# A Bunched Logic for Conditional Independence 

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#### Abstract

Independence and conditional independence are fundamental concepts for reasoning about groups of random variables in probabilistic programs. Verification methods for independence are still nascent, and existing methods cannot handle conditional independence. We extend the logic of bunched implications (BI) with a non-commutative conjunction and provide a model based on Markov kernels; conditional independence can be directly captured as a logical formula in this model. Noting that Markov kernels are Kleisli arrows for the distribution monad, we then introduce a second model based on the powerset monad and show how it can capture join dependency, a non-probabilistic analogue of conditional independence from database theory. Finally, we develop a program logic for verifying conditional independence in probabilistic programs.


## I. Introduction

The study of probabilistic programming languages and their semantics dates back to the 1980s, starting from the seminal work of Kozen [1]. The last decade has seen a surge of richer probabilistic languages [2, 3, 4], motivated by applications in machine learning, and accompanying research into their semantics [5, 6, 7]. This burst of activity has also created new opportunities and challenges for formal verification.

Independence and conditional independence are two fundamental properties that are poorly handled by existing verification methods. Intuitively, two random variables are probabilistically independent if information about one gives no information about the other (for example, the results of two coin flips). Conditional independence is more subtle: two random variables $X$ and $Y$ are independent conditioned on a third variable $Z$ if for every fixed value of $Z$, information about one of $X$ and $Y$ gives no information about the other.

Both forms of independence are useful for modelling and verification. Probabilistic independence enables compositional reasoning about groups of random variables: if a group of random variables are independent, then their joint distribution is precisely described by the distribution of each variable in isolation. It also captures the semantics of random sampling constructs in probabilistic languages, which generate a fresh random quantity that is independent of the program state. Conditional independence often arises in programs with probabilistic control flow, as conditioning models probabilistic branching. Bayesian networks encode conditional independence statements in complex distributions, and conditional independence captures useful properties in many applications.

For instance, criteria ensuring that algorithms do not discriminate based on sensitive characteristics (e.g., gender or race) can be formulated using conditional independence [8].

Aiming to prove independence in probabilistic programs, Barthe et al. [9] recently introduced Probabilistic Separation Logic (PSL) and applied it to formalize security for several well-known constructions from cryptography. The key ingredient of PSL is a new model of the logic of bunched implications (BI), in which separation is interpreted as probabilistic independence. While PSL enables formal reasoning about independence, it does not support conditional independence. The core issue is that the model of BI underlying PSL provides no means to describe the distribution of one set of variables obtained by fixing (conditioning) another set of variables to take specific values. Accordingly, one cannot capture the basic statement of conditional independence- $X$ and $Y$ are independent random variables conditioned on any value of $Z$.

In this paper, we develop a logical framework for formal reasoning about notions of dependence and independence. Our approach is inspired by PSL but the framework is more sophisticated: to express conditional independence, we develop a novel assertion logic extending BI with new connectives-: and its adjoints. The key intuition is that conditional independence can be expressed as independence plus composition of Markov kernels; as our leading example, we give a kernels model of our logic.

Then, we show how to adapt the probabilistic model to other settings. As is well-known in category theory, Markov kernels are the arrows in the Kleisli category of the distribution monad. By varying the monad, our logic smoothly extends to analogues of conditional independence in other domains. To demonstrate, we show how replacing the distribution monad by the powerset monad gives a model where we can capture join/multivalued dependencies in relational algebra and database theory. We also show that the semi-graphoid laws, introduced by Pearl and Paz [10] in their work axiomatizing conditional independence, can be translated into formulas that are valid in both of our models.

The rest of the paper is organized as follows. We give a bird's-eye view in Section [II providing intuitions on our design choices and highlighting differences with existing work. Section III] presents the main contribution: the design of DIBI, a new bunched logic to reason about dependence and independence. We show that the proof system of DIBI is
sound and complete with respect to its Kripke semantics. Then, we present two concrete models in Section IV based on probability distributions and relations. In Section (V) we consider how to express dependencies in DIBI: we show that the same logical formula captures conditional independence and join dependency in our two models, and our models validate the semi-graphoid laws. Finally, in Section VI, we design a program logic with DIBI assertions, and use it to verify conditional independence in two probabilistic programs.

## II. Overview of the contributions

The logic DIBI: The starting point of our work is the logic of bunched implications (BI) [11]. BI extends intuitionistic propositional logic with substructural connectives to facilitate reasoning about sharing and separation of resources, an idea most prominently realized in Separation Logic's handling of heap-manipulating programs [12]. The novel connectives are a separating conjunction $P * Q$, intuitively stating that $P$ and $Q$ hold in separate resources, and its adjoint $\rightarrow$, called magic wand. We will extend BI with a non-commutative conjunction, written $P \stackrel{q}{9}$. Intuitively, $\stackrel{\circ}{9}$ expresses a possible dependency of $Q$ on $P$. The end result is a logic with two conjunctive connectives-* and $\%$ capturing notions of independence and dependence. We call the logic Dependence and Independence Bunched Implications (DIBI).

To give a semantics to our logic, we start from the semantics of BI. The simplest BI models are partial resource monoids: Kripke structures ( $M, \sqsubseteq, \circ, e$ ) in which $\circ$ is an order-preserving, partial, commutative monoid operation with unit $e$. The operation o allows interpreting the separating conjunction $P * Q$ and magic wand $P * Q$. For example, the probabilistic model of BI underlying PSL [9] is a partial resource monoid: by taking $M$ to be the set of distributions over program memories and o to be the independent product of distributions over memories with disjoint variables, the interpretation of $P * Q$ gives the desired notion of probabilistic independence.
This is the first point where we fundamentally differ from PSL. To capture both dependence and independence, we change the structure in which formulas are interpreted. In Section [II], we will introduce a structure $\mathcal{X}=(X, \sqsubseteq, \oplus, \odot, E)$, a DIBI frame, with two operations $\oplus: X^{2} \rightarrow \mathcal{P}(X)$ and $\odot: X^{2} \rightarrow \mathcal{P}(X)$, and a set of units $E \subseteq X$. Three remarks are in order. First, the preorder $\sqsubseteq$ makes DIBI an intuitionistic logic. There are many design trade-offs between intuitionistic and classical, but the most important consideration is that intuitionistic formulas can describe proper subsets of states (e.g., random variables), leaving the rest of the state implicit. Second, DIBI frames contain an additional monoidal operation $\odot$ for interpreting ${ }_{9}(\oplus$ will be used in interpreting $*)$. Third, as the completeness of BI for its simple PCM models is an open problem [13], our models are examples of a broader notion of BI model with non-deterministic operations (following [14, 15]). These models subsume partial resource monoids, and enable our completeness proof of DIBI. While the conditions that DIBI frames must satisfy are somewhat cryptic at first sight, they can be naturally understood as axioms defining
monoidal operations in a partial, non-deterministic setting. E.g., we will require:

```
\((\oplus\) Comm.) \(\quad z \in x \oplus y \rightarrow z \in y \oplus x\);
\((\oplus\) Assoc.) \(\quad w \in t \oplus z \wedge t \in x \oplus y \rightarrow \exists s(s \in y \oplus z \wedge w \in x \oplus s)\);
( \(\odot\) Unit Exist. L\() \quad \exists e \in E .(x \in e \odot x)\)
```

where unbound variables are universally quantified. Crucially, the operation $\odot$ need not be commutative: this operation interprets the dependence conjunction $\stackrel{\circ}{9}$, where commutativity is undesirable. In a DIBI frame, $*$ and $;$ are interpreted as:

$$
\begin{array}{ll}
x \vDash P * Q & \text { iff exists } x^{\prime}, y, z \text { s.t. } x \sqsupseteq x^{\prime} \in y \oplus z, y \vDash P, \text { and } z \vDash Q \\
x \vDash P ; Q & \text { iff exists } y, z \text { s.t. } x \in y \odot z, y \vDash P, \text { and } z \vDash Q
\end{array}
$$

In DIBI, * has a similar reading as in PSL: it states that two parts of a distribution can be combined because they are independent. In contrast, the new conjunction $P \circ Q$ asserts that the $Q$ part of a distribution may depend on the $P$ part. Combined with the separating conjunction $*$, the new conjunction ${ }_{\circ}$ can express more complex dependencies: e.g. $P \circ(Q * R)$ asserts that $Q$ and $R$ both may depend on $P$, and are independent conditioned on $P$.

A sound and complete proof system for DIBI: To reason about DIBI validity, in Section III we also provide a Hilbertstyle proof system for DIBI, and prove soundness and completeness. The proof system extends BI with rules for the new connective ${ }_{9}$, e.g. ${ }_{9}$ Cons, and for the interaction between ${ }_{9}^{\circ}$ and *, e.g., RevEx:

RevEx-reverse-exchange-captures the fundamental interaction between the two conjunctions. Computations $T=P ; Q$ and $U=R ; S$ are built from dependent components, yet $T$ and $U$ are independent and hence can be combined with $*$. We can then infer that the building blocks of $T$ and $U$ must also be pair-wise independent and can be combined, yielding formulas $P * R$ and $Q * S$. These can then be combined with $\circ$ as they retain the dependency of the original building blocks.

Models and applications of DIBI: Separation logics are based on a concrete BI model over program states, together with a choice of atomic assertions. Before explaining the models of DIBI, we recall two prior models of BI.

In the heap model, states are heaps: partial maps from memory addresses to values. Atomic assertions of the form $x \mapsto v$ indicate that the location to which $x$ points has value $v$. Then, $x \mapsto v * y \mapsto u$ states that $x$ points to $v$ and $y$ points to $u$, and $x$ and $y$ do not alias-they must point to different locations. In general, $P * Q$ holds when a heap can be split into two subheaps with disjoint domains, satisfying $P$ and $Q$ respectively.


In PSL, states are distributions over program memories, basic assertions $\mathbf{D}[x]$ indicate that $x$ is a random variable,
and $P * Q$ states that a distribution $\mu$ can be factored into two independent distributions $\mu_{1}$ and $\mu_{2}$ satisfying $P$ and $Q$, respectively. Consider the following simple program:

$$
\begin{equation*}
x \hookleftarrow \mathbf{B}_{1 / 2} ; y \hookleftarrow \mathbf{B}_{1 / 2} ; z \leftarrow x \vee y \tag{1}
\end{equation*}
$$

Here, $x$ and $y$ are Boolean variables storing the result of two fair coin flips and $z$ stores the result of $x \vee y$. The output distribution $\mu$ is a distribution over a memory with variables $x, y$ and $z$ (depicted below on the right). In $\mu$, the variables $x$ and $y$ are independent and $\mathbf{D}[x] * \mathbf{D}[y]$ holds, since the marginal distribution of $\mu$ is a product of $\mu_{1}$ and $\mu_{2}$, which satisfy $\mathbf{D}[x]$ and $\mathbf{D}[y]$ respectively:


In Section IV, we develop two concrete models for DIBI: one based on probability distributions, and one based on relations. Here we outline the probabilistic model, as it generalizes the model of PSL. Let Val be a finite set of values and $S$ a finite set of memory locations. We use $\operatorname{Mem}[S]$ to denote functions $S \rightarrow$ Val, representing program memories. The states in the DIBI probabilistic model, over which the formulas will be interpreted, are Markov kernels on program memories. More precisely, given sets of memory locations $S \subseteq U$, these are functions $f: \operatorname{Mem}[S] \rightarrow \mathcal{D}(\operatorname{Mem}[U])$ that preserve their input. Regular distributions can be lifted to Markov kernels: the distribution $\mu: \mathcal{D}(\operatorname{Mem}[U])$ corresponds to the kernel $f_{\mu}: \operatorname{Mem}[\emptyset] \rightarrow$ $\mathcal{D}(\operatorname{Mem}[U])$ that assigns $\mu$ to the only element in Mem[ $[\emptyset]$. We depict input-preserving Markov kernels as trapezoids, where the smaller side represents the domain and the larger side the range; our basic assertions will track $\operatorname{dom}(f)$ and
 range $(f)$, justifying this simplistic depiction.

Separating and dependent conjunction will be interpreted via $\oplus$ and $\odot$ on Markov kernels. Intuitively, $\oplus$ is a parallel composition that takes union on both domains and ranges, whereas $\odot$ composes the kernels using Kleisli composition.


To demonstrate, recall the simple program (1). In the output distribution $\mu, z$ depends on $x$ and $y$ since $z$ stores $x \vee y$, and $x$ and $y$ are independent. In our setting, this dependency structure can be seen when decomposing $f_{\mu}=\left(f_{\mu_{1}} \oplus f_{\mu_{2}}\right) \odot f_{z}$, where kernel $f_{z}: \operatorname{Mem}[\{x, y\}] \rightarrow \mathcal{D}(\operatorname{Mem}[\{x, y, z\}])$ captures how the value of $z$ depends on the values of $\{x, y\}$ :

$\delta: X \rightarrow \mathcal{D}(X)$ is the Dirac distribution $\delta(v)(w)=1$ if $v=w, 0$ otherwise.

We can then prove:

$$
\begin{equation*}
f_{\mu_{1}} \oplus f_{\mu_{2}} \vDash P_{x * y} \quad \text { and } \quad f_{z} \models Q_{z} \quad \text { implies } \quad f_{\mu} \vDash P_{x * y} \circ Q_{z} \tag{2}
\end{equation*}
$$

When analyzing composition of Markov kernels, the domains and ranges provide key information: the domain determines which variables a kernel may depend on, and the range determines which variables a kernel describes. Accordingly, we use basic assertions of the form $(A \triangleright[B])$, where $A$ and $B$ are sets of memory locations. A Markov kernel $f: \operatorname{Mem}[S] \rightarrow$ $\mathcal{D}(\operatorname{Mem}[T])$ satisfies $(A \triangleright[B])$ if there exists a $f^{\prime} \sqsubseteq f$ with $\operatorname{dom}\left(f^{\prime}\right)=A$ and $\operatorname{range}\left(f^{\prime}\right) \supseteq B$ (we will define $f^{\prime} \sqsubseteq f$ formally later and for now read it as $f$ extends $f^{\prime}$ ). For instance, the kernel $f_{z}$ above satisfies $(\{x, y\} \triangleright[x, y]),(\{x, y\} \triangleright[x, y, z])$, and $(\{x, y\} \triangleright[\emptyset])$. One choice for $P_{x * y}$ and $Q_{z}$ in (2) can be: $P_{x * y}=(\emptyset \triangleright[x]) *(\emptyset \triangleright[y])$ and $\left.Q_{z}=(\{x, y\} \triangleright[x, y, z])\right)$

Formalizing conditional independence: The reader might wonder how to use such simple atomic propositions, which only talk about the domain/range of a kernel and do not describe numeric probabilities, to assert conditional independence. The key insight is that conditional independence can be formulated using sequential ( $\odot$ ) and parallel $(\oplus)$ composition of kernels. In Section V, we show that given $\mu \in \mathcal{D}(\operatorname{Mem}[\mathrm{Var}])$, for any $X, Y, Z \subseteq \operatorname{Var}$, the satisfaction of

$$
\begin{equation*}
f_{\mu} \vDash(\emptyset \triangleright[Z]) \stackrel{q}{g}(Z \triangleright[X]) *(Z \triangleright[Y]) \tag{3}
\end{equation*}
$$

captures conditional independence of $X, Y$ given $Z$ in $\mu$.
Moreover, the formula in (3) smoothly generalizes to other models. In the relational model of DIBI-obtained by switching the distribution monad to the powerset monad-the exact same formula encodes join dependency, a notion of conditional independence from the databases and relational algebra literature. More generally, we also show that the semi-graphoid axioms of Pearl and Paz [10] are valid in these two models, and two of the axioms can be derived in the DIBI proof system.

## III. The Logic DIBI

## A. Syntax and semantics

The syntax of DIBI extends the logic of bunched implications (BI) [11] with a non-commutative conjunctive connective $;$ and its associated implications. Let $\mathcal{A P}$ be a set of propositional atoms. The set of DIBI formulas, Form ${ }_{\text {DIBI }}$, is generated by the following grammar:

$$
\begin{gathered}
P, Q::=p \in \mathcal{A P}|\top| I|\perp| P \wedge Q|P \vee Q| P \rightarrow Q \\
|P * Q| P * Q|P \circ Q| P \multimap Q \mid P \circ Q .
\end{gathered}
$$

DIBI is interpreted on DIBI frames, which extend BI frames.
Definition III. 1 (DIBI Frame). A DIBI frame is a structure $\mathcal{X}=(X, \sqsubseteq, \oplus, \odot, E)$ such that $\sqsubseteq$ is a preorder, $E \subseteq X$, and $\oplus: X^{2} \rightarrow \mathcal{P}(X)$ and $\odot: X^{2} \rightarrow \mathcal{P}(X)$ are binary operations, satisfying the rules in Figure 1

Intuitively, $X$ is a set of states, the preorder $\sqsubseteq$ describes when a smaller state can be extended to a larger state, the binary operators $\odot, \oplus$ offer two ways of combining states, and $E$ is the set of states that act like units with respect to these

| $(\oplus$ Down-Closed) | $z \in x \oplus y \wedge x \sqsupseteq x^{\prime} \wedge y \sqsupseteq y^{\prime}$ | $\rightarrow$ | $\exists z^{\prime}\left(z \sqsupset z^{\prime} \wedge z^{\prime} \in x^{\prime} \oplus y^{\prime}\right) ;$ |
| :---: | :---: | :---: | :---: |
| (¢ Up-Closed) | $z \in x \odot y \wedge z^{\prime} \sqsupseteq z$ | $\rightarrow$ | $\exists x^{\prime}, y^{\prime}\left(x^{\prime} \sqsupseteq x \wedge y^{\prime} \sqsupseteq y \wedge z^{\prime} \in x^{\prime} \odot y^{\prime}\right)$ |
| $(\oplus$ Commutativity) | $z \in x \oplus y$ | $\rightarrow$ | $z \in y \oplus x ;$ |
| $(\oplus$ Associativity) | $w \in t \oplus z \wedge t \in x \oplus y$ | $\rightarrow$ | $\exists s(s \in y \oplus z \wedge w \in x \oplus s) ;$ |
| $(\oplus$ Unit Existence) | $\exists e \in E(x \in e \oplus x)$; |  |  |
| ( $\oplus$ Unit Coherence) | $e \in E \wedge x \in y \oplus e$ | $\rightarrow$ | $x \sqsupseteq y ;$ |
| (¢ Associativity) | $\exists t(w \in t \odot z \wedge t \in x \odot y)$ | $\leftrightarrow$ | $\exists s(s \in y \odot z \wedge w \in x \odot s) ;$ |
| (¢ Unit Existence ${ }_{\text {L }}$ ) | $\exists e \in E(x \in e \odot x) ;$ |  |  |
| ( $\odot$ Unit Existence $\mathrm{R}_{\text {) }}$ ) | $\exists e \in E(x \in x \odot e) ;$ |  |  |
| ( $\odot$ Coherence $_{\text {R }}$ ) | $e \in E \wedge x \in y \odot e$ | $\rightarrow$ | $x \sqsupseteq y ;$ |
| (Unit Closure) | $e \in E \wedge e^{\prime} \sqsupseteq e$ | $\rightarrow$ | $e^{\prime} \in E ;$ |
| (Reverse Exchange) | $x \in y \oplus z \wedge y \in y_{1} \odot y_{2} \wedge z \in z_{1} \odot z_{2}$ | $\rightarrow$ | $\exists u, v\left(u \in y_{1} \oplus z_{1} \wedge v \in y_{2} \oplus z_{2} \wedge x \in u \odot v\right)$. |

Fig. 1: DIBI frame requirements (with outermost universal quantification omitted for readability).
operations. The binary operators return a set of states instead of a single state, and thus can be either deterministic (at most one state returned) or non-deterministic, either partial (empty set returned) or total. The operators in the concrete models below will be deterministic, but the proof of completeness relies on the frame's admission of non-deterministic models, as is standard for bunched logics [14].

The frame conditions define properties that must hold for all models of DIBI. Most of these properties can be viewed as generalizations of familiar algebraic properties to nondeterministic operations, suitably interacting with the preorder. The "Closed" properties give coherence conditions between the order and the composition operators. It is known that having the Associativity frame condition together with either the Up- or Down-Closed property for an operator is sufficient to obtain the soundness of associativity for the conjunction associated with the operator [16, 14]. The choices of Closed conditions match the desired interpretations of $\oplus$ as independence and $\odot$ as dependence: independence should drop down to substates (which must necessarily be independent if the superstates were), while dependence should be inherited by superstates (the source of dependence will still be present in any extensions). Having $\odot$ non-commutative also splits the $\odot$ analogues of $\oplus$ axioms into pairs of axioms, although we note that we exclude the left version of ( $\odot$ Coherence) for reasons we explain in Section 【II-B Finally, the (Reverse Exchange) condition defines the interaction between $\oplus$ and $\odot$.

We will give a Kripke-style semantics for DIBI, much like the semantics for BI [17]. Given a DIBI frame, the semantics defines which states in the frame satisfy each formula. Since the definition is inductive on formulas, we must specify which states satisfy the atomic propositions.

Definition III. 2 (Valuation and model). A persistent valuation is an assignment $\mathcal{V}: \mathcal{A P} \rightarrow \mathcal{P}(X)$ of atomic propositions to subsets of states of a DIBI frame satisfying: if $x \in \mathcal{V}(p)$ and $y \sqsupseteq x$ then $y \in \mathcal{V}(p)$. A DIBI model $(\mathcal{X}, \mathcal{V})$ is a DIBI frame $\mathcal{X}$ together with a persistent valuation $\mathcal{V}$.

Since DIBI is an intuitionistic logic, persistence is necessary for soundness. We can now give a semantics to DIBI formulas in a DIBI model.

Definition III. 3 (DIBI Satisfaction and Validity). Satisfaction at a state $x$ in a model is inductively defined by the clauses in Figure 2, $P$ is valid in a model, $\mathcal{X} \vDash_{\mathcal{V}} P$, iff $x \vDash_{\mathcal{V}} P$ for all $x \in \mathcal{X} . P$ is valid,$\vDash P$, iff $P$ is valid in all models. $P \vDash Q$ iff, for all models, $\mathcal{X} \models_{\mathcal{V}} P$ implies $\mathcal{X} \models_{\mathcal{V}} Q$.

Where the context is clear, we omit the subscript $\mathcal{V}$ on the satisfaction relation. With the semantics in Figure 2, persistence on propositional atoms extends to all formulas:

Lemma III. 1 (Persistence Lemma). For all $P \in$ Form $_{\text {Dibi }}$, if $x \vDash P$ and $y \sqsupseteq x$ then $y \vDash P$.

The reader may note the difference between the semantic clauses for $\stackrel{\circ}{\circ}$ and $*$, and $*$ and - : the satisfaction of the UpClosed (Down-Closed) frame axiom for $\odot(\oplus)$ leads to the persistence and thus the soundness of the simpler clause for $\circ(*)$ [16]. Without the other Closed property, we must use a satisfaction clause which accounts for the order, as in BI.

## B. Proof system

A Hilbert-style proof system for DIBI is given in Figure 3 . This calculus extends a system for BI with additional rules governing the new connectives $\stackrel{\circ}{9}, \multimap$ and $\circ$ : in Section III-C we will prove this calculus is sound and complete. We briefly comment on two important details in this proof system.

Reverse exchange: The proof system of DIBI shares many similarities with Concurrent Kleene Bunched Logic (CKBI) [14], which also extends BI with a noncommutative conjunction. Inspired by concurrent Kleene algebra (CKA) [18], CKBI supports the following exchange axiom, derived from CKA's exchange law:

$$
(P * R) \stackrel{( }{q}(Q * S) \vdash_{\mathrm{CKBI}}(P \circ Q) *(R \stackrel{\circ}{q} S)
$$

In models of CKBI, * describes interleaving concurrent composition, while $\stackrel{\circ}{ }$ describes sequential composition. The exchange rule states that the process on the left has fewer behaviors than the process on the right-e.g., $P \circ Q$ allows fewer behaviors than $P * Q$, so $P \circ Q \vdash_{\text {Скві }} P * Q$ is derivable.

In our models, $*$ has a different reading: it states that two computations can be combined because they are independent (i.e., non-interfering). Accordingly, DIBI replaces Exch by the reversed version $\operatorname{RevEx}$ - the fact that the process on the left


Fig. 2: Satisfaction for DIBI

$$
\begin{aligned}
& \frac{P \vdash Q \quad P \vdash R}{P \vdash Q \wedge R} \wedge 1 \\
& \frac{Q \vdash R}{P \wedge Q \vdash R} \wedge 2 \\
& \frac{P \vdash Q_{1} \wedge Q_{2}}{P \vdash Q_{i}} \wedge 3 / \wedge 4 \\
& \frac{P \wedge Q \vdash R}{P \vdash Q \rightarrow R} \rightarrow \\
& \frac{P \vdash Q \rightarrow R \quad P \vdash Q}{P \vdash R} \mathrm{MP} \\
& \frac{P * Q \vdash R}{P \vdash Q * R} * \\
& \frac{P \vdash Q * R \quad S \vdash Q}{P * S \vdash R} * \mathrm{MP} \\
& \frac{P \% Q+R}{P \vdash Q \multimap R} \multimap \\
& \frac{P \vdash Q \multimap R \quad S \vdash Q}{P \circ S \vdash R} \multimap \mathrm{MP} \\
& \frac{P \vdash R \quad Q \vdash S}{P * Q \vdash R * S} * \text { Cons } \\
& \frac{P \vdash R \quad Q \vdash S}{P \circ Q \vdash R ; S} \fallingdotseq-\mathrm{ConJ} \\
& \frac{P \vdash Q \circ-R \quad S \vdash Q}{S \circ P \vdash R} \circ \mathrm{MP} \\
& \overline{P * Q \vdash Q * P} * \text {-Сомм }
\end{aligned}
$$

$$
\begin{aligned}
& \overline{P \dashv P * I} * \text {-UnIT } \\
& \overline{P \vdash I}{ }_{9} P \text { 冗-Left Unit } \\
& \overline{(P \circ Q) *(R \circ S) \vdash(P * R) \stackrel{\circ}{\circ}(Q * S)} \operatorname{RevEx}
\end{aligned}
$$

Fig. 3: Hilbert system for DIBI
is safe to combine implies that the process on the right is also safe. $P * Q$ is now stronger than $P \circ Q$, and $P * Q \vdash P \circ Q$ is derivable (Theorem A.1).

Left unit: While ${ }_{9}$ has a right unit in our logic, it does not have a proper left unit. Semantically, this corresponds to the lack of a frame condition for $\odot$-Coherence ${ }_{L}$ in our definition of DIBI frames. This difference can also be seen in our proof rules: while ${ }_{q}$-Right Unit gives entailment in both directions, §-Left Unit only shows entailment in one direction-there is no axiom stating $I \stackrel{\circ}{9} P \vdash P$.

We make this relaxation to support our intended models, which we will see in Section IV] In a nutshell, states in our models are Kleisli arrows that preserve their input through to their output-intuitively, in conditional distributions, the variables that have we conditioned on will remain fixed. Our models take $\odot$ to be Kleisli composition, which exhibits an important asymmetry for such arrows: $f$ can always be recovered from $f \odot g$, but not from $g \odot f$. As a result, the set of all arrows naturally serves as the set of right units, but these arrows cannot all serve as left units.

## C. Soundness and Completeness of DIBI

A methodology for proving the soundness and completeness of bunched logics is given by Docherty [14], inspired by the duality-theoretic approach to modal logic [19]. First, DIBI is proved sound and complete with respect to an algebraic semantics obtained by interpreting the rules of the proof system as algebraic axioms. We then establish a representation theorem: every DIBI algebra $\mathbb{A}$ embeds into a DIBI algebra generated by a DIBI frame, that is in turn generated by $\mathbb{A}$.

Soundness and completeness of the algebraic semantics can then be transferred to the Kripke semantics. Omitted details can be found in Appendix B

Definition III. 4 (DIBI Algebra). A DIBI algebra is an algebra $\mathbb{A}=\left(A, \wedge, \vee, \rightarrow, \top, \perp, *, *,{ }_{9}^{\circ}, \rightarrow, \circ-I\right)$ such that, for all $a, b, c, d \in A$ :

- $(A, \wedge, \vee, \rightarrow, \top, \perp)$ is a Heyting algebra;
- $(A, *, I)$ is a commutative monoid;
- $\left(A,{ }_{9}, I\right)$ is a weak monoid: $\circ$ is an associative operation with right unit $I$ and $a \leq I \circ a$;
- $a * b \leq c$ iff $a \leq b * c$;
- $a \circ b \leq c$ iff $a \leq b \multimap c$ iff $b \leq a \circ c$;
- $(a ; b) *(c ; d) \leq(a * c) \circ(b * d)$.

An algebraic interpretation of DIBI is specified by an assignment $\llbracket-\rrbracket: \mathcal{A P} \rightarrow A$. The interpretation is obtained as the unique homomorphic extension of this assignment, and so we use the notation $\llbracket-\rrbracket$ interchangeably for both assignment and interpretation. Soundness and completeness can be established by constructing a term DIBI algebra by quotienting formulas by equiderivability.
Theorem III.2. $P \vdash Q$ is derivable iff $\llbracket P \rrbracket \leq \llbracket Q \rrbracket$ for all algebraic interpretations $\llbracket-\rrbracket$.

We now connect these algebras to DIBI frames. A filter on a bounded distributive lattice $\mathbb{A}$ is a non-empty set $F \subseteq A$ such that, for all $x, y \in A$, (1) $x \in F$ and $x \leq y$ implies $y \in F$; and (2) $x, y \in F$ implies $x \wedge y \in F$. It is a proper filter if it additionally satisfies (3) $\perp \notin F$, and a prime filter if it also
satisfies (4) $x \vee y \in F$ implies $x \in F$ or $y \in F$. We denote the set of prime filters of $\mathbb{A}$ by $\mathbb{P F}_{\mathbb{A}}$.
Definition III. 5 (Prime Filter Frame). Given a DIBI algebra $\mathbb{A}$, the prime filter frame of $\mathbb{A}$ is defined as $\operatorname{Pr}(\mathbb{A})=\left(\mathbb{P F}_{\mathbb{A}}, \subseteq\right.$ , $\left.\oplus_{\mathbb{A}}, \odot_{\mathbb{A}}, E_{\mathbb{A}}\right)$, where $F \oplus_{\mathbb{A}} G:=\left\{H \in \mathbb{P F}_{\mathbb{A}} \mid \forall a \in F, b \in G(a *\right.$ $b \in H)\}, F \odot_{\mathbb{A}} G:=\left\{H \in \mathbb{P F}_{\mathbb{A}} \mid \forall a \in F, b \in G(a \circ b \in H)\right\}$ and $E_{\mathbb{A}}:=\left\{F \in \mathbb{P F}_{\mathbb{A}} \mid I \in F\right\}$.

