*

Example 4.1.8. Consider the torus $\mathbb{T}^2$, which is the quotient space obtained by identifying opposite pairs of edges of a square and so it is a CW complex with one

0-cell, two 1-cells and one 2-cell. Then its universal cover $\tilde{\mathbb{T}}^2 = \mathbb{R}^2$, and again as a CW complex this is given by the tiling of the plane $\mathbb{R}^2$ by the original squares. The sets $C((0,0),m)$ squares of increasing size, as illustrated in Figure 4.3.

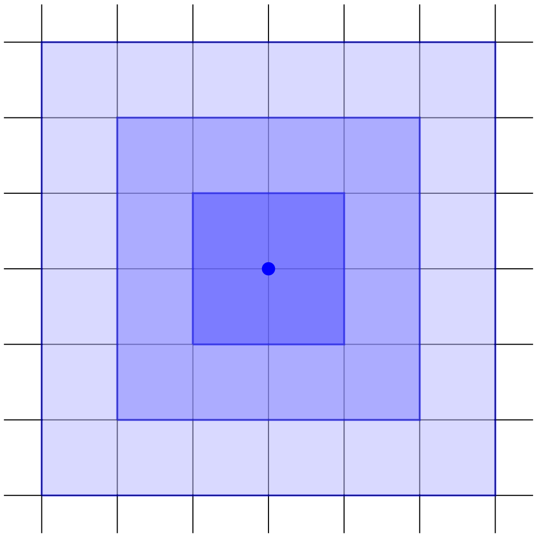


Figure 4.3: These are the sets $C((0,0),0)$, $C((0,0),1)$, $C((0,0),2)$, $C((0,0),3)$ for the torus.

To create the sets C required from Lemma 4.1.5 we will use these compact sets $C(x,m)$ in each vertex of $\mathbb{S}$ and connect them up with edges of $\mathbb{S}$. Some things we need to be careful about are the fact that the set C won't necessarily have $\pi_1(C) \cong \mathbb{Z}$ and we will need the m large enough in each vertex such that C is connected. These are both problems that are easy to deal with.

Lemma 4.1.9. *Let Γ be a relative graph and $\Lambda : \mathbb{S} \to \Gamma$ an indivisible word cover. There exists a nested sequence of compact, connected subcomplexes of $\mathbb{S}$, $\{C_i\}_{i\geq 1}$, satisfying the following conditions:*

- $\bigcup_{i\geq 1} C_i = \mathbb{S}$;
- *C_i contains Edges$(\mathbb{S})$; and*
- *$\pi_1(C_i) \cong \mathbb{Z}$ for each $i \geq 1$.*

Proof. Define C_1 in the following manner. For each vertex V_j of $\mathbb{S}$ choose m_j, x_j so that $C(x_j,m_j)$ contains both endpoints of the two incident edges of $\mathbb{S}$ to V_j. Let C_1'

be the following union of subcomplexes of $\mathbb{S}$

$$C_1' = \text{Edges}(\mathbb{S}) \cup \left(\bigcup_j C(x_j, m_j) \right).$$

If some $C(x_j, m_j)$ is not simply connected, then because vertices are simply connected, $C(x_j, m_j)$ is contained in a compact, simply connected subcomplex of V_j. Replace $C(x_j, m_j)$ in C_1' with this larger complex. Call the resulting complex C_1, by construction it satisfies the second two points above. Define C_i from C_{i-1}

$$C_i' = C_{i-1} \cup \left(\bigcup_j C(x_j, m_j + i) \right),$$

and C_i is again the complex formed after replacing any non simply-connected parts as above. The C_i we have defined are clearly nested and their union must be $\mathbb{S}$, hence the three conditions are satisfied. □

To complete the proof that stackings exist we just need to be able to pass from these compact subsets to the entirety of $\mathbb{S}$. To do this we are going to take a brief detour into formal logic, as it provides a neat way to tie up the proof.

4.2 The Compactness Theorem

Without wanting to go too deep into logic, we wish to introduce some ideas that allow for a neat proof of the existence of stackings. In particular, we want to state and use the Compactness Theorem, which was first proved by Kurt Gödel in 1930. As we don't want to go into too much detail here, we will introduce the relevant definitions only with our purpose in mind. There are many references for the following ideas, a couple of sets of notes that have been used for the following exposition are [24] and [31].

To start with we need to define a set of symbols, which we will combine to produce formulas. In our case we have the following situation. Let $\Lambda : \mathbb{S} \to \Gamma$ be an indivisible word cover of a relative graph Γ. Then our symbols are L_a^b, where a, b are closed 0-, 1- or 2-cells in $\mathbb{S}$ with $\Lambda(a) = \Lambda(b)$. (When thinking of the stacking we will be using L_a^b to mean $a < b$, or the midpoints of these closed cells, according to the height function provided by the stacking). Call our set of symbols $\mathcal{S}$.

Formulas are defined inductively using the relations $\wedge, \vee, \neg, \rightarrow$. In general there are other relations but as we won't use them here we won't include them to keep things simple. So if F, F' are formulas, the following are also formulas: $F \wedge F'$; $F \vee F'$; $\neg F$; and $F \rightarrow F'$.

Define a set, $\hat{\Sigma}$, over our language consisting of the following formulas.

1. $L_a^b \vee L_b^a$;
2. $\neg(L_a^b \wedge L_b^a)$;
3. $((L_a^b \wedge L_b^c) \rightarrow L_a^c)$; and
4. $(L_c^{c'} \rightarrow L_a^b)$ for $a \in \partial c$ and $b \in \partial c'$ being copies of the same boundary cell of $\Lambda(c) = \Lambda(c')$ according to the chosen orientation on ∂c.

The next step is to relate this abstract set of formulas to a stacking. To do so we need to know how to evaluate these formulas. The following few definitions will help us with this.

Definition 4.2.1. A *valuation* on a set of symbols $\mathcal{S}$ is a map $\nu : \mathcal{S} \rightarrow \{0, 1\}$. If Σ is a set of formulas with symbols in $\mathcal{S}$ then we can extend any valuation $\nu : \mathcal{S} \rightarrow \{0, 1\}$ to a *valuation* $\nu : \Sigma \rightarrow \{0, 1\}$ on Σ using the following rules: Let F, F' be formulas in Σ, then

- $\nu(F \vee F') = max\{\nu(F), \nu(F')\}$;
- $\nu(F \wedge F') = min\{\nu(F), \nu(F')\}$;
- $\nu(F \rightarrow F') = 0$ if and only if $\nu(F) = 1$ and $\nu(F') = 0$; and
- $\nu(\neg F) = 1 - \nu(F)$.

Finally, we say a set of formulas is *satisfiable* if there exists a valuation ν on Σ such that $\nu(F) = 1$ for every formula F in Σ.

We can now return to our set of formulas $\hat{\Sigma}$ and show how they relate to a stacking.

Proposition 4.2.2. *$\hat{\Sigma}$ is satisfiable if and only if there exists a stacking of Λ.*

Proof. "$\Leftarrow$" Suppose that $\hat{\Lambda} : \mathbb{S} \hookrightarrow \Gamma \times \mathbb{R}$ is a stacking of Λ. Then we can define a valuation as follows. For any closed cells a, b with $\Lambda(a) = \Lambda(b)$ define $\nu(L_a^b) = 1$ if and only if $\iota(\hat{\Lambda}(a)) < \iota(\hat{\Lambda}(b))$ (or usually we just write $a < b$, and note really we are talking about midpoints if a and b are not 0-cells), where $\iota : \Gamma \times \mathbb{R} \to \mathbb{R}$ is the projection map. Using this definition we see that

1. either $a < b$ or $b < a$ so $\nu(L_a^b \vee L_b^a) = 1$.

2. If $\Lambda(a) = \Lambda(b)$ and $a \neq b$ then at most one of $a < b$ or $b < a$ holds so

$$\nu(\neg(L_a^b \wedge L_b^a)) = 1 - \nu(L_a^b \wedge L_b^a) = 1 - min\{L_a^b, L_b^a\} = 1.$$

3. If $\nu(L_a^b \wedge L_b^c) = 1$ then $a < b$ and $b < c$, which clearly implies $a < c$ since we are in $\mathbb{R}$. So $\nu(L_a^c) = 1$ and thus $\nu((L_a^b \wedge L_b^c) \to L_a^c) = 1$

4. If $\nu(L_c^{c'}) = 1$ then $c < c'$ so there exists paths γ, γ' in $\mathbb{S}$ starting at the midpoint of c and c' respectively and ending at the midpoint of a and b respectively. Moreover, we can choose γ, γ' and a parameterisation such that they have identical image under Λ. Since $\hat{\Lambda}$ is an embedding the ordering of the end-points of γ and γ' must be preserved by $\hat{\Lambda}$ so $a < b$. Thus $\nu(L_a^b) = 1$, so $\nu(L_c^{c'} \to L_a^b) = 1$.

Therefore, ν as defined is a valuation satisfying $\hat{\Sigma}$.

"$\Rightarrow$" The opposite direction works similarly. Let ν be a valuation satisfying $\hat{\Sigma}$. Define a map $\hat{\Lambda} : \mathbb{S} \hookrightarrow \Gamma \times \mathbb{R}$ as follows. Firstly we define the restriction of $\hat{\Lambda}$ to the midpoints of cells. Suppose x, y are the midpoints of cells (or 0-cells), then map to

$(\Lambda(x), h_x), (\Lambda(y), h_y)$ for a choice of h_* such that if $\Lambda(x) = \Lambda(y)$ then $h_x < h_y$ if and only if $\nu(L_x^y) = 1$ (here we really mean the cell containing x, y for L_x^y to make sense). The specific choice of h_* does not matter, only the relative heights. The formulas labelled $1. - 3.$ of $\hat{\Sigma}$ determine an order on all cells in $\Lambda^{-1}(c)$ for any cell of Γ so this map must be well-defined. Furthermore, formula 4. of $\hat{\Sigma}$ tells us that the relative heights of boundary cells agree with the relative heights of the cells themselves, so this map extends to a continuous embedding $\hat{\Lambda} : \mathbb{S} \hookrightarrow \Gamma \times \mathbb{R}$ as we could for instance use barycentric subdivision to then order edges whose endpoints are midpoints of cells and order the remaining 2-cells accordingly. □

We now have a way to turn stackings into a set of satisfiable formulas. This is very useful due to the following theorem relating satisfiability to finite satisfiability.

Definition 4.2.3. A set of formulas Σ is finitely satisfiable if and only if every finite subset $\Sigma_0 \subset \Sigma$ is satisfiable.

Theorem 4.2.4 (The Compactness Theorem (Gödel 1930))**.** *A set of formulas Σ is finitely satisfiable if and only if it is satisfiable.*

See [24] for a proof of the compactness theorem. Using this result we know that we only need to consider finite subsets of $\hat{\Sigma}$. This is exactly why we were using compact sets to exhaust $\mathbb{S}$ as these will cover all finite subsets of $\hat{\Sigma}$.

Theorem 4.2.5. *Let Γ be a relative graph whose vertices have NPI, and let $\Lambda : \mathbb{S} \to \Gamma$ be an indivisible word cover. There exists a stacking, $\hat{\Lambda} : \mathbb{S} \hookrightarrow \Gamma \times \mathbb{R}$, of Λ.*

Proof. By Proposition 4.2.2 the existence of a stacking is equivalent to checking if the set of formulas $\hat{\Sigma}$ is satisfiable. By The Compactness Theorem, this in turn is equivalent to every finite subset $\hat{\Sigma}_0 \subset \Sigma$ being satisfiable. This is the result we will prove here.

Take any finite subset $\hat{\Sigma}_0 \subset \hat{\Sigma}$. Then there are finitely many closed cells of $\mathbb{S}$ that appear as indices in some formula of $\hat{\Sigma}_0$. By Lemma 4.1.9 there exists an infinite nested sequence $C_1 \subset C_2 \subset \cdots$ of connected, compact subcomplexes of $\mathbb{S}$ such that $\bigcup_{i\geq 1} C_i = \mathbb{S}$; each C_i contains Edges($\mathbb{S}$); and $\pi_1(C_i) \cong \mathbb{Z}$ for each $i \geq 1$. Let $\Lambda_i : C_i \to \Gamma$ be the restriction of Λ to C_i for each $i \geq 1$. Since only finitely many cells are used for formulas in $\hat{\Sigma}_0$ there must exist an $i \geq 1$ such that all of these are contained in some C_i. Let $\hat{\Sigma}^i \subset \Sigma$ be the finite set of formulas found by restricting to all formulas using L_a^b for both $a, b \in C_i$. By construction $\hat{\Sigma}_0 \subset \hat{\Sigma}^i \subset \hat{\Sigma}$, so if $\hat{\Sigma}^i$ is satisfiable, then $\hat{\Sigma}_0$ is also satisfiable. The valuation on $\hat{\Sigma}_0$ can be found by simply restricting the valuation we would have on $\hat{\Sigma}^i$.

By Proposition 4.2.2, since there exists a stacking of Λ_i, the set of formulas $\hat{\Sigma}^i$ is satisfiable, so $\hat{\Sigma}_0$ is satisfiable and since $\hat{\Sigma}_0 \subset \hat{\Sigma}$ was arbitrary finite subset, $\hat{\Sigma}$ is finitely satisfiable, completing the proof. □

4.3 Properties of a Stacking

We are now at the point where we know stackings exist for indivisible word covers, but do not know much about what they actually look like. In this section we will look more closely at the properties of stackings. As a start, recall that the proof of the W-cycles conjecture for graphs used the sets that you 'see' from looking at the stacking either from above or below. These sets can be defined exactly as before for relative graphs although their relation to a characteristic is not immediate, as it was before. We will define these sets for subsets of $\mathbb{S}$ and begin by looking at specific compact subsets, remember the embedding for $\mathbb{S}$ was found using compact sets so this is a reasonable approach.

Definition 4.3.1. Let $\hat{\Lambda} : \mathbb{S} \to \Gamma \times \mathbb{R}$ be a stacking of some disjoint union of word covers Λ and let $\iota : \Gamma \times \mathbb{R} \to \mathbb{R}$ be the projection to the second coordinate. For any $C \subseteq \mathbb{S}$ define the sets:

$$\mathcal{A}(\hat{\Lambda}, C) = \{x \in C \mid \forall y \in C, \text{ with } y \neq x, \pi(\hat{\Lambda}(y)) = \pi(\hat{\Lambda}(x)) \Rightarrow \iota(\hat{\Lambda}(y)) < \iota(\hat{\Lambda}(x))\};$$

$$\mathcal{B}(\hat{\Lambda}, C) = \{x \in C \mid \forall y \in C, \text{ with } y \neq x, \pi(\hat{\Lambda}(y)) = \pi(\hat{\Lambda}(x)) \Rightarrow \iota(\hat{\Lambda}(y)) > \iota(\hat{\Lambda}(x))\};$$

Remark 4.3.2. *Notice that the sets $\mathcal{A}(\hat{\Lambda}, C)$ and $\mathcal{B}(\hat{\Lambda}, C)$ need not be simply connected.*

In other words these are the sets of elements of C that are seen when either looking from above or below in the stacking. Recall that vertices of $\mathbb{S}$ (when it is viewed as a relative graph) are universal covers of vertices of Γ. The next few lemmas will restrict our attention to the properties of the stacking in these universal covers. Let $\tilde{X}$ be the universal cover of a finite CW complex X with locally indicable fundamental group. For $x, x' \in \tilde{X}$ with $\pi(x) = \pi(x')$, as in the previous section, we will abuse notation and write $x > x'$ if $\iota(\hat{\Lambda}(x)) > \iota(\hat{\Lambda}(x'))$. We will now determine some properties of the sets $\mathcal{A}(\hat{\Lambda}, C(x,m))$. Note that each result also holds for $\mathcal{B}(\hat{\Lambda}, C(x,m))$ by symmetry.