Proposition III.3. For any DIBI algebra $\mathbb{A}$, the prime filter frame $\operatorname{Pr}(\mathbb{A})$ is a DIBI frame.

In the other direction, DIBI frames generate DIBI algebras.
Definition III. 6 (Complex Algebra). Given a DIBI frame $\mathcal{X}=(X, \sqsubseteq, \oplus, \odot, E)$, the complex algebra of $\mathcal{X}$ is defined to be $\operatorname{Com}(\mathcal{X})=\left(\mathcal{P}_{\sqsubseteq}(X), \cap, \cup, \Rightarrow_{\chi}, X, \emptyset, \bullet_{\chi}, \bullet_{\chi}, \triangleright_{\chi}, \rightarrow_{\chi}, \triangleright_{\chi}, E\right)$ :
$\mathcal{P}_{\sqsubseteq}(X) \quad=\{A \subseteq X \mid$ if $a \in A$ and $a \sqsubseteq b$ then $b \in A\}$
$A \stackrel{\perp}{\Rightarrow} B \quad=\{a \mid$ for all $b$, if $b \sqsupseteq a$ and $b \in A$ then $b \in B\}$
$A \bullet x B=\left\{x \mid\right.$ there exist $x^{\prime}, a, b$ s.t $x \sqsupseteq x^{\prime} \in a \oplus b, a \in A$ and $\left.b \in B\right\}$
$A \rightarrow X^{B} B=\{x \mid$ for all $a, b$, if $b \in x \oplus a$ and $a \in A$ then $b \in B\}$
$A \triangleright{ }^{\prime} B=\{x \mid$ there exist $a, b$ s.t $x \in a \odot b, a \in A$ and $b \in B\}$
$A \rightarrow X B=\left\{x \mid\right.$ for all $x^{\prime}, a, b$, if $x \sqsubseteq x^{\prime}, b \in x^{\prime} \odot a$ and $a \in A$ then $\left.b \in B\right\}$
$A \triangleright-x B=\left\{x \mid\right.$ for all $x^{\prime}, a, b$, if $x \sqsubseteq x^{\prime}, b \in a \odot x^{\prime}$ and $a \in A$ then $\left.b \in B\right\}$.
Proposition III.4. For any DIBI frame $\mathcal{X}$, the complex algebra $\operatorname{Com}(\mathcal{X})$ is a DIBI algebra.

The following main result facilitates transference of soundness and completeness.

Theorem III. 5 (Representation of DIBI algebras). Every DIBI algebra is isomorphic to a subalgebra of a complex algebra: given a DIBI algebra $\mathbb{A}$, the map $\theta_{\mathbb{A}}: \mathbb{A} \rightarrow \operatorname{Com}(\operatorname{Pr}(\mathbb{A}))$ defined by $\theta_{\mathbb{A}}(a)=\left\{F \in \mathbb{P F}_{\mathbb{A}} \mid a \in F\right\}$ is an embedding.

Given the previous correspondence between DIBI algebras and frames, we only need to show that $\theta$ is a monomorphism: the necessary argument is identical to that for similar bunched logics [14, Theorems $6.11,6.25]$. Given $\llbracket-\rrbracket$ on $\mathbb{A}$, the representation theorem establishes that $\mathcal{V}_{\llbracket-\rrbracket}(p):=\theta_{\mathbb{A}}(\llbracket p \rrbracket)$ is a persistent valuation on $\operatorname{Pr}(\mathbb{A})$ such that $F \not \models_{\llbracket-\mathbb{\rrbracket}} P$ iff $\llbracket P \rrbracket \in F$, from which our main theorem can be proved.

Theorem III. 6 (Soundness and Completeness). $P \vdash Q$ is derivable iff $P \vDash Q$.

## IV. Models of DIBI

In this section, we introduce two concrete models of DIBI to facilitate logical reasoning about (in)dependence in probability distributions and relational databases. In both models the operations $\odot$ and $\oplus$ will be deterministic partial functions; we write $h=f \bullet g$ instead of $\{h\}=f \bullet g$, for $\bullet \in\{\odot, \oplus\}$. We start with some preliminaries on memories and distributions.

## A. Memories, distributions, and Markov kernels

Operations on Memories: Let Val be a fixed set of values (e.g., the Booleans), $S$ be a set of variable names, and let $\operatorname{Mem}[S]$ denote the set of functions of type $m: S \rightarrow$ Val. We call such functions memories because we can think of $m$ as assigning a value to each variable in $S$; we will refer to $S$ as
the domain of $m$. The only element in Mem[ $[0]$ is the empty memory, which we write as $\rangle$.

We need two operations on memories. First, a memory $m$ with domain $S$ can be projected to a memory $m^{T}$ with domain $T$ if $T \subseteq S$, defined as $m^{T}(x)=m(x)$ for all $x \in T$. Second, two memories can be combined if they agree on the intersection of their domains: given memories $m_{1} \in \operatorname{Mem}[S], m_{2} \in \operatorname{Mem}[T]$ such that $m_{1}^{S \cap T}=m_{2}^{S \cap T}$, we define $m_{1} \otimes m_{2}: S \cup T \rightarrow$ Val by

$$
m_{1} \otimes m_{2}(x):= \begin{cases}m_{1}(x) & \text { if } x \in S  \tag{4}\\ m_{2}(x) & \text { if } x \in T\end{cases}
$$

Probability distributions and Markov kernels: We use the distribution monad to model distributions over memories. Given a set $X$, let $\mathcal{D}(X)$ denote the set of finite distributions over $X$, i.e., the set containing all finite support functions $\mu: X \rightarrow[0,1]$ satisfying $\sum_{x \in X} \mu(x)=1$. This operation on sets can be lifted to functions $f: X \rightarrow Y$, resulting in a map of distributions $\mathcal{D}(f): \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$ given by $\mathcal{D}(f)(\mu)(y):=\sum_{f(x)=y} \mu(x)$ (intuitively, $\mathcal{D}(f)$ takes the sum of the probabilities of all elements in the pre-image of $y$ ). These operations turn $\mathcal{D}$ into a functor on sets and, further, $\mathcal{D}$ is also a monad [20, 21].

Definition IV. 1 (Distribution Monad). Define unit: $X \rightarrow \mathcal{D}(X)$ as unit $_{X}(x):=\delta_{x}$ where $\delta_{x}$ denotes the Dirac distribution on $x$ : for any $y \in X$, we have $\delta_{x}(y)=1$ if $y=x$, otherwise $\delta_{x}(y)=0$. Further, define bind: $\mathcal{D}(X) \rightarrow(X \rightarrow \mathcal{D}(Y)) \rightarrow \mathcal{D}(Y)$ by $\operatorname{bind}(\mu)(f)(y):=\sum_{p \in \mathcal{D}(Y)} \mathcal{D}(f)(\mu)(p) \cdot p(y)$.

Intuitively, unit embeds a set into distributions over the set, and bind enables the sequential combination of probabilistic computations. Both maps are natural transformations and satisfy the following interaction laws, establishing that $\langle\mathcal{D}$, unit, bind〉 is a monad:

$$
\begin{align*}
& \quad \operatorname{bind}(\operatorname{unit}(x))(f)=f(x), \quad \operatorname{bind}(\mu)(\text { unit })=\mu \\
& \operatorname{bind}(\operatorname{bind}(\mu)(f))(g)=\operatorname{bind}(\mu)(\lambda x \cdot \operatorname{bind}(f(x))(g)) \tag{5}
\end{align*}
$$

The distribution monad has an equivalent presentation in which bind is replaced with a multiplication operation $\mathcal{D D}(X) \rightarrow \mathcal{D}(X)$, which flattens distributions by averaging.

The monad $\mathcal{D}$ gives rise to the Kleisli category of $\mathcal{D}$, denoted $\mathcal{K} \ell(\mathcal{D})$, with sets as objects and arrows of the form $f: X \rightarrow \mathcal{D}(Y)$, also known as Markov kernels [22]. Arrow composition in $\mathcal{K} \ell(\mathcal{D})$ is defined using bind: given $f: X \rightarrow$ $\mathcal{D}(Y), g: Y \rightarrow \mathcal{D}(Z)$, the composition $f \odot g: X \rightarrow \mathcal{D}(Z)$ is:

$$
\begin{equation*}
(f \odot g)(x):=\operatorname{bind}(f(x))(g) \tag{6}
\end{equation*}
$$

Markov kernels generalize distributions: we can lift a distribution $\mu: \mathcal{D}(X)$ to the kernel $f_{\mu}: 1 \rightarrow \mathcal{D}(X)$ assigning $\mu$ to the single element of 1 . Kernels can also encode conditional distributions, which play a key role in conditional independence.

Example IV.1. Consider the program $p$ in Figure 4a where $x, y$, and $z$ are Boolean variables. First, flip a fair coin and store the result in $z$. If $z=0$, flip a fair coin twice, and store the results in $x$ and $y$, respectively. If $z=1$, flip a coin with

$$
\begin{aligned}
& z \stackrel{\mathbf{B}_{1 / 2}}{ } \text {; } \\
& \text { if } z \text { then } \\
& x \Leftarrow \mathbf{B}_{1 / 4} ; \\
& y \stackrel{\leftrightarrow}{\leftarrow} \mathbf{B}_{1 / 4} ;
\end{aligned}
$$

(a) Probabilistic program $p$

| $x$ | $y$ | $z$ | $\mu$ |
| :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | $1 / 8$ |
| 0 | 0 | 1 | $1 / 32$ |
| 1 | 0 | 0 | $1 / 8$ |
| 1 | 0 | 1 | $3 / 32$ |
| 0 | 1 | 0 | $1 / 8$ |
| 0 | 1 | 1 | $3 / 32$ |
| 1 | 1 | 0 | $1 / 8$ |
| 1 | 1 | 1 | $9 / 32$ |

(b) Distribution $\mu$ generated by $p$

| $x$ | $y$ | $\mu_{0}$ |
| :--- | :--- | :--- |
| 0 | 0 | $1 / 4$ |
| 1 | 0 | $1 / 4$ |
| 0 | 1 | $1 / 4$ |
| 1 | 1 | $1 / 4$ |

(c) $\mu$ conditioned on $z=0$

| $x$ | $y$ | $\mu_{1}$ |
| :---: | :---: | :---: |
| 0 | 0 | $1 / 16$ |
| 1 | 0 | $3 / 16$ |
| 0 | 1 | $3 / 16$ |
| 1 | 1 | $9 / 16$ |

(d) $\mu$ conditioned on $z=1$

Fig. 4: From probabilistic programs to kernels
bias $1 / 4$ twice, and store the results in $x$ and $y$. This program produces a distribution $\mu$, shown in Figure 4b,

If we condition $\mu$ on $z=0$, then the resulting distribution $\mu_{0}$ models two independent fair coin flips: $1 / 4$ probability for each possible pair of outcomes (Figure 4c). If we condition on $z=1$, however, then the distribution $\mu_{1}$ will be skewed-there will be a much higher probability that we observe $(1,1)$ than $(0,0)$, but $x$ and $y$ are still independent (Figure 4d).

To connect $\mu_{0}$ and $\mu_{1}$ to the original distribution $\mu$, we package $\mu_{0}$ and $\mu_{1}$ into a Markov kernel $k: \operatorname{Mem}[z] \rightarrow$ $\mathcal{D}(\operatorname{Mem}[\{x, y, z\}])$ given by $k(i)(d)=\mu_{i}\left(d^{\{x, y\}}\right)$. Then, the relation between the conditional and original distributions is $f_{\mu}=f_{\mu_{z}} \odot k$, where $\mu_{z}$ is the projection of $\mu$ on $\{z\}$.

Finite distributions of memories over $U$, denoted $\mathcal{D}(\operatorname{Mem}[U])$, will play a central role in our models. We will refer to maps $f: \operatorname{Mem}[S] \rightarrow \mathcal{D}(\operatorname{Mem}[U])$ as (Markov) kernels, and define $\operatorname{dom}(f)=S$ and range $(f)=U$.

We can marginalize/project kernels to a smaller range.
Definition IV. 2 (Marginalizing kernels). For a Markov kernel $f: \operatorname{Mem}[S] \rightarrow \mathcal{D}(\operatorname{Mem}[U])$ and $V \subseteq U$, the marginalization of $f$ by $V$ is the map $\pi_{V} f: \operatorname{Mem}[S] \rightarrow \mathcal{D}(\operatorname{Mem}[V])$ : $\left(\pi_{V} f\right)(d)(r):=\sum_{m \in \operatorname{Mem}[U \backslash V]} f(d)(r \otimes m)$ for $d \in \operatorname{Mem}[S], r \in$ $\operatorname{Mem}[V]$; undefined terms do not contribute to the sum.

We say a kernel $f: \operatorname{Mem}[S] \rightarrow \mathcal{D}(\operatorname{Mem}[U])$ preserves its input to its output if $S \subseteq U$ and $\pi_{S} f=$ unit $_{\text {Mem[S] }}$. Intuitively, such kernels are suitable for encoding conditional distributions: once a variable has been conditioned on, its value should not change. We can compose these kernels in two ways.
Definition IV. 3 (Composing Markov kernels on memories). Given $f: \operatorname{Mem}[S] \rightarrow \mathcal{D}(\operatorname{Mem}[T])$ and $g: \operatorname{Mem}[U] \rightarrow$ $\mathcal{D}(\operatorname{Mem}[V])$ that preserve their inputs, we define their parallel composition, whenever $S \cap U=T \cap V$, as the map $f \oplus g: \operatorname{Mem}[S \cup U] \rightarrow \mathcal{D}(\operatorname{Mem}[T \cup V])$ given by

$$
(f \oplus g)(d)(m):=f\left(d^{S}\right)\left(m^{T}\right) \cdot g\left(d^{U}\right)\left(m^{V}\right)
$$

If $T=U$, the sequential composition $f \odot g: \operatorname{Mem}[S] \rightarrow$ $\mathcal{D}(\operatorname{Mem}[V])$ is just Kleisli composition (Eq. (6)).

## B. A concrete probabilistic model of DIBI

We now have all the ingredients to define a first concrete model: states are Markov kernels that preserve their input; $\oplus$
(resp. $\odot$ ) will be parallel (resp. sequential) composition. The use of $\oplus$ to model independence generalizes the approach in Barthe et al. [9]. Combining both compositions-sequential and parallel-enables capturing conditional independence.

Definition IV. 4 (Probabilistic frame). We define the frame ( $M^{D}, \sqsubseteq, \oplus, \odot, M^{D}$ ) as follows:

- Let $M^{D}$ consist of Markov kernels that preserve their input to their output;
- $\oplus, \odot$ are parallel and sequential composition of kernels;
- Given $f, g \in M^{D}, f \sqsubseteq g$ if there exist $R \subseteq$ Val, $h \in M^{D}$ such that $g=\left(f \oplus\right.$ unit $\left._{\text {Mem }[R]}\right) \odot h$.

We make two remarks. First, $f \sqsubseteq g$ holds when $g$ can be obtained from extending $f$ : compose $f$ in parallel with unit $_{\text {Mem }[R]}$, then extend the range via composition with $h$. We can recover $f$ from $g$ by marginalizing $g$ to range $(f) \cup R$, then ignoring the $R$ portion. Second, the definition of $f \odot g$ on $M^{D}$ can be simplified. Given $f: \operatorname{Mem}[S] \rightarrow \mathcal{D}(\operatorname{Mem}[T])$ and $g: \operatorname{Mem}[T] \rightarrow \mathcal{D}(\operatorname{Mem}[V])$, Eq. (6) yields the formula:

$$
(f \odot g)(d)(m):=\sum_{m^{\prime} \in \operatorname{Mem}[T]} f(d)\left(m^{\prime}\right) \cdot g\left(m^{\prime}\right)(m) .
$$

Since $f, g \in M^{D}$ preserve input to output, this reduces to

$$
\begin{equation*}
(f \odot g)(d)(m)=f(d)\left(m^{T}\right) \cdot g\left(m^{T}\right)\left(m^{V}\right) \tag{7}
\end{equation*}
$$

We show that our probabilistic frame is indeed a DIBI frame.

Theorem IV.1. $\left(M^{D}, \sqsubseteq, \oplus, \odot, M^{D}\right)$ is a DIBI frame.
Proof sketch. First, we show that $M^{D}$ is closed under $\oplus$ and $\odot$, and $\sqsubseteq$ is transitive and reflexive. The frame axioms are mostly straightforward, but some conditions rely on a property of our model we call Exchange Equality: if both $\left(f_{1} \oplus f_{2}\right) \odot\left(f_{3} \oplus f_{4}\right)$ and $\left(f_{1} \odot f_{3}\right) \oplus\left(f_{2} \odot f_{4}\right)$ are defined, then they are equal, and if the second is defined, then so is the first. For example:
( $\oplus$ Unit Coherence): The unit set in this frame is the entire state space $M^{D}$ : we must show that for any $f_{1}, f_{2} \in M^{D}$, if $f_{1} \oplus f_{2}$ is defined, then $f_{1} \sqsubseteq f_{1} \oplus f_{2}$ :

$$
\begin{aligned}
f_{1} \oplus f_{2} & =\left(f_{1} \odot \text { unit }_{\text {range }}\left(f_{1}\right)\right) \oplus\left(\text { unit }_{\operatorname{dom}\left(f_{2}\right)} \odot f_{2}\right) \\
& =\left(f_{1} \oplus \text { unit }_{\operatorname{dom}\left(f_{2}\right)}\right) \odot\left(\text { unit }_{\text {range }\left(f_{1}\right)} \oplus f_{2}\right) \quad \text { (Exch. Eq.) } \\
& =\left(f_{1} \oplus \text { unit }_{\operatorname{dom}\left(f_{2}\right)}\right) \odot\left(f_{2} \oplus \text { unit }_{\text {range }\left(f_{1}\right)}\right) \quad(\oplus \text { Comm. })
\end{aligned}
$$

We present the complete proof in Appendix $\mathbf{C}$
Example IV. 2 (Kernel decomposition). Recall the distribution $\mu$ on $\operatorname{Mem}[\{x, y, z\}]$ from Example IV. 1 Let $k_{x}: \operatorname{Mem}[z] \rightarrow$ $\mathcal{D}(\operatorname{Mem}[\{x, z\}])$ encode the conditional distribution of $x$ given $z$, and let $k_{y}: \operatorname{Mem}[z] \rightarrow \mathcal{D}(\operatorname{Mem}[\{y, z\}])$ encode the conditional distribution of $y$ given $z$. Explicitly, for $v=x$ or $y$,

$$
\begin{array}{ll}
k_{v}(z=0)(v=1, z=0)=1 / 2 & k_{v}(z=0)(v=0, z=0)=1 / 2 \\
k_{v}(z=1)(v=1, z=1)=1 / 4 & k_{v}(z=1)(v=0, z=1)=3 / 4
\end{array}
$$

Since $k_{x}, k_{y}$ include $z$ in their range, $k_{x} \oplus k_{y}$ is defined. A small calculation shows that $k_{x} \oplus k_{y}=k$, where $k: \operatorname{Mem}[z] \rightarrow$ $\mathcal{D}(\operatorname{Mem}[\{x, y, z\}])$ is the conditional distribution of $(x, y, z)$ given $z$. This decomposition shows that $x$ and $y$ are independent conditioned on $z$ (we shall formally prove this later in Section $V-A$.

## C. Relations, join dependency, and powerset kernels

We developed the probabilistic model in the previous section using operations from the distribution monad $\mathcal{D}$. Instantiating our definitions with operations from other monads gives rise to other interesting models of DIBI. In this section, we develop a relational model based on the powerset monad $\mathcal{P}$, and show how our logic can be used to reason about join dependency properties of tables from database theory. Before we present our relational model, we introduce some notations and basic definitions on relations.

Tables are often viewed as relations-sets of tuples where each component of the tuple corresponds to an attribute. Formally, a relation $R$ over a set of attributes $S$ is a set of tuples indexed by $S$. Each tuple maps an attribute in $S$ to a value in Val, and hence can be seen as a memory in $\operatorname{Mem}[S]$, as defined in Section IV-A. The projection and $\otimes$ operations on Mem[S] from Equation (4) can be lifted to relations.

Definition IV. 5 (Projection and Join). The projection of a relation $R$ over attributes $X$ to $Y \subseteq X$ is given by $R^{Y}:=\left\{r^{Y} \mid r \in R\right\}$. The natural join of relations $R_{1}$ and $R_{2}$ over attributes $X_{1}$ and $X_{2}$, respectively, is the relation $R_{1} \bowtie R_{2}:=\left\{m_{1} \otimes m_{2} \mid m_{1} \in\right.$ $R_{1}$ and $\left.m_{2} \in R_{2}\right\}$ over attributes $X_{1} \cup X_{2}$.

Since tables can often be very large, finding compact representations for them is useful. These representations can leverage additional structure common in real-world databases; for instance, the value of one attribute might determine the value of another, a so-called functional dependency. Other dependency structures can enable a large relation to be factored as a combination of smaller ones. A classical example is on join dependency, a relational analogue of conditional independence.

Definition IV. 6 (Join dependency [23, 24]). A relation $R$ over attribute set $X_{1} \cup X_{2}$ satisfies the join dependency $X_{1} \bowtie X_{2}$ if $R=\left(R^{X_{1}}\right) \bowtie\left(R^{X_{2}}\right)$.
Example IV. 3 (Decomposition). Consider the relation $R$ in Figure 5, with three attributes: Researcher, Field, and Conference. $R$ contains triple $(a, b, c)$ if and only if researcher $a$
works in field $b$ and attends conference $c$. If we know that researchers in the same field all have a shared set of conferences they attend, then we can recover $R$ by joining two relations: one associating researchers to their fields, and another associating fields to conferences. As shown below, $R$ satisfies the join dependency \{Researcher, Field\} $\bowtie\{$ Conference, Field\}. While the factored form is only a bit smaller ( 12 entries instead of 15 ), savings can be significant for larger relations.

Powerset monad and kernels: Much like how we decomposed distributions as Markov kernels-Kleisli arrows for the distribution monad-we will decompose relations using Kleisli arrows for the powerset monad, $\mathcal{K} \ell(\mathcal{P})$.

Definition IV. 7 (Powerset monad). Let $\mathcal{P}$ be the endofunctor Set $\rightarrow$ Set mapping every set to the set of its subsets $\mathcal{P}(X)=$ $\{U \mid U \subseteq X\}$. We define unit ${ }_{X}: X \rightarrow \mathcal{P}(X)$ mapping each $x \in X$ to the singleton $\{x\}$, and bind: $\mathcal{P}(X) \rightarrow(X \rightarrow \mathcal{P}(Y)) \rightarrow \mathcal{P}(Y)$ by $\operatorname{bind}(U)(f):=\cup\{y \mid \exists x \in U . f(x)=y\}$.

The triple $\langle\mathcal{P}$, unit, bind $\rangle$ forms a monad, and obeys the laws in Equation (5). We overload the use of unit and bind as it will be clear from the context which monad, powerset or distribution, we are considering. The Kleisli category $\mathcal{K} \ell(\mathcal{P})$ is defined analogously as for $\mathcal{D}$, with sets as objects and arrows $X \rightarrow \mathcal{P}(Y)$, and composition given as in Equation (6).

Like before, we consider maps $\operatorname{Mem}[S] \rightarrow \mathcal{P}(\operatorname{Mem}[T])$, which we call powerset kernels in analogy to Markov kernels, or simply kernels when the monad is clear from the context. Powerset kernels can also be projected to a smaller range.
Definition IV. 8 (Marginalization). Suppose that $T \subseteq U$. A map $f$ of type $\operatorname{Mem}[S] \rightarrow \mathcal{P}(\operatorname{Mem}[U])$ can be marginalized to $\pi_{T} f: \operatorname{Mem}[S] \rightarrow \mathcal{P}(\operatorname{Mem}[T])$ by defining: $\left(\pi_{T} f\right)(s):=f(s)^{T}$

We need two composition operations on powerset kernels. We say that powerset kernel $f: \operatorname{Mem}[S] \rightarrow \mathcal{P}(\operatorname{Mem}[S \cup T])$ preserves input to output if $\pi_{S} f=$ unit $_{\text {Mem }[S]}$.
Definition IV. 9 (Composition of powerset kernels). Given kernels $f: \operatorname{Mem}[S] \rightarrow \mathcal{P}(\operatorname{Mem}[T])$ and $g: \operatorname{Mem}[U] \rightarrow$ $\mathcal{P}(\operatorname{Mem}[V])$ that preserve input to output, we define their parallel composition whenever $T \cap V=S \cap U$ as the map $f \oplus g: \operatorname{Mem}[S \cup U] \rightarrow \mathcal{P}(\operatorname{Mem}[T \cup V])$ given by $(f \oplus g)(d):=$ $f\left(d^{S}\right) \bowtie g\left(d^{U}\right)$. Whenever $T=U$ we define the sequential composition $f \odot g: \operatorname{Mem}[S] \rightarrow \mathcal{P}(\operatorname{Mem}[V])$ using Kleisli composition. Explicitly: $(f \odot g)(s)=\{v \mid u \in f(s)$ and $v \in g(u)\}$.

## D. A concrete relational model of DIBI

We can now define the second concrete model of DIBI: states will be powerset kernels, and we will use the parallel and sequential composition in a construction similar to $M^{D}$.

Definition IV. 10 (Relational frame). We define the frame $\left(M^{P}, \sqsubseteq, \oplus, \odot, M^{P}\right)$ as follows:

- $M^{P}$ consists of powerset kernels preserving input to output;
- $\oplus, \odot$ are parallel and sequential composition of powerset kernels;
- Given $f, g \in M^{P}, f \sqsubseteq g$ if there exist $R \subseteq$ Val, $h \in M^{P}$ such that $g=\left(f \oplus\right.$ unit $\left._{\text {Mem }[R]}\right) \odot h$.
$\underbrace{\left(\begin{array}{lll}\text { Researcher } & \text { Field } & \text { Conference } \\ \text { Alice } & \text { Theory } & \text { LICS } \\ \text { Alice } & \text { Theory } & \text { ICALP } \\ \text { Bob } & \text { Theory } & \text { LICS } \\ \text { Bob } & \text { Theory } & \text { ICALP } \\ \text { Alice } & \text { DB } & \text { PODS }\end{array}\right)}_{R}=\underbrace{\left(\begin{array}{lll}\text { Field } & \text { Conference } \\ \text { Theory } & \text { LICS } \\ \text { Theory } & \text { ICALP } \\ \text { DB } & \text { PODS }\end{array}\right)}_{R_{1}} \bowtie \underbrace{\left(\begin{array}{ll}\text { Field } & \text { Researcher } \\ \text { Theory } & \text { Alice } \\ \text { Theory } & \text { Bob } \\ \text { DB } & \text { Alice }\end{array}\right.}_{R_{2}})$

Fig. 5: Factoring a relation

Like in $M^{D}, f \sqsubseteq g$ iff $g$ can be obtained from $f$ by adding attributes that are preserved from domain to range, and then mapping tuples in the range to relations over a larger set of attributes. We can recover $f$ from $g$ by marginalizing to range $(f) \cup R$, and then ignoring the attributes in $R$.
$M^{P}$ is also a DIBI frame.
Theorem IV.2. $\left(M^{P}, \sqsubseteq, \oplus, \odot, M^{P}\right)$ is a DIBI frame.
Proof sketch.. The proof follows Theorem IV. 1 quite closely, since $M^{P}$ also satisfies Exchange equality. We present the full proof in Appendix D

## V. Application: Modeling Conditional and Join Dependencies

In our concrete models, distributions and relations can be factored into simpler parts. Here, we show how DIBI formulas capture conditional independence and join dependency.

## A. Conditional independence

Conditional independence (CI) is a well-studied notion in probability theory and statistics [25]. While there are many interpretations of CI, a natural reading is in terms of irrelevance: $X$ and $Y$ are independent conditioned on $Z$ if knowing the value of $Z$ renders $X$ irrelevant to $Y$-observing one gives no further information about the other.

Before defining CI, we introduce some notations. Let $\mu \in$ $\mathcal{D}(\operatorname{Mem}[\mathrm{Var}])$ be a distribution. For any subset $S \subseteq \operatorname{Var}$ and assignment $s \in \operatorname{Mem}[S]$, we write:

$$
\mu(S=s):=\sum_{m \in \operatorname{Mem}[\operatorname{Var}]} \mu(s \otimes m)
$$

Terms with undefined $s \otimes m$ contribute zero to the sum. We can now define conditional probabilities:

$$
\mu\left(S=s \mid S^{\prime}=s^{\prime}\right):=\frac{\mu\left(S=s, S^{\prime}=s^{\prime}\right)}{\mu\left(S^{\prime}=s^{\prime}\right)}
$$

where $\mu\left(S=s, S^{\prime}=s^{\prime}\right):=\mu\left(S \cup S^{\prime}=s \otimes s^{\prime}\right)$. Intuitively, this ratio is the probability of $S=s$ given $S^{\prime}=s^{\prime}$, and it is only defined when the denominator is non-zero and $s, s^{\prime}$ are consistent (i.e., $s \otimes s^{\prime}$ is defined). CI can be defined as follows.

Definition V. 1 (Conditional independence). Let $X, Y, Z \subseteq$ Var. $X$ and $Y$ are independent conditioned on $Z$, written $X \Perp Y \mid Z$, if for all $x \in \operatorname{Mem}[X], y \in \operatorname{Mem}[Y]$, and $z \in \operatorname{Mem}[Z]$ :

$$
\mu(X=x \mid Z=z) \cdot \mu(Y=y \mid Z=z)=\mu(X=x, Y=y \mid Z=z)
$$

When $Z=\emptyset$, we say $X$ and $Y$ are independent, written $X \Perp Y$.
Example V.1. We give two simple examples of CI.

Chocolate and Nobel laureates: Researchers found a strong positive correlation between a nation's per capita Nobel laureates number and chocolate consumption. But the correlation may be due to other factors, e.g., a nation's economic status. A simple check is to see if the two are conditionally independent fixing the third factor.

Algorithmic fairness: To prevent algorithms from discriminating based on sensitive features (e.g., race and gender), researchers formalized notions of fairness using conditional independence [8]. For instance, let $A$ be the sensitive features, $Y$ be the target label, and $\widehat{Y}$ be the algorithm's prediction for $Y$. Considering the joint distribution of $(A, Y, \widehat{Y})$, an algorithm satisfies equalized odds if $\widehat{Y} \Perp A \mid Y$; calibration if $Y \Perp A \mid \widehat{Y}$.

We will define a DIBI formula $P$ such that a distribution $\mu$ satisfies $X \Perp Y \mid Z$ if and only if its lifted kernel $f_{\mu}:=\langle \rangle \mapsto f$ satisfies $P$. For this, we will need a basic atomic proposition which describes the domain and range of kernels.

Definition V. 2 (Basic atomic proposition). For sets of variables $A, B \subseteq$ Var, a basic atomic proposition has the form $(A \triangleright[B])$. We give the following semantics to these formulas:

$$
\begin{aligned}
& f \vDash(A \triangleright[B]) \text { iff there exists } f^{\prime} \sqsubseteq f \\
& \quad \text { such that } \operatorname{dom}\left(f^{\prime}\right)=A \text { and } \operatorname{range}\left(f^{\prime}\right) \supseteq B .
\end{aligned}
$$

For example, $f: \operatorname{Mem}[y] \rightarrow \mathcal{D}(\operatorname{Mem}[y, z])$ defined by $f(y \mapsto v):=\operatorname{unit}(y \mapsto v, z \mapsto v)$ satisfies $(y \triangleright[y])$, $(y \triangleright[z]),(y \triangleright[\emptyset]),(y \triangleright[y, z]),(\emptyset \triangleright[\emptyset])$, and no other atomic propositions.
Theorem V.1. Given distribution $\mu \in \mathcal{D}(\mathbf{M e m}[\mathrm{Var}])$, then for any $X, Y, Z \subseteq \operatorname{Var}$,

$$
\begin{equation*}
f_{\mu} \vDash(\emptyset \triangleright[Z]) \stackrel{(Z \triangleright[X]) *(Z \triangleright[Y])}{ } \tag{8}
\end{equation*}
$$

if and only if $X \Perp Y \mid Z$ and $X \cap Y \subseteq Z$ are both satisfied.
The restriction $X \cap Y \subseteq Z$ is harmless: when $X \Perp Y \mid Z$ but $X \cap Y \nsubseteq Z, X \cap Y$ must be deterministic given $Z$ (see Theorem A.12), and it suffices to check $X \Perp Y \mid Z \cup(X \cap Y)$. For simplicity, we abbreviate the formula $(\emptyset \triangleright[Z]) \circ((Z \triangleright[X]) *$ $(Z \triangleright[Y])$ ) as $[Z] \circ([X] *[Y])$.

Proof sketch. For the forward direction, suppose $f_{\mu}$ satisfies 8 . Then by Theorem A. 38 there exist $f, g$, and $h$ in $M^{D}$ with $f \odot$ $(g \oplus h) \sqsubseteq f_{\mu}$, where $f: \operatorname{Mem}[\emptyset] \rightarrow \mathcal{D}(\operatorname{Mem}[Z]), g: \operatorname{Mem}[Z] \rightarrow$ $\mathcal{D}(\operatorname{Mem}[Z \cup X])$, and $h: \operatorname{Mem}[Z] \rightarrow \mathcal{D}(\operatorname{Mem}[Z \cup Y])$; we also
have $X \cap Y \subseteq Z$ as $f \odot(g \oplus h)$ is defined. Since $\operatorname{dom}\left(f_{\mu}\right)=$ $\operatorname{Mem}[\emptyset], f \odot(g \oplus h) \sqsubseteq f_{\mu}$ implies:

$$
f \odot(g \oplus h)=\pi_{Z \cup X \cup Y} f_{\mu} \quad \text { and } \quad f=\pi_{Z} f_{\mu}
$$

Further, we can show that $f \odot(g \oplus h)=f \odot g \odot\left(\right.$ unit $\left._{X} \oplus h\right)=$ $f \odot h \odot\left(\right.$ unit $\left._{Y} \oplus g\right)$, and thus:

$$
f \odot g=\pi_{Z \cup X} f_{\mu} \quad \text { and } \quad f \odot h=\pi_{Z \cup Y} f_{\mu}
$$

These imply that $g$ ( $h$ resp.) encodes the conditional distributions of $X$ ( $Y$ resp.) given $Z$, and $g \oplus h$ encodes the conditional distribution of $(X, Y)$ given $Z$. Hence, the conditional distribution of ( $X, Y$ ) given $Z$ is equal to the product distribution of $X$ given $Z$ and $Y$ given $Z$, and so $X \Perp Y \mid Z$ holds in $\mu$.

For the reverse direction, suppose that (a) $X \Perp Y \mid Z$ holds in $\mu$ and (b) $X \cap Y \subseteq Z$. Now, consider $\pi_{X \cup Y \cup Z} f_{\mu}$, the marginal distribution on $(X, Y, Z)$ encoded as a kernel, and observe that $\pi_{X, Y, Z} f_{\mu}=f \odot f^{\prime}$, where $f$ encodes the marginal distribution of $Z$, and $f^{\prime}$ is the conditional distribution of $(X, Y)$ given values of $Z$. From (a), the conditional distribution of $(X, Y)$ given $Z$ is the product of the conditional distributions of $X$ given $Z$, and $Y$ given $Z$, that is $f^{\prime}=g \oplus h$, where $g$ (resp. $h$ ) encode the conditional distribution of $X$ (resp. $Y$ ) given $Z$. Then by (b), $f \odot(g \oplus h)$ is defined and $f \odot(g \oplus h)=\pi_{X \cup Y \cup Z} f_{\mu} \sqsubseteq f_{\mu}$. It is straightforward to see that $f \odot(g \oplus h)$ satisfies $[Z] \circ([X] *[Y])$. Hence, persistence shows that $f_{\mu}$ also satisfies $[Z] \stackrel{\circ}{\circ}([X] *[Y])$.

See Theorem A. 11 for details.

## B. Join dependency

Recall that a relation $R$ over attributes $X \cup Y$ satisfies the Join Dependency (JD) $X \bowtie Y$ if $R=R^{X} \bowtie R^{Y}$. As we illustrated through the Researcher-Field-Conference example in Section IV join dependencies can enable a relation to be represented more compactly. By interpreting the atomic propositions in the relational model, JD is captured by the same formula we used for CI.

Theorem V.2. Let $R \in \mathcal{P}(\operatorname{Mem}[\mathrm{Var}])$ and $X, Y$ be sets of attributes such that $X \cup Y=$ Var. The lifted relation $f_{R}=\langle \rangle \mapsto$ $R$ satisfies $f_{R} \vDash[X \cap Y] \stackrel{( }{ }([X] *[Y])$ iff $R$ satisfies the join dependency $X \bowtie Y$.

JD is a special case of Embedded Multivalued Dependency (EMVD), where the relation $R$ may have more attributes than $X \cup Y$. It is straightforward to encode EMVD in our logic, but for simplicity we stick with JD.

Proof sketch. For the forward direction, by Theorem A.38, there exist $f, g$, and $h \in M^{P}$ such that $f: \operatorname{Mem}[\emptyset] \rightarrow$ $\mathcal{P}(\operatorname{Mem}[X \cap Y]), g: \operatorname{Mem}[X \cap Y] \rightarrow \mathcal{P}(\operatorname{Mem}[X]), h: \operatorname{Mem}[X \cap$ $Y] \rightarrow \mathcal{P}(\operatorname{Mem}[Y])$, and $f \odot(g \oplus h) \sqsubseteq f_{R}$. Since by assumption $X \cup Y=$ Var, we must have $f \odot(g \oplus h)=f_{R}$.

Unfolding $\oplus$ and $\odot$ and using the fact that range $(f)=$ $\boldsymbol{\operatorname { d o m }}(g)=\boldsymbol{\operatorname { d o m }}(h)$, we can show:
$f \odot(g \oplus h)\left(\rangle)=\left\{u \bowtie\left(v_{1} \bowtie v_{2}\right) \mid u \in f(\langle \rangle), v_{1} \in g(u), v_{2} \in h(u)\right\}\right.$.

Since $\bowtie$ is commutative, associative and idempotent, we have:

$$
\begin{aligned}
f \odot(g \oplus h)(\rangle) & =\left\{\left(u \bowtie v_{1}\right) \bowtie\left(u \bowtie v_{2}\right) \mid u \in f(\langle \rangle), v_{1} \in g(u), v_{2} \in h(u)\right\} \\
& =f \odot g(\langle \rangle) \bowtie f \odot h(\langle \rangle) .
\end{aligned}
$$

We can also convert the parallel composition of $g, h$ into sequential composition by padding to make the respective domain and range match: $f \odot(g \oplus h)=f \odot g \odot\left(\right.$ unit $\left._{X} \oplus h\right)=$ $f \odot h \odot\left(\right.$ unit $\left._{Y} \oplus g\right)$. Hence $f \odot g=\pi_{X} f_{R}$ and $f \odot h=\pi_{Y} f_{R}$, which implies $f \odot g\left(\rangle)=R^{X}\right.$ and $f \odot h\left(\rangle)=R^{Y}\right.$. Thus:

$$
R=f \odot(g \oplus h)(\langle \rangle)=f \odot g(\langle \rangle) \bowtie f \odot h(\langle \rangle)=R^{X} \bowtie R^{Y},
$$

so $R$ satisfies the join dependency $X \bowtie Y$. The reverse direction is analogous to Theorem V.1. See Theorem A. 14 for details.

## C. Proving and validating the semi-graphoid axioms

Conditional independence and join dependency are closely related in our models. Indeed, there is a long line of research on generalizing these properties to other independence-like notions, and identifying suitable axioms. Graphoids are perhaps the most well-known approach [10]; Dawid [26] has a similar notion called separoids.
Definition V. 3 (Graphoids and semi-graphoids). Suppose that $I(X, Z, Y)$ is a ternary relation on subsets of $\operatorname{Var}$ (i.e., $X, Z, Y \subseteq$ Var). Then $I$ is a graphoid if it satisfies:

$$
\begin{array}{lr}
I(X, Z, Y) \Leftrightarrow I(Y, Z, X) & \text { (Symmetry) } \\
I(X, Z, Y \cup W) \Rightarrow I(X, Z, Y) \wedge I(X, Z, W) & \text { (Decomposition) } \\
I(X, Z, Y \cup W) \Rightarrow I(X, Z \cup W, Y) & \text { (Weak Union) } \\
I(X, Z, Y) \wedge I(X, Z \cup Y, W) \Leftrightarrow I(X, Z, Y \cup W) & \text { (Contraction) } \\
I(X, Z \cup W, Y) \wedge I(X, Z \cup Y, W) \Rightarrow I(X, Z, Y \cup W) & \text { (Intersection) }
\end{array}
$$

If $I$ satisfies the first four properties, then it is a semi-graphoid.
Intuitively, $I(X, Z, Y)$ states that knowing $Z$ renders $X$ irrelevant to $Y$. If we fix a distribution over $\mu \in \mathcal{D}(\mathbf{M e m}[$ Var]), then taking $I(X, Z, Y)$ to be the set of triples such that $X \Perp Y \mid Z$ holds (in $\mu$ ) defines a semi-graphoid. Likewise, if we fix a relation $R \in \mathcal{P}(\mathbf{M e m}[\mathrm{Var}])$, then the triples of sets of attributes such that $R$ satisfies an Embedded Multivalue Dependency (EMVD) forms a semi-graphoid [23, 27].

Previously, we showed that the DIBI formula $[Z] \stackrel{ }{\circ}([X] *[Y])$ asserts conditional independence of $X$ and $Y$ given $Z$ in $M^{D}$, and join dependency $X \bowtie Y$ in $M^{P}$ when $Z=X \cap Y$. Here, we show that the semi-graphoid axioms can be naturally translated into valid formulas in our concrete models.

Theorem V.3. Given a model $M$, define $I(X, Z, Y)$ iff $M \vDash[Z]$ g ([X] * [Y]). Then, Symmetry Decomposition Weak Union and Contraction are valid when $M$ is the probabilistic or the relational model. Furthermore, SYMMETRY is derivable in the proof system, and DECOMPOSITION is derivable given the following axiom, valid in both models:

$$
(Z \triangleright[Y \cup W]) \leftrightarrow(Z \triangleright[Y]) \wedge(Z \triangleright[W])
$$

(Split)
Proof sketch. We comment on the derivable axioms. To derive SyMMETRY we use the $*$-Сомm proof rule to commute the
separating conjunction. The proof of Decomposition uses the axiom Split to split up $Y \cup W$, and then uses proof rules $\wedge 3$ and $\wedge 4$ to prove the two conjuncts. We show derivations (Theorems A. 15 and A.16) and prove validity (Theorems A. 17 and A.18) in Appendix G.

## VI. Application: Conditional Probabilistic Separation Logic

As our final application, we design a separation logic for probabilistic programs. We work with a simplified probabilistic imperative language with assignments, sampling, sequencing, and conditionals; our goal is to show how a DIBI-based program logic could work in the simplest setting. For lack of space, we only show a few proof rules and example programs here; we defer the full presentation of the separation logic, the metatheory, and the examples to Appendix H

Proof rules: CPSL includes novel proof rules for randomized conditionals and inherits the frame rule from PSL [9]. Here, we show two of the rules and explain how to use them in the simple program from Eq. (1), reproduced here:

$$
\text { Simple }:=x \stackrel{\leftrightarrow}{\leftarrow} \mathbf{B}_{1 / 2} ; y \hookleftarrow \mathbf{B}_{1 / 2} ; z \leftarrow x \vee y
$$

CPSL has Hoare-style rules for sampling and assignments:

$$
\begin{aligned}
& \operatorname{SAMP} \frac{x \notin \mathrm{FV}(d) \cup \mathrm{FV}(P)}{\vdash\{P\} x \stackrel{S}{ }(P\{P \circ(\mathrm{FV}(d) \triangleright[x])\}} \\
& \text { Assn } \frac{x \notin \mathrm{FV}(e) \cup \mathrm{FV}(P)}{\vdash\{P\} x \leftarrow e\{P ;(\mathrm{FV}(e) \triangleright[x])\}}
\end{aligned}
$$

Using SAMP and the fact that the coin-flip distribution $\mathbf{B}_{1 / 2}$ has no free variables, we can infer:

$$
\vdash\{T\} x \stackrel{\leftrightarrow}{s} \mathbf{B}_{1 / 2}\{(\emptyset \triangleright[x])\} \quad \vdash\{T\} y \stackrel{\left(\mathbf{B}_{1 / 2}\{(\emptyset \triangleright[y])\}, ~\right.}{ }
$$

Applying a variant of the frame rule, we are able to derive:

$$
\vdash\{T\} x \longleftarrow \mathbf{B}_{1 / 2} ; y \hookleftarrow \mathbf{B}_{1 / 2}\{(\emptyset \triangleright[x]) *(\emptyset \triangleright[y])\}
$$

Using Assn on $P=(\emptyset \triangleright[x]) *(\emptyset \triangleright[y])$ and the fact that $z$ is not a free variable in either $P$ or $x \vee y$ :

$$
\vdash\{P\} z \leftarrow x \vee y\{P \circ(\{x, y\} \triangleright[z])\}
$$

Putting it all together, we get the validity of triple:

$$
\vdash\{丁\} \text { Simple }\{((\emptyset \triangleright[x]) *(\emptyset \triangleright[y])) \stackrel{(\{x, y\} \triangleright[z])\}}{ }
$$

stating that $z$ depends on $x$ and $y$, which are independent.
Example programs: Figure 6 introduces two example programs. CommonCause (Figure 6a) models a distribution where two random observations share a common cause. Specifically, we consider $z, x$, and $y$ to be independent random samples, and $a$ and $b$ to be values computed from ( $x, z$ ) and $(y, z)$, respectively. Intuitively, $z, x, y$ could represent independent noisy measurements, while $a$ and $b$ could represent quantities derived from these measurements. Since $a$ and $b$ share a common source of randomness $z$, they are not independent. However, $a$ and $b$ are independent conditioned on the value of $z$-this is a textbook example of conditional

| $\begin{aligned} & z \stackrel{\&}{s} \mathbf{B}_{1 / 2} ; \\ & x \leftarrow \mathbf{B}_{1 / 2} ; \\ & y \leftarrow \mathbf{B}_{122} ; \\ & a \leftarrow x \vee z ; \\ & b \leftarrow y \vee z \end{aligned}$ |
| :---: |
|  |  |
|  |  |
|  |  |

(a) CommonCause

$$
\begin{aligned}
& z \hookleftarrow \mathbf{B}_{1 / 2} ; \\
& \text { if } z \text { then } \\
& \quad x \hookleftarrow \mathbf{B}_{p} ; y \hookleftarrow \mathbf{B}_{p} \\
& \text { else } \quad x \hookleftarrow \mathbf{B}_{q} ; y \hookleftarrow \mathbf{B}_{q}
\end{aligned}
$$

(b) CondSamples

Fig. 6: Example programs
independence. Our program logic can establish the following judgment capturing this fact:

$$
\vdash\{T\} \text { CommonCause }\{(\emptyset \triangleright[z]) \stackrel{\circ}{9}((z \triangleright[a]) *(z \triangleright[b]))\}
$$

The program CondSamples (Figure 6b) is a bit more complex: it branches on a random value $z$, and then assigns $x$ and $y$ with two independent samples from $\mathbf{B}_{p}$ in the true branch, and $\mathbf{B}_{q}$ in the false branch. While we might think that $x$ and $y$ are independent at the end of the program since they are independent at the end of each branch, this is not true because their distributions are different in the two branches. For example, suppose that $p=1$ and $q=0$. Then at the end of the first branch $(x, y)=(t t, t t)$ with probability 1 , while at the end of the second branch $(x, y)=(f f, f f)$ with probability 1. Thus observing whether $x=t t$ or $x=f f$ determines the value of $y$-clearly, $x$ and $y$ can't be independent. However, $x$ and $y$ are independent conditioned on $z$. Using our program logic's proof rules for conditionals, we are able to prove the following judgment capturing this fact:

$$
\vdash\{T\} \text { CondSamples }\{(\emptyset \triangleright[z]) \stackrel{q}{q}((z \triangleright[x]) *(z \triangleright[y]))\}
$$

The full development of the separation logic, consisting of a proof system, a soundness theorem, along with the detailed verification of the two examples above, can be found in Appendix H

## VII. Related Work

Bunched implications and other non-classical logics: DIBI extends the logic of bunched implications (BI) [11], and shares many similarities: DIBI can be given a Kripkestyle resource semantics, just like BI, and our completeness proof relies on a general framework for proving completeness for bunched logics [14]. The non-commutative conjunction and exchange rules are inspired by the logic CKBI [14]. The main difference is that our exchange rule is reversed, due to our reading of separating conjunction $*$ as "can be combined independently", rather than "interleaved". In terms of models, the probabilistic model of DIBI can be seen as a natural extension of the probabilistic model for BI [9]-by lifting distributions to kernels, DIBI is able to reason about dependencies, while probabilistic BI is not.

There are other non-classical logics that aim to model dependencies. Independence-friendly (IF) logic [28] and dependence logic [29] introduce new quantifiers and propositional atoms to state that a variable depends, or does not depend, on another variable; these logics are each equivalent in expressivity to existential second-order logic. More recently,

Durand et al. [30] proposed a probabilistic team semantics for dependence logic, and Hannula et al. [31] gave a descriptive complexity result connecting this logic to real-valued Turing machines. Under probabilistic team semantics, the universal and existential quantifiers bear a resemblance to our separating and dependent conjunctions, respectively. It would be interesting to understand the relation between these two logics, akin to how the semantics of propositional IF forms a model of BI [32]

Conditional independence, join dependency, and logic: There is a long line of research on logical characterizations of conditional independence and join dependency. The literature is too vast to survey here. On the CI side, we can point to work by Geiger and Pearl [33] on graphical models; on the JD side, the survey by Fagin and Vardi [34] describes the history of the area in database theory. There are several broadly similar approaches to axiomatizing the general properties of conditional dependence, including graphoids [10] and separoids [26].

Categorical probability: The view of conditional independence as a factorization of Markov kernels has previously been explored [35, 36, 37]. Taking a different approach, Simpson [38] has recently introduced category-theoretic structures for modeling conditional independence, capturing CI and JD as well as analogues in heaps and nominal sets [39]. Roughly speaking, conditional independence in heaps requires two disjoint portions except for a common overlap contained in the part that is conditioned; this notion can be smoothly accommodated in our framework as a DIBI model where kernels are Kleisli arrows for the identity monad ([40]) also consider a similar notion of separation). Simpson's notion of conditional independence in nominal sets suggests that there might be a DIBI model where kernels are Kleisli arrows for some monad in nominal sets, although the appropriate monad is unclear.

Program logics: Bunched logics are well-known for their role in separation logics, program logics for reasoning about heap-manipulating [12] and concurrent programs [41, 42]. Recently, separation logics have been developed for probabilistic programs. Our work is most related to PSL [9], where separation models probabilistic independence. Batz et al. [43] gives a different, quantitative interpretation to separation in their logic QSL, and uses it to verify expected-value properties of probabilistic heap-manipulating programs. Finally, there are more traditional program logics for probabilistic program. The Ellora logic by Barthe et al. [44] has assertions for modeling independence, but works with a classical logic. As a result, basic structural properties of independence must be introduced as axioms, rather than being built-in to the logical connectives.

## VIII. Discussion and Future Directions

We have presented DIBI, a new bunched logic to reason about dependence and independence, together with its Kripke semantics and a sound and complete proof system. We provided two concrete models, based on Markov and powerset kernels, that can capture conditional independencelike notions. We see several directions for further investigation.

Generalizing the two models: The probabilistic and relational models share many similarities: both $M^{D}$ and $M^{P}$ are sets of Kleisli arrows, and use Kleisli composition to interpret $\odot$; both $\oplus$ operators correspond to parallel composition. Since both the distribution and powerset monads are commutative strong monads [45, 46], which come with a double strength bi-functor $s t_{A, B}: T(A) \times T(B) \rightarrow T(A \times B)$ that seems suitable for defining $\oplus$, it is natural to consider more general models based on Kleisli arrows for such monads. Indeed, variants of conditional independence could make sense in other settings; taking the multiset monad instead of the powerset monad would lead to a model where we can assert join dependency in bags, rather than relations, and the free vector space monad could be connected to subspace models of the graphoid axioms [47].

However, it is not easy to define an operation generalizing $\oplus$ from our concrete models. The obvious choice-taking $\oplus$ as $f_{1} \oplus f_{2}=\left(f_{1} \otimes f_{2}\right)$; st-gives a total operation, but in our concrete models $\oplus$ is partial, since it is not possible to compose two arrows that disagree on their domain overlap. For instance in the probabilistic model, there is no sensible way to use $\oplus$ to combine a kernel encoding the normal distribution $\mathcal{N}(0,1)$ on $x$ with another encoding the Dirac distribution of $x=1$. We do not know how to model such coherence requirements between two Kleisli arrows in a general categorical model, and we leave this investigation to future work.

Restriction and intuitionistic DIBI: A challenge in designing the program logic is ensuring that formulas in the assertion logic satisfy restriction (see Appendix (J), and one may wonder if a classical version of DIBI would be more suitable for the program logic-if assertions were not required to be preserved under kernel extensions, it might be easier to show that they satisfy restriction. However, a classical logic would require assertions to specify the dependence structure of all variables, which can be quite complicated. Moreover, intuitionistic logics like probabilistic BI can also satisfy the restriction property, so the relevant design choice is not classical versus intuitionistic.

Rather, the more important point appears to be whether the preorder can extend a kernel's domain. If this is allowedas in DIBI-then kernels satisfying an assertion may have extraneous variables in the domain. However, this choice also makes the dependent conjunction $P \% Q$ more flexible: $Q$ does not need to exactly describe the domain of the second kernel, which is useful since the range of the first kernel cannot be constrained by $P$. This underlying tension-allowing the range to be extended, while restricting the domain-is an interesting subject for future investigation.

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## Appendix

## A. Section IIT. omitted proof

Lemma A.1. $P * Q \vdash P \stackrel{\circ}{q} Q$
Proof. For better readability, we break the proof tree down into two components.

With $P * Q \vdash(P * I) \stackrel{q}{(I * Q) \text {, we construct the following }}$

This proof uses the admissible rule Cut, which can be derived as follows:

$$
\frac{\frac{Q \vdash R}{\frac{P_{\wedge} Q \vdash R}{P \vdash Q \rightarrow R}} \wedge 2}{} \rightarrow \quad P \vdash Q(\mathrm{MP}
$$

## B. Section III-C Soundness and Completeness: Omitted Details

Theorem III.2. $P \vdash Q$ is derivable iff $\llbracket P \rrbracket \leq \llbracket Q \rrbracket$ for all algebraic interpretations $\llbracket-\rrbracket$.
Proof. Soundness can be established by a straightforward induction on the proof rules. For completeness, we can define a Lindenbaum-Tarski algebra by quotienting Form DIBI by the equivalence relation $P \equiv Q$ iff $P \vdash Q$ and $Q \vdash P$ derivable. This yields a DIBI algebra, and moreover, $[P]_{\equiv} \leq[Q]_{\equiv}$ iff $[P \rightarrow Q]_{\equiv}=[\top]_{\equiv}$ iff $\top \vdash P \rightarrow Q$ derivable iff $P \vdash Q$ derivable. Hence for any $P, Q$ such that $P \vdash Q$ is not derivable, in the Lindenbaum-Tarski algebra (with the canonical interpretation sending formulas to their equivalence class) $[P]_{\equiv} \nsubseteq[Q]_{\equiv}$ holds, establishing completeness.

A filter on a bounded distributive lattice $\mathbb{A}$ is a non-empty set $F \subseteq A$ such that, for all $x, y \in A$, (1) $x \in F$ and $x \leq y$ implies $y \in F$; and (2) $x, y \in F$ implies $x \wedge y \in F$. It is a proper filter if it additionally satisfies (3) $\perp \notin F$, and a prime filter if in addition it also satisfies (4) $x \vee y \in F$ implies $x \in F$ or $y \in F$. The order dual version of these definitions gives the notions of ideal, proper ideal and prime ideal. We denote the sets of proper and prime filters of $\mathbb{A}$ by $\mathbb{F}_{\mathbb{A}}$ and $\mathbb{P F}_{\mathbb{A}}$ respectively, and the sets of proper and prime ideals of $\mathbb{A}$ by $\mathbb{I}_{\mathbb{A}}$ and $\mathbb{P}_{\mathbb{A}}$ respectively.

To prove that prime filter frames are DIBI frames we require an auxiliary lemma that can be used to establish the existence of prime filters. First some terminology: a $\subseteq$-chain is a sequence of sets $\left(X_{\alpha}\right)_{\alpha<\lambda}$ such that $\alpha \leq \alpha^{\prime}$ implies $X_{\alpha} \subseteq X_{\alpha^{\prime}}$. A basic fact about proper filters (ideals) is that the union of a $\subseteq$-chain of proper filters (ideals) is itself a proper filter (ideal). We lift the terminology to $n$-tuples of sets by determining $\left(X_{\alpha}^{1}, \ldots, X_{\alpha}^{n}\right)_{\alpha<\lambda}$ to be a $\subseteq$-chain if each $\left(X_{\alpha}^{i}\right)_{\alpha<\lambda}$ is a $\subseteq$-chain.

Definition A. 1 (Prime Predicate). A prime predicate is a map $P: \mathbb{F}_{\mathbb{A}}^{n} \times \mathbb{I}_{\mathbb{A}}^{m} \rightarrow\{0,1\}$, where $n, m \geq 0$ and $n+m \geq 1$, such that
a) Given a $\subseteq$-chain $\left(F_{\alpha}^{0}, \ldots, F_{\alpha}^{n}, I_{\alpha}^{0}, \ldots, I_{\alpha}^{m}\right)_{\alpha<\lambda}$ of proper filters/ideals,

$$
\min \left\{P\left(F_{\alpha}^{0}, \ldots, I_{\alpha}^{m}\right) \mid \alpha<\lambda\right\} \leq P\left(\bigcup_{\alpha} F_{\alpha}^{0}, \ldots, \bigcup_{\alpha} I_{\alpha}^{m}\right)
$$

b) $P\left(\ldots, H_{0} \cap H_{1}, \ldots\right) \leq \max \left\{P\left(\ldots, H_{0}, \ldots\right), P\left(\ldots, H_{1}, \ldots\right)\right\}$.

Intuitively, a prime predicate is a property of proper filter/ideal sequences whose truth value is inherited by unions of chains, and is witnessed by one of $H_{0}$ or $H_{1}$ whenever witnessed by $H_{0} \cap H_{1}$. The proof of the next lemma can be found in [14].
Lemma $\mathbf{A} .2$ (Prime Extension Lemma [14, Lemma 5.7]). If $P$ is an $(n+m)$-ary prime predicate and $F_{0}, \ldots, F_{n}, I_{0}, \ldots, I_{m}$ an $(n+m)$-tuple of proper filters and ideals such that $P\left(F_{0}, \ldots, F_{n}, I_{0}, \ldots, I_{m}\right)=1$ then there exists $a(n+m)$-tuple of prime filters and ideals $F_{0}^{p r}, \ldots, F_{n}^{p r}, I_{0}^{p r}, \ldots I_{m}^{p r}$ such that $P\left(F_{0}^{p r}, \ldots, F_{n}^{p r}, I_{0}^{p r}, \ldots I_{m}^{p r}\right)=1$.

Now, whenever prime filters are required that satisfy a particular property (for example, an existentially quantified consequent of a frame axiom), it is sufficient to show that the property defines a prime predicate and there exists proper filters satisfying it. We also note the following useful properties of DIBI algebras, which are special cases of those found in [14, Proposition 6.2].