The following lemma is an easy result that follows from the stacking and the structure of the sets $C(x,m)$. We include the hypothesis that vertices have locally indicable fundamental as this is generally assumed for one-relator products, even though we have not proved that a stacking necessarily exists in this case. However, the alternative condition that vertices have NPI does produce some alternative results, and we have already proved that indivisible word covers admit a stacking in this situation, so we include those as well.

Lemma 4.3.3. *Let X be a finite CW complex with $\pi_1(X)$ locally indicable (resp. X has NPI), let $\Lambda : \tilde{X} \to X$ be its universal cover and suppose there exists an embedding $\hat{\Lambda} : \tilde{X} \to X \times \mathbb{R}$ such that $\pi\hat{\Lambda} = \Lambda$. If $x \in \tilde{X}^0$, then either $\pi_1(X) \cong \{1\}$ (resp. X is contractible), or there exists $M \in \mathbb{N}$ such that for any $m \geq M$ no connected component of the set $\mathcal{A}(\hat{\Lambda}, C(x,m))$ is closed in $\tilde{X}$.*

Proof. Suppose that a component of $\mathcal{A}(\hat{\Lambda}, C(x,m))$ is closed and not equal to $C(x,m)$. Since $C(x,m)$ is connected there exists $c, c' \in \tilde{X}$ such that $c \in \mathcal{A}(\hat{\Lambda}, C(x,m))$, $c' \in C(x,m) \setminus \mathcal{A}(\hat{\Lambda}, C(x,m))$ and $c \in \partial c'$ (in words there is a cell outside $\mathcal{A}(\hat{\Lambda}, C(x,m))$ whose boundary intersects $\mathcal{A}(\hat{\Lambda}, C(x,m))$). Since $c' \notin \mathcal{A}(\hat{\Lambda}, C(x,m))$ there exists $d' \in \mathcal{A}(\hat{\Lambda}, C(x,m))$ such that $d' > c'$. Since $d' > c'$

the cells d' and c' have the same image under Λ. After choosing an orientation on the boundary, let d be the copy of c in $\partial d'$ corresponding to same boundary cell of d' as c does for c'. In particular, $d \neq c$ since covering maps are immersions. Then $d \in C(x,m)$ because $d \in \partial d' \in C(x,m)$, so $d < c$ since $c \in \mathcal{A}(\hat{\Lambda}, C(x,m))$. Let γ be an embedded path from the centre of c' to c and let γ' be the copy of γ from the centre of d' to d. These paths are disjoint in $\tilde{X}$ with identical images in X and they must intersect under the image of $\hat{\Lambda}$, which contradicts the fact that $\hat{\Lambda}$ is an embedding, see Figure 4.4 for an illustration of this argument. Thus, for a component to be closed we must have $\mathcal{A}(\hat{\Lambda}, C(x,m)) = C(x,m)$. Furthermore, since $C(x,m)$ is closed this is an if and only if.

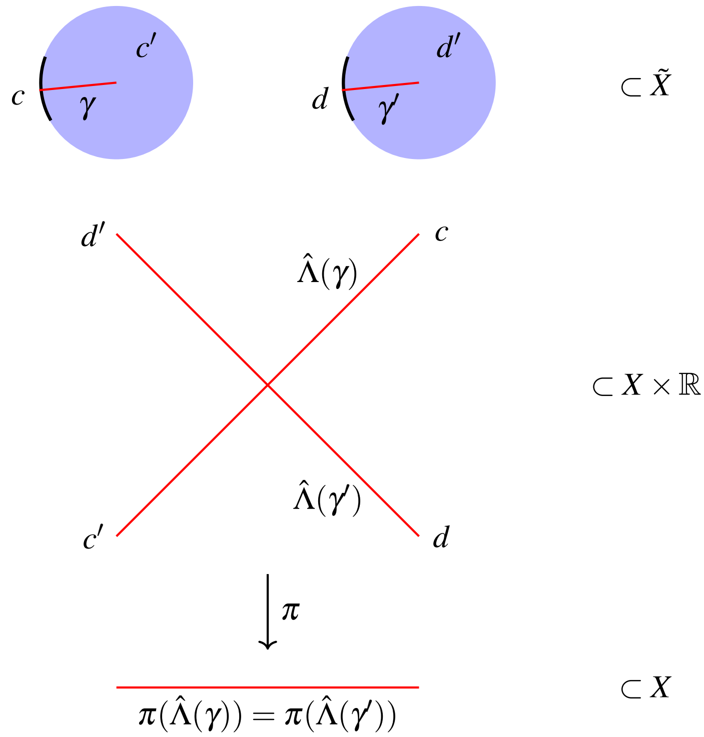


Figure 4.4: This is an illustration of the proof of Lemma 4.3.3. If a component of $\mathcal{A}(\hat{\Lambda}, C(x,m))$ is closed and not equal to $C(x,m)$ we can find disjoint paths γ, γ' in $\tilde{X}$ that would have to intersect in the stacking.

Notice that if for some $m \in \mathbb{N}$, no connected component of $\mathcal{A}(\hat{\Lambda}, C(x,m))$ is closed. Then $\mathcal{A}(\hat{\Lambda}, C(x,m)) \neq C(x,m)$, and since the sets $C(x,m)$ are nested this implies for any $M \geq m$, $\mathcal{A}(\hat{\Lambda}, C(x,M)) \neq C(x,M)$, so no component of $\mathcal{A}(\hat{\Lambda}, C(x,M))$ is closed. Suppose instead that for every $m \in \mathbb{N}$, $\mathcal{A}(\hat{\Lambda}, C(x,m)) = C(x,m)$. Then

$$\bigcup_{m \in \mathbb{N}} C(x,m) = \tilde{X},$$

and for every $m \in \mathbb{N}$ the map $\Lambda|_{C(x,m)}$ is injective. Hence Λ is injective, but Λ is a cover so it must be a homeomorphism and $\pi_1(\tilde{X}) \cong \{1\} \Rightarrow \pi_1(X) \cong \{1\}$. In the case of X having NPI we can note that $\chi(\tilde{X}) = \beta_0(\tilde{X}) - \beta_1(\tilde{X}) + \beta_2(\tilde{X}) \geq \beta_0(\tilde{X}) + \beta_2(\tilde{X}) \geq 1$, so $\tilde{X}$ is contractible because covers are immersions and thus X is also contractible. $\square$

The next lemma tells us how the set $\mathcal{A}(\hat{\Lambda}, C(x,m))$ changes as we increase m, which will allow us to use induction. This result is essentially saying that there are no maximal cells in a stacking of $\mathbb{R} \to S^1$.

Lemma 4.3.4. *Let X be a finite CW complex with $\pi_1(X)$ locally indicable, let $\Lambda : \tilde{X} \to X$ be its universal cover and suppose there exists an embedding $\hat{\Lambda} : \tilde{X} \to X \times \mathbb{R}$ such that $\pi\hat{\Lambda} = \Lambda$. If $x \in \tilde{X}^0$ and $c \in \mathcal{A}(\hat{\Lambda}, C(x,m))$ such that $\partial c \not\subset \mathcal{A}(\hat{\Lambda}, C(x,m))$ then $c \notin \mathcal{A}(\hat{\Lambda}, C(x,m+1))$. Furthermore, if $e \in \partial c \cap \mathcal{A}(\hat{\Lambda}, C(x,m))$, then $e \notin \mathcal{A}(\hat{\Lambda}, C(x,m+1))$.*

Proof. Let $d \in \partial c \setminus \mathcal{A}(\hat{\Lambda}, C(x,m))$ then there exists $d' > d$ with $d' \in \mathcal{A}(\hat{\Lambda}, C(x,m))$. After orienting the boundary of c we can find a copy c' of c in $\tilde{X}$ such that d' corresponds to the same boundary cell of c' as d does to c (covers are immersions so $c' \neq c$). We can use the same argument as illustrated in Figure 4.4 to see that since $\hat{\Lambda}$ is an embedding we must have $c' > c$ because $d' > d$, and thus $c' \notin C(x,m)$. However, since $d' \in C(x,m)$ and $d' \in \partial c'$, then $c' \in C(x,m+1)$, so we must have $c \notin \mathcal{A}(\hat{\Lambda}, C(x,m+1))$ because $c < c'$.

For the second statement notice that c' contains a boundary cell e' corresponding to e (not equal) and since $\hat{\Lambda}$ is an embedding $e' > e$ and $e' \in C(x,m+1)$. $\square$

Now we can prove the result we were aiming for. This tells us that the sets

$\mathcal{A}(\hat{\Lambda}, C(x,m))$ 'move outwards' from x as we increase m. In particular for any choice of cell y in $\tilde{X}$ we can always choose m large enough so that y is not in $\mathcal{A}(\hat{\Lambda}, C(x,m))$.

Lemma 4.3.5. *Let X be a finite CW complex with $\pi_1(X)$ locally indicable (resp. X has NPI). Let $\Lambda : \tilde{X} \to X$ be its universal cover and suppose there exists an embedding $\hat{\Lambda} : \tilde{X} \to X \times \mathbb{R}$ such that $\pi\hat{\Lambda} = \Lambda$. If $x \in \tilde{X}^0$, then either $\pi_1(X) \cong \{1\}$ (resp. X is contractible), or for any cell y in $\tilde{X}$ there exists $M \in \mathbb{N}$ such that for every $m \geq M$, $y \notin \mathcal{A}(\hat{\Lambda}, C(x,m))$.*

Proof. Take any cell y in $\in \tilde{X}$ (note y can be a 0-, 1- or 2-cell). Firstly, note that if $y \notin \mathcal{A}(\hat{\Lambda}, C(x,m))$ for some $m \in \mathbb{N}$ then $y \notin \mathcal{A}(\hat{\Lambda}, C(x,m'))$ for any $m' \geq m$. Suppose $\pi_1(X) \ncong \{1\}$ (resp. X not contractible) then by Lemma 4.3.3 there exists $m_0 \in \mathbb{N}$ such that for any $m \geq m_0$ no component of $\mathcal{A}(\hat{\Lambda}, C(x,m))$ is closed. If $y \notin \mathcal{A}(\hat{\Lambda}, C(x,m_0))$ we are done so we can assume $y \in \mathcal{A}(\hat{\Lambda}, C(x,m_0))$.

Consider the component A of $\mathcal{A}(\hat{\Lambda}, C(x,m_0))$ containing y. A is not closed so there exists a cell $c \in A$ with $\partial c \not\subset A$. Since A is connected there exists a sequence of cells $y_0, \ldots, y_k$ such that

1. $y_i \in A \; \forall i$;
2. $\partial y_i \subset A$ for each $0 \leq i \leq k-1$ and $\partial y_k \not\subset A$;
3. $\partial y_{i-1} \cap \partial y_i \neq \emptyset$ for each $1 \leq i \leq k$.

For notational purposes if v is a 0-cell we say $\partial v = \{v\}$. Call a sequence satisfying these conditions with $y_0 = y$ an *outward-sequence for y*. See Figure 4.5 for an illustration of an outward sequence. The length of such a sequence is k. Proceed by induction on the minimum length of outward-sequences for y.

Suppose the minimum length of an outward sequence for y in A is $k = 0$. Then y is not a 0-cell otherwise $\partial y = \{y\} \subset A$ contradicting point two in the definition of an outward sequence. Lemma 4.3.4 tells us $y \notin C(x, m_0 + 1)$ so we can take

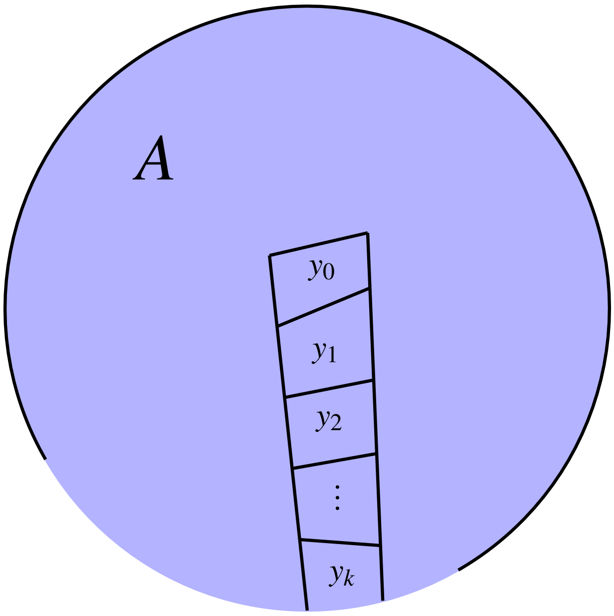


Figure 4.5: Illustration of an outward-sequence, this is a way to find paths from a particular cell, y_0, to the boundary of A.

$M = m_0 + 1$. If y is a 0-cell we need the base case for the minimum length $k = 1$. In this case $y \in \partial y_1$ so $y \notin C(x, m_0 + 1)$ by the second statement of Lemma 4.3.4.

Assume true for outward-sequences of minimum length at most $n-1$. and consider any outward-sequence for y of length n. If this is not minimum then take a shorter sequence. Otherwise, $\partial y_n \not\subset A$ so Lemma 4.3.4 tells us $y_n \notin \mathcal{A}(\hat{\Lambda}, C(x, m_0 + 1))$ and $\partial y_{n-1} \cap \partial y_n \notin \mathcal{A}(\hat{\Lambda}, C(x, m_0 + 1))$. Therefore, in $C(x, m_0 + 1)$ if y is still in $\mathcal{A}$ it has an outward-sequence of length strictly less than n (possibly shorter than $n-1$). By induction there exists some M such that $y \notin \mathcal{A}(\hat{\Lambda}, C(x, m_0 + 1 + M))$. Since X is a finite CW complex, the set $\mathcal{A}(\hat{\Lambda}, C(x, m))$ contains finitely many cells meaning outward sequences are always finite and the inductive argument holds. □

This allows us the deal with compact subsets of $\mathbb{S}$, but not yet about the whole of $\mathbb{S}$. To this end, we can consider the sets $\mathcal{A}(\hat{\Lambda}, \mathbb{S})$ exactly as in the compact case, just by replacing C in the definition with $\mathbb{S}$ (we can obviously replace C with any subset of $\mathbb{S}$). The previous lemma leads us to the following interesting result about these sets. In particular, the following lemma tells us that if V is a vertex of $\mathbb{S}$ corresponding to a universal cover of a vertex in Γ with locally indicable fundamental group (or NPI), then $\mathcal{A}(\hat{\Lambda}, V) = \emptyset$. Again, notice that the same applies for $\mathcal{B}(\hat{\Lambda}, \mathbb{S})$ by symmetry.