Lemma A.3. Given any DIBI algebra $\mathbb{A}$, for all $a, b, c \in A$ and $\circ \in\left\{*,{ }_{9}^{\circ}\right\}$, the following properties hold:

$$
\begin{array}{ll}
\quad(a \vee b) \circ c=(a \circ c) \vee(b \circ c) & a \circ(b \vee c)=(a \circ b) \vee(a \circ c) \\
a \leq a^{\prime} \text { and } b \leq b^{\prime} \text { implies } a \circ b \leq a^{\prime} \circ b^{\prime} & \perp \circ a=\perp=a \circ \perp
\end{array}
$$

Proposition A.4. For any DIBI algebra $\mathbb{A}$, the prime filter frame $\operatorname{Pr}(\mathbb{A})$ is a DIBI frame.
Proof. All but one of the frame axioms can be verified in an identical fashion to the analogous proof for BI [14, Lemma 6.24], and $\oplus_{\mathbb{A}}$ and $\odot_{\mathbb{A}}$ are both Up-Closed and Down-Closed. We focus on the novel frame axiom: Reverse Exchange. For readability we omit the $\mathbb{A}$ subscripts on operators. Assume there are prime filters such that $F_{x} \supseteq F_{x}^{\prime} \in F_{y} \oplus F_{z}, F_{y} \in F_{y_{1}} \odot F_{y_{2}}$ and $F_{z} \in F_{z_{1}} \odot F_{z_{2}}$. We will prove that

$$
P(F, G)= \begin{cases}1 & \text { if } F_{x} \in F \odot G \text { and } F \in F_{y_{1}} \oplus F_{z_{1}} \text { and } G \in F_{y_{2}} \oplus F_{z_{2}} \\ 0 & \text { otherwise }\end{cases}
$$

is a prime predicate, abusing notation to allow $\odot$ and $\oplus$ to be defined for non-prime filters.
For a), suppose $\left(F_{\alpha}, G_{\alpha}\right)_{\alpha \leq \lambda}$ is a $\subseteq$-chain such that for all $\alpha, P\left(F_{\alpha}, G_{\alpha}\right)=1$. Call $F=\bigcup_{\alpha} F_{\alpha}$ and $G=\bigcup_{\alpha} G_{\alpha}$. We must show that $P(F, G)=1$. Let $a \in F, b \in G$. Then $a \in F_{\alpha}, b \in G_{\beta}$ for some $\alpha, \beta$. Wolog, we may assume $\alpha \leq \beta$. Then since $F_{x} \in F_{\beta} \odot G_{\beta}$, we have that $a ; b \in F_{x}$ as required, so $F_{x} \in F \odot G . F \in F_{y_{1}} \oplus F_{z_{1}}$ and $G \in F_{y_{2}} \oplus F_{z_{2}}$ hold trivially.

For b), suppose for contradiction that $P\left(F \cap F^{\prime}, G\right)=1, P(F, G)=0$ and $P\left(F^{\prime}, G\right)=0$. From $P\left(F \cap F^{\prime}, G\right)=1$ we know $F, F^{\prime} \in F_{y_{1}} \oplus F_{y_{2}}$ : for all $a \in F_{y_{1}}, b \in F_{y_{2}}, a * b \in F \cap F^{\prime} \subseteq F, F^{\prime}$. So the only way this can be the case is if $F_{x} \notin F \odot G$ and $F_{x} \notin F^{\prime} \odot G$. Hence there exists $a \in F, b \in G$ such that $a \circ b \notin F_{x}$, and $a^{\prime} \in F^{\prime}, b^{\prime} \in G$ such that $a^{\prime}$; $b^{\prime} \notin F_{x}$. It follows by properties of filters that $a \vee a^{\prime} \in F \cap F^{\prime}$ and $b^{\prime \prime}=b \wedge b^{\prime} \in G$. Hence $\left(a \vee a^{\prime}\right) * b^{\prime \prime} \in F_{x}$ by assumption, and $\left(a \vee a^{\prime}\right) * b^{\prime \prime}=\left(a * b^{\prime \prime}\right) \vee\left(a^{\prime} * b^{\prime \prime}\right)$. Since $F_{x}$ is prime, this means either $a * b^{\prime \prime} \in F_{x}$ or $a^{\prime} * b^{\prime \prime} \in F_{x}$. But that's not possible: $a * b^{\prime \prime} \leq a * b$ and $a^{\prime} * b^{\prime \prime} \leq a^{\prime} * b^{\prime}$, so whichever holds results in a contradiction. Hence either $P(F, G)=1$ or $P\left(F^{\prime}, G\right)=1$ as required. The argument for the second component is similar.

Now consider $F=\left\{c \mid \exists a \in F_{y_{1}}, b \in F_{z_{1}}(c \geq a * b)\right\}$ and $G=\left\{c \mid \exists a \in F_{y_{2}}, b \in F_{z_{2}}(c \geq a * b)\right\}$. These are both proper filters. Focusing on $F$ (both arguments are essentially identical), it is clearly upwards-closed. Further, it is closed under $\wedge$ : if $c, c^{\prime} \in F$ because $c \geq a * b$ and $c^{\prime} \geq a^{\prime} * b^{\prime}$ for $a, a^{\prime} \in F_{y_{1}}$ and $b, b^{\prime} \in F_{z_{1}}$ then $c \wedge c^{\prime} \geq(a * b) \wedge\left(a^{\prime} * b^{\prime}\right) \geq\left(a \wedge a^{\prime}\right) *\left(b \wedge b^{\prime}\right)$, with $a \wedge a^{\prime} \in F_{y_{1}}$ and $b \wedge b^{\prime} \in F_{z_{1}}$. It is proper, because if $\perp \in F$, then there exists $a \in F_{y_{1}}$ and $b \in F_{z_{1}}$ such that $a * b=\perp$. Let $c \in F_{y_{2}}$ and $d \in F_{z_{2}}$ be arbitrary. Then by our initial assumption, $a \circ c \in F_{y}$ and $b \circ d \in F z$. Hence $(a \circ c) *(b \circ d) \in F_{x^{\prime}} \subseteq F_{x}$. However, by the Reverse Exchange algebraic axiom, $(a \circ c) *(b ; d) \leq(a * b) \circ(c * d)=\perp \circ(c * d)=\perp$. By upwards-closure, $\perp \in F_{x}$, which is supposed to be a prime, and therefore proper, filter, which gives a contradiction.

By definition, $F \in F_{y_{1}} \oplus F_{z_{1}}$ and $G \in F_{y_{2}} \oplus F_{z_{2}}$. To see that $F_{x} \in F \odot G$, let $c \in F$ (with $c \geq a * b$ for some $a \in F_{y_{1}}$ and $b \in F_{z_{1}}$ ) and $c^{\prime} \in G$ (with $c^{\prime} \geq a^{\prime} * b^{\prime}$ for some $a^{\prime} \in F_{y_{2}}$ and $b \in F_{z_{2}}$ ). By assumption $a ; a^{\prime} \in F_{y}$ and $b ; b^{\prime} \in F_{z}$, and so $\left(a \circ a^{\prime}\right) *\left(b \circ b^{\prime}\right) \in F_{x^{\prime}} \subseteq F_{x}$. By the algebraic Reverse Exchange axiom, we obtain $(a * b) \%\left(a^{\prime} * b^{\prime}\right) \in F_{x}$, and by monotonicity of $;$ and upwards-closure of $F_{x}$ we obtain $c ; c^{\prime} \in F_{x}$. Hence $P(F, G)=1$ and by Lemma A. 2 there are prime $F$, $G$ with $P(F, G)=1$. This verifies that the Reverse Exchange frame axiom holds.

Proposition A.5. For any DIBI frame $\mathcal{X}$, the complex algebra $\operatorname{Com}(\mathcal{X})$ is a DIBI algebra.
Proof. We focus on the Reverse Exchange algebraic axiom (the other DIBI algebra properties can be proven in identical fashion to the analogous proof for BI [14, Lemma 6.22]). Suppose $x \in(A \triangleright B) \bullet(C \triangleright D)$. Then there exists $x^{\prime}, y, z$ such that $x \sqsupseteq x^{\prime} \in y \oplus z$, with $y \in A \triangleright B$ and $z \in C \triangleright D$. In turn, there thus exists $y_{1}, y_{2}, z_{1}, z_{2}$ such that $y \in y_{1} \odot y_{2}$ and $z \in z_{1} \odot z_{2}$ with $y_{1} \in A, y_{2} \in B, z_{1} \in C$ and $z_{2} \in D$. By the Reverse Exchange frame axiom, there exist $u, v$ such that $u \in y_{1} \oplus z_{1}, v \in y_{2} \oplus z_{2}$ and $x^{\prime} \in u \odot v$. Hence $u \in A \bullet C, v \in B \bullet D$ and $x^{\prime} \in(A \bullet C) \triangleright(B \bullet D)$. Since $x^{\prime} \sqsubseteq x$ and $(A \bullet C) \triangleright(B \bullet D)$ is an upwards-closed set, $x \in(A \bullet C) \triangleright(B \bullet D)$ as required.

Now clearly every persistent valuation $\mathcal{V}$ on a Kripke frame $\mathcal{X}$ generates an algebraic interpretation $\llbracket-\rrbracket_{\mathcal{V}}$ on $\operatorname{Com}(\mathcal{X})$ with the property that $x \models_{\mathcal{V}} P$ iff $x \in \llbracket P \rrbracket$ (note that the complex algebra operations are defined precisely as the corresponding semantic clauses). Similarly, by the Representation Theorem, given an algebraic interpretation $\llbracket-\rrbracket$ on $\mathbb{A}$, a persistent valuation $\mathcal{V}_{\llbracket-\rrbracket}$ on $\operatorname{Pr}(\mathbb{A})$ can be defined by $\mathcal{V}_{\llbracket-\rrbracket}(p)=\left\{F \in \mathbb{P F}_{\mathbb{A}} \mid \llbracket p \rrbracket \in F\right\}=\theta_{\mathbb{A}}(\llbracket p \rrbracket)$. That $\theta$ is a monomorphism into $\operatorname{Com}(\operatorname{Pr}(\mathbb{A}))$ establishes that, for all $P \in$ Form $_{\text {Dibi }}, F \vDash_{\mathcal{V}_{\llbracket-\rrbracket}} P$ iff $\llbracket P \rrbracket \in F$.
Theorem III. 6 (Soundness and Completeness). $P \vdash Q$ is derivable iff $P \vDash Q$.
Proof. Assume $P \not \vDash Q$. Then there exists a DIBI model $(\mathcal{X}, \mathcal{V})$ and a state $x \in X$ such that $x \vDash P$ but $x \not \vDash Q$. Hence $\llbracket P \rrbracket_{\mathcal{V}} \nsubseteq \llbracket Q \rrbracket_{\mathcal{V}}$ in $\operatorname{Com}(\mathcal{X})$, so, by Theorem【II.2 $P \vdash Q$ is not derivable. Now assume $P \vdash Q$ is not derivable. By Theorem III. 2 there exists a DIBI algebra $\mathbb{A}$ and an interpretation $\llbracket-\rrbracket$ such that $\llbracket P \rrbracket \nsubseteq \llbracket Q \rrbracket$. From this it can be established that there is a prime filter $F$ on $\mathbb{A}$ such that $\llbracket P \rrbracket \in F$ and $\llbracket Q \rrbracket \notin F$. Hence $F \vDash_{\mathcal{V}_{\llbracket-\mathbb{1}}} P$ but $F \not{\forall \mathcal{V}_{\llbracket-\rrbracket}} Q$, so $P \not \vDash Q$.
C. Section IV-B probabilistic model: omitted proofs
a) Remark: In the following, we sometimes abbreviate $\operatorname{dom}\left(f_{i}\right)$ as $D_{i}$ and range $\left(f_{i}\right)$ as $R_{i}$.

In the proof of Theorem IV.1 we use that $M^{D}$ is closed under $\oplus$ and $\odot$, which we prove next.
Lemma A.6. $M^{D}$ is closed under $\oplus$ and $\odot$.
Proof. For any $f_{1}, f_{2} \in M^{D}$, we need to show that

- If $f_{1} \oplus f_{2}$ is defined, then $f_{1} \oplus f_{2} \in M^{D}$. Recall that $f_{1} \oplus f_{2}$ is defined if and only if $R_{1} \cap R_{2}=D_{1} \cap D_{2}$, which implies that $\left(R_{1} \cup R_{2}\right) \backslash\left(D_{1} \cup D_{1}\right)=\left(R_{1} \backslash D_{1}\right) \cup\left(R_{2} \backslash D_{2}\right)$, and $\left(R_{1} \backslash D_{1}\right) \cap\left(R_{2} \backslash D_{2}\right)=\emptyset$.
So we can split any memory assignment on $\left(R_{1} \cup R_{2}\right) \backslash\left(D_{1} \cup D_{2}\right)$ into two disjoint parts, one on $R_{1} \backslash D_{1}$, another on $R_{2} \backslash D_{2}$. State $f_{1} \oplus f_{2}$ preserves the input because for any $d \in \operatorname{Mem}\left[D_{1} \cup D_{2}\right]$, we can obtain $(\star)$ :

$$
\begin{aligned}
& \left(\pi_{D_{1} \cup D_{2}}\left(f_{1} \oplus f_{2}\right)\right)(d)(d) \\
& =\sum_{x}\left(f_{1} \oplus f_{2}\right)(d)(d \bowtie x) \quad\left(x \in \operatorname{Mem}\left[\left(R_{1} \cup R_{2}\right) \backslash\left(D_{1} \cup D_{2}\right)\right]\right) \\
& \stackrel{\ddagger}{=} \sum_{x_{1}, x_{2}} f_{1}\left(d^{D_{1}}\right)\left(d^{D_{1}} \bowtie x_{1}\right) \cdot f_{2}\left(d^{D_{2}}\right)\left(d^{D_{2}} \bowtie x_{2}\right) \quad\left(x_{1} \in \operatorname{Mem}\left[R_{1} \backslash D_{1}\right], x_{2} \in \operatorname{Mem}\left[R_{2} \backslash D_{2}\right]\right) \\
& =\left(\sum_{x_{1} \in \operatorname{Mem}\left[R_{1} \backslash D_{1}\right]} f_{1}\left(d^{D_{1}}\right)\left(d^{D_{1}} \bowtie x_{1}\right)\right) \cdot\left(\sum_{x_{2} \in \operatorname{Mem}\left[R_{2} \backslash D_{2}\right]} f_{2}\left(d^{D_{2}}\right)\left(d^{D_{2}} \bowtie x_{2}\right)\right) \\
& =1 \cdot 1=1 \quad \quad\left(\operatorname{Using} f_{1}, f_{2} \in M^{D}\right)
\end{aligned}
$$

Step $\dagger$ follows using $\left(R_{1} \cup R_{2}\right) \backslash\left(D_{1} \cup D_{1}\right)=\left(R_{1} \backslash D_{1}\right) \cup\left(R_{2} \backslash D_{2}\right)$ and $\left(R_{1} \backslash D_{1}\right) \cap\left(R_{2} \backslash D_{2}\right)=\emptyset$. Then, for any $d \in \operatorname{Mem}\left[D_{1} \cup D_{2}\right]$, $\left(f_{1} \oplus f_{2}\right)(d)$ is a distribution since:

$$
\begin{aligned}
& \sum_{m \in \operatorname{Mem}\left[R_{1} \cup R_{2}\right]}\left(f_{1} \oplus f_{2}\right)(d)(m) \\
& =\sum_{m \in \operatorname{Mem}\left[R_{1} \cup R_{2}\right]} f_{1}\left(d^{D_{1}}\right)\left(m^{R_{1}}\right) \cdot f_{2}\left(d^{D_{2}}\right)\left(m^{R_{2}}\right) \\
& \stackrel{\ddagger}{=} \sum_{x_{1}, x_{2}} f_{1}\left(d^{D_{1}}\right)\left(d^{D_{1}} \bowtie x_{1}\right) \cdot f_{2}\left(d^{D_{2}}\right)\left(d^{D_{2}} \bowtie x_{2}\right. \\
& =1
\end{aligned}
$$

$$
\stackrel{\ddagger}{=} \sum_{x_{1}, x_{2}} f_{1}\left(d^{D_{1}}\right)\left(d^{D_{1}} \bowtie x_{1}\right) \cdot f_{2}\left(d^{D_{2}}\right)\left(d^{D_{2}} \bowtie x_{2}\right) \quad\left(x_{1} \in \operatorname{Mem}\left[R_{1} \backslash D_{1}\right], x_{2} \in \operatorname{Mem}\left[R_{2} \backslash D_{2}\right]\right)
$$

(Using ( $\star$ ) )
Step $\ddagger$ follows using $\left(R_{1} \backslash D_{1}\right) \cap\left(R_{2} \backslash D_{2}\right)=\emptyset$, and the $f_{i}$ term is 0 when $d^{D_{i}} \neq m^{D_{i}}$.
Thus, $f_{1} \oplus f_{2}$ is a kernel in $M^{D}$.

- If $f_{1} \odot f_{2}$ is defined, then $f_{1} \odot f_{2} \in M^{D}$. Recall that $f_{1} \odot f_{2}: \operatorname{Mem}\left[D_{1}\right] \rightarrow \mathcal{D}\left(\operatorname{Mem}\left[R_{2}\right]\right)$ is defined iff $R_{1}=D_{2} . f_{1} \odot f_{2}$ preserves the input because for any $d \in \operatorname{Mem}\left[D_{1}\right]$, we can obtain ( $\uparrow$ )

$$
\begin{aligned}
& \left(\pi_{D_{1}} f_{1} \odot f_{2}\right)(d)(d) \\
& =\sum_{x \in \operatorname{Mem}\left[R_{2} \backslash D_{1}\right]}\left(f_{1} \odot f_{2}\right)(d)(d \bowtie x) \\
& =\sum_{x \in \operatorname{Mem}\left[R_{2} \backslash D_{1}\right]} f_{1}(d)\left(d \bowtie x^{R_{1} \backslash D_{1}}\right) \cdot f_{2}\left(d \bowtie x^{R_{1} \backslash D_{1}}\right)(d \bowtie x) \\
& =\sum_{x_{1}} f_{1}(d)\left(d \bowtie x_{1}\right) \cdot\left(\sum_{x_{2}} f_{2}\left(d \bowtie x_{1}\right)\left(d \bowtie x_{1} \bowtie x_{2}\right)\right) \quad\left(x_{1} \in \operatorname{Mem}\left[R_{1} \backslash D_{1}\right], x_{2} \in \operatorname{Mem}\left[R_{2} \backslash R_{1}\right]\right) \\
& =\sum_{x_{1} \in \operatorname{Mem}\left[R_{1} \backslash D_{1}\right]}\left(f_{1}(d)\left(d \bowtie x_{1}\right) \cdot 1\right) \quad \quad\left(\operatorname{Using} f_{2} \in M^{D}\right) \\
& =1
\end{aligned}
$$

Then, for any $d \in D_{1},\left(f_{1} \odot f_{2}\right)(d)$ is a distribution as

$$
\begin{align*}
\sum_{m \in R_{2}}\left(f_{1} \odot f_{2}\right)(d)(m) & =\sum_{m \in R_{2}} f_{1}(d)\left(m^{R_{1}}\right) \cdot f_{2}\left(m^{R_{1}}\right)(m)  \tag{7}\\
& \stackrel{\ominus}{=} \sum_{x \in R_{2} \backslash D_{1}} f_{1}(d)\left(d \bowtie x^{R_{1} \backslash D_{1}}\right) \cdot f_{2}\left(d \bowtie x^{R_{1} \backslash D_{1}}\right)(d \bowtie x) \\
& =1
\end{align*}
$$

(Using ( $\uparrow$ ))

Step $\triangleright$ follows since the $f_{i}$ term is 0 when $d^{D_{i}} \neq m^{D_{i}}$.
Thus $f_{1} \odot f_{2}$ is a kernel in $M^{D}$.

Lemma A.7. The probabilistic model $M^{D}$ is a $\mathcal{T}$-model defined in Definition A.9 for $\mathcal{T}=\mathcal{D}$.
Proof. $M^{D}$ satisfies condition (1)-(4) and (10) by construction, so we only prove (5)-(9).
(5) We show that when $(f \oplus g) \oplus h$ and $f \oplus(g \oplus h)$ are defined, $(f \oplus g) \oplus h=f \oplus(g \oplus h)$. Consider $f: \operatorname{Mem}[S] \rightarrow$ $\mathcal{D}(\operatorname{Mem}[S \cup T]), g: \operatorname{Mem}[U] \rightarrow \mathcal{D}(\operatorname{Mem}[U \cup V])$, and $h: \operatorname{Mem}[W] \rightarrow \mathcal{D}(\operatorname{Mem}[W \cup X])$. For any $d \in \operatorname{Mem}[S \cup U \cup W]$, and $m \in \operatorname{Mem}[S \cup T \cup U \cup V \cup W \cup X]$,

$$
\begin{align*}
((f \oplus g) \oplus h)(d)(m) & =\left(f\left(d^{S}\right)\left(m^{S \cup T}\right) \cdot g\left(d^{U}\right)\left(m^{U \cup V}\right)\right) \cdot h\left(d^{W}\right)\left(m^{W \cup X}\right) \\
& =f\left(d^{S}\right)\left(m^{S \cup T}\right) \cdot\left(g\left(d^{U}\right)\left(m^{U \cup V}\right) \cdot h\left(d^{W}\right)\left(m^{W \cup X}\right)\right) \\
& =(f \oplus(g \oplus h))(d)(m)
\end{align*}
$$

(6) When $f_{1} \oplus f_{2}$ and $f_{2} \oplus f_{1}$ defined, $f_{1} \oplus f_{2}=f_{2} \oplus f_{1}$.

For any $d \in \operatorname{Mem}\left[D_{1} \cup D_{2}\right], m \in \mathcal{D}\left(\operatorname{Mem}\left[R_{1} \cup R_{2}\right]\right)$ such that $d \bowtie m$ is defined,

$$
\left(f_{1} \oplus f_{2}\right)(d)(m):=f_{1}\left(d^{D_{1}}\right)\left(m^{R_{1}}\right) \cdot f_{2}\left(d^{D_{2}}\right)\left(m^{R_{2}}\right)=f_{2}\left(d^{D_{2}}\right)\left(m^{R_{2}}\right) \cdot f_{1}\left(d^{D_{1}}\right)\left(m^{R_{1}}\right)=\left(f_{2} \oplus f_{1}\right)(d)(m)
$$

Thus, $f_{1} \oplus f_{2}=f_{2} \oplus f_{1}$.
(7) For any $f: \operatorname{Mem}[A] \rightarrow \mathcal{D}(\operatorname{Mem}[A \cup X]) \in M$, and any $S \subseteq A$, we must show

$$
f \oplus \text { unit }_{S}=f
$$

Since $S \subseteq A$, we have $\operatorname{dom}\left(f \oplus\right.$ unit $\left._{S}\right)=A \cup S=A=\operatorname{dom}(f)$ and $\operatorname{range}\left(f \oplus\right.$ unit $\left._{S}\right)=A \cup X \cup S=A \cup X=\operatorname{range}(f)$. For any $d \in \operatorname{Mem}[A]$, and any $r \in \operatorname{Mem}[A \cup X]$ such that $d \otimes r$ is defined, we have

$$
\begin{aligned}
\left(f \oplus \text { unit }_{S}\right)(d)(r) & =f(d)(r) \cdot \operatorname{unit}\left(d^{S}\right)\left(r^{S}\right) \\
& =f(d)(r) \cdot 1=f(d)(r)
\end{aligned}
$$

Hence, $f \oplus$ unit $_{s}=f$.
(8) We show that when both $\left(f_{1} \oplus f_{2}\right) \odot\left(f_{3} \oplus f_{4}\right)$ and $\left(f_{1} \odot f_{3}\right) \oplus\left(f_{2} \odot f_{4}\right)$ are defined, it hold that

$$
\left(f_{1} \oplus f_{2}\right) \odot\left(f_{3} \oplus f_{4}\right)=\left(f_{1} \odot f_{3}\right) \oplus\left(f_{2} \odot f_{4}\right)
$$

First note that the well-definedness of both terms we can conclude that $D_{1} \subseteq R_{1}=D_{3} \subseteq R_{3}, D_{2} \subseteq R_{2}=D_{4} \subseteq R_{4}$, where $D_{i}=\operatorname{dom}\left(f_{i}\right)$ and $R_{i}=\operatorname{range}\left(f_{i}\right)$. Moreover, both terms are of type $\operatorname{Mem}\left[D_{1} \cup D_{2}\right] \rightarrow \mathcal{D}\left(\operatorname{Mem}\left[R_{3} \cup R_{4}\right]\right)$, and, for any $d \in D_{1} \cup D_{2}$ and $m \in R_{3} \cup R_{4}$ :

$$
\begin{aligned}
\left(\left(f_{1} \oplus f_{2}\right) \odot\left(f_{3} \oplus f_{4}\right)\right)(d)(m) & =\left(f_{1} \oplus f_{2}\right)(d)\left(m^{R_{1} \cup R_{2}}\right) \cdot\left(f_{3} \oplus f_{4}\right)\left(m^{D_{3} \cup D_{4}}\right)(m) \\
& =\left(f_{1}\left(d^{D_{1}}\right)\left(m^{R_{1}}\right) \cdot f_{2}\left(d^{D_{2}}\right)\left(m^{R_{2}}\right)\right) \cdot\left(f_{3}\left(m^{D_{3}}\right)\left(m^{R_{3}}\right) \cdot f_{4}\left(m^{D_{4}}\right)\left(m^{R_{4}}\right)\right) \\
\left(\left(f_{1} \odot f_{3}\right) \oplus\left(f_{2} \odot f_{4}\right)\right)(d)(m) & =\left(f_{1} \odot f_{3}\right)\left(d^{D_{1}}\right)\left(m^{R_{3}}\right) \cdot\left(f_{2} \odot f_{4}\right)\left(d^{D_{2}}\right)\left(m^{R_{3}}\right) \\
& =\left(f_{1}\left(d^{D_{1}}\right)\left(m^{R_{1}}\right) \cdot f_{3}\left(d^{D_{3}}\right)\left(m^{R_{3}}\right)\right) \cdot\left(f_{2}\left(d^{D_{2}}\right)\left(m^{R_{2}}\right) \cdot f_{4}\left(d^{D_{4}}\right)\left(m^{R_{4}}\right)\right) \\
& =\left(f_{1}\left(d^{D_{1}}\right)\left(m^{R_{1}}\right) \cdot f_{2}\left(d^{D_{2}}\right)\left(m^{R_{2}}\right)\right) \cdot\left(f_{3}\left(m^{D_{3}}\right)\left(m^{R_{3}}\right) \cdot f_{4}\left(m^{D_{4}}\right)\left(m^{R_{4}}\right)\right)
\end{aligned}
$$

Thus, $\left(f_{1} \odot f_{3}\right) \oplus\left(f_{2} \odot f_{4}\right)=\left(f_{1} \oplus f_{2}\right) \odot\left(f_{3} \oplus f_{4}\right)$.
(9) Proved in Theorem 6

Theorem IV.1. $\left(M^{D}, \sqsubseteq, \oplus, \odot, M^{D}\right)$ is a DIBI frame.
Proof. By Theorem A. 37 that all $\mathcal{T}$-models are DIBI frames and by Theorem A. 7 that $M^{D}$ is a $\mathcal{T}$-model, $M^{D}$ is a DIBI frame.

## D. Section IV-D relational model: omitted proofs

For the proof of Theorem IV.2 we need the following closure property.
Lemma A.8. $M^{P}$ is closed under $\oplus$ and $\odot$.
Proof. For any $f_{1}, f_{2} \in M^{P}$, we need to show that :

- If $f_{1} \oplus f_{2}$ is defined, then $f_{1} \oplus f_{2} \in M^{P}$. Recall that $f_{1} \oplus f_{2}$ is defined if and only if $R_{1} \cap R_{2}=D_{1} \cap D_{2}$, which implies that

$$
\begin{aligned}
& \left(D_{1} \cup D_{2}\right) \cap R_{1}=\left(D_{1} \cap R_{1}\right) \cup\left(D_{2} \cap R_{1}\right)=D_{1} \cup\left(D_{2} \cap D_{1}\right)=D_{1} \\
& \left(D_{1} \cup D_{2}\right) \cap R_{2}=\left(D_{1} \cap R_{2}\right) \cup\left(D_{2} \cap R_{2}\right)=\left(D_{1} \cap D_{2}\right) \cup D_{2}=D_{2}
\end{aligned}
$$

We show $f_{1} \oplus f_{2}$ also preserves the input: for any $d \in \operatorname{Mem}\left[D_{1} \cup D_{2}\right]$,

$$
\begin{aligned}
\left(\pi_{D_{1} \cup D_{2}}\left(f_{1} \oplus f_{2}\right)\right)(d) & =\pi_{D_{1} \cup D_{2}}\left(\left(f_{1} \oplus f_{2}\right)(d)\right) \\
& =\pi_{D_{1} \cup D_{2}} f_{1}\left(d^{D_{1}}\right) \bowtie f_{2}\left(d^{D_{2}}\right) \\
& \stackrel{\ddagger}{=} \pi_{D_{1}} f_{1}\left(d^{D_{1}}\right) \bowtie \pi_{D_{1}} f_{2}\left(d^{D_{2}}\right) \\
& =\left\{d^{D_{1}}\right\} \bowtie\left\{d^{D_{2}}\right\} \\
& =\{d\} .
\end{aligned}
$$

$$
=\left\{d^{D_{1}}\right\} \bowtie\left\{d^{D_{2}}\right\} \quad \text { (Because } f_{1}, f_{2} \in M^{P} \text { ) }
$$

Step $\dagger$ follows because $\left(D_{1} \cup D_{2}\right) \cap R_{1}=D_{1}$ and $\left(D_{1} \cup D_{2}\right) \cap R_{2}=D_{2}$.

- If $f_{1} \odot f_{2}$ is defined, then $f_{1} \odot f_{2} \in M^{P}$. Recall $f_{1} \odot f_{2}$ is defined iff $R_{1}=D_{2}$, and gives a map of type $\operatorname{Mem}\left[D_{1}\right] \rightarrow \mathcal{D}\left(\operatorname{Mem}\left[R_{2}\right]\right)$. We show that $f_{1} \odot f_{2}$ preserves the input: for any $d \in \operatorname{Mem}\left[D_{1}\right]$,

$$
\begin{array}{rlr}
\left(\pi_{D_{1}} f_{1} \odot f_{2}\right)(d) & =\left(\pi_{D_{1}} f_{1}\right)(d) \quad \quad\left(\text { Because } D_{1} \subseteq R_{1}=D_{2}\right) \\
& =\operatorname{unit}_{D_{1}}(d)
\end{array}
$$

Thus, $\pi_{D_{1}} f_{1} \odot f_{2}=$ unit $_{D_{1}}$ and hence $f_{1} \odot f_{2}$ preserves the input.
Lemma A.9. The relational model $M^{P}$ is a $\mathcal{T}$-model Definition A.9 for the monad $\mathcal{T}=\mathcal{P}$.
Proof. $M^{P}$ satisfies conditions (1)-(4) and (10) by construction, so we only prove (5)-(9).
(5) We show that when both $(f \oplus g) \oplus h$ and $f \oplus(g \oplus h)$ are defined, $(f \oplus g) \oplus h=f \oplus(g \oplus h)$. Consider $f: \operatorname{Mem}[S] \rightarrow \mathcal{P}(\operatorname{Mem}[S \cup T])$, $g: \operatorname{Mem}[U] \rightarrow \mathcal{P}(\operatorname{Mem}[U \cup V])$, and $h: \operatorname{Mem}[W] \rightarrow \mathcal{P}(\operatorname{Mem}[W \cup X])$. For any $d \in \operatorname{Mem}[S \cup U \cup W]$,

$$
\begin{aligned}
((f \oplus g) \oplus h)(d) & =\left(f\left(d^{S}\right) \bowtie f_{2}\left(d^{U}\right)\right) \bowtie f_{3}\left(d^{V}\right) \\
& =f\left(d^{S}\right) \bowtie\left(g\left(d^{U}\right) \bowtie h\left(d^{V}\right)\right) \\
& =(f \oplus(g \oplus h))(d)
\end{aligned}
$$

$$
=f\left(d^{S}\right) \bowtie\left(g\left(d^{U}\right) \bowtie h\left(d^{V}\right)\right) \quad \quad(\text { By associativity of } \bowtie)
$$

(6) When both $f_{1} \oplus f_{2}$ and $f_{2} \oplus f_{1}$ are defined, they are equal.

Analogous to $M^{D}$, instead of followed from the commutativity of $\cdot$, it follows from the commutativity of $\bowtie$.
(7) For any $f: \operatorname{Mem}[A] \rightarrow \mathcal{P}(\operatorname{Mem}[A \cup X])$, and any $S \subseteq A$, we must show

$$
f \oplus \text { unit }_{S}=f
$$

Since $S \subseteq A$, so $\operatorname{dom}\left(f \oplus\right.$ unit $\left._{S}\right)=A \cup S=A=\operatorname{dom}(f)$, and $\operatorname{range}\left(f \oplus\right.$ unit $\left._{S}\right)=A \cup X \cup S=A \cup X=\operatorname{range}(f)$. For any $d \in \operatorname{Mem}[A]$, we have

$$
\left(f \oplus \operatorname{unit}_{S}\right)(d)=f(d) \bowtie \operatorname{unit}_{S}\left(d^{S}\right)=f(d) \bowtie\left\{d^{S}\right\}=f(d)
$$

Hence, $f \oplus$ unit $_{s}=f$.
(8) We show that when both $\left(f_{1} \oplus f_{2}\right) \odot\left(f_{3} \oplus f_{4}\right)$ and $\left(f_{1} \odot f_{3}\right) \oplus\left(f_{2} \odot f_{4}\right)$ are defined, it hold that

$$
\left(f_{1} \oplus f_{2}\right) \odot\left(f_{3} \oplus f_{4}\right)=\left(f_{1} \odot f_{3}\right) \oplus\left(f_{2} \odot f_{4}\right)
$$

Take $D_{i}=\boldsymbol{\operatorname { d o m }}\left(f_{i}\right)$ and $R_{i}=\operatorname{range}\left(f_{i}\right)$ and note that well-definedness of the above terms implies that $R_{1}=D_{3}$ and $R_{2}=D_{4}$. Both terms have type $\operatorname{Mem}\left[D_{1} \cup D_{2}\right] \rightarrow \mathcal{P}\left(\operatorname{Mem}\left[R_{3} \cup R_{4}\right]\right)$, and, for any $d \in D_{1} \cup D_{2}$ :

$$
\begin{align*}
\left(\left(f_{1} \oplus f_{2}\right) \odot\left(f_{3} \oplus f_{4}\right)\right)(d) & =\left\{v \mid u \in\left(f_{1} \oplus f_{2}\right)(d), v \in\left(f_{3} \oplus f_{4}\right)(u)\right\} \\
& =\left\{v \mid u \in f_{1}\left(d^{D_{1}}\right) \bowtie f_{2}\left(d^{D_{2}}\right), v \in f_{3}\left(u^{D_{3}}\right) \bowtie f_{4}\left(u^{D_{4}}\right)\right\} \\
& =\left\{v \mid v \in f_{3}(x) \bowtie f_{4}(y), x \in f\left(d^{D_{1}}\right), y \in g\left(d^{D_{2}}\right)\right\} \\
& =\left\{v_{1} \bowtie v_{2} \mid v_{1} \in f_{3}(x), v_{2} \in f_{4}(y), x \in f\left(d^{D_{1}}\right), y \in g\left(d^{D_{2}}\right)\right\} \\
\left(\left(f_{1} \odot f_{3}\right) \oplus\left(f_{2} \odot f_{4}\right)\right)(d) & =\left(f_{1} \odot f_{3}\right)\left(d^{D_{1}}\right) \bowtie\left(f_{2} \odot f_{4}\right)\left(d^{D_{2}}\right) \\
& =\left\{v_{1} \mid u_{1} \in f_{1}\left(d^{D_{1}}\right), v_{1} \in f_{3}\left(u_{1}\right)\right\} \bowtie\left\{v_{2} \mid u_{2} \in f_{2}\left(d^{D_{2}}\right), v_{2} \in f_{4}\left(u_{2}\right)\right\} \\
& =\left\{v_{1} \bowtie v_{2} \mid v_{1} \in f_{3}\left(u_{1}\right), v_{2} \in f_{4}\left(u_{2}\right), u_{1} \in f_{1}\left(d^{D_{1}}\right), u_{2} \in f_{2}\left(d^{D_{2}}\right)\right\}
\end{align*}
$$

(Def. $\otimes$ )

The step marked with ( $\star$ ) follows from the fact that $R_{1}=D_{3}$ and $R_{2}=D_{4}$ implies that for any $u \in f\left(d^{D_{1}}\right) \bowtie g\left(d^{D_{2}}\right)$, we have $u^{D_{3}}=x \in f_{1}\left(d^{D_{1}}\right)$ and $u^{D_{3}}=y \in f_{1}\left(d^{D_{1}}\right)$.
(9) Proved in Theorem A. 8

Theorem IV.2. $\left(M^{P}, \sqsubseteq, \oplus, \odot, M^{P}\right)$ is a DIBI frame.
Proof. By Theorem A. 37 that all $\mathcal{T}$-models are DIBI frames and by Theorem A. 9 that $M^{P}$ is a $\mathcal{T}$-model, $M^{P}$ is a DIBI frame.

## E. Section V-A. Conditional Independence: Omitted Details

First, we prove Theorem A. 10 so we can use Theorem A. 38 for $M^{D}$.
Lemma A. 10 (Disintegration). If $f=f_{1} \odot f_{2}$, then $\pi_{R_{1}} f=f_{1}$. Conversely, if $\pi_{R_{1}} f=f_{1}$, then there exists $g$ such that $f=f_{1} \odot g$. Proof. For the forwards direction, suppose that $f=f_{1} \odot f_{2}$. Then,

$$
\pi_{R_{1}} f=\pi_{R_{1}}\left(f_{1} \odot f_{2}\right)=f_{1} \odot\left(\pi_{R_{1}} f_{2}\right)=f_{1} \odot \text { unit }_{\text {Mem }\left[R_{1}\right]}=f_{1}
$$

Thus, $\pi_{R_{1}} f=f_{1}$. For the converse, assume $\pi_{R_{1}} f=f_{1}$. Define $g: \operatorname{Mem}\left[R_{1}\right] \rightarrow \mathcal{D}(\operatorname{Mem}[\operatorname{range}(f)])$ such that for any $r \in$ $\operatorname{Mem}\left[R_{1}\right], m \in \operatorname{Mem}[\operatorname{range}(f)]$ such that $r \bowtie m$ is defined, let

$$
g(r)(m):= \begin{cases}\frac{f\left(r r^{D_{1}}\right)(m)}{\left(\pi_{R_{1}} f\right)\left(r^{D_{1}}\right)(r)} & :\left(\pi_{R_{1}} f\right)\left(r^{D_{1}}\right)(r) \neq 0 \\ 0 & :\left(\pi_{R_{1}} f\right)\left(r^{D_{1}}\right)(r)=0\end{cases}
$$

We need to check that $g \in M^{D}$. Fixing any $r \in \operatorname{Mem}\left[R_{1}\right]$, denote the distribution $\operatorname{Pr}_{f\left(r^{\left.D_{1}\right)}\right.}$ as $\mu_{r}$, then

$$
\begin{aligned}
\left(\pi_{R_{1}} f\right)\left(r^{D_{1}}\right)(r) & =\frac{\mu_{r}(\operatorname{range}(f)=m)}{\mu_{r}\left(R_{1}=r\right)}=\mu_{r}\left(\operatorname{range}(f)=m \mid R_{1}=r\right) \quad\left(\text { if }\left(\pi_{R_{1}} f\right)\left(r^{D_{1}}\right)(r) \neq 0\right) \\
\sum_{m \in \operatorname{Mem}[\operatorname{range}(g)]} g(r)(m) & =\sum_{m \in \operatorname{Mem}[\operatorname{range}(g)]} \mu_{r}\left(\operatorname{range}(f)=m \mid R_{1}=r\right)=1
\end{aligned}
$$

so $g$ does map any input to a distribution, and $g$ preserves the input.
By their types, $f_{1} \odot g$ is defined, and for any $d \in \operatorname{Mem}\left[D_{1}\right], m \in \operatorname{Mem}[\operatorname{range}(f)]$ such that $d \bowtie m$ is defined. If $\left(\pi_{R_{1}} f\right)(d)\left(m^{R_{1}}\right) \neq$ 0 , then

$$
\begin{aligned}
\left(f_{1} \odot g\right)(d)(m)=f_{1}(d)\left(m^{R_{1}}\right) \cdot g\left(m^{R_{1}}\right)(m) & =f_{1}(d)\left(m^{R_{1}}\right) \cdot \frac{f\left(m^{D_{1}}\right)(m)}{\left(\pi_{R_{1}} f\right)\left(m^{D_{1}}\right)(m)} \\
& =f_{1}(d)\left(m^{R_{1}}\right) \cdot \frac{f\left(m^{D_{1}}\right)(m)}{f_{1}\left(m^{D_{1}}\right)\left(m^{R_{1}}\right)} \\
& =f(d)(m) \quad\left(d \bowtie m \text { is defined iff } d=m^{D_{1}}\right)
\end{aligned}
$$

If $\left(\pi_{R_{1}} f\right)(d)\left(m^{R_{1}}\right) \neq 0$, then $f(d)(m)=0$, and $\left(f_{1} \odot g\right)(d)(m)=f_{1}(d)\left(m^{R_{1}}\right) \cdot g\left(m^{R_{1}}\right)(m)=0=f(d)(m)$. Thus, $f_{1} \odot g=f$.
Theorem V.1. Given distribution $\mu \in \mathcal{D}(\operatorname{Mem}[\operatorname{Var}])$, then for any $X, Y, Z \subseteq \operatorname{Var}$,

$$
\begin{equation*}
f_{\mu} \vDash(\emptyset \triangleright[Z]) \stackrel{\circ}{\circ}(Z \triangleright[X]) *(Z \triangleright[Y]) \tag{8}
\end{equation*}
$$

if and only if $X \Perp Y \mid Z$ and $X \cap Y \subseteq Z$ are both satisfied.
Proof. This result follows by combining Theorem A. 11 and Theorem A.38,
Lemma A.11. For a distribution $\mu$ on $\operatorname{Var}, S, X, Y \subseteq \operatorname{Var}$, there exist $f_{1}: \operatorname{Mem}[\emptyset] \rightarrow \mathcal{D}(\operatorname{Mem}[S]), f_{2}: \operatorname{Mem}[S] \rightarrow \mathcal{D}(\operatorname{Mem}[S \cup$ $X])$, $f_{3}: \operatorname{Mem}[S] \rightarrow \mathcal{D}(\operatorname{Mem}[S \cup Y])$, such that $f_{1} \odot\left(f_{2} \oplus f_{3}\right) \sqsubseteq f_{\mu}$, if and only if $X \Perp Y \mid S$ and also $X \cap Y \subseteq S$.

Proof. Forward direction: Assume the existence of $f_{1}, f_{2}, f_{3}$ satisfying $f_{1} \odot\left(f_{2} \oplus f_{3}\right) \sqsubseteq f_{\mu}$. We must prove $X \Perp Y \mid S$ and $X \cap Y \subseteq S$.

1) $X \cap Y \subseteq S: f_{2} \oplus f_{3}$ defined implies $(X \cup S) \cap(Y \cup S) \subseteq S \cap S$. Thus, $X \cap Y \subseteq S$
2) $X \Perp Y \mid S$ : By assumption, $f_{1} \odot\left(f_{2} \oplus f_{3}\right) \sqsubseteq f_{\mu}$. Theorem A. 10 gives us $f_{1} \odot\left(f_{2} \oplus f_{3}\right)=\pi_{S \cup X \cup Y}\left(f_{\mu}\right)$, and $f_{1}=\pi_{S}\left(f_{\mu}\right)$. For any $m \in \operatorname{Mem}[X \cup Y \cup S], m^{X} \bowtie m^{Y} \bowtie m^{S}$ is defined. Thus,

$$
\begin{aligned}
\mu\left(X=m^{X}, Y=m^{Y}, S=m^{S}\right) & \left.=\left(\pi_{X \cup Y \cup S} \mu\right)\left(m^{X} \bowtie m^{Y} \bowtie m^{S}\right) \quad \quad \text { (By definition } \mu\right) \\
& =\pi_{X \cup Y \cup S}\left(f_{\mu}\right)(\langle \rangle)\left(m^{X} \bowtie m^{Y} \bowtie m^{S}\right) \\
& =f_{1} \odot\left(f_{2} \oplus f_{3}\right)(\langle \rangle)\left(m^{X} \bowtie m^{Y} \bowtie m^{S}\right)
\end{aligned}
$$

Similarly, $\mu\left(S=m^{S}\right):=\left(\pi_{S} \mu\right)\left(m^{S}\right)$. We have $f_{1}=\pi_{S}\left(f_{\mu}\right)$, and so

$$
\begin{equation*}
\mu\left(S=m^{S}\right)=\left(\pi_{S} \mu\right)\left(m^{S}\right)=\left(\pi_{S}\left(f_{\mu}\right)\right)(\langle \rangle)\left(m^{S}\right)=f_{1}(\langle \rangle)\left(m^{S}\right) \tag{9}
\end{equation*}
$$

By definition of conditional probability, when $\mu\left(S=m^{S}\right) \neq 0$,

$$
\begin{aligned}
\mu\left(X=m^{X}, Y=m^{Y} \mid S=m^{S}\right) & =\frac{\mu\left(X=m^{X}, Y=m^{Y}, S=m^{S}\right)}{\mu\left(S=m^{S}\right)} \\
& =\frac{f_{1} \odot\left(f_{2} \oplus f_{3}\right)(\langle \rangle)\left(m^{S} \bowtie m^{X} \bowtie m^{Y}\right)}{f_{1}(\langle \rangle)\left(m^{S}\right)}
\end{aligned}
$$

By Eq. (7): $f_{1} \odot\left(f_{2} \oplus f_{3}\right)(\langle \rangle)\left(m^{S} \bowtie m^{X} \bowtie m^{Y}\right)=f_{1}(\langle \rangle)\left(m^{S}\right) \cdot\left(f_{2} \oplus f_{3}\right)\left(m^{S}\right)\left(m^{S} \bowtie m^{X} \bowtie m^{Y}\right)$. Thus,

$$
\begin{align*}
\mu\left(X=m^{X}, Y=m^{Y} \mid S=m^{S}\right) & =\frac{f_{1} \odot\left(f_{2} \oplus f_{3}\right)(\langle \rangle)\left(m^{S} \bowtie m^{X} \bowtie m^{Y}\right)}{f_{1}(\langle \rangle)\left(m^{S}\right)} \\
& =\left(f_{2} \oplus f_{3}\right)\left(m^{S}\right)\left(m^{S} \bowtie m^{X} \bowtie m^{Y}\right) \\
& =f_{2}\left(m^{S}\right)\left(m^{X \cup S}\right) \cdot f_{3}\left(m^{S}\right)\left(m^{Y \cup S}\right) \tag{10}
\end{align*}
$$

Let $f_{2}^{\prime}=f_{2} \oplus$ unit $_{\text {Mem }[Y]}, f_{3}^{\prime}=f_{3} \oplus$ unit $_{\text {Mem }[X]}$. By Theorem A.35,

$$
\begin{aligned}
& f_{1} \odot\left(f_{2} \oplus f_{3}\right)=f_{1} \odot f_{2} \odot\left(f_{3} \oplus \text { unit }_{\mathbf{M e m}[X]}\right)=f_{1} \odot f_{2} \odot f_{3}^{\prime} \\
& f_{1} \odot\left(f_{2} \oplus f_{3}\right)=f_{1} \odot\left(f_{3} \oplus f_{2}\right)=f_{1} \odot f_{3} \odot\left(f_{2} \oplus \text { unit }_{\mathbf{M e m}[Y]}\right)=f_{1} \odot f_{3} \odot f_{2}^{\prime}
\end{aligned}
$$

Theorem A. 10 gives us $\pi_{X \cup S}\left(f_{\mu}\right)=f_{1} \odot f_{2}$, and $\pi_{Y \cup S}\left(f_{\mu}\right)=f_{1} \odot f_{3}$, Therefore,

$$
\begin{align*}
\mu\left(X=m^{X}, S=m^{S}\right) & :=\left(\pi_{X \cup S} \mu\right)\left(m^{S} \otimes m^{X}\right) \\
& =\left(\pi_{X \cup S}\left(f_{\mu}\right)\right)(\langle \rangle)\left(m^{S} \otimes m^{X}\right) \\
& =\left(f_{1} \odot f_{2}\right)(\langle \rangle)\left(m^{S} \otimes m^{X}\right)  \tag{11}\\
& =f_{1}(\langle \rangle)\left(m^{S}\right) \cdot f_{2}\left(m^{S}\right)\left(m^{S} \otimes m^{X}\right) \\
\mu\left(Y=m^{Y}, S=m^{S}\right) & :=\left(\pi_{Y \cup S} \mu\right)\left(m^{S} \otimes m^{Y}\right) \\
& =\left(\pi_{Y \cup S}\left(f_{\mu}\right)(\langle \rangle)\left(m^{S} \otimes m^{Y}\right)\right. \\
& =\left(f_{1} \odot f_{3}\right)(\langle \rangle)\left(m^{S} \otimes m^{Y}\right)  \tag{12}\\
& =f_{1}(\langle \rangle)\left(m^{S}\right) \cdot f_{3}\left(m^{S}\right)\left(m^{S} \otimes m^{Y}\right)
\end{align*}
$$

Thus, by definition of conditional probability.

$$
\begin{align*}
\mu\left(X=m^{X} \mid S=m^{S}\right) & =\frac{\mu\left(X=m^{X}, S=m^{S}\right)}{\mu\left(S=m^{S}\right)} \\
& =\frac{f_{1}(\langle \rangle)\left(m^{S}\right) \cdot f_{2}\left(m^{S}\right)\left(m^{S \cup X}\right)}{f_{1}(\langle \rangle)\left(m^{S}\right)} \\
& =f_{2}\left(m^{S}\right)\left(m^{S \cup X}\right)  \tag{13}\\
\mu\left(X=m^{Y} \mid S=m^{S}\right) & =\frac{\mu\left(X=m^{X}, S=m^{S}\right)}{\mu\left(S=m^{S}\right)} \\
& =\frac{f_{1}(\langle \rangle)\left(m^{S}\right) \cdot f_{3}\left(m^{S}\right)\left(m^{S \cup Y}\right)}{f_{1}(\langle \rangle)\left(m^{S}\right)} \\
& =f_{3}\left(m^{S}\right)\left(m^{S \cup Y}\right) \tag{14}
\end{align*}
$$

Substituting Eq. (13) and Eq. (14) into the equation Eq. (10), we have

$$
\left.\mu\left(X=m^{X}, Y=m^{Y} \mid S=m^{S}\right)=\mu\left(X=m^{X} \mid S=m^{S}\right) \cdot \mu\left(X=m^{Y} \mid S=m^{S}\right)\right)
$$

Thus, $X, Y$ are conditionally independent given $S$. This completes the proof for the first direction.
Backward direction: We want to show that if $X \Perp Y \mid S$ and $X \cap Y \subseteq S$ then $f_{1} \odot\left(f_{2} \oplus f_{3}\right) \sqsubseteq f_{\mu}$. Given $\mu$, we define $f_{1}=\pi_{S}\left(f_{\mu}\right)$ and construct $f_{2}, f_{3}$ as follows:

Let $f_{2}: \operatorname{Mem}[S] \rightarrow \mathcal{D}(\operatorname{Mem}[S \cup X])$. For any $s \in \operatorname{Mem}[S], x \in \operatorname{Mem}[X]$ such that $s \otimes x$ is defined, when $f_{1}(\langle \rangle)(s) \neq 0$, let

$$
f_{2}(s)(s \otimes x):=\frac{\left(\pi_{S \cup X} f_{\mu}\right)(\langle \rangle)(s \otimes x)}{f_{1}(\langle \rangle)(s)}
$$

(When $f_{1}(\langle \rangle)(s)=0$, we can define $f_{2}(s)(s \otimes x)$ arbitrarily as long as $f_{2}(s)$ is a distribution, because that distribution will be zeroed out in $f_{1} \odot\left(f_{2} \oplus f_{3}\right)$ anyway. $)$

Similarly, let $f_{3}: \operatorname{Mem}[S] \rightarrow \mathcal{D}(\operatorname{Mem}[S \cup Y])$. For any $s \in \operatorname{Mem}[S], x \in \operatorname{Mem}[Y]$ such that $s \otimes y$ is defined, when $f_{1}(\langle \rangle)(s) \neq 0$, let

$$
f_{3}(s)(s \otimes y):=\frac{\left(\pi_{S \cup Y} f_{\mu}\right)(s \otimes y)}{f_{1}(\langle \rangle)(s)}
$$

By construction, $f_{1}, f_{2}, f_{3}$ each has the type needed for the lemma. We are left to prove that given any $s \in \operatorname{Mem}[S], f_{2}$ and $f_{3}$ are kernels in $M^{D}, f_{1} \odot\left(f_{2} \oplus f_{3}\right)$ is defined, and $f_{1} \odot\left(f_{2} \oplus f_{3}\right) \sqsubseteq f_{\mu}$.

- State $f_{2}$ is in $M^{D}$.

We need to show that for any $s \in \operatorname{Mem}[S], f_{2}(s)$ forms a distribution, and also $f_{2}$ preserves the input. For any $s \in \operatorname{Mem}[S]$, by equation Eq. (9), $f_{1}(\langle \rangle)(s)=\mu(S=s)$.
If $f_{1}(\langle \rangle)(s)=0$, then we define $f_{2}(s)$ arbitrarily but make sure $f_{2}(s)$ is a distribution.
If $f_{1}(\langle \rangle)(s) \neq 0$ : for any $x \in \operatorname{Mem}[X]$ such that $s \otimes x$ is defined, $\left(\pi_{S \cup X} f_{\mu}\right)(\rangle)(s \otimes x)=\mu(S=s, X=x)$, so

$$
\begin{aligned}
& f_{2}(s)(s \otimes x)=\frac{\left(\pi_{S \cup X} f_{\mu}\right)(\langle \rangle)(s \otimes x)}{f_{1}(\langle \rangle)(s)} \\
= & \frac{\mu(S=s, X=x)}{\mu(S=s)}=\mu(X=x \mid S=s)
\end{aligned}
$$

Thus, $f_{2}(s)$ is a distribution for any $s \in \operatorname{Mem}[S]$.
Also, $f_{2}(s)(s \otimes x)$ is non-zero only when $s \otimes x$ is defined, i.e., when $(s \otimes x)^{S}=s$. $\operatorname{So}\left(\pi_{S} f_{2}\right)(s)(s)=\sum_{x \in \operatorname{Mem}[X]} f_{2}(s)(s \otimes x)=1$, and thus $\pi_{S} f_{2}=$ unit $_{\text {Mem }[S]}$. Therefore, $f_{2}$ preserves the input.
Therefore, $f_{2} \in M^{D}$.

- State $f_{3}$ is in $M^{D}$. Similar as above.
- State $f_{1} \odot\left(f_{2} \oplus f_{3}\right)$ is defined.
$f_{2} \oplus f_{3}$ is defined because $R_{2} \cap R_{3}=(S \cup X) \cap(S \cup Y)=S \cup(X \cap Y)$, and by assumption, $X \cap Y \subseteq S$, so $S \cup(X \cap Y)=S=D_{2} \cap D_{3}$. Then $f_{1} \odot\left(f_{2} \oplus f_{3}\right)$ is defined because $\operatorname{dom}\left(f_{2} \oplus f_{3}\right)=D_{2} \cup D_{3}=S \cup S=S=\operatorname{range}\left(f_{1}\right)$.
- State $f_{1} \odot\left(f_{2} \oplus f_{3}\right) \sqsubseteq f_{\mu}$.

It suffices to show that there exists $g$ such that $\left(f_{1} \odot\left(f_{2} \oplus f_{3}\right)\right) \odot g=f_{\mu}$.
For any $s \in \operatorname{Mem}[S], x \in \operatorname{Mem}[X], y \in \operatorname{Mem}[Y]$ such that $s \otimes x \otimes y$ is defined,

$$
\begin{align*}
f_{1} \odot\left(f_{2} \oplus f_{3}\right)(\langle \rangle)(s \otimes x \otimes y) & =f_{1}(\langle \rangle)(s) \cdot f_{2} \oplus f_{3}(s)(s \otimes x \otimes y) \\
& =f_{1}(\langle \rangle)(s) \cdot\left(f_{2}(s)(s \otimes x) \cdot f_{3}(s)(s \otimes y)\right) \\
& =\mu(S=s) \cdot(\mu(X=x \mid S=s) \cdot \mu(Y=y \mid S=s)) \tag{15}
\end{align*}
$$

Because $X, Y$ are conditionally independent given $S$ in the distribution $q$, so

$$
\begin{equation*}
\mu(X=x \mid S=s) \cdot \mu(Y=y \mid S=s)=\mu(X=x, Y=y \mid S=s) \tag{16}
\end{equation*}
$$

Substituting Eq. (16) into Eq. (15), we have

$$
\begin{aligned}
f_{1} \odot\left(f_{2} \oplus f_{3}\right)(\langle \rangle)(s \otimes x \otimes y) & =\mu(S=s) \cdot \mu(X=x, Y=y \mid S=s) \\
& =\mu(X=x, Y=y, S=s)
\end{aligned}
$$

Let $g: \operatorname{Mem}[X \cup Y \cup S] \rightarrow \mathcal{D}(\operatorname{Mem}[\mathbf{V a l}])$ such that for any $d \in \operatorname{Mem}[X \cup Y \cup S], m \in \operatorname{Mem}[\mathbf{V a l}]$ such that $d \otimes m$ is defined, let

$$
g(d)(m)=\mu(\mathbf{V a l}=m \mid X \cup Y \cup S=d)
$$

Then, $\left(f_{1} \odot\left(f_{2} \oplus f_{3}\right)\right) \odot g$ is defined, and

$$
\begin{aligned}
\left(f_{1} \odot\left(f_{2} \oplus f_{3}\right) \odot g\right)(\rangle)(m) & =\left(f_{1} \odot\left(f_{2} \oplus f_{3}\right)\right)(\langle \rangle)\left(m^{X \cup Y \cup S}\right) \cdot g\left(m^{X \cup Y \cup S}\right)(m) \\
& =\mu(\mathbf{V a l}=m)
\end{aligned}
$$

Thus, $\left(f_{1} \odot\left(f_{2} \oplus f_{3}\right)\right) \odot g=f_{\mu}$, and therefore $f_{1} \odot\left(f_{2} \oplus f_{3}\right) \sqsubseteq f_{\mu}$.
This completes the proof for the backwards direction.
Lemma A.12. If $X, Y$ are conditionally independent given $S$, then values on $X \cap Y$ is determined given values on $S$.

Proof. For any $x \in \operatorname{Mem}[X], y \in \operatorname{Mem}[Y], s \in \operatorname{Mem}[S], m \in \operatorname{Mem}[M]$, when $\mu(X=x, Y=y, M=m \mid S=s) \neq 0$, it must $x \otimes y \otimes s \otimes m$ is defined. Note that $x \otimes y \otimes s \otimes m$ defined only if $m=\pi_{M} x=\pi_{M} y$, which implies that $m \otimes x=x, m \otimes y=y$, $m \otimes x \otimes y=x \otimes y$.