Corollary 4.3.6. *Let X be a finite CW complex with $\pi_1(X)$ locally indicable (resp. X has NPI). Let $\Lambda : \tilde{X} \to X$ be its universal cover and suppose there exists an embedding $\hat{\Lambda} : \tilde{X} \to X \times \mathbb{R}$ such that $\pi\hat{\Lambda} = \Lambda$. Then either $\pi_1(X) \cong \{1\}$ (resp. X is contractible), or $\mathcal{A}(\hat{\Lambda}, \tilde{X})$ is empty.*

Proof. Choose some $x \in \tilde{X}^0$. Take any cell $y \in \tilde{X}$, then by Lemma 4.3.5, either $\pi_1(\tilde{X}) = \{1\}$ or there exists $M \in \mathbb{N}$ such that for every $m \geq M$, $y \notin \mathcal{A}(\hat{\Lambda}, C(x, m))$, i.e. there exists some $y' \in \tilde{X}$ with $y' > y$. Therefore, $y \notin \mathcal{A}(\hat{\Lambda}, \tilde{X})$, so $\mathcal{A}(\hat{\Lambda}, \tilde{X}) = \emptyset$. □

Example 4.3.7. Let $\Lambda : \mathbb{R}^2 \to \mathbb{T}^2$ be the universal covering map of a torus and suppose we have an embedding $\hat{\Lambda} : \mathbb{R}^2 \hookrightarrow \mathbb{T}^2 \times \mathbb{R}$ such that $\pi\hat{\Lambda} = \Lambda$. Then without loss of generality we can assume that $\mathcal{A}(\hat{\Lambda}, C((0,0),1))$ has the form shown in Figure 4.6.

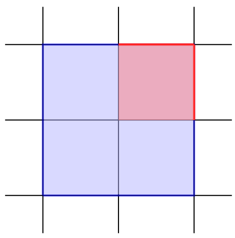


Figure 4.6: The set $\mathcal{A}(\hat{\Lambda}, C((0,0),1))$ is shaded red.

Once we know this initial set it is easy to see what happens as we increase m, see Figure 4.7.

We now have enough information to relate the number of components of the sets $\mathcal{A}$ and $\mathcal{B}$ to the relative characteristic of the relative graph. Open arcs in this case are not going to be literal arcs in the sense they were for graphs since if a vertex of the relative graph is compact (e.g. a sphere) its universal cover will also be compact and $\mathcal{A}$ can contain the entire compact 2-complex, although if we assume vertices have

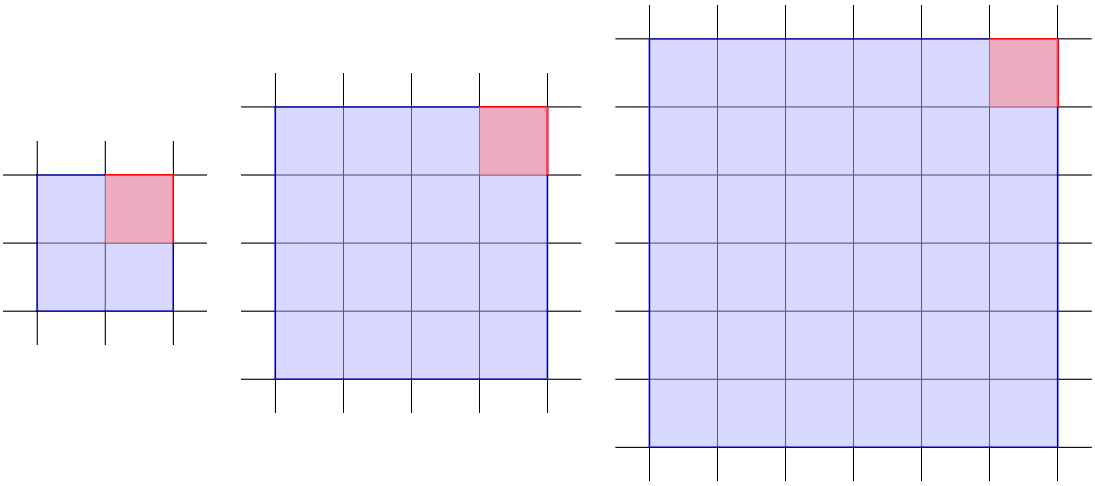


Figure 4.7: This diagram shows the sets $\mathcal{A}(\hat{\Lambda}, C((0,0),m))$ for $m \in \{1,2,3\}$. Notice that as m increases this set moves "further" away from $(0,0)$, this is the idea behind Lemma 4.3.5.

NPI, then any vertex of an open arc that is not a point will have to be contractible. It is however still easy to define open arcs in the way we want.

Definition 4.3.8. Let Γ be a relative graph. An *open arc* s of Γ is a connected, simply connected open subset of Γ such that $s \cap \mathrm{Ell}(\Gamma) = \emptyset$ and s is the union of vertices and interiors of edges of Γ.

The following lemma tells us that components of $\mathcal{A}(\hat{\Lambda}, \mathbb{S})$ are open arcs. Again, an identical result holds for $\mathcal{B}$ by symmetry.

Lemma 4.3.9. *Let $\Lambda : \mathbb{S} \to \Gamma$ be a surjective disjoint union of word covers for a relative graph Γ whose vertices have locally indicable fundamental group. Suppose there exists a stacking $\hat{\Lambda} : \mathbb{S} \hookrightarrow \Gamma \times \mathbb{R}$ of Λ. Then the components of $\mathcal{A}(\hat{\Lambda}, \mathbb{S})$ are either open arcs in $\mathbb{S}$ or components of $\mathbb{S}$. Furthermore, the number of open arcs in $\mathcal{A}(\hat{\Lambda}, \mathbb{S})$ is exactly $-\chi_r(\Gamma)$.*

Proof. By Corollary 4.3.6 if a cell within a vertex is contained in a component of $\mathcal{A}(\hat{\Lambda}, \mathbb{S})$, then the entire vertex must be in $\mathcal{A}(\hat{\Lambda}, \mathbb{S})$ and it must have trivial fundamental group. Additionally, if a vertex is in $\mathcal{A}(\hat{\Lambda}, \mathbb{S})$, then its incident edges must also be in $\mathcal{A}(\hat{\Lambda}, \mathbb{S})$ using the same argument as in the proof of Lemma 4.3.3.

The second statement is an observation following from the fact that if we collapsed all the vertices in $\mathrm{Ell}(\Gamma)$ we would expect to have exactly $-\chi(\Gamma/\mathrm{Verts}(\Gamma))$ open arcs, but for each element of $\mathrm{Ell}(\Gamma)$ we have in effect broken one of these arcs (by removing a vertex), increasing the number of open arcs by $|\mathrm{Ell}(\Gamma)|$. □

Example 4.3.10. Let Γ be the relative graph with one edge and $\mathrm{Verts}(\Gamma) = \mathrm{Ell}(\Gamma) = \{\mathbb{T}^2, S^1\}$, i.e. Γ is homotopy equivalent to $\mathbb{T}^2 \vee S^1$. If a, b are two generators of $\pi_1(\mathbb{T}^2) \cong \mathbb{Z}^2$ and c is a generator of $\pi_1(S^1) \cong \mathbb{Z}$, let $w = acbc$. Define the set $C(x_1, x_2, m)$ to be the subcomplex of $\mathbb{S}$ (the word cover of w) containing each edge of $\mathbb{S}$ where the vertices are given by the intersections of vertices of $\mathbb{S}$ with the sets $C(x_i, m)$. Then by taking large enough m we see that $\mathcal{A}(\hat{\Lambda}, C(x_1, x_2, m))$ has exactly $|\mathrm{Ell}(\Gamma)| - \chi(\Gamma/\mathrm{Verts}(\Gamma)) = 1$ open arc, see Figure 4.8 for a possibility for the structure of the stacking.

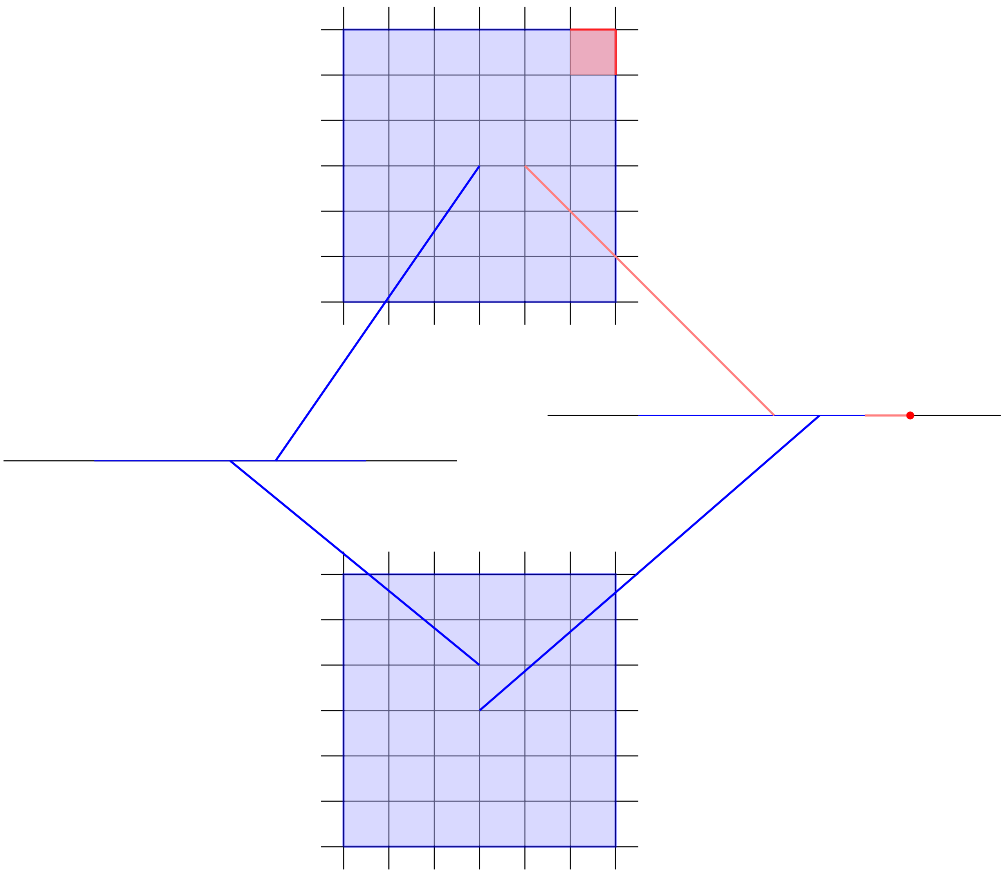


Figure 4.8: $\Gamma = \mathbb{T}^2 \vee S^1$, by taking m large enough (here $m = 3$) we have exactly one open arc in $\mathcal{A}(\hat{\Lambda}, C(x_1, x_2, m))$, where this set is shaded red.

Definition 4.3.11. An indivisible word cover $\Lambda : \mathbb{S} \to \Gamma$ is *reducible* if there is an edge of Γ that is traversed at most once by Λ.

Definition 4.3.12. A stacking $\hat{\Lambda}$ of a disjoint union of word covers $\Lambda : \mathbb{S} \to \Gamma$ for a relative graph Γ is called *good* if both $\mathcal{A}(\hat{\Lambda}, \mathbb{S})$ and $\mathcal{B}(\hat{\Lambda}, \mathbb{S})$ intersect each component of $\mathbb{S}$.

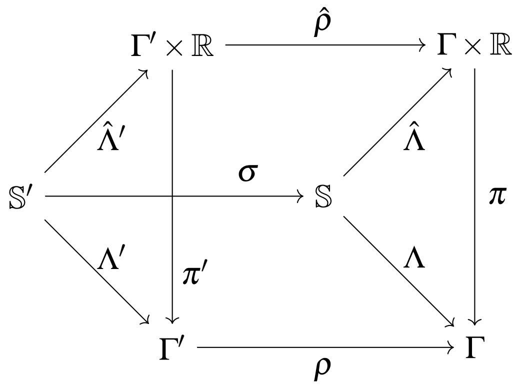
Notice that if $\mathbb{S}$ contains a single component then any stacking must be good. Using the sets $\mathcal{A}$ and $\mathcal{B}$ it is easy to determine when Λ is reducible (this result is a relative version of Lemma 9 in [17]).

Lemma 4.3.13. *$\mathcal{A}(\hat{\Lambda}, \mathbb{S}) \cap \mathcal{B}(\hat{\Lambda}, \mathbb{S})$ contains the interior of an edge if and only if Λ is reducible. Moreover, if the stacking is good and $\mathcal{A}(\hat{\Lambda}, \mathbb{S})$ or $\mathcal{B}(\hat{\Lambda}, \mathbb{S})$ contains a component of $\mathbb{S}$ then Λ is reducible.*

Proof. Let e be the interior of an edge in $\text{Edges}(\mathbb{S})$. If $e \in \mathcal{A}(\hat{\Lambda}, \mathbb{S}) \cap \mathcal{B}(\hat{\Lambda}, \mathbb{S})$, then there is no $e' \in \text{Edges}(\mathbb{S})$, with $e' \neq e$ such that $\iota\hat{\Lambda}(e') > \iota\hat{\Lambda}(e)$ or $\iota\hat{\Lambda}(e') < \iota\hat{\Lambda}(e)$. In particular there is no $e' \in \mathbb{S}$ with $e' \neq e$ such that $\Lambda(e) = \Lambda(e')$ otherwise they would have some ordering by the stacking. So $\Lambda(e)$ is an edge that is traversed exactly once by Λ, meaning Λ is reducible. The second statement follows easily from the first: If S is a component of $\mathbb{S}$ in $\mathcal{A}(\hat{\Lambda}, \mathbb{S})$, then since the stacking is good there exists a cell c in S contained in $\mathcal{B}(\hat{\Lambda}, \mathbb{S})$. Since components of $\mathcal{B}(\hat{\Lambda}, \mathbb{S})$ are either open arcs or entire components of $\mathbb{S}$ we can assume that c is an edge of $\mathbb{S}$, so $c \in \mathcal{A}(\hat{\Lambda}, \mathbb{S}) \cap \mathcal{B}(\hat{\Lambda}, \mathbb{S})$ and Λ is reducible. By symmetry the same result holds if a component of $\mathcal{B}(\hat{\Lambda}, \mathbb{S})$ is a component of $\mathbb{S}$. □

Lemma 4.3.14. *Let $\rho : \Gamma' \looparrowright \Gamma$ be an immersion, where Γ, Γ' are relative graphs. If $\hat{\Lambda}$ is a stacking then so is the pullback $\hat{\Lambda}' : \mathbb{S}' \to \Gamma' \times \mathbb{R}$, where $\mathbb{S}'$ is the pullback $\Gamma' \times_\Gamma \mathbb{S}$. Furthermore, if $\hat{\Lambda}$ is good, then $\hat{\Lambda}'$ is also good.*

Proof. We have the following commutative diagram:



The aim is to show that $\hat{\Lambda}'$ is an embedding. Suppose there is some $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{S}'$ with $\hat{\Lambda}'(x) = \hat{\Lambda}'(y)$, where $x_1, y_1 \in \Gamma'$ and $x_2, y_2 \in \mathbb{S}$. Then $\hat{\Lambda}\sigma(x) = \hat{\rho}\hat{\Lambda}'(x) = \hat{\rho}\hat{\Lambda}'(y) = \hat{\Lambda}\sigma(y)$, by commutativity, but $\hat{\Lambda}$ is an embedding so $\sigma(x) = \sigma(y)$ and thus $x_2 = y_2$. Furthermore, by commutativity we see that $\Lambda'(x) = \pi'\hat{\Lambda}'(x) = \pi'\hat{\Lambda}'(y) = \Lambda'(y)$, and so $x_1 = y_1$, therefore $x = y$. Notice that for an edge $e \in \mathcal{A}(\hat{\Lambda}, \mathbb{S})$ every edge in $\bar{\sigma}^{-1}(e)$ is contained in $\mathcal{A}(\hat{\Lambda}', \mathbb{S}')$ since if there exists $e' \in \mathbb{S}'$ with $e' > e$ then using the above diagram $\hat{\Lambda}\bar{\sigma}(e') = \hat{\rho}\hat{\Lambda}'(e')$, so since $\Lambda'(e') = \Lambda'(e)$, $\hat{\Lambda}\bar{\sigma}(e') = (\rho(\Lambda(e)), h')$ and $\hat{\Lambda}\bar{\sigma}(e) = (\rho(\Lambda(e)), h)$ for $h' > h$ so $\bar{\sigma}(e') > \bar{\sigma}(e)$. Therefore, if $\hat{\Lambda}$ is good, every component of $\mathbb{S}$ intersects $\mathcal{A}(\hat{\Lambda}, \mathbb{S})$ and so every component of $\mathbb{S}'$ also intersects $\mathcal{A}(\hat{\Lambda}', \mathbb{S}')$. The same is true for $\mathcal{B}$ by symmetry. □

4.4 W-cycles

We are now able to prove the main result of this chapter, which is a version of the W-cycles conjecture for free products of groups with NPI. This proof follows the argument of the proof of the W-cycles conjecture for free groups from [17].