Let $M=X \cap Y, \widehat{X}=X \backslash Y, \widehat{Y}=Y \backslash X$. By assumption, $X, Y$ are conditionally independent given $S$, so $x \in \operatorname{Mem}[X]$, $y \in \operatorname{Mem}[Y], s \in \operatorname{Mem}[S], m \in \operatorname{Mem}[M]$

$$
\mu(X=x \mid S=s) \cdot \mu(Y=y \mid S=s)=\mu(X=x, Y=y \mid S=s)
$$

which implies that, if we denote $x^{\prime}=\pi_{\widehat{X}} x, y^{\prime}=\pi_{\bar{Y}} y$,

$$
\begin{equation*}
\mu\left(\widehat{X}=x^{\prime}, M=m \mid S=s\right) \cdot \mu\left(\widehat{Y}=y^{\prime}, M=m \mid S=s\right)=\mu\left(\widehat{X}=x^{\prime}, \widehat{Y}=y^{\prime}, M=m \mid S=s\right) \tag{17}
\end{equation*}
$$

For any probabilistic events $E_{1}, E_{2}, E_{3}, \mu\left(E_{1}, E_{2} \mid E_{3}\right)=\mu\left(E_{1} \mid E_{2}, E_{3}\right) \cdot \mu\left(E_{2} \mid E_{3}\right)$. Thus, Eq. (17) implies that

$$
\begin{equation*}
\mu\left(\widehat{X}=x^{\prime} \mid M=m, S=s\right) \cdot \mu\left(\widehat{Y}=y^{\prime} \mid M=m, S=s\right) \cdot \mu(M=m \mid S=s)=\mu\left(\widehat{X}=x^{\prime}, \widehat{Y}=y^{\prime} \mid M=m, S=s\right) \tag{18}
\end{equation*}
$$

Then, for any $s \in \operatorname{Mem}[S], m \in \operatorname{Mem}[M]$ such that $m \otimes s$ is defined and $\mu(M=m, S=s) \neq 0$,

$$
\begin{align*}
& \sum_{x^{\prime} \in \operatorname{Mem}[\widehat{X X}], y^{\prime} \in \operatorname{Mem}[\widehat{Y}]} \mu\left(\widehat{X}=x^{\prime} \mid M=m, S=s\right) \cdot \mu\left(\widehat{Y}=y^{\prime} \mid M=m, S=s\right) \cdot \mu(M=m \mid S=s) \\
&= \sum_{x^{\prime} \in \operatorname{Mem}[\widehat{X}], y^{\prime} \in \operatorname{Mem}[\widehat{Y}]} \mu\left(\widehat{X}=x^{\prime}, \widehat{Y}=y^{\prime} \mid M=m, S=s\right) \quad \quad \text { (Because of Eq. (18)) } \\
&=1
\end{align*}
$$

Meanwhile, for any $s \in \operatorname{Mem}[S], m \in \operatorname{Mem}[M]$ such that $m \otimes s$ is defined and $\mu(M=m, S=s) \neq 0$,

$$
\begin{align*}
& \sum_{x^{\prime} \in \operatorname{Mem}[\widehat{X}], y^{\prime} \in \operatorname{Mem}[\widehat{Y}]} \mu\left(\widehat{X}=x^{\prime} \mid M=m, S=s\right) \cdot \mu\left(\widehat{Y}=y^{\prime} \mid M=m, S=s\right) \cdot \mu(M=m \mid S=s) \\
= & \left(\sum_{x^{\prime} \in \operatorname{Mem}[\widehat{X}], y^{\prime} \in \operatorname{Mem}[\widehat{Y}]} \mu\left(\widehat{X}=x^{\prime} \mid M=m, S=s\right) \cdot \mu\left(\widehat{Y}=y^{\prime} \mid M=m, S=s\right)\right) \cdot \mu(M=m \mid S=s) \\
= & \left(\sum_{x^{\prime} \in \operatorname{Mem}[\widehat{X}]} \mu\left(\widehat{X}=x^{\prime} \mid M=m, S=s\right)\right) \cdot\left(\sum_{y^{\prime} \in \operatorname{Mem}[\widehat{Y}]} \mu\left(\widehat{Y}=y^{\prime} \mid M=m, S=s\right)\right) \cdot \mu(M=m \mid S=s) \\
= & 1 \cdot \mu(M=m \mid S=s) \tag{20}
\end{align*}
$$

Combining Eq. (20) and Eq. (19), we derive $\mu(M=m \mid S=s)=1$. That is, when $X \Perp Y \mid S$, whether $M \supseteq S$ or not, $m \otimes s$ is defined and $\mu(M=m, S=s) \neq 0$ implies $\mu(M=m \mid S=s)=1$. Thus, $X \Perp Y \mid S$ renders values on $X \cap Y$ deterministic given values on $S$.

## F. Section $[\nabla-B$ Join Dependency: Omitted Details

We again prove a disintegration lemma for $M^{P}$ Theorem A. 13 so that we can use Theorem A. 38 on $M^{P}$.
Lemma A. 13 (Disintegration). If $f=f_{1} \odot f_{2}$ and $D_{2}=R_{1}$, then $\pi_{R_{1}} f=f_{1}$. Conversely, if $\pi_{R_{1}} f=f_{1}$, then there exists $g$ such that $f=f_{1} \odot g$.
Proof. Assume $f=f_{1} \odot f_{2}$ and $D_{2}=R_{1}$. Then,

$$
\pi_{R_{1}} f=\pi_{R_{1}}\left(f_{1} \odot f_{2}\right)=f_{1} \odot\left(\pi_{R_{1}} f_{2}\right)=f_{1} \odot \text { unit }_{\operatorname{Mem}\left[R_{1}\right]}=f_{1}
$$

Conversely, assume $\pi_{R_{1}} f=f_{1}$. Define $g: \operatorname{Mem}\left[R_{1}\right] \rightarrow \mathcal{P}\left(\operatorname{Mem}\left[R_{2}\right]\right)$ by $g(r)=\left\{s \otimes r \mid s \in f\left(r^{D_{1}}\right)\right\}$.

$$
\left(f_{1} \odot g\right)(d)=\left\{u \mid u \in g(r), r \in f_{1}(d)\right\}=\left\{s \otimes r \mid s \in f\left(r^{D_{1}}\right), r \in \pi_{R_{1}} f(d)\right\}=\{s \mid s \in f(d)\}=f(d)
$$

Theorem V.2. Let $R \in \mathcal{P}(\operatorname{Mem}[\mathrm{Var}])$ and $X, Y$ be sets of attributes such that $X \cup Y=$ Var. The lifted relation $f_{R}=\langle \rangle \mapsto R$ satisfies $f_{R} \vDash[X \cap Y] \circ([X] *[Y])$ iff $R$ satisfies the join dependency $X \bowtie Y$.
Proof. The result follows from combining Theorem A. 14 and Theorem A. 38 ,
Lemma A.14. For a relation $R$ on Val, $X, Y \subseteq$ Val, there exists $f_{1}: \operatorname{Mem}[\emptyset] \rightarrow \mathcal{P}(\operatorname{Mem}[X \cap Y]), f_{2}: \operatorname{Mem}[X \cap Y] \rightarrow \mathcal{P}(\operatorname{Mem}[X])$, $f_{3}: \operatorname{Mem}[X \cap Y] \rightarrow \mathcal{P}(\operatorname{Mem}[Y])$, such that $f_{1} \odot\left(f_{2} \oplus f_{3}\right) \sqsubseteq f_{R}$, if and only if $R^{X \cup Y}=R^{X} \bowtie R^{Y}$.

Proof. Forward Direction: Assuming there exist $f_{1}, f_{2}, f_{3}$ such that $f_{1} \odot\left(f_{2} \oplus f_{3}\right) \sqsubseteq f_{R}$, we want to show that $R^{X \cup Y}=R^{X} \bowtie R^{Y}$.
We have $f_{1} \odot\left(f_{2} \oplus f_{3}\right) \sqsubseteq f_{R}$ and $f_{R}$ with empty domain. Hence, there exists $h \in M^{P}$ such that

$$
f_{R}=\left(f_{1} \odot\left(f_{2} \oplus f_{3}\right)\right) \odot h
$$

Thus, $f_{1} \odot\left(f_{2} \oplus f_{3}\right)=\pi_{X \cup Y} f_{R}$, and so $f_{1} \odot\left(f_{2} \oplus f_{3}\right)(\langle \rangle)=R^{X \cup Y}$.
Similarly to the reasoning in Theorem A.11 by Theorem A.35 we have

$$
\begin{aligned}
& f_{1} \odot f_{2} \sqsubseteq f_{1} \odot\left(f_{2} \oplus f_{3}\right) \\
& f_{1} \odot f_{3} \sqsubseteq f_{1} \odot\left(f_{2} \oplus f_{3}\right)
\end{aligned}
$$

Then, as above, $f_{1} \odot f_{2}=\pi_{X} f_{R}, f_{1} \odot f_{3}=\pi_{Y}\left(f_{R}\right)$. So, $f_{1} \odot f_{2}(\langle \rangle)=R^{X}, f_{1} \odot f_{3}(\langle \rangle)=R^{Y}$.
By definition of $\oplus$ and $\odot$,

$$
\begin{aligned}
f_{1} \odot\left(f_{2} \oplus f_{3}\right)(\langle \rangle) & =\left\{u \bowtie v \mid u \in f_{1}(\langle \rangle) \text { and } v \in f_{2} \oplus f_{3}(u)\right\} \\
& =\left\{u \bowtie v \mid u \in f_{1}(\langle \rangle) \text { and } v \in\left\{v_{1} \bowtie v_{2} \mid v_{1} \in f_{2}(u) \text { and } v_{2} \in f_{3}(u)\right\}\right\} \\
& =\left\{u \bowtie\left(v_{1} \bowtie v_{2}\right) \mid u \in f_{1}(\langle \rangle) \text { and } v_{1} \in f_{2}(u) \text { and } v_{2} \in f_{3}(u)\right\}
\end{aligned}
$$

Since $\bowtie$ is idempotent, i.e., $u \bowtie u=u$, commutative and associative, we have

$$
u \bowtie(v \bowtie w)=(u \bowtie u) \bowtie(v \bowtie w)=(u \bowtie v) \bowtie(u \bowtie w) .
$$

Therefore, we can convert the previous equality into

$$
\begin{aligned}
& f_{1} \odot\left(f_{2} \oplus f_{3}\right)(\langle \rangle)=\left\{\left(u \bowtie v_{1}\right) \bowtie\left(u \bowtie v_{2}\right) \mid u \in f_{1}(\langle \rangle) \text { and } v_{1} \in f_{2}(u) \text { and } v_{2} \in f_{3}(u)\right\} \\
= & \left\{u \bowtie v_{1} \mid u \in f_{1}(\langle \rangle) \text { and } v_{1} \in f_{2}(u)\right\} \bowtie\left\{u \bowtie v_{2} \mid u \in f_{1}(\langle \rangle) \text { and } v_{2} \in f_{3}(u)\right\} \\
= & \left(f_{1} \odot f_{2}\right)\left(\rangle) \bowtie\left(f_{1} \odot f_{3}\right)(\rangle)\right.
\end{aligned}
$$

Thus, $R^{X \cup Y}=R^{X} \bowtie R^{Y}$.
This completes the proof for the first direction.
Backward direction: If $R^{X \cup Y}=R^{X} \bowtie R^{Y}$, then we want to show that there exist $f_{1}: \operatorname{Mem}[\emptyset] \rightarrow \mathcal{P}(\operatorname{Mem}[X \cap Y]), f_{2}: \operatorname{Mem}[X \cap$ $Y] \rightarrow \mathcal{P}(\operatorname{Mem}[X]), f_{3}: \operatorname{Mem}[X \cap Y] \rightarrow \mathcal{P}(\operatorname{Mem}[Y])$, such that $f_{1} \odot\left(f_{2} \oplus f_{3}\right) \sqsubseteq f_{R}$.

Let $f_{1}=f_{R}^{X \cap Y}$ and define $f_{2}: \operatorname{Mem}[X \cap Y] \rightarrow \mathcal{P}(\operatorname{Mem}[X])$ by having

$$
f_{2}(s):=\left\{r \in R^{X} \mid r^{X \cap Y}=s\right\}
$$

for all $s \in \operatorname{Mem}[X \cap Y]$. Define $f_{3}: \operatorname{Mem}[X \cap Y] \rightarrow \mathcal{P}(\operatorname{Mem}[Y])$ by having

$$
f_{3}(s)=\left\{r \in R^{Y} \mid r^{X \cap Y}=s\right\}
$$

for all $s \in \operatorname{Mem}[X \cap Y]$.

- By construction, $f_{1}, f_{2}, f_{3}$ have the desired types.
- States $f_{2}, f_{3}$ are both in $M^{P}$.
$f_{2}$ preserves the input because for any $s \in \operatorname{Mem}[X \cap Y], f_{2}(s)$ as a relation only includes tuples whose projection to $X \cap Y$ is equals to $s$. Thus, $f_{2}$ is in $M^{P}$.
Similarly, $f_{3}$ is in $M^{P}$.
- $f_{1} \odot\left(f_{2} \oplus f_{3}\right) \sqsubseteq f_{R}$.

First, by their types, $f_{1} \odot\left(f_{2} \oplus f_{3}\right)$ is defined, and

$$
\begin{align*}
f_{1} \odot\left(f_{2} \oplus f_{3}\right)(\langle \rangle) & =\left\{u \bowtie v \mid u \in f_{1}(\langle \rangle) \text { and } v \in\left(f_{2} \oplus f_{3}\right)(u)\right\} \\
& =\left\{u \bowtie v \mid u \in f_{1}(\langle \rangle) \text { and } v \in f_{2}\left(u^{D_{2}}\right) \bowtie f_{3}\left(u^{D_{3}}\right)\right\} \\
& \left.=\left\{u \bowtie v \mid u \in f_{1}(\langle \rangle) \text { and } v \in f_{2}(u) \bowtie f_{3}(u)\right\} \quad \text { (By } D_{2}=D_{3}=X \cap Y\right) \\
& =\left\{u \bowtie\left(v_{i} \bowtie v_{j}\right) \mid u \in f_{1}(\langle \rangle) \text { and } v_{i} \in f_{2}(u) \text { and } v_{j} \in f_{3}(u)\right\} \\
& =\left\{\left(u \bowtie v_{i}\right) \bowtie\left(u \bowtie v_{j}\right) \mid u \in f_{1}(\langle \rangle) \text { and } v_{i} \in f_{2}(u) \text { and } v_{j} \in f_{3}(u)\right\} \\
& =\left\{u \bowtie v_{i} \mid u \in f_{1}(\langle \rangle) \text { and } v_{i} \in f_{2}(u)\right\} \bowtie\left\{u \bowtie v_{j} \mid u \in f_{1}(\langle \rangle) \text { and } v_{j} \in f_{3}(u)\right\}
\end{align*}
$$

Recall that we define $f_{1}$ such that $f_{1}(\langle \rangle)=R^{X \cap Y}$, and $f_{2}(s):=\left\{r \in R \mid r^{X \cap Y}=s\right\}$, so

$$
\begin{align*}
\left\{u \bowtie v_{i} \mid u \in R^{X \cap Y} \text { and } v_{i} \in f_{2}(u)\right\} & =\left\{\left(u \bowtie v_{i}\right) \mid u \in R^{X \cap Y} \text { and } v_{i} \in\left\{r \in R^{X} \mid r^{X \cap Y}=u\right\}\right\} \\
& =\left\{v_{i} \mid v_{i} \in\left\{r \in R^{X} \mid r^{X \cap Y} \in R^{X \cap Y}\right\}\right\} \\
& =R^{X} \tag{23}
\end{align*}
$$

$f_{1} \odot\left(f_{2} \oplus f_{3}\right) \sqsubseteq f_{\mu}$ Analogously,

$$
\begin{equation*}
\left\{u \bowtie v_{j} \mid u \in f_{1}(\langle \rangle) \text { and } v_{j} \in f_{3}(u)\right\}=R^{Y} \tag{24}
\end{equation*}
$$

Substituting Eq. (23) and Eq. (24) into Eq. (22), we have

$$
f_{1} \odot\left(f_{2} \oplus f_{3}\right)(\langle \rangle)=R^{X} \bowtie R^{Y}
$$

By assumption, $R^{X} \bowtie R^{Y}=R^{X \cup Y}$. Thus, $f_{1} \odot\left(f_{2} \oplus f_{3}\right)(\langle \rangle)=R^{X \cup Y}$, and $f_{1} \odot\left(f_{2} \oplus f_{3}\right)=\pi_{X \cup Y} f_{R}$. By Theorem A.13, this implies that $f_{1} \odot\left(f_{2} \oplus f_{3}\right) \sqsubseteq f_{R}$.
Thus, the constructed $f_{1}, f_{2}, f_{3}$ satisfy all requirements.
G. Section $V-C$ graphoid axioms: Omitted Details

Lemma A.15. The following judgment is derivable in DIBI:

$$
\vdash P \circ(Q * R) \rightarrow P \circ(R * Q) .
$$

Proof. We have the derivation:

Lemma A.16. The following judgment is derivable in DIBI:

$$
\vdash P ;(Q *(R \wedge S)) \rightarrow P ;(Q * R) \wedge P_{9}^{\circ}(Q * S) .
$$

Proof. We have the derivation:

Lemma $\mathbf{A . 1 7}$ (Weak Union). The following judgment is valid in any $\mathcal{T}$-model where Disintegration holds (see Theorem A. 10 and Theorem A.13 for Disintegration):

$$
\vDash[Z] \stackrel{\circ}{\circ}([X] *[Y \cup W]) \rightarrow[Z \cup W] \stackrel{ }{9}([X] *[Y])
$$

Proof. Let $M$ be a $\mathcal{T}$-model. If $f \vDash[Z] \stackrel{ }{\circ}([X] *[Y \cup W])$, by Theorem A.38, there exist $f_{1}, f_{2}, f_{3} \in M$ such that $f_{1} \odot\left(f_{2} \oplus f_{3}\right) \sqsubseteq f$, $f_{1}: \operatorname{Mem}[\emptyset] \rightarrow \mathcal{T}(\operatorname{Mem}[Z]), f_{2}: \operatorname{Mem}[Z] \rightarrow \mathcal{T}(\operatorname{Mem}[Z \cup X]), f_{3}: \operatorname{Mem}[Z] \rightarrow \mathcal{T}(\operatorname{Mem}[Z \cup Y \cup W])$.

Let $f_{3}^{1}=\pi_{Z \cup W} f_{3}$, then by Disintegration there exists $f_{3}^{2} \in M$ such that $f_{3}=f_{3}^{1} \odot f_{3}^{2}$.
Since $f_{1} \odot\left(f_{2} \oplus f_{3}\right) \sqsubseteq f$, and $f$ has empty domain, there must exists $v \in M$ such that

$$
\begin{align*}
f & =f_{1} \odot\left(f_{2} \oplus f_{3}\right) \odot v \\
& =f_{1} \odot f_{3} \odot\left(\text { unit }_{Z \cup Y \cup W} \oplus f_{2}\right) \odot v \\
& =f_{1} \odot f_{3} \odot\left(\text { unit }_{Y \cup W} \oplus f_{2}\right) \odot v \\
& =f_{1} \odot\left(f_{3}^{1} \odot f_{3}^{2}\right) \odot\left(\text { unit }_{Y \cup W} \oplus f_{2}\right) \odot v \\
& =f_{1} \odot f_{3}^{1} \odot\left(f_{3}^{2} \odot\left(\text { unit }_{Y \cup W} \oplus f_{2}\right)\right) \odot v \\
& =f_{1} \odot f_{3}^{1} \odot\left(\left(f_{2} \oplus \text { unit }_{W}\right) \oplus f_{3}^{2}\right) \odot v
\end{align*}
$$

where $\dagger$ follows from Theorem $\mathbf{A . 3 4}$ and $\boldsymbol{\operatorname { d o m }}\left(f_{2} \oplus\right.$ unit $\left._{W}\right)=Z \cup W \subseteq \operatorname{range}\left(f_{3}^{1}\right)$.
Thus, $f_{1} \odot f_{3}^{1} \odot\left(\left(f_{2} \oplus\right.\right.$ unit $\left.\left._{W}\right) \oplus f_{3}^{2}\right) \sqsubseteq f$.
Note that $f_{1} \odot f_{3}^{1}$ has type $\operatorname{Mem}[\emptyset] \rightarrow \mathcal{T} \operatorname{Mem}[Z \cup W]$, so $f_{1} \odot f_{3}^{1} \vDash(\emptyset \triangleright[Z \cup W])$.
State $f_{2} \oplus$ unit $_{W}$ has type $\operatorname{Mem}[Z \cup W] \rightarrow \mathcal{T}\left(\operatorname{Mem}[Z \cup W \cup X]\right.$, so $f_{2} \oplus$ unit $_{W} \vDash(Z \cup W \triangleright[X])$.
State $f_{3}^{2}$ has type $\operatorname{Mem}[Z \cup W] \rightarrow \mathcal{T}(\operatorname{Mem}[Z \cup W \cup Y])$, so $f_{3}^{2} \vDash(Z \cup W \triangleright[Y])$.

By persistence, $f \vDash[Z \cup W] ;([X] *[Y])$, and Weak Union is valid.
Lemma A. 18 (Contraction). The following judgment is valid in any $\mathcal{T}$-model:

$$
\vDash([Z] ;([X] *[Y])) \wedge([Z \cup Y] ;([X] *[W])) \rightarrow[Z] ;([X] *[Y \cup W])
$$

Proof. Let $M$ be a $\mathcal{T}$-model. If $h \vDash([Z] \stackrel{\rho}{\rho}([X] *[Y])) \wedge([Z \cup Y] \rho([X] *[W]))$, then
 $\mathcal{T}(\operatorname{Mem}[Z \cup X]), f_{3}: \operatorname{Mem}[Z] \rightarrow \mathcal{T}(\operatorname{Mem}[Z \cup Y])$, and $f_{1} \odot\left(f_{2} \oplus f_{3}\right) \sqsubseteq h$.
Note $f_{1} \odot\left(f_{2} \oplus f_{3}\right)$ has type $\operatorname{Mem}[\emptyset] \rightarrow \mathcal{T}(\operatorname{Mem}[Z \cup Y \cup Z])$.
 $\operatorname{Mem}[Z \cup Y] \rightarrow \mathcal{T}(\operatorname{Mem}[Z \cup Y \cup X]), g_{3}: \operatorname{Mem}[Z \cup Y] \rightarrow \mathcal{T}(\operatorname{Mem}[Z \cup Y \cup W])$, and $g_{1} \odot\left(g_{2} \oplus g_{3}\right) \sqsubseteq h$.
Note $g_{1} \odot g_{2}$ has type $\operatorname{Mem}[\emptyset] \rightarrow \mathcal{T}(\operatorname{Mem}[Z \cup Y \cup X])$.
By Theorem A. $39 f_{1} \odot\left(f_{2} \oplus f_{3}\right)=g_{1} \odot g_{2}$.

$$
\begin{aligned}
g_{1} \odot\left(g_{2} \oplus g_{3}\right) & =g_{1} \odot\left(g_{2} \oplus \text { unit }_{\text {zuY }}\right) \odot\left(\text { unit }_{\text {ZUYUX }} \oplus g_{3}\right) \\
& =g_{1} \odot g_{2} \odot\left(\text { unit }_{\text {ZUX }} \oplus g_{3}\right) \\
& =f_{1} \odot\left(f_{2} \oplus f_{3}\right) \odot\left(\text { unitzuX }_{\text {zuX }} \oplus g_{3}\right) \\
& =f_{1} \odot\left(\left(f_{2} \odot \text { unit }_{\text {ZuX }}\right) \oplus\left(f_{3} \odot g_{3}\right)\right) \\
& =f_{1} \odot\left(f_{2} \oplus\left(f_{3} \odot g_{3}\right)\right)
\end{aligned}
$$

(By!Theorem A.35)
(Because $Z \cup Y \subseteq \operatorname{dom}\left(g_{2}\right), Y \subseteq \operatorname{dom}\left(g_{3}\right)$ )
$\left(f_{1} \odot\left(f_{2} \oplus f_{3}\right)=g_{1} \odot g_{2}\right)$
(By Exchange equality)

By their types, it is easy to see that $f_{1} \vDash(\emptyset \triangleright[Z]), f_{2} \vDash(Z \triangleright[X]), f_{3} \odot g_{3} \vDash(Z \triangleright[Y \cup W])$. So,

$$
f_{1} \odot\left(f_{2} \oplus\left(f_{3} \odot g_{3}\right)\right) \vDash[Z] \stackrel{([X] *[Y \cup W]) .}{ }
$$

Also, note that $h \sqsupseteq g_{1} \odot\left(g_{2} \oplus g_{3}\right)=f_{1} \odot\left(f_{2} \oplus\left(f_{3} \odot g_{3}\right)\right)$, so by persistence,

$$
h \vDash(\emptyset \triangleright[Z]) \stackrel{\rho}{((Z \triangleright[X]) *(Z \triangleright[Y \cup W])) .}
$$

## H. Section VI Conditional Probabilistic Separation Logic

As our final application, we design a separation logic for probabilistic programs. We work with a simplified probabilistic imperative language with assignments, sampling, sequencing, and conditionals; our goal is to show how a DIBI-based program logic could work in the simplest setting. Following the design of PSL [9], a richer program logic could also layer on constructs for deterministic assignment and deterministic control flow (conditionals and loops) at the cost of increasing the complexity of the programming language and semantics. We do not foresee difficulties in implementing these extensions, and we leave them for future work.

## I. A basic probabilistic programming language

Program syntax: Let Var be a fixed, finite set of program variables. We will consider the following programming language:

$$
\begin{aligned}
\operatorname{Exp} \ni e::= & x \in \operatorname{Var}|t t| f f\left|e \wedge e^{\prime}\right| e \vee e^{\prime} \mid \cdots \\
\operatorname{Com} \ni c::= & \operatorname{skip}|x \leftarrow e| x \longleftarrow \mathbf{B}_{p} \quad(p \in[0,1]) \\
& \left|c ; c^{\prime}\right| \mathbf{i f} x \text { then } c \text { else } c^{\prime}
\end{aligned}
$$

We assume that all variables and expressions are Boolean-valued, for simplicity. The only probabilistic command is $x \longleftarrow \mathbf{B}_{p}$, which draws from a $p$-biased coin flip (i.e., probability of $t t$ is $p$ ) and stores the result in $x$; for instance, $x \longleftarrow \mathbf{B}_{1 / 2}$ samples from a fair coin flip.

$$
\begin{aligned}
\llbracket x \leftarrow e \rrbracket \mu & :=\operatorname{bind}(\mu, m \mapsto \operatorname{unit}(m[x \mapsto \llbracket e \rrbracket(m)])) \\
\llbracket x \hookleftarrow \mathbf{B}_{p} \rrbracket \mu & :=\operatorname{bind}\left(\mu, m \mapsto \operatorname{bind}\left(\operatorname{Bern}_{p}, v \mapsto \operatorname{unit}(m[x \mapsto v])\right)\right) \\
\llbracket c ; c^{\prime} \rrbracket \mu & : \llbracket c^{\prime} \rrbracket(\llbracket c \rrbracket \mu) \\
\llbracket \text { if } b \text { then } c \text { else } c^{\prime} \rrbracket \mu & :=(\llbracket c \rrbracket \mu \mid \llbracket b=t t \rrbracket) \oplus_{p}\left(\llbracket c^{\prime} \rrbracket \mu \mid \llbracket b=f f \rrbracket\right) \quad \text { where } p:=\mu(\llbracket b=t t \rrbracket)
\end{aligned}
$$

Fig. 7: Program semantics

| $z \stackrel{s}{s} \mathbf{B}_{1 / 2} ;$ | $z \stackrel{s}{s} \mathbf{B}_{1 / 2} ;$ |
| :---: | :---: |
| $x \stackrel{s}{s} \mathbf{B}_{1 / 2} ;$ | if $z$ then |
| $y \leftarrow \mathbf{B}_{1 / 2} ;$ | $x \stackrel{s}{ } \mathbf{B}_{p} ; y \stackrel{s}{ } \mathbf{B}_{p}$ |
| $a \leftarrow x \vee z ;$ | else |
| $b \leftarrow y \vee z$ | $x \longleftarrow \mathbf{B}_{q} ; y \stackrel{s}{s} \mathbf{B}_{q}$ |

(a) CommonCause
(b) CondSamples

Fig. 8: Example programs
Program semantics: Following Kozen [1], we give programs a denotational semantics as distribution transformers $\llbracket c \rrbracket$ : $\mathcal{D}(\operatorname{Mem}[\mathrm{Var}]) \rightarrow \mathcal{D}(\operatorname{Mem}[\mathrm{Var}])$, see Figure 7] To define the semantics of randomized conditionals, we will use operations for conditioning to split control flow, and convex combinations to merge control flow. More formally, let $\mu \in \mathcal{D}(A)$ be a distribution, let $S \subseteq A$ be an event, and let $\mu(S)$ be the probability of $S$ in $\mu$. Then the conditional distribution of $\mu$ given $S$ is:

$$
(\mu \mid S)(a):= \begin{cases}\frac{\mu(a)}{\mu(S)} & : a \in S, \mu(S) \neq 0 \\ 0 & : a \notin S\end{cases}
$$

For convex combination, let $p \in[0,1]$ and $\mu_{1}, \mu_{2} \in \mathcal{D}(A)$. We define:

$$
\left(\mu_{1} \oplus_{p} \mu_{2}\right)(a):=p \cdot \mu_{1}(a)+(1-p) \cdot \mu_{2}(a)
$$

When $p=0$ or $p=1$, we define $\oplus_{p}$ lazily: $\mu_{1} \oplus_{0} \mu_{2}:=\mu_{2}$ and $\mu_{1} \oplus_{1} \mu_{2}:=\mu_{1}$. Conditioning and convex combination are inverses in the following sense: $\mu=(\mu \mid S) \oplus_{\mu(S)}(\mu \mid \bar{S})$.

Example A.1. Figure 8 introduces two more example programs. The program CommonCause (Figure 8a) generates a distribution where two random observations share a common cause. Specifically, $z, x$, and $y$ are independent random samples, and $a$ and $b$ are values computed from $(x, z)$ and $(y, z)$, respectively. Intuitively, $z, x, y$ could represent independent noisy measurements, while $a$ and $b$ could represent quantities derived from these measurements. Since $a$ and $b$ share a common source of randomness $z$, they are not independent. However, $a$ and $b$ are independent conditioned on the value; this is a textbook example of conditional independence.

The program CondSamples (Figure 8b) is a bit more complex: it branches on a random value $z$, and then assigns $x$ and $y$ with two independent samples from $\mathbf{B}_{p}$ in the true branch, and $\mathbf{B}_{q}$ in the false branch. While we might think that $x$ and $y$ are independent at the end of the program since they are independent at the end of each branch, this is not true because their distributions are different in the two branches. For example, suppose that $p=1$ and $q=0$. Then at the end of the first branch $(x, y)=(t t, t t)$ with probability 1 , while at the end of the second branch $(x, y)=(f f, f f)$ with probability 1 . Thus observing whether $x=t t$ or $x=f f$ determines the value of $y$-clearly, $x$ and $y$ can't be independent. However, $x$ and $y$ are independent conditioned on $z$. Verifying this example relies on the proof rule for conditionals.

## J. CPSL: Assertion Logic

Like all program logics, CPSL is constructed in two layers: the assertion logic describes program states-here, probability distributions-while the program logic describes probabilistic programs, using the assertion logic to specify pre- and postconditions. Our starting point for the assertion logic is the probabilistic model of DIBI introduced in Section IV, with atomic assertions as in Section $\nabla$ However, it turns out that the full logic DIBI is not suitable for a program logic. The main problem is that not all formulas in DIBI satisfy a key technical condition, known as restriction.

Definition A. 2 (Restriction). A formula $P$ satisfies restriction if: a Markov kernel $f$ satisfies $P$ if and only if there exists $f^{\prime} \sqsubseteq f$ such that range $\left(f^{\prime}\right) \subseteq \mathrm{FV}(P)$ and $f^{\prime} \vDash P$.