Theorem 4.4.1. *Let Γ, Γ' be relative graphs such that every vertex of Γ has NPI. Let $\Gamma' \looparrowright \Gamma$ be an immersion, and let $\Lambda : \mathbb{S} \to \Gamma$ be an indivisible word cover. Let $\mathbb{S}'$ be the pullback to $\Gamma' \times_\Gamma \mathbb{S}$, with word cover $\Lambda' : \mathbb{S}' \to \Gamma'$, and map $\sigma : \mathbb{S}' \to \mathbb{S}$. Then $\bar{\sigma} : \mathbb{S}'/Verts(\mathbb{S}') \to \mathbb{S}/Verts(\mathbb{S})$ is a cover and if Λ' is not reducible then*

$$deg(\bar{\sigma}) \leq -\chi_r(\Gamma').$$

Proof. Suppose that the map Λ' is not reducible. We can assume it is surjective. Since Λ is indivisible it has a stacking $\hat{\Lambda} : \mathbb{S} \hookrightarrow \Gamma \times \mathbb{R}$ by Theorem 4.2.5, since vertices of Γ have NPI. By Lemma 4.3.14, Λ' has a stacking $\hat{\Lambda}' : \mathbb{S}' \hookrightarrow \Gamma' \times \mathbb{R}$. Note that if we restrict to edges then $\sigma^{-1}(\mathcal{A}(\hat{\Lambda}), \mathbb{S}) \subset \mathcal{A}(\hat{\Lambda}', \mathbb{S}')$, and the same holds for $\mathcal{B}$ (see the proof of Lemma 4.3.14). Let e be an edge of $\mathcal{A}(\hat{\Lambda}, \mathbb{S})$, which exists since the definition of a word cover ensures edges are crossed, and consider its deg$(\bar{\sigma})$ pre-images, $\{e_i \,|\, 1 \leq i \leq deg\bar{\sigma}\}$, in $\mathcal{A}(\hat{\Lambda}, \mathbb{S}')$. If $deg(\bar{\sigma}) > -\chi_r(\Gamma')$ then by Lemma 4.3.9 there are exactly $-\chi_r(\Gamma')$ open arcs in $\mathcal{A}(\hat{\Lambda}, \mathbb{S}')$, and so, by the pigeonhole principle, since there are more pre-images of e than there are open arcs, there must exist e_i, e_j contained in the same open arc of $\mathcal{A}(\hat{\Lambda}, \mathbb{S}')$. However, this open arc, s, must hit every edge of $\mathbb{S}$ since it contains two pre-images of e (so the image of the arc must wrap around $\mathbb{S}$). In particular it hits an edge f in $\mathcal{B}(\hat{\Lambda}, \mathbb{S})$ since $\hat{\Lambda}$ is good. Let $f' \in s \cap \bar{\sigma}^{-1}(f)$, then f' is an edge of $\mathbb{S}'$ contained in $\mathcal{A}(\hat{\Lambda}', \mathbb{S}') \cap \mathcal{B}(\hat{\Lambda}, \mathbb{S}')$, contradicting the fact that Λ' is not reducible by Lemma 4.3.13. □

Observe that this statement is a generalisation of the original W-cycles result from [17] for free groups, since if we take all elements of Verts(Γ), Verts(Γ') to be points (i.e. the relative graph is in fact a graph) then $\bar{\sigma} = \sigma$ and the result reads $deg(\sigma) \leq -\chi(\Gamma')$, which is exactly the result for free groups. Similarly if we decomposed a free group $\mathbb{F}_p$ for $p \geq 2$ into a relative graph by taking vertex spaces to be circles, the result reads $deg(\bar{\sigma}) \leq p - 1$, which is exactly minus the Euler characteristic of Γ' if we instead view it as a graph.

An initial result that follows from Theorem 4.4.1 is a non-positive immersions property for torsion-free one-relator products of groups. This result has also recently been proved by Howie and Short in [12], we will give a brief indication of their proof, which differs from ours. Take a relative graph whose vertices have NPI, and attach a 2-cell α to create a one-relator product (call the 2-complex X) and consider an immersion $Y \looparrowright X$. Now consider a subcomplex of Y, say Z, found by removing all open 2-cells in the pre-image of α together with a 'highest edge' for each (they use the fact that locally indicable groups have a left order for this, but the main thing to note is that since $Y \looparrowright X$ is an immersion the 'highest edges' are distinct). Howie and Short then prove that if a component T of Z has trivial fundamental group, Y Nielsen reduces to T. The proof works by showing that you can keep adding 2-cells (and their 'highest edges') to T so that the resulting complex admits a Nielsen reduction via this additional 2-cell. Since Z and Y have the same Euler characteristic and X has NPI it is easy to then see that either $\chi(Y) \leq 0$ or Z has a component of positive Euler characteristic and is hence contractible so Y is contractible. Here we show how non-positive immersions can also be proved using stackings.

Corollary 4.4.2. *Let Γ be a relative graph whose vertices are freely indecomposable and have NPI. Let $\Gamma \cup_w \{\alpha\}$ be the 2-complex formed by attaching a 2-cell, α, such that if $w : \partial\alpha \looparrowright \Gamma$ is the attaching map of w, then the homotopy class $[w] \in \pi_1(\Gamma)$ is not a proper power and is not conjugate into a vertex of Γ. Then $\Gamma \cup_w \{\alpha\}$ has NPI.*

Proof. Take any immersion $f : Y \looparrowright \Gamma \cup_w \{\alpha\}$ for a connected, compact 2-complex Y. Let $Y' = f^{-1}(\Gamma)$ (i.e. remove the pre-images of the interior of α), then the restriction $f|_{Y'} : Y' \looparrowright \Gamma$ is an immersion. Each vertex of Γ has NPI and since Y' immerses in Γ then each vertex of Y' (as a relative graph) also has NPI. Notice further that Y' has NPI by Lemma 3.2.2, this is the base case for induction (where there are no 2-cells in $f^{-1}(\alpha)$). Let $\Lambda : \mathbb{S} \to \Gamma$ be the indivisible word cover for w (which exists since w is not a proper power and is not conjugate into a vertex). Then we can take $\mathbb{S}'$, $\Lambda' : \mathbb{S}' \to Y'$ as in Theorem 4.4.1. If Λ' is reducible then there exists some edge of the relative graph Y' crossed exactly once by a 2-cell in $f^{-1}(\alpha)$, so Y Nielsen reduces (free-face collapse) to a 2-complex that immerses into X and $f^{-1}(\alpha)$ contains fewer 2-cells. If Λ' is not reducible then by Theorem 4.4.1 $deg(\bar{\sigma}) \leq -\chi_r(Y')$, where $\bar{\sigma} : \mathbb{S}'/\text{Verts}(\mathbb{S}') \to \mathbb{S}/\text{Verts}(\mathbb{S})$ is the natural cover. Notice that if n is the number of 2-cells in $f^{-1}(\alpha)$ then $deg(\bar{\sigma}) \geq n$, so $n \leq -\chi_r(Y')$. Thus, $\chi_r(Y') + n \leq 0$, but Y' has NPI so by Lemma 3.2.4, $\chi_r(Y') \geq \chi(Y')$, hence

$$\chi(Y) = \chi(Y') + n \leq \chi_r(Y') + n \leq 0.$$

Therefore, by induction Y either has non-positive Euler characteristic or Y Nielsen reduces to a point. $\square$

Chapter 5

Discussion and Areas of Future Work

5.1 Comparison of Chapter 4 to Results of Howie and Short

In their paper on Coherence of one-relator products with torsion (they prove that a one-relator product of coherent locally indicable groups where the word w is a proper power is coherent), [12], Howie and Short use different methods to prove torsion-free one-relator products have NPI. Here we will dig a little deeper into how the stackings method differs. In order to prove coherence, they are looking for bounds on the number of 2-cells in the pre-image of the defining 2-cell for the original one relator product, we will show how Theorem 4.4.1 produces an improved bound on this when vertices have NPI (although if Proposition 4.2.5 is proved for relative graphs whose vertices have locally indicable fundamental group, then Theorem 4.4.1 would also hold for relative graphs whose vertices have locally indicable fundamental group and the improved bound would apply in this case too). To do so we need to introduce some of their notation for context as we are now working with torsion.

Let Γ be a relative graph whose vertices each have NPI. We wish to consider one-relator products where the word is a proper power w^n for some $n \geq 2$, and not conjugate into a vertex. Let $n > 1$ and define a complex $\hat{Y}_n$ (referred to in [12] as the n-fold branched cover) as follows. Let $\hat{Y}_n = \Gamma \cup \alpha_n$ where α_n is a 2-cell whose attaching path is the nth power of a path α, where the free homotopy class of α is conjugate to the word w. We now consider immersions into $\hat{Y}_n$.

In [12], Howie and Short prove the following theorem that bounds the number of 2-cells (not attached along nth powers) we can have in Y' if we have an immersion $Y' \to \hat{Y}_n$. Their proof uses more of a local argument than the one we provide. As in their proof of NPI, they again consider the complex found by removing 2-cells from Y' whose image is α_n together with their 'highest edges' (in this case each cell would have multiple 'highest edges' due to torsion). They then prove that components of this complex with trivial fundamental group have at least 5 incident half-edges of 'highest edges' and go on to calculate a bound on the first Betti number using this fact.

Theorem 5.1.1 ([12] Theorem 3.3)**.** *Let $n > 1$ and suppose $f : Y' \looparrowright \hat{Y}_n$ is an immersion, where Y' is compact and connected with first Betti number β. Suppose that none of the 2-cells in $f^{-1}(\alpha_n)$ are attached along an nth power. If Y' has no free edges then the number of 2-cells in $f^{-1}(\alpha_n)$ is at most 5β.*

Using stackings we can use a more global picture to improve this bound by removing the factor of 5. In order to use stackings we need to instead view $\hat{Y}_n$ as an orbicomplex, so α_n is a 2-cell with a central degree n cone point. In general, for us an orbicomplex will be a 2-complex whose 2-cells have a cone point. A morphism of orbicomplexes is a map $Y \to X$ that sends 0-,1-cells homeomorphically to 0-,1-cells respectively and 2-cells restrict to a map $p_k : D \to D'$, given by $p_k(z) = z^k$, such that if D has a cone point of degree d and D' has a cone point of degree d', then $d' = dk$. A morphism of orbicomplexes is an immersion if it is locally injective away from cone points. Notice that any immersion into one defintion of $\hat{Y}_n$ can easily be translated into the other definition, so the bounds above relate to either definition. Here if $f : Y \looparrowright \hat{Y}_n$ is an immersion then 2-cells in $f^{-1}(\alpha_n)$ will be 2-cells with cone points whose degree divides n, in the Howie-Short version the two cells are attached along powers whose degree divides n. Notice that our result varies slightly from the above theorem in that we are allowing 2-cells in Y' to have degree n cone points. This is because if we don't assume this we cannot apply our W-cycles result due to the possibility that we may have free faces. However, our result will still allow us to improve the bound on the total number of 2-cells (which is a corollary of the following proposition), which was the aim for the theorem above anyway. Note again that for our version the vertices of the relative graph Γ have NPI rather than locally indicable fundamental group.

Proposition 5.1.2. *Let $n > 1$ and suppose $f : Y' \looparrowright \hat{Y}_n$ is an immersion, where Y' is a compact and connected orbicomplex with first Betti number β. Suppose that K of the 2-cells in $f^{-1}(\alpha_n)$ have a cone point of degree strictly less than n. If no 2-cell in $f^{-1}(\alpha_n)$ has a free face, then $K \leq \beta$.*

Proof. Γ is the relative graph in $\hat{Y}_n$ before attaching α_n. Let Γ' be the relative graph found by removing the interiors of 2-cells in $f^{-1}(\alpha_n)$ from Y' (as usual vertices/edges of Γ' are found by taking components of pre-images of vertices/edges of Γ). Then $\Gamma' \looparrowright \Gamma$ is an immersion by construction. As in our proof of NPI, if w^n is the attaching path of α_n, we can take the indivisible word cover $\Lambda : \mathbb{S} \to \Gamma$ for the word w (not the power since we are using the orbicomplex $\hat{Y}_n$). Again we can form $\mathbb{S}'$ as the pullback and the immersion $\mathbb{S}' \looparrowright \Gamma'$ as well as the cover $\bar{\sigma}$. We may assume the map Λ' is surjective, otherwise we can replace Y' with a subcomplex so that it is surjective (if this disconnects Y' we can treat components separately). Since no 2-cell in $f^{-1}(\alpha_n)$ has a free face, Λ' is not reducible. By Theorem 4.4.1, $deg(\bar{\sigma}) \leq -\chi_r(\Gamma')$. We need to compute $deg(\bar{\sigma})$. Let $\{\alpha_1, \ldots, \alpha_K, \alpha_{K+1}, \ldots, \alpha_l\}$ be the set of all 2 -cells in $f^{-1}(\alpha_n)$, where the first K 2-cells are those with cone points of degree strictly less than n. If p_i is the degree of the cone point of α_i for each $1 \leq i \leq l$ then there exists q_i such that $p_i q_i = n$. It is this q_i that is relating to the degree of the cover $\bar{\sigma}$. In particular, we see that

$$deg(\bar{\sigma}) = \sum_{i=1}^{l} q_i.$$

By assumption for each $1 \leq i \leq K$, $p_i < n$, so $q_i \geq 2$, and for $K+1 \leq i \leq l$ $p_i = n$ so $q_i = 1$. Therefore

$$deg(\bar{\sigma}) = \sum_{i=1}^{l} q_i \geq 2K + l - K.$$

This means that $K + l \leq -\chi_r(\Gamma')$ and since we obtain Y' by adding l 2-cells to Γ' we must have

$$K + l \leq -\chi_r(\Gamma') \leq \beta_1(\Gamma') \leq \beta_1(Y') + l = \beta + l,$$

so $K \leq \beta$. □

Compare this to the one-relator group case, see [16], where it can be shown that we can factor our immersion through an 'unwrapped cover' of Γ which means all p_i are in fact 1 and hence $deg(\bar{\sigma}) = nk$. We are unable to find this 'unwrapped cover' here hence the weaker result and the need to deal with 'nth powers' separately. In [12], Howie and Short are able to show that the number of 2-cells in Y' with a cone point of degree n is at most the number of generators of $\pi_1(Y')$ (this is a consequence of Theorem 3.2 in [12]). This allows them to prove that the total number of 2-cells in $f^{-1}(\alpha_n)$, with the set up as in Proposition 5.1.2 is at most $11k$, where $\pi_1(Y')$ can be generated by k elements. Using Proposition 5.1.2 and their bound on the number of 2-cells attached along nth powers, we can easily improve this bound (if again vertices are assumed to have NPI). The following corollary is our version of Corollary 3.4 from [12], where we have improved the bound from $11k$ to $2k$.