The reverse direction is immediate by persistence, but the forward direction is more delicate. Restriction was first considered by Barthe et al. [9] while developing PSL: formulas satisfying restriction are preserved if the program does not modify variables appearing in the formula. This technical property is crucial to supporting Frame-like rules in PSL, which are also used to derive
general versions of rules for assignment and sampling, so failure of the restriction property imposes severe limitations on the program logic. In PSL, assertions were drawn from BI with atomic formulas for modeling random variables. Using properties specific to probability distributions, they showed that their logic is well-behaved with respect to restriction: all formulas satisfy this property. However, DIBI is richer than BI, and there are simple formulas where restriction fails.

Example A. 2 (Failure of restriction). Consider the formula $P:=\top \circ(x \triangleright[x])$, and consider the kernel $f: \operatorname{Mem}[z] \rightarrow$ $\mathcal{D}(\operatorname{Mem}[x, z])$ with $f(z \mapsto c):=\operatorname{unit}(x \mapsto c, z \mapsto c)$. Letting $f_{1}: \operatorname{Mem}[z] \rightarrow \mathcal{D}(\operatorname{Mem}[x, z])$ and $f_{2}: \operatorname{Mem}[x, z] \rightarrow \mathcal{D}(\operatorname{Mem}[x, z])$ with $f_{1}(z \mapsto c):=\operatorname{unit}(x \mapsto c, z \mapsto c) \vDash \top$ and $f_{2}:=$ unit $_{\operatorname{Mem}[x]} \oplus$ unit $_{\operatorname{Mem}[z]} \vDash(x \triangleright[x])$, we have $f=f_{1} \odot f_{2} \vDash P$. Any subkernel $f^{\prime} \sqsubseteq f$ satisfying $P$ and witnessing restriction must be of type $f^{\prime}: \operatorname{Mem}[x] \rightarrow \mathcal{D}(\operatorname{Mem}[x])$, but it is not hard to check that there is no such subkernel.

To address this problem, we will identify a fragment of DIBI that satisfies restriction and is sufficiently rich to support an interesting program logic. Intuitively, restriction may fail for $P$ when a kernel satisfying $P$ (i) implicitly requires unexpected variables in its domain, or (ii) does not describe needed variables in its range. Thus, we employ syntactic conditions to approximate which variables may appear in the domain $\left(\mathrm{FV}_{\mathrm{D}}\right)$, and which variables must appear in the range ( $\mathrm{FV} \mathrm{V}_{\mathrm{R}}$ ).

Definition A. 3 ( $\mathrm{FV}_{\mathrm{D}}$ and $\mathrm{FV}_{\mathrm{R}}$ ). For the formulas in Form ${ }_{\text {RDIBI }}$ generated by probabilistic atomic propositions, conjunctions $(\wedge, *, \stackrel{\circ}{\circ})$ and disjunction $(\vee)$, we define two sets of variables:

$$
\begin{aligned}
\mathrm{FV}_{\mathrm{D}}(\mathrm{~T})=\mathrm{FV}_{\mathrm{D}}(\perp) & :=\emptyset \\
\mathrm{FV}_{\mathrm{D}}(A \triangleright B) & :=\mathrm{FV}(A) \\
\mathrm{FV}_{\mathrm{D}}(P \wedge Q) & :=\mathrm{FV}_{\mathrm{D}}(P) \cup \mathrm{FV}_{\mathrm{D}}(Q) \\
\mathrm{FV}_{\mathrm{D}}(P * Q) & :=\mathrm{FV}_{\mathrm{D}}(P) \cup \mathrm{FV}_{\mathrm{D}}(Q) \\
\mathrm{FV}_{\mathrm{D}}(P ; Q) & :=\mathrm{FV}_{\mathrm{D}}(P) \cup \mathrm{FV}_{\mathrm{D}}(Q) \\
\mathrm{FV}_{\mathrm{D}}(P \vee Q) & :=\mathrm{FV}_{\mathrm{D}}(P) \cup \mathrm{FV}_{\mathrm{D}}(Q)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{FV}_{\mathrm{R}}(\mathrm{~T})=\mathrm{FV}_{\mathrm{R}}(\perp) & :=\emptyset \\
\mathrm{FV}_{\mathrm{R}}(A \triangleright B) & :=\mathrm{FV}(A) \cup \mathrm{FV}(B) \\
\mathrm{FV}_{\mathrm{R}}(P \wedge Q) & :=\mathrm{FV}_{\mathrm{R}}(P) \cup \mathrm{FV}_{\mathrm{R}}(Q) \\
\mathrm{FV}_{\mathrm{R}}(P * Q) & :=\mathrm{FV}_{\mathrm{R}}(P) \cup \mathrm{FV}_{\mathrm{R}}(Q) \\
\mathrm{FV}_{\mathrm{R}}(P ; Q) & :=\mathrm{FV}_{\mathrm{R}}(P) \cup \mathrm{FV}_{\mathrm{R}}(Q) \\
\mathrm{FV}_{\mathrm{R}}(P \vee Q) & :=\mathrm{FV}_{\mathrm{R}}(P) \cap \mathrm{FV}_{\mathrm{R}}(Q)
\end{aligned}
$$

Now, we have all the ingredients to introduce our assertions. The logic RDIBI is a fragment of DIBI with atomic propositions $\mathcal{A P}$, with formulas Form ${ }_{\text {RDIBI }}$ defined by the following grammar:

$$
\begin{aligned}
P, Q & ::=\mathcal{A P}|\mathrm{\top}| \perp|P \vee Q| P * Q \\
& \mid P \circ Q \quad\left(\mathrm{FV}_{\mathrm{D}}(Q) \subseteq \mathrm{FV}_{\mathrm{R}}(P)\right) \\
& \mid P \wedge Q \quad\left(\mathrm{FV}_{\mathrm{R}}(P)=\mathrm{FV}_{\mathrm{R}}(Q)=\mathrm{FV}(P)=\mathrm{FV}(Q)\right)
\end{aligned}
$$

The side-condition for $P \circ Q$ ensures that variables used by $Q$ are described by $P$. The side-condition for $P \wedge Q$ is the most restrictive-to understand why we need it, consider the following example.

Example A. 3 (Failure of restriction for $\wedge$ ). Consider the formula $P:=(\emptyset \triangleright[x]) \wedge(\emptyset \triangleright[y])$, and kernel $f: \operatorname{Mem}[z] \rightarrow$ $\mathcal{D}(\operatorname{Mem}[x, y, z])$ with $f(z \mapsto t t)$ being the distribution with $x$ a fair coin flip, $y=x$, and $z=t t$, and $f(z \mapsto f f)$ being the distribution with $x$ a fair coin flip, $y=\neg x$, and $z=f f$. Then, there exist $f_{1}: \operatorname{Mem}[\emptyset] \rightarrow \mathcal{D}(\operatorname{Mem}[x])$ and $f_{2}: \operatorname{Mem}[\emptyset] \rightarrow \mathcal{D}(\operatorname{Mem}[y])$ such that $f_{1} \sqsubseteq f$ and $f_{2} \sqsubseteq f$. Since $f_{1} \vDash(\emptyset \triangleright[x])$ and $f_{2} \vDash(\emptyset \triangleright[y])$, it follows $f \vDash P$. But, because $z$ is correlated with $(x, y)$, there is no kernel $f^{\prime}: \operatorname{Mem}[\emptyset] \rightarrow \mathcal{D}(\operatorname{Mem}[x, y])$ satisfying $P$ such that $f^{\prime} \sqsubseteq f$.

When we take atomic propositions from Section $\overline{\mathrm{V}}$ formulas are pairs of sets of variables: $(A \triangleright[B])$ where $A, B \subseteq$ Var. With these atoms, all formulas in RDIBI satisfy restriction. Before showing this property, however, we will enrich the atomic propositions to describe more fine-grained information about the domain and range of kernels:
Domain. Given a kernel $f$, the existing atomic propositions can only describe properties that hold for all (well-typed) inputs $m$ to $f$. We would like to be able to describe properties that hold for only certain inputs, e.g., for memories $m$ where a variable $z$ is true.
Range. Given any input $m$ to a kernel $f$, the existing atomic propositions can only guarantee the presence of variables in the output distribution $f(m)$. We would like describe more precise information about $f(m)$, e.g., that certain variables are independent conditioned on a particular value of $m$, rather on all values of $m$.
Our strategy will be to extend atomic propositions to all pairs of logical formula $(D \triangleright R)$, where $D$ is a logical formula over the kernel domain (i.e., memories), while $R$ is a logical formula over the kernel range (i.e., distributions over memories).

To describe memories, we take a simple propositional logic for the domain logic.
Definition A. 4 (Domain logic). The domain logic has formulas $D$ of the form $S: p_{d}$, where $S \subseteq$ Var is a subset of variables and: $p_{d}::=x=e|\top| \perp\left|p_{d} \wedge p_{d}^{\prime}\right| p_{d} \vee p_{d}^{\prime}$. A formula $S: p_{d}$ is satisfied in $m \in \operatorname{Mem}[T]$, written $m \vDash_{d} S: p_{d}$, if $S=T$ and $p_{d}$ holds in $m$.

We can read $S: p_{d}$ as "memories over $S$ such that $p_{d}$ " and abbreviate $S:$ 〒 as just $S$. To describe distributions over memories, we adapt probabilistic BI [9] for the range logic.

Definition A. 5 (Range logic). The range logic has the following formulas from probabilistic BI:

$$
p_{r}::=[S] \quad(S \subseteq \operatorname{Var})|x \sim d| x=e|\top| \perp\left|p_{r} \wedge p_{r}^{\prime}\right| p_{r} * p_{r}^{\prime} .
$$

We give a semantics where states are distributions over memories: $M_{r}=\{\mu: \mathcal{D}(\operatorname{Mem}[S]) \mid S \subseteq \operatorname{Var}\}$. We define a preorder on states via $\mu_{1} \sqsubseteq_{r} \mu_{2}$ if and only if $\operatorname{dom}\left(\mu_{1}\right) \subseteq \operatorname{dom}\left(\mu_{2}\right)$ and $\pi_{\operatorname{dom}\left(\mu_{1}\right)} \mu_{2}=\mu_{1}$, and we define a partial binary operation on states: if $\operatorname{dom}\left(\mu_{1}\right)=S_{1} \cup T$ and $\operatorname{dom}\left(\mu_{2}\right)=S_{2} \cup T$ with $S_{1}, S_{2}, T$ disjoint, and $\pi_{T} \mu_{1}=\pi_{T} \mu_{2}=\operatorname{unit}(m)$ for some $m \in \operatorname{Mem}[T]$, then

$$
\mu_{1} \oplus_{r} \mu_{2}:=\pi_{S_{1}} \mu_{1} \otimes \operatorname{unit}(m) \otimes \pi_{S_{2}} \mu_{2}
$$

where $\otimes$ takes the independent product of two distributions over disjoint domains; otherwise $\oplus_{r}$ is not defined. This operation generalizes the monoid from probabilistic BI to allow combining distributions with overlapping domains if the distributions over the overlap are deterministic and equal; this mild generalization is useful for our setting, where distributions often have deterministic variables (e.g., variables corresponding to the input of kernels).

Then, we define the semantics of the range logic as:

$$
\begin{array}{ll}
\mu \vDash_{r} \top & \quad \text { always } \quad \mu \vDash_{r} \perp \quad \text { never } \\
\mu \models_{r}[S] & \text { iff } S \subseteq \operatorname{dom}(\mu) \\
\mu \vDash_{r} x \sim d & \text { iff } x \in \operatorname{dom}(\mu) \text { and } \pi_{x} \mu=\llbracket d \rrbracket m_{v}, \text { where unit }\left(m_{v}\right)=\pi_{\mathrm{FV}(d)} \mu \\
\mu \vDash_{r} x=e & \text { iff }\{x\}, \mathrm{FV}(e) \subseteq \operatorname{dom}(\mu) \text { and } \mu(\llbracket x=e \rrbracket)=1 \\
\mu \vDash_{r} p_{r} \wedge p_{r}^{\prime} & \text { iff } \mu \models_{r} p_{r} \text { and } \mu \vDash_{r} p_{r}^{\prime} \\
\mu \vDash_{r} p_{r} * p_{r}^{\prime} & \text { iff there exists } \mu_{1} \oplus_{r} \mu_{2} \sqsubseteq \mu \text { with } \mu_{1} \vDash_{r} p_{r} \text { and } \mu_{2} \vDash_{r} p_{r}^{\prime} .
\end{array}
$$

Now, we can give a semantics to our enriched atomic propositions.
Definition A.6. Given a kernel $f$ and atomic proposition $(D \triangleright R)$, we define a persistent semantics:

$$
f \models(D \triangleright R) \text { iff there exists } f^{\prime} \sqsubseteq f \text { such that } m \models_{d} D \text { implies } m \in \operatorname{dom}\left(f^{\prime}\right) \text { and } f(m) \models_{r} R \text {. }
$$

Atomic propositions satisfy the following axiom schemas, inspired by Hoare logic.
Proposition A.19. The following axiom schemas for atomic propositions are sound.

$$
\begin{align*}
& \left(S: p_{d} \triangleright p_{r}\right) \wedge\left(S: p_{d}^{\prime} \triangleright p_{r}^{\prime}\right) \rightarrow\left(S: p_{d} \wedge p_{d}^{\prime} \triangleright p_{r} \wedge p_{r}^{\prime}\right) \quad \text { if } F V\left(p_{r}\right)=F V\left(p_{r}^{\prime}\right)  \tag{AP-And}\\
& \left(S: p_{d} \triangleright p_{r}\right) \wedge\left(S: p_{d}^{\prime} \triangleright p_{r}^{\prime}\right) \rightarrow\left(S: p_{d} \vee p_{d}^{\prime} \triangleright p_{r} \vee p_{r}^{\prime}\right)  \tag{AP-OR}\\
& \left(S: p_{d} \triangleright p_{r}\right) *\left(S^{\prime}: p_{d}^{\prime} \triangleright p_{r}^{\prime}\right) \rightarrow\left(S \cup S^{\prime}: p_{d} \wedge p_{d}^{\prime} \triangleright p_{r} * p_{r}^{\prime}\right)  \tag{AP-PAR}\\
& p_{d}^{\prime} \rightarrow p_{d} \text { and } \models_{r} p_{r} \rightarrow p_{r}^{\prime} \text { implies } \vDash\left(S: p_{d} \triangleright p_{r}\right) \rightarrow\left(S: p_{d}^{\prime} \triangleright p_{r}^{\prime}\right)
\end{align*}
$$

(AP-Імр)
Finally, formulas in RDIBI satisfy restriction.
Theorem A. 20 (Restriction in RDIBI). Let $P \in$ Form $_{\text {RDIBI }}$ with atomic propositions $(D \triangleright R)$, as described above. Then $f \vDash P$ if and only if there exists $f^{\prime} \sqsubseteq f$ such that range $\left(f^{\prime}\right) \subseteq F V(P)$ and $f^{\prime} \models P$.

Proof sketch.. By induction on $P$, proving a stronger statement: $f \vDash P$ if and only if there exists $f^{\prime} \sqsubseteq f$ such that $\operatorname{dom}\left(f^{\prime}\right) \subseteq$ $\mathrm{FV}_{\mathrm{D}}(P)$, and $\mathrm{FV}_{\mathrm{R}}(P) \subseteq \operatorname{range}\left(f^{\prime}\right) \subseteq \mathrm{FV}(P)$.

## K. CPSL: program logic

With the assertion logic set, we are now ready to introduce our program logic. Judgments in CPSL have the form $\{P\} c\{Q\}$, where $c \in \mathrm{Com}$ is a probabilistic program and $P, Q \in \mathrm{Form}_{\text {RDIBI }}$ are restricted assertions. As usual, a program in a judgment maps states satisfying the pre-condition to states satisfying the post-condition.

Definition A. 7 (CPSL Validity). A CPSL judgment $\{P\} c\{Q\}$ is valid, written $\vDash\{P\} c\{Q\}$, if for every input distribution $\mu \in \mathcal{D}(\operatorname{Mem}[\operatorname{Var}])$ such that the lifted input $f_{\mu}: \operatorname{Mem}[\emptyset] \rightarrow \mathcal{D}(\operatorname{Mem}[\operatorname{Var}])$ satisfies $f_{\mu} \vDash P$, the lifted output satisfies $f_{\llbracket c \rrbracket \mu} \vDash Q$.

The proof rules of CPSL are presented in Figure 9 Note that all rules implicitly require that assertions are from RDIBI, e.g., the rule Asss requires that the post-condition $P ;(\mathrm{FV}(e) \triangleright x=e)$ is a formula in RDIBI, which in turn requires that $\mathrm{FV}(e)=\mathrm{FV}_{\mathrm{D}}(\mathrm{FV}(e) \triangleright x=e) \subseteq \mathrm{FV}_{\mathrm{R}}(P)$.

The rules Skip, Seqn, Weak are standard, we comment on the other, more interesting rules. Assn and Samp allow forward reasoning across assignments and random sampling commands. In both cases, a pre-condition that does not mention the assigned variable $x$ is augmented with new information tracking the value or distribution of $x$, and variables $x$ may depend on.

$$
\begin{aligned}
& \text { Assn } \frac{x \notin \mathrm{FV}(e) \cup \mathrm{FV}(P)}{\vdash\{P\} x \leftarrow e\{P ;(\mathrm{FV}(e) \triangleright x=e)\}} \\
& \text { Skip } \overline{\vdash\{P\} \text { skip }\{P\}} \\
& \text { SAMP } \frac{x \notin \mathrm{FV}(d) \cup \mathrm{FV}(P)}{\vdash\{P\} x \Leftarrow d\{P \circ(\mathrm{FV}(d) \triangleright x \sim d)\}} \\
& \text { SEQN } \frac{\vdash\{P\} c\{Q\} \quad \vdash\{Q\} c^{\prime}\{R\}}{\vdash\{P\} c ; c^{\prime}\{R\}}
\end{aligned}
$$

$$
\begin{aligned}
& W_{\text {EAK }} \frac{\begin{array}{c}
\vdash\{P\} c\{Q\} \\
\vDash P^{\prime} \rightarrow P \wedge Q \rightarrow Q^{\prime}
\end{array}}{\vdash\left\{P^{\prime}\right\} c\left\{Q^{\prime}\right\}} \\
& \begin{array}{c}
\stackrel{\vdash\{P\} c\{Q\} \quad \mathrm{FV}(R) \cap \mathrm{MV}(c)=\emptyset}{\mathrm{FV}(Q) \subseteq \mathrm{FV}_{\mathrm{R}}(P) \cup \mathrm{WV}(c) \quad \mathrm{RV}(c) \subseteq \mathrm{FV}_{\mathrm{R}}(P)} \\
\text { FRAME }^{\mathrm{FV}(P * R\} c\{Q * R\}}
\end{array}
\end{aligned}
$$

Fig. 9: Proof rules: CPSL
DCond allows reasoning about probabilistic control flow, and the ensuing conditional dependence that may result. The main pre-condition $P$ is allowed to depend on the guard variable $b$-recalling that $\mathrm{FV}_{\mathrm{D}}(P) \subseteq \mathrm{FV}_{\mathrm{R}}(\emptyset \triangleright[b])$-and $P$ is preserved as a pre-condition for both branches. The post-conditions allows introducing new facts ( $b: b=t t \triangleright Q_{1}$ ) and $\left(b: b=t t \triangleright Q_{2}\right)$, which are then combined in the post-condition of the entire conditional command. As in PSL, the rule for conditionals does not allow the branches to modify the guard $b$-this restriction is needed to accurately associate each post-condition to each branch.

Finally, Frame is the frame rule for CPSL. Much like in PSL, the rule involves three classes of variables: MV $(c)$ is the set of variables that $c$ may write to, $\mathrm{RV}(c)$ is the set of variables that $c$ may read from the input, and $\mathrm{WV}(c)$ is the set of variables that $c$ must write to; these variable sets are defined in Appendix $\mathbf{N}$ Then, Frame is essentially the same as in PSL. The first side-condition $\mathrm{FV}(R) \cap \mathrm{MV}(c)$ ensures that the framing condition is not modified-this condition is fairly standard. The second and third side-conditions are more specialized. First, the variables described by $Q$ in the post-condition are either already described by $P$ in the pre-condition, or are written by $c$. Second, the variables read by $c$ must be described by $P$ in the pre-condition. These two side-conditions ensure that variables mentioned by $Q$ that were not already independent of $R$ are freshly written, and freshly written variables are derived from variables that were already independent of $R$.

Theorem A. 21 (CPSL Soundness). CPSL is sound: derivable judgments are valid.
Proof sketch. By induction on the proof derivation. The restriction property is used repeatedly to constrain the domains and ranges of kernels witnessing different sub-assertions, ensuring that pre-conditions about unmodified variables continue to hold in the post-condition.

## L. Example: proving conditional independence for programs

Now, we show how to use CPSL to verify our two example programs in Figure 8 In both cases, we will prove a conditional independence assertion as the post-condition. We will need some axioms for implications between formulas in RDIBI; these axioms are valid in our probabilistic model $M^{D}$.

Proposition A.22. (Axioms for RDIBI) The following axioms are sound, assuming both precedent and antecedent are in Form RDIBI-

$$
\begin{align*}
& (P \circ Q) \stackrel{\circ}{\circ} \rightarrow P \circ(Q * R)  \tag{IndEP-1}\\
& P \circ Q \rightarrow P * Q  \tag{Indep-2}\\
& P \circ Q \rightarrow P \circ(Q *(S \triangleright[S]))  \tag{PAD}\\
& (P * Q) \stackrel{ }{\circ}(R * S) \rightarrow(P \circ R) *(Q \circ S) \tag{RestExch}
\end{align*}
$$

We briefly explain the axioms. InDEP-1 holds because $P \circ(Q * R) \in$ Form $_{\text {RDIBI }}$ implies that $R$ only mentions variables that are guaranteed to be in $P$. InDEP-2 holds because any kernel witnessing $Q$ depends on no variables and thus independent of any kernel witnessing $P$. PAD allows conjoining $(S \triangleright[S])$ to the second conjunct; since $P \circ(Q *(S \triangleright[S])$ ) is in RDIBI, $S$ can only mention variables that are already in $P$. Finally, RESTEXCH shows that the standard exchange law holds for restricted assertions. We defer the proof to Appendix 0 .

We also need the following axioms for a particular form of atomic propositions, in addition to the axioms for general atomic propositions in Theorem A. 19

$$
\begin{align*}
& (S \triangleright[A] *[B]) \rightarrow(S \triangleright[A]) *(S \triangleright[B]) \quad \text { if } A \cap B \subseteq S \\
& \text { (RevPar) } \\
& (S \triangleright[A] *[B]) \rightarrow(S \triangleright[A \cup B])  \tag{UnionRan}\\
& (A \triangleright[B]) \stackrel{\circ}{(B \triangleright[C]) \rightarrow(A \triangleright[C])} \\
& (A \triangleright[B]) \rightarrow(A \triangleright[A]) \stackrel{(A \triangleright[B])}{ } \\
& (A \triangleright[B]) \rightarrow(A \triangleright[B]) \stackrel{\circ}{ }(B \triangleright[B]) \\
& \text { (AtomSeq) } \\
& \text { (UnitL) } \\
& \text { (UnitR) }
\end{align*}
$$

We defer the proof to Appendix 0 .
Now, we will describe how to verify our example programs, CommonCause and CondSamples. Throughout, we must ensure that all formulas used in CPSL rules or RDIBI axioms are in Form RDibI . The product $\stackrel{\text { raises a tricky point: Form }}{\text { RDIBI }}$ is not closed under reassociating $\stackrel{\circ}{9}$, so we add parentheses for formulas that must be in RDIBI. However, we may soundly use the full proof system of DIBI when proving implications between RDIBI assertions, since RDIBI is a fragment of DIBI.

Verification of CommonCause: We aim to prove the following judgment:

$$
\vdash\{T\} \text { CommonCause }\{(\emptyset \triangleright[z]) \stackrel{q}{( }(z \triangleright[a]) *(z \triangleright[b]))\}
$$

By Theorem V.1, this shows that $a, b$ are conditionally independent given $z$ at the end of the program. Using Samp to handle the sampling for $z, x, y$, we can prove the assertion: $(\emptyset \triangleright[z]) \stackrel{\circ}{\circ}(\emptyset \triangleright[x]) \stackrel{ }{(\emptyset}(\emptyset \triangleright[y])$. Using Axioms Pad UnitL, AP-PaR, UnionRan, and $\stackrel{\circ}{ }$ Assoc, this assertion implies $(\emptyset \triangleright[z]) \stackrel{\circ}{g}(z \triangleright[z, x]) \stackrel{ }{q}(z \triangleright[z, y])$. We take this as the pre-condition before assigning to $a$ and assigning to $b$. After the assignments, Assn proves:

$$
(((\emptyset \triangleright z) \stackrel{q}{q}(z \triangleright[z, x]) \stackrel{q}{q}(z \triangleright[z, y])) \stackrel{q}{ }(z, x \triangleright[a])) \dot{q}(z, y \triangleright[b]) .
$$

Then, we can apply INDEP-1 to derive: $(\emptyset \triangleright[z]) \stackrel{\circ}{g}((z \triangleright[z, x]) \dot{g}(z, x \triangleright[a])) *((z \triangleright[z, y]) \dot{q}(z, y \triangleright[b]))$. By Axiom AtomSeQ we obtain the desired post-condition: $(\emptyset \triangleright[z]) \stackrel{\circ}{( }(z \triangleright[a]) *(z \triangleright[b]))$.

Verification of CondSamples: We aim to show the following judgment:

$$
\vdash\{T\} \text { CondSamples }\{(\emptyset \triangleright[z]) \stackrel{q}{g}((z \triangleright[x]) *(z \triangleright[y]))\}
$$

By Theorem V.1, this shows that $x, y$ are conditionally independent given $z$ at the end of the program. Starting with the sampling statement for $z$, applying SAMP and Axiom Indep-2 gives:

$$
\vdash\{T\} z \hookleftarrow \mathbf{B}_{1 / 2}\{(\emptyset \triangleright[z]) \stackrel{\circ}{9}\}
$$

To reason about the branching, we use DCond. We start with the first branch. By Samp, Weak and Seq, we have $\vdash\{(0 \triangleright$ $z=t t) \circ T\} x \leftrightarrow \mathbf{B}_{p} \circ y \leftrightarrow \mathbf{B}_{p}\{(\emptyset \triangleright z=t t) \dot{q}(\emptyset \triangleright[x]) \circ(\emptyset \triangleright[y])\}$. As before, Axioms Pad UNITL AP-PaR, UNIONRaN together with $\stackrel{\circ}{9}$ Assoc give the post-condition

$$
(\emptyset \triangleright z=t t) \stackrel{q}{q}(z \triangleright[z, x]) \stackrel{(z \triangleright[z, y]) .}{ }
$$

Applying Axiom InDEP-1 we can show $(\emptyset \triangleright z=t t) ;((z \triangleright[z, x]) *(z \triangleright[z, y]))$ at the end of the branch. Thus: $\vdash\{(\emptyset \triangleright$


Applying DCond, we have:

$$
\vdash\{(\emptyset \triangleright[z])\} \text { CondSAMPLES }\{(\emptyset \triangleright[z]) \stackrel{q}{9}((z: z=t t \triangleright[z, x] *[z, y]) \wedge(z=f f \triangleright[z, x] *[z, y]))\}
$$

 logic, we have: $\vDash_{d} z: \top \rightarrow z:(z=t t \vee z=f f)$ and

$$
\vDash_{r}[z, x] *[z, y] \vee[z, x] *[z, y] \rightarrow[z, x] *[z, y]
$$

So AP-ImP implies $(\emptyset \triangleright[z]) \stackrel{\circ}{q}(z \triangleright[z, x] *[z, y])$. We can then apply RevPAR because $\{z, x\} \cap\{z, y\}=z$, deriving the postcondition $(\emptyset \triangleright[z]) \stackrel{\circ}{9}((z \triangleright[z, x]) *(\mathrm{z} \triangleright[z, y]))$. By Axiom SpLIT , we obtain the desired post-condition: $(\emptyset \triangleright[z]) \stackrel{\circ}{\circ}((z \triangleright[x]) *(\mathrm{z} \triangleright[y]))$.

## M. Section J atomic propositions: Omitted Details

As we described in Appendix J, atomic formulas for CPSL are of the form $(D \triangleright R)$. The domain assertions $D$ are of the form $S: \phi_{d}$, where $S$ is a set of variables and $\phi_{d}$ describes memories, and the range assertions $R$ are of the form $\phi_{r}$, where $\phi_{r}$ is from a fragment of probabilistic BI.