Corollary 5.1.3. *Let $n > 1$ and suppose $f : Y' \looparrowright \hat{Y}_n$ is an immersion, where Y' is a compact and connected orbicomplex such that $\pi_1(Y')$ can be generated by k elements. If no 2-cell in $f^{-1}(\alpha_n)$ has a free face, then the number of 2-cells in $f^{-1}(\alpha_n)$ is at most $2k$.*

Proof. Using Theorem 3.2 from [12], it can be shown that there are at most k 2-cells in $f^{-1}(\alpha_n)$ with cone points of degree n. Additionally, by Proposition 5.1.2, there are at most $\beta_1(Y')$ 2-cells in $f^{-1}(\alpha_n)$ with a cone point of degree strictly less than

n. Since this covers all possible 2-cells in $f^{-1}(\alpha_n)$ and $\beta_1(Y') \leq k$, the number of 2-cells in $f^{-1}(\alpha_n)$ is at most $k+k=2k$. □

This shows our W-cycles result can be used to improve the bounds found on the number of 2-cells in a very straightforward manner. For the remainder of the proof of coherence, see Theorem 3.5 of [12].

5.2 Which One-Relator Groups are Hyperbolic?

Throughout this thesis we have been discussing the notion of non-positive immersions (NPI). There is another related notion that we mentioned in Chapter 2 called negative immersions (NI) and we stated Theorem 2.2.9 from [15] telling us that for one-relator groups this property is related to the primitivity rank of the word. In this chapter we will discuss how the notion of NI is related to a conjecture about hyperbolicity of one-relator groups and discuss some possible starting points for generalising these ideas to one-relator products.

In Chapter 2 we talked about the question asking whether every one-relator group not containing a Baumslag-Solitar group is hyperbolic. The following conjecture of Louder and Wilton in [15] suggests a possible solution to this question.

Conjecture 5.2.1. *Let F be a free group and $w \in F \setminus \{1\}$. If $\pi(w) \neq 2$, where $\pi(w)$ is the primitivity rank, then the group $F/\langle\langle w\rangle\rangle$ is hyperbolic.*

Notice that if $\pi(w) = 1$ then w is a proper power, so this part is already proven, and $\pi(w) > 2$ is equivalent to $F/\langle\langle w\rangle\rangle$ having NI, as proved in [15]. One way to consider hyperbolicity of groups is by looking at disk diagrams. We give a brief introduction to the idea of disk diagrams (or van Kampen diagrams, as they were initially defined by Egbert van Kampen [26]). A good reference for the following results about disk diagrams is [20].

Definition 5.2.2. A *disk diagram* is a contractible, finite 2-complex with a fixed embedding in the plane. The *area*, *Area*(D), of a disk diagram, D, is given by the number of 2-cells. The boundary cycle, denoted ∂D, of a disk diagram D is the closed path around the complement of D.

Definition 5.2.3. Let X be the presentation 2-complex of a group G and let D be a disk diagram. A map $D \to X$ is said to be *reduced* if whenever two 2-cells share an edge in D, the boundary cycles of these 2-cells do not read the same word when starting at this edge.

Definition 5.2.4. $D \to X$ is a disk diagram for a word $v \in G$ if it is a reduced map and ∂D maps to a representative of the free homotopy class of v in X. The map $D \to X$ is said to have *minimal area*, if $Area(D)$ is minimal over all such disk diagrams for v.

The following lemma tells us that disk diagrams exist for trivial words, so talking about minimal area does indeed make sense.

Lemma 5.2.5 (Van Kampen's Lemma)**.** *Let $G \cong \langle S \mid R \rangle$ be a group presentation, and v a word in $F(S)$, the free group on S. The following are equivalent:*

1. $v =_G 1$.

2. *There exists a disk diagram $D \to X$ for the word v, where X is the presentation 2-complex of G.*

Example 5.2.6. Let $G = \langle a, b \mid abb \rangle$, and let $v = aabAb$, where A denotes the inverse of a. Then Figure 5.1 shows a minimal area disk diagram for v.

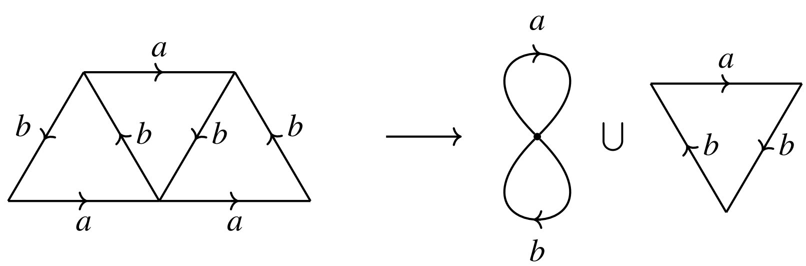


Figure 5.1: Diagram showing a minimal area disk diagram for the word $v = aabAb$ in $G = \langle a, b \mid abb \rangle$. In this case $Area(v) = 3$

Definition 5.2.7. A group G satisifies a *linear isoperimetric inequality* if there exists $k > 0$ such that

$$Area(v) \leq k|v|$$

for any $v =_G 1$, where $Area(v)$ is the area of a reduced minimal area disk diagram for v.

It turns out that hyperbolicity is determined by these isoperimetric inequalities.

Theorem 5.2.8 (Gromov [7]). *A group G is hyperbolic if and only if it satisfies a linear isoperimetric inequality.*

We mentioned earlier that Baumslag-Solitar groups are not hyperbolic. The following easy example will demonstrate this for $BS(1,2)$ using the above theorem.

Example 5.2.9. Consider the Baumslag-Solitar group

$$BS(1,2) = \langle a, b \,|\, bab^{-1} = a^2 \rangle,$$

we will describe how to construct a sequence of reduced disk diagrams whose area increases exponentially and boundary length increases linearly. See Figure 5.2 for an illustration of the following construction. We will use capitalised letters for inverses of elements. Firstly, for $n \geq 1$, let A_n be the disk diagram for the word $w_n = a^2 B^n A^{2^{n+1}} b^n$ constructed inductively. So A_1 consists of two 2-cells attached along a single b edge and A_n is produced from A_{n-1} by gluing a line of $2n$ 2-cells consecutively attached to each other by a single b edge to the path of A_{n-1} labelled a^{2^n}. So A_n has a path labelled by $a^{2^{n+1}}$ and we construct a disk diagram D_n by gluing two copies of A_n together along this path with a shift of exactly one edge.

Observe that the boundary of D_n reads the word $v_n = a^2 B^n a b^n A^2 B^n A b^n$, which has length $4n + 6$ so it is increasing linearly. However

$$Area(D_n) = \sum_{i=1}^{n} 2^{i+1} = 4 \cdot (2^n - 1),$$

so the area is increasing exponentially. Thus, we have a sequence of reduced disk diagrams whose area increases exponentially but perimeter increases linearly.

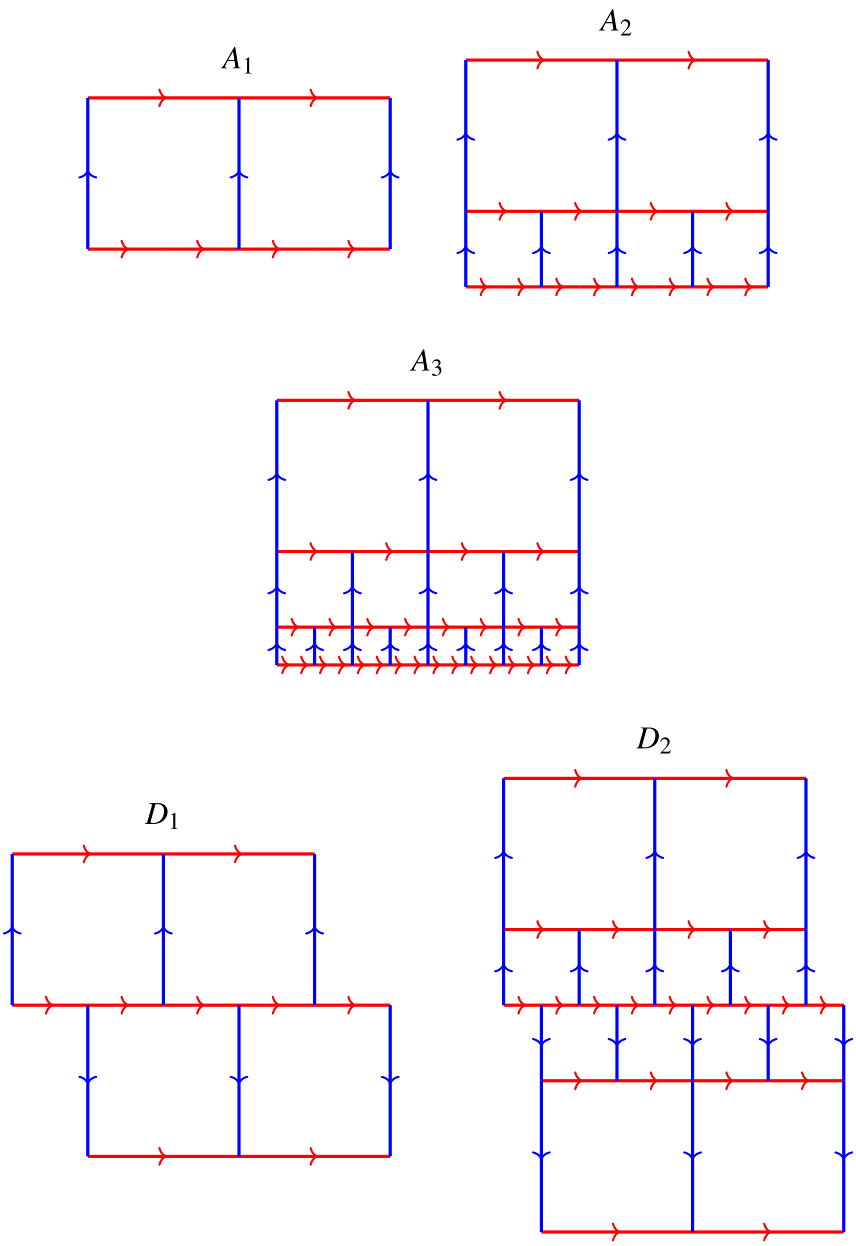


Figure 5.2: Diagram showing the construction of a sequence of disk diagrams D_n for $BS(1,2)$ whose area increases exponentially with n and perimeter increases linearly with n.

Let F_n be a free group and $w \in F_n$ a primitive element. Then it is easy to see that the one-relator group $G = F/\langle\langle w \rangle\rangle$ is a free group and therefore hyperbolic. We know from the above theorem that a group is hyperbolic if and only if it admits a linear isoperimetric inequality. Therefore, for any minimal area disk diagram $D \to X$ where X is the presentation 2-complex of G we have the inequality,

$$|\partial D| \geq C_w \cdot \text{Area}(D),$$

where C_w is a constant depending only on w. We conjecture that $\sup_w C_w < \infty$ or possibly at most 1, i.e. there is a universal bound for all such one-relator groups. The idea behind the use of this conjecture was that we could use the bound to show that one-relator groups with NI also satisfy a linear isoperimetric inequality although we have not proved either conjecture yet. However, on the way to this statement we have constructed a way to study the disk diagrams where w is primitive via the stackings of w. This makes use of Whitehead automorphisms and peak reduction.

5.3 Whitehead Automorphisms

In this section we will describe an algorithm that determines primitivity of elements in a free group. Let F_n be a free group of rank n and let $A = \{a_1, \ldots, a_n\}$ be a basis for F_n, let $\bar{A} = \{a_1^{-1}, \ldots, a_n^{-1}\}$. J. H. C. Whitehead defined and used the following generators of the automorphism group to prove results about elements of a free group in [27].

Definition 5.3.1. Partition $A \cup \bar{A}$ into two disjoint sets X, Y such that there is some $v \in X$ with $v^{-1} \in Y$. The *Whitehead automorphism* $\varphi_{(X,Y,v)}$ defined on $A \cup \bar{A}$ is:

$$\varphi_{(X,Y,v)}(z) = \begin{cases} vzv^{-1} & \text{if } z \in X, z^{-1} \in X \\ vz & \text{if } z \in X, z^{-1} \in Y, z \neq v^{\pm 1} \\ zv^{-1} & \text{if } z^{-1} \in X, z \in Y, z \neq v^{\pm 1} \\ z & \text{otherwise.} \end{cases}$$

Let $\mathcal{W}$ be the set of all such Whitehead automorphisms.

Let $w \in F_n$ and define the Whitehead graph of w, $Wh_A(w)$ as follows.

- $V(Wh_A(w)) = A \cup \bar{A}$

- $x, y \in A \cup \bar{A}$ are connected by an edge for each occurrence of xy^{-1} or yx^{-1} as a subword of the cyclic word w.

Example 5.3.2. Let $A = \{a, b\}$ and let $w = abababbababb$, then $Wh_A(w)$ is the graph in Figure 5.3.

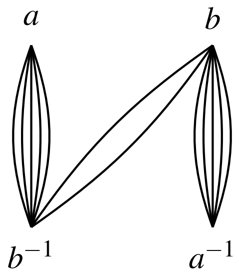


Figure 5.3: $Wh(abababbababb)$.

Notice that this Whitehead graph has both b and b^{-1} as cut-vertices. In fact, as we shall see, this choice of w is primitive in F_2.

The following result of J. H. C. Whitehead helps to check whether an element is primitive.

Theorem 5.3.3 (Whitehead's Lemma [27]). *If $w \in F_n$ is primitive, then $Wh(w)$ is either disconnected or has a cut-vertex.*

5.3.1 Peak Reduction

Let $w \in F_n$ be a primitive element. Then by definition there exists an automorphism Φ of F_n such that $|\Phi(w)| = 1$. The following theorem, now known as peak reduction tells us that w can be mapped to a word of length one by a sequence of Whitehead automorphisms that strictly decrease the length of w. For a proof see [9].

Theorem 5.3.4 (Peak Reduction [9]). *If w and w' are cyclic words equivalent under an automorphism of $\mathbb{F}_2$ and w' has minimum length for the equivalence class, then $\exists T_1, \ldots, T_n \in \mathcal{W} = \{$Whitehead Automorphisms$\}$ such that writing $w_i = T_i \circ \cdots \circ T_1(w)$, $w_n = w'$, $|w_1|, \ldots, |w_n| \leq |w|$ and strict inequality holds unless w has minimum length.*

Whitehead's Lemma is used to find the sequence of Whitehead automorphisms described in the Peak Reduction Theorem. To do so partition $A \cup \bar{A}$ into X, Y where either in $Wh(w)$ these sets are determined by the cut-vertex (where the cut-vertex is in the set not containing its inverse) or if $Wh(w)$ is disconnected we take X, Y so that there are no edges between them and there is a $v \in X$ with $v^{-1} \in Y$. The Whitehead automorphism here will then decrease the length of w so we then repeat.

Example 5.3.5. Returning to Example 5.3.2 we will demonstrate the above algorithm to show that w is in fact a primitive word. Firstly, take the automorphism $\varphi_1 = \varphi_{(\{a,b^{-1}\},\{b,a^{-1}\},b^{-1})}$, then $\varphi_1(w) = b^{-1}abb^{-1}abb^{-1}abbb^{-1}abb^{-1}abb = aaabaab$. Then $Wh_A(\varphi_1(w))$ has cut-vertices at a and a^{-1} so take $\varphi_2 = \varphi_{(\{b,a^{-1}\},\{a,b^{-1}\},a^{-1})}$. Then $\varphi_2(\varphi_1(w)) = aaaa^{-1}baaa^{-1}b = aabab$. Again $Wh(\varphi_2(\varphi_1(w)))$ has cut-vertices a and a^{-1} so we again use φ_2. Then $\varphi_2^2\varphi_1(w) = abb$, and the Whitehead graph has cut-vertices b and b^{-1} so we use φ_1 again giving $\varphi_1\varphi_2^2\varphi_1(w) = ab$ which produces a disconnected Whitehead graph and we can choose either φ_1 or φ_2, for example $\varphi_1^2\varphi_2^2\varphi_1(w) = a$, meaning that w is primitive. This process is depicted in Figure 5.4.

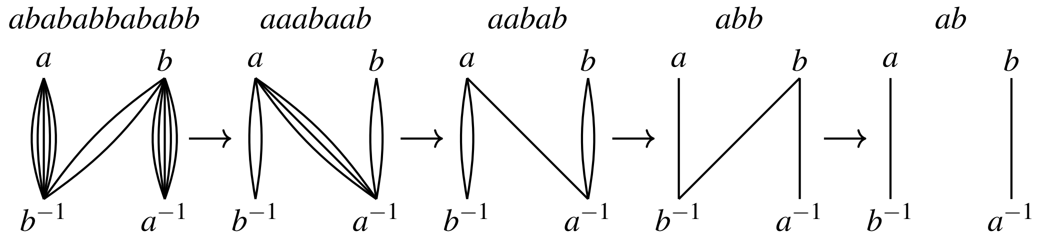


Figure 5.4: Reduction of $w = abababbababb$ via Whitehead automorphisms.

5.3.2 Peak Reduction of Disk Diagrams

Let $w \in F_n$ be primitive, $v \in \langle\langle w \rangle\rangle$ and let $D \to X$ be a minimal area disk diagram for v. If φ is any Whitehead automorphism then we can use it to define a map of minimal area disk diagrams $\varphi : D \to D^{\varphi}$ by mapping each 2-cell to a 2-complex with boundary $\varphi(w)$ (these will be a single 2-cell with some leaves) and gluing back together as expected (notice that before removing any leaves, the one-skeleton of the disk diagram will remain unchanged after applying a Whitehead automorphism).

Notice that D^{φ} is minimal area since otherwise there would be a D' of smaller area with boundary reading $\varphi(v)$, and then $(D')^{\varphi^{-1}}$ is a disk diagram for v with smaller area than D.

By Peak Reduction there is a sequence $\varphi_1, \dots, \varphi_k \in \mathcal{W}$ such that $|w| > |\varphi_1(w)| > \cdots > |\varphi_k \circ \cdots \circ \varphi_1(w)| = 1$. Thus $D^{\varphi_k \circ \cdots \circ \varphi_1}$ has a tree-like structure and all 2-cells have boundary length one, therefore it has boundary length larger than its area. The problem is how to keep track of the boundary length throughout this process as it can both increase and decrease. We will do this via the use of stackings, as introduced earlier.

Example 5.3.6. Let $w = abb$ and let $v = aaba^{-1}b$. Let $\varphi_1 = \varphi_{(\{a,b^{-1}\},\{b,a^{-1}\},b^{-1})}$ and $\varphi_{(\{b,a^{-1}\},\{a,b^{-1}\},a^{-1})}$. Then a sequence of Whitehead moves that reduces w to a length one word is $\varphi_1, \varphi_2, \varphi_1$. Figure 5.5 shows the reduction of a minimal area disk diagram D_1 for v via this sequence of automorphisms.

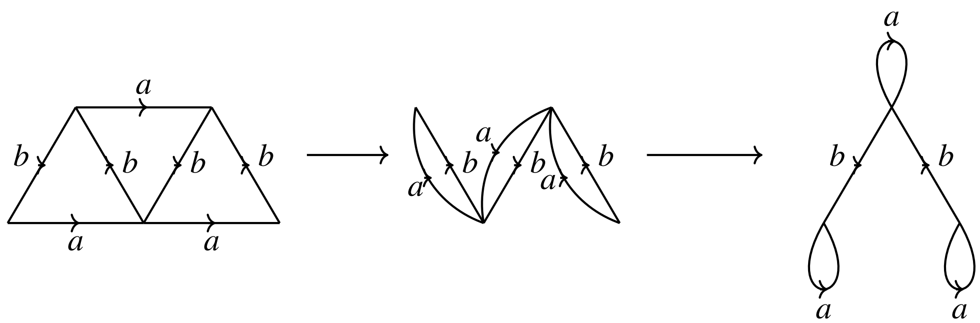


Figure 5.5: Reduction of a minimal area disk diagram for $v = aaba^{-1}b$ via the reduction of $w = abb$.

Notice in Figure 5.5 that the length of the boundary increases as we decrease the length of w, although it is always larger than the area.

5.3.3 Stackings and Disk Diagrams

For now we will work in $F_2 = \langle a, b \rangle$. Let $w \in \langle a, b \rangle$ be primitive, $v \in \langle\langle w \rangle\rangle$ and let $D \to X$ be a minimal area disk diagram, where X is the presentation complex of $\langle a, b \,|\, w \rangle$. Let $\varphi_1, \dots, \varphi_k$ be a sequence of Whitehead automorphisms that strictly decrease the length of w. Notice that since we are in F_2 then without loss of generality we can assume that each φ_i is either $\varphi_{(\{b^{-1},a\},\{a^{-1},b\},b^{-1})}$ or $\varphi_{(\{b,a^{-1}\},\{b^{-1},a\},a^{-1})}$, and that w is a word using only a and b letters. We now define an ordering on vertices and edges of w induced by the Whitehead automorphisms in the increasing direction. Suppose we are at some level w_i and the next Whitehead automorphism is $\varphi^{-1}_{(\{b^{-1},a\},\{a^{-1},b\},b^{-1})}$. Then as words, before each edge labelled a we add an edge labelled b. So let $a_1 < \cdots < a_l$ be the ordered a edges and $b_1 < \cdots < b_m$ the ordered b edges.

The added b edges are $b'_1 < \cdots < b'_l$, which correspond to the a edges. Define the new order on b edges as $b'_1 < \cdots b'_l < b_1 < \cdots < b_m$. To order the vertices we take a copy of each of the initial vertices of the a edges and place them below all the current vertices, the original new vertices become the terminal vertices of the new b edges and the initial vertices of the original a edges. The initial vertices of the new b edges are the remaining valence one vertices, assigned according to the order of the b edges. For the other Whitehead automorphism, place the new edges above the current a edges.

Lemma 5.3.7. *The algorithm described above produces a stacking on w.*

Proof. The above description provides an ordering on preimages of edges and vertices in the rose. Thus it is enough to check that the new ordering is consistent with the maps ι, τ, but this is exactly how we defined the ordering on the vertices. Therefore, it does indeed provide a stacking. □

Example 5.3.8. Let $w = aabab$, Figure 5.6 shows how we build the stacking on w using a sequence of Whitehead automorphisms. Edges with the label a are coloured red and edges with the label b are coloured blue.

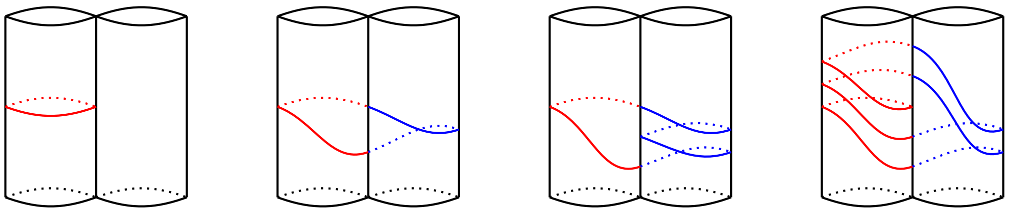


Figure 5.6: Constructing the stacking for $w = ababa$ via Whitehead automorphisms.

Let Γ be the rose with two edges and let $w : S \looparrowright \Gamma$. Let P be the disjoint union of $n = Area(D)$ copies of S, let D^1 be the 1-skeleton of D. Let $\sigma : P \to S$ be the identity on each copy of S and let $\lambda : P \to D^1$ be the map defined by taking the n copies of S to be the boundaries of each 2-cell in D. Define W to be the adjunction space,

$$W = (D^1 \sqcup S) \cup_f (P \times [-1, 1]),$$

where $f : P \times \{-1, 1\} \to S \sqcup D^1$ is defined by $f(y, -1) = \lambda(y)$ and $f(y, 1) = \sigma(y)$. Then we can view W as a graph of graphs, which we can formally define as follows.

Definition 5.3.9. A *graph* is a tuple $G = (V(G), E(G), \iota, \tau)$ where $V(G), E(G)$ are sets and $\iota, \tau : E(G) \to V(G)$ are incidence maps. A *morphism of graphs* is a map $f : G \to H$ such that $f(V(G)) \subseteq V(H)$, $f(E(G)) \subseteq E(H)$ and $f \circ \alpha = \alpha \circ f$ for $\alpha \in \{\iota, \tau\}$. A morphism of graphs $f : G \to H$ is an *immersion* if for any edges $e \neq e'$ of G, $\alpha(e) = \alpha(e') \Rightarrow f(e) \neq f(e')$. All graphs are assumed to be connected and finite.

Definition 5.3.10. A *graph of graphs* X over a graph F_X is a graph
$X = (V(X), E(X), \iota_X, \tau_X)$, where

$$V(X) = \{X_v \mid v \in V(F_X)\}, \quad E(X) = \{X_e \mid e \in E(F_X)\},$$

where each X_v, X_e is a graph and $\iota_X : X_e \to X_{\iota(e)}$, $\tau_X : X_e \to X_{\tau(e)}$ are morphisms of graphs that are injective on $E(X_e)$, but not necessarily embeddings.

The realisation of a graph of graphs X, also denoted here by X, is the space:

$$X = \frac{\bigsqcup_{v \in V(F_X)} X_v \sqcup \bigsqcup_{e \in E(F_X)} X_e \times [-1,1]}{(x,-1) \sim \iota_X(x),\ (x,1) \sim \tau_X(x)}.$$

We will interchangeably use X for the graph of graphs and its realisation. For a graph of graphs X with edge spaces $\{X_e\}$, vertex spaces $\{X_v\}$ and underlying graph F_X, consider the following chain complex C_X.

$$C_X: \quad 0 \to \bigoplus_{e \in E(F_X)} H_1(X_e) \to \bigoplus_{v \in V(F_X)} H_1(X_v) \to 0.$$

If we define the characteristic of X (it is easy to see this is the Euler characteristic of the realisation of X) to be

$$\chi(X) = \sum_{v \in V(F_X)} \chi(X_v) - \sum_{e \in E(F_X)} \chi(X_e).$$

Then we can see that

$$\chi(X) = \chi(F_X) - \chi(C_X).$$

In our case we view W as a graph of graphs whose vertex spaces are components of pre-images of vertices in Γ and whose edge spaces are the components of pre-images of midpoints of edges in Γ. Then as above we define the chain complexes C_W. Notice that $\chi(W) = 1-n$ and if $\chi(F_W) = 0$ then the underlying graph is a circle, but then all our edge spaces can only have a single vertex and so $\chi(C_W) = 0$, but this contradicts the fact that $\chi(W) = \chi(F_W) - \chi(C_W)$ and therefore, $\chi(F_W) \leq -1$, $\chi(C_W) \leq n-2$.

Additionally, using stackings we can define a pair of chain complexes with the same characteristic as C_W.

Since w is primitive, it is indivisible and so we have a stacking of S over Γ, which we can pull back to a stacking over F_W. Let $W_f = (D^1_f \sqcup S_f, P_f, \lambda, \sigma)$ be an edge or vertex graph of W, where $D^1_f = D^1 \cap W_f$, $S_f = S \cap W_f$ and $P_f = P \cap W_f$. For $s \in S$, let $P_s = \sigma^{-1}(s)$, and let $\{\leq_f\}$ be a stacking of S over F. For $s \in S_f$ define,

$$W^+_f(s) = D^1_f \cup \{s' \in S_f \mid s' \leq_f s\} \cup \{p \in P_f \mid \sigma(p) \leq_f s\}, \text{ and}$$

$$W^-_f(s) = D^1_f \cup \{s' \in S_f \mid s' \geq_f s\} \cup \{p \in P_f \mid \sigma(p) \geq_f s\}.$$

By Mayer-Vietoris, if we let $A^\pm(s) = \ker(\lambda_s : P_s \to H_0(W^\pm_f(s \mp 1)))$, we can think of $A^\pm(s)$ as representing the additional H_1 when moving from $W^\pm_f(s \mp 1)$ to $W^\pm_f(s)$ since,

$$H_1(W^\pm_f(s)) \cong H_1(W^\pm_f(s \mp 1)) \oplus A^\pm(s).$$

By defining a pair of chain complexes $C^\pm_W$ as

$$C^\pm_W : \quad 0 \to \bigoplus_{s \in E(S)} A^\pm(s) \to \bigoplus_{v \in V(S)} A^\pm(v) \to 0,$$

it is straightforward to show that $\chi(C^\pm_W) = \chi(C_W)$ and as there is an edge $e \in E(S)$ with $dim A^+(e) = 0$ (and same for $-$) it is also easy to see that

$$\chi(C^\pm_W) \geq max\{dim A^\pm(s) \mid s \in S\}.$$

In particular, this means that for each $s \in S$, $\dim A^\pm(s) \leq n-2$. We recall the Up-down lemma from [15].

Lemma 5.3.11 ([15]). *With W as above then each edge or vertex space has at least two valence one vertices.*

Proof. Let W_u be an edge or vertex space and let $m, M \in S$ be the minimal and maximal elements of W_u respectively. Recall that $\chi(C_W) \leq n-2$ and therefore $\dim A^\pm(m) \leq n-2$ and similarly for M. This means that $W_u \setminus \{m\}$ must have at least two connected components, as does $W_u \setminus \{M\}$. Choose the component of $W_u \setminus$

$\{m\}$ not containing M and repeat. At some point the connected component chosen must in fact be a single vertex and therefore we find a vertex in W_u of valence one. Performing the same process with M finds another valence one vertex. □

The following is an easy observation following immediately from the Up-down lemma. Note this can also be proved easily without stackings but it does at least provide a starting point for results using stackings.

Lemma 5.3.12. *Let D be a minimal area disk diagram, over the relator w, such that there exists an edge space W_e of W with $|W_e \cap S| = 1$. If φ is a Whitehead automorphism such that $|\varphi(w)| > |w|$, then $|\partial D^{\varphi}| \geq Area(D^{\varphi}) + 2$.*

Proof. Let $Area(D) = Area(D^{\varphi}) = n$. By the Up-down lemma we know that each edge space must have at least two valence one vertices (valence one vertices in edge spaces correspond to the boundary of D). Suppose $|\partial W| = n + k$ ($k \geq 2$ since we have at least two edge spaces and by assumption one of them has n leaves). When attaching the edge spaces together we can glue onto at most $k-1$ vertices (otherwise there would exists some edge $e \in S$ with either $\dim A^{+}(s) = n-1$ or $\dim A^{-}(s) = n-1$). Thus, $|\partial D| \geq k + (n+k) - 2(k-1) = n+2$. □

The following lemma provides some simple observations about how $C_W^{\pm}$ changes after applying a Whitehead automorphism. These follow from the way we produced the stacking on w. If we are to prove results for free groups of general rank it is likely that we will need to know more about properties of the stacking, so these simple results may be a starting point.

Lemma 5.3.13. *Suppose that $\varphi : a \mapsto ba, b \mapsto b$ then for each edge $e \in S$:*

- *$\dim A^{-}(\tau(e)) = \dim A^{-}_{\varphi(W)}(\varphi(\tau(e)))$;*
- *if e is labelled by a then $\dim A^{\pm}(e) = \dim A^{\pm}_{\varphi(W)}(\varphi(e))$;*
- *if e is labelled by b then $\dim A^{-}(e) = \dim A^{-}_{\varphi(W)}(\varphi(e))$.*

A similar result holds for $\varphi : b \mapsto ab, a \mapsto a$.

Proof. This is due to the fact that a copy of the edge spaces corresponding to the a labelled edge is added below the existing b labelled edge spaces, as well as below the existing vertex spaces. □

Remark 5.3.14. *Notice that since in the rank two case we have set Whitehead automorphisms used at each step. If when performing a move we add a valence one vertex to the target space, the corresponding valence one vertex in the initial space will have valence one throughout the entire process (this is not true for higher rank free groups).*

It is possible that a combinatorial proof may exist for there being a universal constant C such that $|\partial D| \geq C \cdot Area(D)$ in the rank 2 case at least, although this is not obvious. However, it also seems possible that using more information about the stacking could provide a proof but again exactly which information to use is not obvious.

Conjecture 5.3.15. *Let F be a free group. There exists a $C > 0$ such that for any primitive $w \in F$, if X is the presentation complex of $F/\langle\langle w \rangle\rangle$ and $D \to X$ is a minimal area disk diagram then $|\partial D| \geq C \cdot Area(D)$.*

5.4 Negative Immersions and One-Relator Products Discussion

Given that we were able to use the generalised version of a stacking to prove results about NPI in one-relator products, it is not ridiculous to expect we could use these stackings to generalise further results. In this section we will discuss some attempts to characterise which one-relator products have NI. In [15] an improvement on the W-cycles conjecture is found (Theorem 2.7 of [15]). This is proved using adjunction spaces and filtering these using stackings as a 'height function' similarly to the description in the previous section. Here we describe a possible way to construct a similar space for one-relator products and provide a conjecture for a generalised version of this theorem to relative graphs. The following uses the ideas from [15] in the setting of relative graphs.

Let Ω be a relative graph such that every vertex has NPI and finitely generated, freely indecomposable fundamental group. Let $w : \mathbb{S}_w \to \Omega$ be an indivisible word cover in Ω, and let $S \subset \mathbb{S}_w$ be a compact, connected subcomplex of $\mathbb{S}_w$ such that $\mathrm{Edges}(\mathbb{S}_w) \subset S$ and $\pi_1(S) \cong \mathbb{Z}$. Let $\rho : \Gamma \to \Omega$ be a morphism of relative graphs and let P be the pullback $\Gamma \times_\Omega S$, where

$$\Gamma \times_\Omega S = \{(x,y) \in \Gamma \times S \mid \rho(x) = w(y)\}.$$

Let $\lambda : P \to \Gamma$ and $\sigma : P \to S$ be the two projection maps.

Let W be the adjunction space

$$(\Gamma \sqcup S) \cup_f (P \times [-1,1]),$$

where $f : P \times [-1,1]$ sends $(x,-1) \mapsto \lambda(x)$ and $(x,1) \mapsto \sigma(x)$. The maps $S, P, \Gamma \to \Omega$ determine a map of sets $p : W \to \Omega$. Let Γ_U be the complex whose cells are the connected components of pre-images of midpoints of the cells in Ω (these components are bipartite graphs sitting vertically in W: The vertices of these graphs are cells in Γ and S and the edges are cells in P). The boundary maps of cells in Γ_U are determined by the boundary maps of cells in W. The map $p : W \to \Omega$ factors through Γ_U. Also Γ_U can be viewed as a relative graph where points of Γ_U are in some $Y \in \mathrm{Verts}(\Gamma_U)$ if and only if their image in Ω is in some $Z \in \mathrm{Verts}(\Omega)$, i.e. Y is a connected component of the pre-image of a vertex of Ω, with the same statement for $\mathrm{Edges}(\Gamma_U)$. We will refer to Γ_U as the underlying relative graph of W. If a is a cell in Γ_U we write W_a to be the vertical graph in W that maps to a by the map $W \to \Gamma_U$.

Define the boundary of W to be $\partial W := \{e \in \mathrm{Edges}(\Gamma) \, : \, |\lambda^{-1}(e)| = 1\}$, i.e. we are counting the number of leaves in the vertical graphs that sit over relative edges of Ω, and define the characteristic of W to be

$$\chi(W) = \sum_{v \in V(\Gamma_U)} \chi(W_v) - \sum_{e \in E(\Gamma_U)} \chi(W_e) + \sum_{f \in F(\Gamma_U)} \chi(W_f),$$

where $V(\Gamma_U), E(\Gamma_U), F(\Gamma_U)$ are the sets of 0-, 1-, 2-cells of Γ_U respectively. Notice that $\chi(W) = \chi(S) + \chi(\Gamma) - \chi(P)$ and $\chi(W)$ really is the Euler characteristic of the realisation of W. If we rewrite $\chi(W)$, using the fact that each W_* is a connected graph, as

$$\chi(W) = \sum_{v \in V(\Gamma_U)} (1 - \beta_1(W_v)) - \sum_{e \in E(\Gamma_U)} (1 - \beta_1(W_e)) + \sum_{f \in F(\Gamma_U)} (1 - \beta_1(W_f)),$$

where β_1 denotes the first Betti number we can see that

$$\chi(W) = \chi(\Gamma_U) - \chi(C),$$

with C being the chain complex

$$C : 0 \to \bigoplus_{f \in F(\Gamma_U)} H_1(W_f) \to \bigoplus_{e \in E(\Gamma_U)} H_1(W_e) \to \bigoplus_{v \in V(\Gamma_U)} H_1(W_v) \to 0.$$

Definition 5.4.1. The map $\lambda : P \to \Gamma$ is *independent* if $\partial W \neq \emptyset$, otherwise it is called *dependent*. Furthermore, if $|\partial W \cap W_e| \geq 2$ for every $e \in \mathrm{Edges}(\Gamma_U)$, then λ is called *strongly independent*, otherwise λ is *weakly dependent*.

Definition 5.4.2. W is *diagrammatically irreducible* if the restriction $\mathrm{Edges}(P) \to \mathrm{Edges}(\Gamma) \times \mathrm{Edges}(S)$ is an embedding and the maps σ and w are immersions.

Recall the equivalent definition of a stacking of $w : S_w \to \Omega$ obtained from Proposition 4.2.2, where a stacking is a collection of orders on w-preimages of cells in Ω, such that the orders are preserved by boundary maps. By Theorem 4.2.5, since w is indivisible, we know a stacking exists. We will denote the partial ordering by $\leq$ or

$<$ if the elements are distinct.

Let W be diagrammatically irreducible (and assume $\lambda : P \to \Gamma$ is surjective), we will replace C by a pair of easily computable chain complexes $C^{\pm}$ indexed by S, in the same manner as the previous section for disk diagrams, these were defined for free groups in [15]. For each $x \in \Gamma_U$, we produce a pair of filtrations of the vertical graph W_x. W_x is a bipartite graph with vertices $\Gamma_x \sqcup S_x$, where $\Gamma_x = \Gamma \cap W_x$ and $S_x = S \cap W_x$, and edges P_x, where $P_x = P \cap W_x$. Incidence maps are given by λ and σ. As in the previous section define,

$$W_x^+(s) = \Gamma_x \cup \{s' \in S_x \,|\, s' \leq s\} \cup \{p \in P_x \,|\, \sigma(p) \leq s\},$$

$$W_x^-(s) = \Gamma_x \cup \{s' \in S_x \,|\, s' \geq s\} \cup \{p \in P_x \,|\, \sigma(p) \geq s\}.$$

Then if $S_x = \{s, s+1, \dots, s+n\}$ with $s < s+1 < \cdots < s+n$ we have

$$\Gamma_x \subset W_x^+(s) \subset W_x^+(s+1) \subset \cdots \subset W_x^+(s+n) = W_x,$$

and the reverse is true for W_x^-. For general $s \in S_x$, we will use $s-1$ as the predecessor of s and $s+1$ as the successor. If s is minimal then define $W_x^+(s-1) = \Gamma_x$ and if s is maximal $W_x^-(s+1) = \Gamma_x$.

For each $s \in S_x$ define

$$A^{\pm}(s) = \frac{H_1(W_x^{\pm}(s))}{H_1(W_x^{\pm}(s-1))}.$$

We will mostly be interested in computing the dimensions of these groups, notice that the quotient group $A^{\pm}(s)$ represents the additional first homology gained when moving from $W_x^{\pm}(s-1)$ to $W_x^{\pm}(s)$, so the dimension of $A^{\pm}$ is the dimension of this homology gained. By summing over S_x,

$$H_1(W_x) = \bigoplus_{s \in S_x} A^{\pm}(s),$$

so

$$\dim H_1(W_x) = \sum_{s \in S_x} \dim(A^{\pm}(s)).$$

Therefore,

$$\chi(C) = \sum_{v \in V(\Gamma_U)} \sum_{s \in S_v} \dim(A^{\pm}(s)) - \sum_{e \in E(\Gamma_U)} \sum_{s \in S_e} \dim(A^{\pm}(s)) + \sum_{f \in F(\Gamma_U)} \sum_{s \in S_f} \dim(A^{\pm}(s))$$

but $V(S) = \bigsqcup_{v \in V(\Gamma_U)} S_v$, where $V(S)$ is the set of 0-cells of S and the same for $E(S)$ (1-cells) and $F(S)$ (2-cells) so

$$\chi(C) = \sum_{s \in V(S)} \dim(A^{\pm}(s)) - \sum_{s \in E(S)} \dim(A^{\pm}(s)) + \sum_{s \in F(S)} \dim(A^{\pm}(s)).$$

As mentioned in the previous section, for the one-relator group case (see [15]) it is easy to find a nice lower bound on $\chi(C)$ as the maximum of the dimensions of the groups $A^{\pm}(s)$. This is not possible immediately in the more general case, which is a major sticking point in an attempt to prove a general result, however this is likely because these calculations have not yet passed to relative characteristic. We conjecture that the following version of Theorem 2.7 in [15] should hold (the difference being the use of relative characteristic as opposed to Euler characteristic for a graph). This conjecture is a stronger version of the W-cycles conjecture we proved in Chapter 4.

Conjecture 5.4.3. *With notation as described above. Suppose that W is diagrammatically irreducible. If λ is weakly dependent then*

$$\chi_r(\Gamma) + deg(\overline{\sigma}) - 1 \leq \chi_r(\Gamma_U),$$

where $\overline{\sigma} : P/Verts(P) \to S/Verts(S)$ is the covering map obtained from σ after collapsing vertices.

Remark 5.4.4. *In [14] Louder uses stars of vertices to define a different graph of* *graphs, and defines values called q-excess/deficiency. These share some properties with the chain complex defined above. In particular, although not explicitly stated it follows immediately, by summing over stars, from Lemma 3.11 in [14] that if* Δ *is the value which we are not defining here (related to these excess and deficiencies) then,*

$$\Delta = \chi(\Gamma/Verts(\Gamma)) - \chi(\Gamma_U/Verts(\Gamma_U)).$$

This suggests something closer to a 'relative' version of the chain complex C *and can be constructed locally (i.e. within stars of vertices) in a way similar to the stacked chain complexes* $C^{\pm}$*. The problem with using stackings here however, is that the space created will not necessarily be consistent with any stacking we have, so the local pictures don't necessarily match up.*

5.5 Primitivity Rank and NI Conjecture

In this section we will show that were our conjecture true we would be closer to classifying which one-relator products have NI, in a similar way to one-relator groups.

We will now go through Section 3 of [15] in the setting of free products of groups rather than just free groups. The idea is to link Conjecture 5.4.3 to complexes associated to one-relator products of groups. We first need to define branched maps of 2-complexes. Let $D \subset \mathbb{C}$ be the unit disc, and let $p_n : D \to D$ be the map $z \mapsto z^n$. Notice the relation between this and the discussion on [12] at the start of Chapter 5.

Definition 5.5.1. A cellular map of 2-complexes $f : Y \to X$ is a *branched map* if

1. f restricts to a homeomorphism on each 1-cell of Y;

2. f induces an immersion on the link of each 0-cell of Y;

3. for each 2-cell, e, of Y, there is a 2-cell e' of X such that $f(e) \subset e'$ and e and e' can be parameterised so that $f|_e$ agrees with some p_n.

Let $f : Y \to X$ be a branched map and e a 2-cell of Y, with e' the corresponding cell in X. The *degree of branching of* e is the number n_e such that $e \to e'$ is parameterised as $z \mapsto z^{n_e}$. Then,

Let X be a relative graph where each non-empty vertex is freely indecomposable and has NPI. Let w be an indivisible element of $\pi_1(X)$ and α a 2-cell. Define $X_\alpha = X \cup_w \alpha$.

Let $f : Y \to X$ be a branched immersion. Construct W and Γ_U as in the previous section. The *one-relator pushout*, $\hat{Y}$ of Y is the one-relator product defined by gluing a 2-cell to Γ_U using the map $w : S^1 \to \Gamma_U$. The *immersed one-relator pushout of Y*, $\hat{Y}^I$, is the result of folding (in the relative graph sense) $\hat{f}^{-1}(X) \subset \hat{Y}$ to an immersion of relative graphs.

Corollary 5.5.2. *Let $f : Y \to X_\alpha$ be a branched map from a compact connected one or two-complex Y to X_α. If $f|_{\partial Y \cap f^{-1}(\alpha)} : \partial Y \cap f^{-1}(\alpha) \to w(S)$ is not at least two-to-one, if D is the preimage of the 2-cell α in $\hat{Y}$ and if Conjecture 5.4.3 holds, then*

$$\chi(Y) + \sum(n_e - 1) \leq \chi_r(\hat{Y} \setminus \mathring{D}) + 1.$$

Proof. Let k be the number of 2-cells in $f^{-1}(\alpha)$. We know that

$$\sum(n_e - 1) = deg(\overline{\sigma}) - k.$$

If $\chi(Y) + \sum(n_e - 1) > \chi_r(\hat{Y} \setminus \mathring{D})$, then

$$\chi(Y) + deg(\overline{\sigma}) - k > \chi_r(\hat{Y} \setminus \mathring{D}) + 1.$$

So

$$\chi(\Gamma) + deg(\bar{\sigma}) > \chi_r(\hat{Y} \setminus \mathring{D}) + 1,$$

where Γ is the relative graph $f^{-1}(X)$. Clearly, $\chi_r(\hat{Y} \setminus \mathring{D}) + 1 = \chi_r(\Gamma_U) + 1$. Vertices of X have NPI so by Lemma 3.2.4, $\chi_r(\Gamma) \geq \chi(\Gamma)$ and

$$\chi_r(\Gamma) + deg(\bar{\sigma}) - 1 > \chi_r(\Gamma_U).$$

If we assume Conjecture 5.4.3 holds then, for each relative edge e of Γ_U, $|\partial W \cap W_e| \geq 2$ since λ must be strongly independent. Therefore the map $\partial W \to w(S)$ must be at least two-to-one and $\partial W = \partial Y \cap f^{-1}(\alpha)$. □

Remark 5.5.3. *Lemma 3.2.9 tells us that* $\chi_r(\hat{Y}^I \setminus \mathring{D}^I) \geq \chi_r(\hat{Y} \setminus \mathring{D})$*, since folding increases relative characteristic.*

Definition 5.5.4 (Nielsen Equivalence)**.** Let Y be a 2-complex. An *edge collapse* of Y is a surjection $f : Y \to Z$ of 2-complexes such that there are finitely many 0-cells $z_1, \ldots, z_n$ of Z such that $f^{-1}(\{z_i\})$ is a disjoint union of closed, embedded 1-cells of Y and $|f^{-1}(z)| = 1$ for $z \neq z_i$.

A *face collapse* of Y is an inclusion $f : Z \hookrightarrow Y$ such that $Y \setminus Z$ consists of a disjoint collection of open 1-cells $e_1, \ldots, e_n$, disjoint open 2-cells $g_1, \ldots, g_n$ such that the attaching map for g_i crosses e_i exactly once and e_i is traversed only by g_i. Both edge collapses and face collapses are homotopy equivalences.

Let $\to_n$ be the reflexive and transitive relation generated by:

1. $Y \to_n Z$ if there is an edge collapse $Y \to Z$ or $Z \to Y$;

2. $Y \to_n Z$ if there is a face collapse $Z \hookrightarrow Y$.

If $Y \to_n Z$ say that Y *Nielsen reduces* to Z.

Suppose that G is a finitely generated group. Let $G \cong G_1 * \cdots * G_p * F_q$ be a Grushko decomposition of G. Recall the numbers p, q are unique and define $\chi_r(G) = 1 - p - q$. Notice this coincides with the original definition of relative characteristic if we assume vertices of a relative graph are freely indecomposable.

Definition 5.5.5. Let G be a finitely generated group. Say that $w \in G$ is *primitive* if there exists a Nielsen transformation $\varphi : G \to G$ such that $\varphi(G) = \langle \varphi(w) \rangle * G'$ for some finitely generated group G'.

Remark 5.5.6. *Notice that if G is a free group, any two bases of G are Nielsen* *equivalent so a word is primitive in the usual sense if and only if it is primitive in the above sense. Also if $q = 0$ in the Grushko decomposition, G will not have primitive elements.*

We will now introduce two candidates for a general version of primitivity rank $\pi(w)$ used in [15]. In the remainder of the section we will explain the reasons why each might be a candidate for determining which one-relator products have NI.

Definition 5.5.7. For any $w \in G$, the *relative primitivity rank* of $w \in G$, $P_1^G(w)$ is

$$P_1^G(w) = min\{1 - \chi_r(K) \mid w \in K \leq G, K \text{ is f.p. and } w \text{ is not primitive in } K\}.$$

If no such K exists then $P_1^G(w) = \infty$, at times we may write $P_1^G(w) = P_1(w)$ if the group G is clearly identified.

For any $w \in G$, the *primitivity rank* of $w \in G$, $P_2^G(w)$ is

$$P_2^G(w) = min\{rank(K) \mid w \in K \leq G, K \text{ is f.p. and } w \text{ is not primitive in } K\},$$

where the rank of a finitely generated group K is defined to be the size of a smallest generating set for K. Again, if no such K exists then $P_2^G(w) = \infty$, and if the group G is clearly identified, $P_2^G(w) = P_2(w)$.

Remark 5.5.8. *In a free group, $G \cong F_q$, $1 - \chi_r(G) = 1 - (1 - q) = q = rank(G)$, so either definition of primitivity rank agrees with the definition in free groups. Also for free groups, any finitely generated subgroup is finitely generated and free, hence finitely presented. This means in the case that G is free we could omit the condition that K is finitely presented.*

The two definitions of primitivity need not be equal as there may be situations where the rank of a vertex in the decomposition is strictly greater than one, i.e. we could have $P_2(w) > P_1(w)$. We note some observations about these definitions.

Lemma 5.5.9. *Let X be a relative graph whose vertices have freely indecomposable* *and locally indicable fundamental groups.*

1. $P_1(w) = 1$ *if and only if w is a proper power or w is non-trivial and conjugate into a vertex of X.*

2. $P_2(w) = 1$ *if and only if w is a proper power.*

3. *If* $P_1(w) = \infty$ *or* $P_2(w) = \infty$ *then w is primitive in* $\pi_1(X)$.

Proof. 1. The reverse direction is clear. In the other direction, if $P_1(w) = 1$ then there exists $K \leq \pi_1(X)$ finitely presented with $w \in K$ not primitive and $1 - \chi_r(K) = 1$, i.e. $\chi_r(K) = 0$. This can only happen if K has a single freely indecomposable factor and either this is $\mathbb{Z}$ in which case w not primitive tells us w is a proper power, or $K \ncong \mathbb{Z}$ in which case since K is freely indecomposable it is conjugate into a freely indecomposable factor of $\pi_1(X)$, i.e. w is conjugate into a vertex of X.

2. This is similar to the first point but here $rank(K) = 1$, and since vertices have locally indicable fundamental group $K \cong \mathbb{Z}$ and w is a proper power.

3. Since $\pi_1(X)$ is finitely presented, if w is not primitive in $\pi_1(X)$, both $P_1(w)$ and $P_2(w)$ have a finite upper bound achieved by $\pi_1(X)$.

□

Lemma 5.5.10. *Let X be a relative graph such that every non-trivial vertex of X has negative immersions. Then X has negative immersions.*

Proof. X has NPI so it is enough to look at $Y \looparrowright X$ with $\chi(Y) = 0$. It is easy to see that

$$0 = \chi(Y) = \chi(Y/\text{Verts}(Y)) + \sum_{V \in \text{Verts}(Y)} (\chi(V) - 1).$$

If there exists a vertex $V \in \text{Verts}(Y)$ with $\chi(V) \leq 0$ then since $Y/\text{Verts}(Y)$ is a graph and any vertex with positive Euler characteristic Nielsen reduces to a point we must have $Y/\text{Verts}(Y)$ being a tree and V being the only vertex with non-positive Euler characteristic ($\chi(V) = 0$). But V immerses in a vertex of X, and vertices of X have NI so V Nielsen reduces to a circle. Otherwise all vertices of Y Nielsen reduce to a point so Y Nielsen reduces to a graph trivially. □

We can now provide a suggestion for which one-relator products have NI in a similar way to the case for one-relator groups. This will use the definition of primitivity rank $P_2(w)$. Later we will show how if Conjecture 5.4.3 holds then the opposite direction is true but only with the relative primitivity rank $P_1(w)$ and we will briefly discuss the gap. For the proof we first need to use the following lemma from [15].

Lemma 5.5.11 ([15])**.** *A 2-complex Y Nielsen reduces to a graph if and only if the conjugacy classes represented by the attaching maps for the two-cells of Y have representatives which are a sub-basis of the free group $\pi_1(Y^{(1)})$.*

Proposition 5.5.12. *Let X be a 2-dimensional relative graph whose vertices have freely indecomposable fundamental group. Let $w : S^1 \to X$ be a representative of the free homotopy class of a word $w \in \pi_1(X)$ and let X_α be the CW complex formed by gluing a 2-cell, α, to X with attaching path w. If X_α has NI then every vertex of X has NI and $P_2^{\pi_1(X)}(w) > 2$.*

Proof. Firstly, if a vertex of X does not have NI then since inclusions of sub-complexes and compositions of immersions are immersions X_α cannot have NI, so since X_α has NI we may assume every vertex of X has NI. Suppose that $P_2(w) := P_2^{\pi_1(X)}(w) \leq 2$. By definition there exists a subgroup $H \leq \pi_1(X)$ such that $w \in H$, H is finitely presented, w is not primitive in H and $rank(H) = P_2(w)$. By Lemma 3.1.4 there exists a combinatorial map $Y \to X$ such that $\pi_1(Y) \cong H$. By Lemma 3.1.3, the map $Y \to X$ factors through an immersion $Y \to Z \looparrowright X$ such that $Y \to Z$ is surjective and π_1-surjective. Thus $rank(\pi_1(Z)) \leq rank(H)$ so since Z is connected

$$\chi(Z) = \beta_0(Z) - \beta_1(Z) + \beta_2(Z) \geq 1 - \beta_1(Z).$$

Clearly, $\beta_1(Z) \leq rank(\pi_1(Z))$ so

$$\chi(Z) \geq 1 - rank(\pi_1(Z)) \geq 1 - rank(H) = 1 - P_2(w) \geq 1 - 2 = -1.$$

Now attach a 2-cell to Z using the attaching path w, then $Z \cup_w D$ immerses in X_α and $\chi(Z \cup_w D) = 0$. Furthermore since $Y \to X$ factors through Z, w is not primitive in $\pi_1(Z)$ so by Lemma 5.5.11, $Z \cup_w D$ does not Nielsen reduce to a graph and therefore X_α does not have NI. □

Notice that the above proof actually doesn't need to use a primitivity rank definition quite as strong as P_2, if we were to instead use P_1 we would need the vertices of Z to not have negative Euler characteristic, which may not necessarily be true. The problem is the possibility that a word has $P_2(w) > 2$ and $P_1(w) = 2$.

In order to attempt a proof in the opposite direction we need the following lemma from [15].

Lemma 5.5.13 ([15])**.** *Let Y be a 2-complex. If $U \looparrowright Y$ is an immersion of 2-complexes and Y Nielsen reduces to Z then there is a two-complex V immersing in Z such that U Nielsen reduces to V. In particular, if U immerses in Y and Y Nielsen reduces to a graph then U Nielsen reduces to a graph.*

If we use the P_1 version of primitivity rank and assume Conjecture 5.4.3 holds we can get a partial result in the opposite direction.

Proposition 5.5.14. *Suppose that Conjecture 5.4.3 holds. Let X be a relative graph whose vertices have freely indecomposable fundamental group, let $w \in \pi_1(X)$ and let X_α be the complex formed by attaching a 2-cell α to X along a path representing the free homotopy class of w. If each vertex of X has NI and $P_1(w) > 2$ then X_α has NI.*

Proof. By Lemma 5.5.9 since $P_1(w) > 2$, w is not a proper power and is not conjugate into a vertex of X. Therefore, by Corollary 4.4.2, X_α has NPI meaning it is enough to consider immersions $Y \looparrowright X_\alpha$ where Y is a compact, connected 2-complex with $\chi(Y) = 0$. Let $Y \looparrowright X_\alpha$ be such an immersion. Corollary 5.5.2 tells us that either Y has free faces, in which case we can Nielsen reduce to remove these, or $\chi_r(\hat{Y}^I) \geq \chi(Y) = 0$, where $\hat{Y}^I$ is the immersed one-relator pushout defined earlier. Furthermore, if $\bar{Y}$ is the complex formed by removing the interior of the 2-cell in $\hat{Y}^I$ whose image is α, then $\pi_1(\bar{Y}) \leq \pi_1(X)$ is a finitely presented subgroup containing w and $1 - \chi_r(\pi_1(\bar{Y})) = 1 - \chi_r(\bar{Y}) = 2 - \chi_r(\hat{Y}^I) \leq 2$. Therefore, w is primitive in $\pi_1(\bar{Y})$ since $P_1(w) > 2$ so $\hat{Y}^I$ Nielsen reduces by collapsing the single 2-cell mapping to α. Therefore, by Lemma 5.5.13, Y Nielsen reduces to a complex Y' that contains no pre-images of α. Since Y' is a Nielsen reduction of Y we must have $\chi(Y') = \chi(Y) = 0$ and Y' immerses in X. Since each vertex of X has NI, X has NI by Lemma 5.5.10. Thus, Y' Nielsen reduces to a graph and so Y must also Nielsen reduce to a graph, meaning X_α has NI. □

As a final remark we note that we would actually only need a slightly weaker version of Conjecture 5.4.3 in the proof of the above proposition. This relies on a result of Howie and Short (Theorem 4.1 or [12]), which we briefly discussed at the start of Chapter 5 but will quickly describe again here. Firstly for the 2-cell α, we choose a 'highest edge' (Howie and Short use a left order induced by locally indicable groups but we could use a stacking) and for each 2-cell in $f^{-1}(\alpha)$ we remove the interior of the 2-cell from Y and the corresponding 'highest edge' (these are distinct since the map is an immersion). Let Z be the complex formed in this manner from Y. The theorem says that if a component T of Z has $f_*(\pi_1(T)) = 1$ then Y Nieslen reduces to T. The above proposition could therefore be proved if we further assume $f^{-1}(X)$ is homotopy equivalent to a graph. This is because if we have Y with $\chi(Y) = 0$, then $\chi(Z) = \chi(Y) = 0$ and X has NPI so if a component of Z has positive Euler characteristic it is contractible and hence by the Theorem, Y is contractible. Therefore, each component of Z has Euler characteristic 0, but X has NI so each component can be Nielsen reduced to a circle and so we only need Conjecture 5.4.3 to hold if $f^{-1}(X) \subset Y$ Nielsen reduces to a graph.

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