Proposition A.24. The following axiom schemas for atomic propositions are sound.

$$
\begin{align*}
& \left(S: p_{d} \triangleright p_{r}\right) \wedge\left(S: p_{d}^{\prime} \triangleright p_{r}^{\prime}\right) \rightarrow\left(S: p_{d} \wedge p_{d}^{\prime} \triangleright p_{r} \wedge p_{r}^{\prime}\right) \quad \text { if } F V\left(p_{r}\right)=F V\left(p_{r}^{\prime}\right)  \tag{AP-AND}\\
& \left(S: p_{d} \triangleright p_{r}\right) \wedge\left(S: p_{d}^{\prime} \triangleright p_{r}^{\prime}\right) \rightarrow\left(S: p_{d} \vee p_{d}^{\prime} \triangleright p_{r} \vee p_{r}^{\prime}\right)  \tag{AP-OR}\\
& \left(S: p_{d} \triangleright p_{r}\right) *\left(S^{\prime}: p_{d}^{\prime} \triangleright p_{r}^{\prime}\right) \rightarrow\left(S \cup S^{\prime}: p_{d} \wedge p_{d}^{\prime} \triangleright p_{r} * p_{r}^{\prime}\right)  \tag{AP-PAR}\\
& p_{d}^{\prime} \rightarrow p_{d} \text { and } \models_{r} p_{r} \rightarrow p_{r}^{\prime} \text { implies } \vDash\left(S: p_{d} \triangleright p_{r}\right) \rightarrow\left(S: p_{d}^{\prime} \triangleright p_{r}^{\prime}\right)
\end{align*}
$$

(AP-Iмp)
Proof. We check each of the axioms.
Case: AP-AND. Suppose that $w \vDash\left(S: p_{d} \triangleright p_{r}\right) \wedge\left(S: p_{d}^{\prime} \triangleright p_{r}^{\prime}\right)$. By semantics of atomic propositions, there exists $w_{1} \sqsubseteq_{k} w$ and $w_{2} \sqsubseteq_{k} w$ such that for all $m \in \operatorname{Mem}[S]$ such that $m \vDash_{d} p_{d} \wedge p_{d}^{\prime}$, we have $w_{1}(m) \vDash_{r} p_{r}$ and $w_{2}(m) \vDash_{r} p_{r}^{\prime}$. By restriction (Theorem A.20), we may assume that range $\left(w_{1}\right)=\mathrm{FV}\left(p_{r}\right)=\mathrm{FV}\left(p_{r}^{\prime}\right)=\operatorname{range}\left(w_{2}\right)$. Thus, Theorem A.25 implies that $w_{1}=w_{2}$, and so $w \vDash\left(S: p_{d} \wedge p_{d}^{\prime} \triangleright p_{r} \wedge p_{r}^{\prime}\right)$.
Case: AP-OR Immediate, by semantics of $\vee$.
Case: AP-PAR Suppose that $w \vDash\left(S: p_{d} \triangleright p_{r}\right) *\left(S^{\prime}: p_{d}^{\prime} \triangleright p_{r}^{\prime}\right)$. We will show that $w \vDash\left(S \cup S^{\prime}: p_{d} * p_{d}^{\prime} \triangleright p_{r} * p_{r}^{\prime}\right)$.
By semantics of atomic propositions, there exists $w_{1} \sqsubseteq_{k} w$ and $w_{2} \sqsubseteq_{k} w$ such that $w_{1} \oplus w_{2} \sqsubseteq w$, and for all $m_{1} \in \operatorname{Mem}[S]$ such that $m_{1} \vDash_{d} p_{d}$, we have $w_{1}\left(m_{1}\right) \vDash_{r} p_{r}$, and for all $m_{2} \in \operatorname{Mem}\left[S^{\prime}\right]$ such that $m_{2} \vDash_{d} p_{d}^{\prime}$, we have $w_{2}\left(m_{2}\right) \vDash_{r} p_{r}^{\prime}$.
Now for any $m \in \operatorname{Mem}\left[S \cup S^{\prime}\right]$ such that $m \vDash_{d} p_{d} \wedge p_{d}^{\prime}$, we have $m^{S} \vDash_{d} p_{d}$ and $m^{S^{\prime}} \vDash_{d} p_{d}^{\prime}$. Thus $w_{1}\left(m^{S}\right) \not \vDash_{r} p_{r}$ and $w_{2}\left(m^{S^{\prime}}\right) \vDash_{r} p_{r}^{\prime}$. Letting $T=S \cap S^{\prime}$ and $T_{1}=S \backslash T ; T_{2}=S^{\prime} \backslash T$ be disjoint sets, and noting that $w_{1}, w_{2}$ both preserve inputs on $T$, we have:

$$
\begin{aligned}
w_{1} \oplus w_{2}(m) & =\pi_{T_{1}} w_{1}\left(m^{S}\right) \otimes \operatorname{unit}\left(m^{T}\right) \otimes \pi_{T_{2}} w_{2}\left(m^{S^{\prime}}\right) \\
& =\left(\pi_{T_{1}} w_{1}\left(m^{S}\right) \otimes \operatorname{unit}\left(m^{T}\right)\right) \oplus_{r}\left(\operatorname{unit}\left(m^{T}\right) \otimes \pi_{T_{2}} w_{2}\left(m^{S^{\prime}}\right)\right) \\
& =w_{1}\left(m^{S}\right) \oplus_{r} w_{2}\left(m^{S^{\prime}}\right) \\
& =_{r} p_{r} * p_{r}^{\prime}
\end{aligned}
$$

Thus, $w \vDash\left(S \cup S^{\prime}: p_{d} * p_{d}^{\prime} \triangleright p_{r} * p_{r}^{\prime}\right)$.
Case: AP-IMP, Immediate, by semantics of $\rightarrow$.

For the proof of Theorem A.20 we need the following characterization of $g \sqsubseteq f$.
Proposition A.25. Let $f$ be a Markov kernel, and let $D \subseteq \operatorname{dom}(f) \subseteq R \subseteq \operatorname{range}(f)$. Then we have $\pi_{R}(f(m))=g\left(m^{\prime}\right)$ for all $m^{\prime} \in \operatorname{Mem}[D], m \in \operatorname{Mem}[\operatorname{dom}(f)]$ such that $m^{D}=m^{\prime}$ if and only if $g \sqsubseteq f$ and $\operatorname{dom}(g)=D, \operatorname{range}(g)=R$.

Proof. For the reverse direction, suppose that $f=\left(g \oplus\right.$ unit $\left._{S}\right) \odot v$, with $S$ disjoint from $\operatorname{dom}(g)$. Since range $(g) \subseteq \operatorname{dom}(v)$, we have:

$$
\begin{aligned}
\pi_{R}(f(m)) & =\pi_{R}\left(\left(g \oplus \text { unit }_{S}\right)(m)\right) \\
& =\pi_{R}\left(g\left(m^{D}\right) \oplus \text { unit }_{S}\left(m^{S}\right)\right) \\
& =\pi_{R}\left(g\left(m^{D}\right)\right) \otimes \pi_{R}\left(\text { unit }_{S}\left(m^{S}\right)\right) \\
& =g\left(m^{D}\right) \\
& =g\left(m^{\prime}\right) .
\end{aligned}
$$

For the forward direction, evidently $\operatorname{dom}(g)=D$ and $\operatorname{range}(g)=R$. Since $f$ preserves input to output, we have $\pi_{\operatorname{dom}(f)}\left(g\left(m^{\prime}\right)\right)=$ $\pi_{\operatorname{dom}(f)}(f(m))=\operatorname{unit}\left(m^{\prime}\right)$ so $g$ preserves input to output and $g$ is a Markov kernel. We claim that $g \sqsubseteq f$. First, consider $g \oplus$ unit $_{\operatorname{dom}(f) \backslash D}$; write $D^{\prime}=\operatorname{dom}(f) \backslash D$. For any $m \in \operatorname{Mem}[\operatorname{dom}(f)]$, we have:

$$
\begin{aligned}
\pi_{D^{\prime} \cup R}(f(m)) & =\pi_{R}(f(m)) \otimes \pi_{D^{\prime}}(f(m)) \\
& =g\left(m^{D}\right) \otimes \text { unit }_{D^{\prime}}\left(m^{D^{\prime}}\right) \\
& =\left(g \oplus \text { unit }_{D^{\prime}}\right)(m) .
\end{aligned}
$$

So by Theorem A.10, for every $m \in \operatorname{Mem}[\operatorname{dom}(f)]$ there exists a family of kernels $g_{m}^{\prime}: \operatorname{Mem}\left[D^{\prime} \cup R\right] \rightarrow \mathcal{D}(\operatorname{Mem}[\operatorname{range}(f)])$ such that

$$
f(m)=\operatorname{bind}\left(\left(g \oplus \operatorname{unit}_{D^{\prime}}\right)(m), g_{m}^{\prime}\right)
$$

Defining $g^{\prime}(m) \triangleq g_{m^{\operatorname{dom}(f)}}^{\prime}(m)$, we have:

$$
f(m)=\left(\left(g \oplus \text { unit }_{D^{\prime}}\right) \odot g^{\prime}\right)(m)
$$

and so $g \sqsubseteq f$.
We prove that all assertions in the restricted logic RDIBI satisfy restriction.
Theorem A. 20 (Restriction in RDIBI). Let $P \in$ Form $_{\text {RDIBI }}$ with atomic propositions $(D \triangleright R)$, as described above. Then $f \vDash P$ if and only if there exists $f^{\prime} \sqsubseteq f$ such that range $\left(f^{\prime}\right) \subseteq F V(P)$ and $f^{\prime} \vDash P$.
Proof. The reverse direction is immediate from persistence. For the forward direction, we argue by induction with a stronger hypothesis. If $f \vDash P$, we call a state $f^{\prime}$ a witness of $f \vDash P$ if $f^{\prime} \sqsubseteq f, \mathrm{FV}_{\mathrm{R}}(P) \subseteq \operatorname{range}\left(f^{\prime}\right) \subseteq \mathrm{FV}(P), \operatorname{dom}\left(f^{\prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(P)$, and $f^{\prime} \vDash P$. We show that $f \vDash P$ implies that there is a witness $f^{\prime} \vDash P$, by induction on $P$.
Case $(D \triangleright R)$ : We will use two basic facts, both following from the form of the domain and range assertions:

1) If $m \not \vDash_{d} D$, then $\operatorname{dom}(m)=\mathrm{FV}(D)$.
2) If $\mu \models_{r} R$, then $\operatorname{dom}(\mu) \supseteq \operatorname{FV}(D)$.
$f \vDash(D \triangleright R)$ implies that there exists $f^{\prime} \sqsubseteq f$ such that for any $m \in M_{d}$ such that $m \vDash_{d} D, f^{\prime}(m)$ is defined and $f^{\prime}(m) \vDash_{r} R$.
Let $T=\operatorname{range}\left(f^{\prime}\right) \cap(\mathrm{FV}(D) \cup \mathrm{FV}(R))$. We claim that $\pi_{T} f^{\prime}$ is the desired witness for $f \vDash P$.

- $\pi_{T} f^{\prime}$ is defined and $\pi_{T} f^{\prime} \sqsubseteq f$ because:

$$
\begin{aligned}
\operatorname{dom}\left(f^{\prime}\right) & =\operatorname{dom}(m) \\
& =\mathrm{FV}(D) \\
& \subseteq T
\end{aligned}
$$

(for any $m \in M_{d}$ such that $m \not \models_{d} D$ )

Thus $\pi_{T} f^{\prime}$ is defined, and $\pi_{T} f^{\prime} \sqsubseteq f^{\prime} \sqsubseteq f$.

- range $\left(\pi_{T} f^{\prime}\right)=T \subseteq \mathrm{FV}(D) \cup \mathrm{FV}(R)=\mathrm{FV}(P)$.
- $\pi_{T} f^{\prime} \vDash(D \triangleright R)$ : For any $m \in M_{d}$ such that $m \not{ }_{d} D, f^{\prime}(m)$ is a distribution. Based on the restriction theorem for probabilistic BI, $\pi_{\mathrm{FV}(R) \cap \operatorname{range}\left(f^{\prime}\right)}\left(f^{\prime}(m)\right) \vDash R$ too. Since $T \supseteq \mathrm{FV}(R) \cap \operatorname{range}\left(f^{\prime}\right)$, persistence in $M_{r}$, implies $\pi_{T}\left(f^{\prime}(m)\right) \vDash R$. By definition of marginalization on kernels, $\left(\pi_{T} f^{\prime}\right)(m)=\pi_{T}\left(f^{\prime}(m)\right)$. Since $\left(\pi_{T} f^{\prime}\right)(m) \vDash R$, we have $\pi_{T} f^{\prime} \vDash(D \triangleright R)$ as well.
- $\mathrm{FV}_{\mathrm{D}}(P)=\mathrm{FV}(D)$, so $\operatorname{dom}\left(\pi_{T} f^{\prime}\right)=\operatorname{dom}(m)=\mathrm{FV}(D)=\mathrm{FV}_{\mathrm{D}}(P)$.
- $\mathrm{FV}_{\mathrm{R}}(P)=\mathrm{FV}(D \triangleright R)=\mathrm{FV}(D) \cup \mathrm{FV}(R)$, so

$$
\begin{aligned}
\operatorname{range}\left(\pi_{T} f^{\prime}\right) & \supseteq \operatorname{dom}\left(\left(\pi_{T} f^{\prime}\right)(m)\right) \\
& \supseteq \mathrm{FV}(D) \cup \mathrm{FV}(R) \\
& =\mathrm{FV}(P)
\end{aligned}
$$

(for any $m \in M_{d}$ such that $m \not \models_{d} D$ )
$\left(\right.$ By $\left.\left(\pi_{T} f^{\prime}\right)(m) \vDash R\right)$
so $\pi_{T} f^{\prime}$ is a desired witness for $f \vDash P$.
Case $Q \wedge R$ : Assuming $\mathrm{FV}_{\mathrm{R}}(Q)=\mathrm{FV}(Q)=\mathrm{FV}_{\mathrm{R}}(R)=\mathrm{FV}(R)$. By definition, $f \vDash Q \wedge R$ implies that $f \vDash Q$ and $f \vDash R$. By induction, there exists $f^{\prime} \sqsubseteq f$ such that $\mathrm{FV}(Q)=\operatorname{range}\left(f^{\prime}\right)=\mathrm{FV}(Q), \operatorname{dom}\left(f^{\prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(Q)$, and $f^{\prime} \vDash Q$, and there exists $f^{\prime \prime} \sqsubseteq f$ such that $\mathrm{F} \mathrm{V}_{\mathrm{R}}(R)=\operatorname{range}\left(f^{\prime \prime}\right)=\mathrm{FV}(R), \operatorname{dom}\left(f^{\prime \prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(R)$ and $f^{\prime \prime} \vDash R$. Thus, range $\left(f^{\prime}\right)=\operatorname{range}\left(f^{\prime \prime}\right)$.
Note that $\operatorname{dom}\left(f^{\prime}\right)=\operatorname{dom}(f) \cap \operatorname{range}\left(f^{\prime}\right)$ because in our models, $f^{\prime} \sqsubseteq f$ implies that there exists $S$ and some $v$ such that $f=\left(f^{\prime} \oplus \eta_{S}\right) \odot v$, and we can make $S$ disjoint of $\operatorname{dom}\left(f^{\prime}\right)$ and range $\left(f^{\prime}\right)$ wolog. Then, $\operatorname{dom}(f)=\operatorname{dom}\left(f^{\prime} \oplus S\right)=\operatorname{dom}\left(f^{\prime}\right) \cup S$, and range $\left(f^{\prime}\right)=\operatorname{range}\left(f^{\prime} \oplus S\right) \backslash S$, so $\operatorname{dom}(f) \cup \operatorname{range}\left(f^{\prime}\right) \subseteq \operatorname{dom}\left(f^{\prime}\right)$. Meanwhile, since $\operatorname{dom}\left(f^{\prime}\right) \subseteq \operatorname{dom}(f)$ and $\operatorname{dom}\left(f^{\prime}\right) \subseteq$ $\operatorname{range}\left(f^{\prime}\right)$, $\operatorname{dom}\left(f^{\prime}\right) \subseteq \operatorname{dom}(f) \cap \operatorname{range}\left(f^{\prime}\right)$. $\operatorname{So} \operatorname{dom}\left(f^{\prime}\right)=\operatorname{dom}(f) \cap \operatorname{range}\left(f^{\prime}\right)$. Similarly, $\operatorname{dom}\left(f^{\prime \prime}\right) \subseteq \operatorname{dom}(f) \cap \operatorname{range}\left(f^{\prime \prime}\right)$, so $\operatorname{range}\left(f^{\prime}\right)=\operatorname{range}\left(f^{\prime \prime}\right)$ implies that $\operatorname{dom}\left(f^{\prime}\right)=\operatorname{dom}\left(f^{\prime}\right)$.
Since $\operatorname{dom}\left(f^{\prime}\right)=\operatorname{dom}\left(f^{\prime \prime}\right)$ and range $\left(f^{\prime}\right)=\operatorname{range}\left(f^{\prime \prime}\right)$, Theorem A.25 implies that $f^{\prime}=f^{\prime \prime}$. This is the desired witness: $f^{\prime}=f^{\prime \prime} \vDash Q$ and $f^{\prime}=f^{\prime \prime} \vDash R$.
Case $Q \vee R$ : $f \vDash Q \vee R$ implies that $f \vDash Q$ or $f \vDash R$.
Without loss of generality, suppose $f \vDash Q$. By induction, there exists $f^{\prime} \sqsubseteq f$ such that $\mathrm{FV}_{\mathrm{R}}(Q) \subseteq$ range $\left(f^{\prime}\right) \subseteq \mathrm{FV}(Q)$, $\operatorname{dom}\left(f^{\prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(Q)$. Then:

$$
\begin{aligned}
& \operatorname{range}\left(f^{\prime}\right) \subseteq \mathrm{FV}(Q) \cup \mathrm{FV}(R)=\mathrm{FV}(P) \\
& \operatorname{range}\left(f^{\prime}\right) \supseteq \mathrm{FV}_{\mathrm{R}}(Q) \cap \mathrm{FV}_{\mathrm{R}}(R)=\mathrm{FV}_{\mathrm{R}}(P) \\
& \operatorname{dom}\left(f^{\prime}\right) \subseteq \mathrm{FV}(Q) \cup \mathrm{FV}(R)=\mathrm{FV}_{\mathrm{D}}(P)
\end{aligned}
$$

Thus, $f^{\prime}$ is a desired witness.
Case $Q \stackrel{\circ}{ } R$ : Assuming $\mathrm{FV}_{\mathrm{D}}(R) \subseteq \mathrm{FV}_{\mathrm{R}}(Q)$.
$f \vDash Q{ }_{9} R$ implies that there exists $f_{1}, f_{2}$ such that $f_{1} \odot f_{2}=f, f_{1} \vDash Q$, and $f_{2} \vDash R . f_{1} \odot f_{2}$ is defined so range $\left(f_{1}\right)=\operatorname{dom}\left(f_{2}\right)$. By induction, there exists $f_{1}^{\prime} \sqsubseteq f_{1}$ such that $f_{1}^{\prime} \models Q, \mathrm{FV}_{\mathrm{R}}(Q) \subseteq \operatorname{range}\left(f_{1}^{\prime}\right) \subseteq \mathrm{FV}(Q)$ and $\operatorname{dom}\left(f_{1}^{\prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(Q)$, and there exists $f_{2}^{\prime} \sqsubseteq f_{2}$ such that $f_{2}^{\prime} \vDash Q, \mathrm{FV}_{\mathrm{R}}(R) \subseteq \operatorname{range}\left(f_{2}^{\prime}\right) \subseteq \mathrm{FV}(R)$, and $\operatorname{dom}\left(f_{2}^{\prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(R)$.
Now, $\widehat{f}=f_{1}^{\prime} \odot\left(f_{2}^{\prime} \oplus\right.$ unit $\left._{\text {range }\left(f_{1}^{\prime}\right) \backslash \operatorname{dom}\left(f_{2}^{\prime}\right)}\right)$ is defined because $\operatorname{dom}\left(f_{2}^{\prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(R) \subseteq \mathrm{FV}_{\mathrm{R}}(Q) \subseteq \operatorname{range}\left(f_{1}^{\prime}\right)$. Then, we have

$$
\begin{aligned}
\widehat{f} & \models Q \circ R \\
\operatorname{range}(\widehat{f)}) & =\operatorname{range}\left(f_{1}^{\prime}\right) \cup \operatorname{range}\left(f_{2}^{\prime}\right) \subseteq \mathrm{FV}(Q) \cup \mathrm{FV}(R)=\mathrm{FV}(P) \\
\operatorname{range}(\widehat{f)}) & =\operatorname{range}\left(f_{1}^{\prime}\right) \cup \operatorname{range}\left(f_{2}^{\prime}\right) \supseteq \mathrm{FV}_{\mathrm{R}}(Q) \cup \mathrm{FV}_{\mathrm{R}}(R)=\mathrm{FV}_{\mathrm{R}}(P) \\
\operatorname{dom}(\widehat{f}) & =\operatorname{dom}\left(f_{1}^{\prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(Q)=\mathrm{FV}_{\mathrm{D}}(P) .
\end{aligned}
$$

$f_{1}^{\prime} \sqsubseteq f, f_{2}^{\prime} \oplus$ unit $_{\text {range }\left(f_{1}^{\prime}\right) \backslash \operatorname{dom}\left(f_{2}^{\prime}\right)} \oplus f_{2}$, so by Theorem A.36 $\widehat{f}=f_{1}^{\prime} \odot\left(f_{2}^{\prime} \oplus\right.$ unit $\left._{\text {range }\left(f_{1}^{\prime}\right) \backslash \operatorname{dom}\left(f_{2}^{\prime}\right)}\right) \sqsubseteq f_{1} \odot f_{2}=f$.
Thus, $\widehat{f}$ is a desired witness.
Case $Q * R$ : $f \vDash Q * R$ implies that there exists $f_{1}, f_{2}$ such that $f_{1} \oplus f_{2} \sqsubseteq f, f_{1} \vDash Q$, and $f_{2} \vDash R$.
By induction, there exists $f_{1}^{\prime} \sqsubseteq f_{1}$ such that $f_{1}^{\prime} \models Q, \mathrm{FV}_{\mathrm{R}}(Q) \subseteq \operatorname{range}\left(f_{1}^{\prime}\right) \subseteq \mathrm{FV}(Q)$ and $\operatorname{dom}\left(f_{1}^{\prime}\right) \subseteq \mathrm{FV} \mathrm{D}_{\mathrm{D}}(Q)$, and there exists $f_{2}^{\prime} \sqsubseteq f_{2}$ such that $f_{2}^{\prime} \vDash Q, \mathrm{FV}_{\mathrm{R}}(R) \subseteq \operatorname{range}\left(f_{2}^{\prime}\right) \subseteq \mathrm{FV}(R)$, and $\operatorname{dom}\left(f_{2}^{\prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(R)$. By downwards closure of $\oplus, f_{1}^{\prime} \oplus f_{2}^{\prime}$ is defined and $f_{1}^{\prime} \oplus f_{2}^{\prime} \sqsubseteq f_{1} \oplus f_{2} \sqsubseteq f$. We have $f_{1}^{\prime} \oplus f_{2}^{\prime} \vDash Q * R$, and

$$
\begin{aligned}
\operatorname{range}\left(f_{1}^{\prime} \oplus f_{2}^{\prime}\right) & =\operatorname{range}\left(f_{1}^{\prime}\right) \cup \operatorname{range}\left(f_{2}^{\prime}\right) \subseteq \mathrm{FV}(Q) \cup \mathrm{FV}(R)=\mathrm{FV}(P) \\
\text { range }\left(f_{1}^{\prime} \oplus f_{2}^{\prime}\right) & =\operatorname{range}\left(f_{1}^{\prime}\right) \cup \operatorname{range}\left(f_{2}^{\prime}\right) \supseteq \mathrm{FV}_{\mathrm{R}}(Q) \cup \mathrm{FV}_{\mathrm{R}}(R)=\mathrm{FV}_{\mathrm{R}}(P) \\
\operatorname{dom}\left(f_{1}^{\prime} \oplus f_{2}^{\prime}\right) & =\operatorname{dom}\left(f_{1}^{\prime}\right) \cup \operatorname{dom}\left(f_{2}^{\prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(Q) \cup \mathrm{FV}_{\mathrm{D}}(R)=\mathrm{FV}_{\mathrm{D}}(P)
\end{aligned}
$$

Thus, $f_{1}^{\prime} \oplus f_{2}^{\prime}$ is a desired witness.

## N. Section $\mathbb{K}$ CPSL: Omitted Details

To prove soundness for CPSL (Theorem A.21), we rely on a few lemmas about program semantics.
Lemma A.26. Suppose that $e$ is an expression not containing $x$, and let $\mu \in \mathcal{D}(\operatorname{Mem}[\operatorname{Var}])$. Then:

$$
f_{\llbracket x \leftarrow e \rrbracket \mu}=f_{\mu} \odot\left(m \mapsto \operatorname{unit}\left(m^{\operatorname{Var} \backslash\{x\rfloor}\right)\right) \odot\left(\left(m_{1} \mapsto \operatorname{unit}\left(m_{1} \cup\left(x \mapsto \llbracket e \rrbracket\left(m_{1}\right)\right)\right)\right) \oplus\left(m_{2} \mapsto \operatorname{unit}\left(m_{2}\right)\right)\right)
$$

where $m_{1} \in \operatorname{Mem}[\operatorname{Var} \backslash\{x\}]$ and $m_{2} \in \operatorname{Mem}[\operatorname{Var} \backslash\{x\} \backslash F V(e)]$.
Lemma A.27. Suppose that $d$ is a distribution expression not containing $x$, and let $\mu \in \mathcal{D}$ (Mem[Var]). Then:

$$
f_{\llbracket x \leftrightarrow d \rrbracket \mu}=f_{\mu} \odot\left(m \mapsto \operatorname{unit}\left(m^{\operatorname{Var} \backslash\{x\rangle}\right)\right) \odot\left((\llbracket d \rrbracket \odot(v \mapsto[x \mapsto v\rfloor)) \oplus\left(m_{1} \mapsto \text { unit }\left(m_{1}\right)\right) \oplus\left(m_{2} \mapsto \operatorname{unit}\left(m_{2}\right)\right)\right)
$$

where $m_{1} \in \operatorname{Mem}[\operatorname{Var} \backslash\{x\}]$ and $m_{2} \in \operatorname{Mem}[\operatorname{Var} \backslash\{x\} \backslash F V(d)]$, and $\llbracket d \rrbracket: \operatorname{Mem}[F V(d)] \rightarrow \mathcal{D}(\operatorname{Val})$.
The rule Frame relies on simple syntactic conditions for approximating which variables may be read, which variables must be written before they are read, and which variables may be modified.

Definition A.8. RV, WV, MV are defined as follows:

$$
\operatorname{RV}(x \leftarrow e):=\mathrm{FV}(e) \quad \operatorname{RV}(x \longleftarrow d):=\mathrm{FV}(d)
$$

$\operatorname{RV}\left(c ; c^{\prime}\right):=\mathrm{RV}(c) \cup\left(\mathrm{RV}\left(c^{\prime}\right) \backslash \mathrm{WV}(c)\right)$
$\operatorname{RV}\left(\right.$ if $b$ then $c$ else $\left.c^{\prime}\right):=\mathrm{FV}(b) \cup \mathrm{RV}(c) \cup \mathrm{RV}\left(c^{\prime}\right)$

$$
\mathrm{WV}(x \leftarrow e):=\{x\} \backslash \mathrm{FV}(e) \quad \mathrm{WV}(x \longleftarrow d):=\{x\} \backslash \mathrm{FV}(d)
$$

$\mathrm{WV}\left(c ; c^{\prime}\right):=\mathrm{WV}(c) \cup\left(\mathrm{WV}\left(c^{\prime}\right) \backslash \mathrm{RV}(c)\right) \quad \mathrm{WV}\left(\right.$ if $b$ then $c$ else $\left.c^{\prime}\right):=\left(\mathrm{WV}(c) \cap \mathrm{WV}\left(c^{\prime}\right)\right) \backslash \mathrm{FV}(b)$

$$
\begin{array}{cc}
\operatorname{MV}(x \leftarrow e):=\{x\} & \operatorname{MV}(x \hookleftarrow d):=\{x\} \\
\operatorname{MV}\left(c ; c^{\prime}\right):=\operatorname{MV}(c) \cup \operatorname{MV}\left(c^{\prime}\right) & \operatorname{MV}\left(\text { if } b \text { then } c \text { else } c^{\prime}\right):=\operatorname{MV}(c) \cup \operatorname{MV}\left(c^{\prime}\right)
\end{array}
$$

Other analyses are possible, so long as non-modified variables are preserved from input to output, and output modified variables depend only on input read variables.
Lemma $\mathbf{A . 2 8}$ (Soundness for RV, WV, MV [9]). Let $\mu^{\prime}=\llbracket c \rrbracket \mu$, and let $R=R V(c), W=W V(c), C=\operatorname{Var} \backslash M V(c)$. Then:

1) Variables outside of $M V(c)$ are not modified: $\pi_{C}\left(\mu^{\prime}\right)=\pi_{C}(\mu)$.
2) The sets $R$ and $W$ are disjoint.
3) There exists $f: \operatorname{Mem}[R] \rightarrow \mathcal{D}(\operatorname{Mem}[M V(c)])$ with $\mu^{\prime}=\operatorname{bind}\left(\mu, m \mapsto f\left(\pi_{R}(m)\right) \otimes\right.$ unit $\left.\left(\pi_{C}(m)\right)\right)$.

We recall the definition of validity in CPSL.
Definition A. 7 (CPSL Validity). A CPSL judgment $\{P\} c\{Q\}$ is valid, written $\vDash\{P\} c\{Q\}$, if for every input distribution $\mu \in \mathcal{D}(\operatorname{Mem}[\mathrm{Var}])$ such that the lifted input $f_{\mu}: \operatorname{Mem}[\emptyset] \rightarrow \mathcal{D}(\operatorname{Mem}[\operatorname{Var}])$ satisfies $f_{\mu} \vDash P$, the lifted output satisfies $f_{\llbracket c \rrbracket \mu} \vDash Q$.

Now, we are ready to prove soundness of CPSL.
Theorem A. 21 (CPSL Soundness). CPSL is sound: derivable judgments are valid.
Proof. By induction on the derivation. Throughout, we write $\mu: \mathcal{D}$ (Mem[Var]) for the input and $f: \operatorname{Mem}[\emptyset] \rightarrow \mathcal{D}(\operatorname{Mem}[V a r])$ for the lifted input, and we assume that $f$ satisfies the pre-condition of the conclusion.
Case: Assn. By restriction (Theorem A.20), there exists $k_{1} \subseteq f$ such that $\mathrm{FV}(e) \subseteq S \mathrm{FV}(P) \subseteq$ range $\left(k_{1}\right) \subseteq \mathrm{FV}(P)$; let $K=\operatorname{range}\left(k_{1}\right)$. Since $f$ has empty domain, we have $f=k_{1} \odot k_{2}$ for some $k_{2}: \operatorname{Mem}[K] \rightarrow \mathcal{D}(\operatorname{Mem}[\operatorname{Var}])$. Let $f^{\prime}=f_{\llbracket x \leftarrow e \rrbracket \mu}$ be the lifted output. By Theorem A. 26 and associativity, we have:

$$
\begin{aligned}
f^{\prime} & =f \odot\left(m \mapsto \operatorname{unit}\left(m^{\operatorname{Var} \backslash\{x\rangle}\right)\right) \odot\left(\left(m_{1} \mapsto \operatorname{unit}\left(m_{1} \cup\left(x \mapsto \llbracket e \rrbracket\left(m_{1}\right)\right)\right)\right) \oplus\left(m_{2} \mapsto \operatorname{unit}\left(m_{2}\right)\right)\right) \\
& =\underbrace{k_{1} \odot k_{2} \odot\left(m \mapsto \operatorname{unit}\left(m^{\operatorname{Var} \backslash\{x\rangle}\right)\right)}_{j} \odot(\underbrace{m_{1} \mapsto \operatorname{unit}\left(m_{1} \cup\left(x \mapsto \llbracket e \rrbracket\left(m_{1}\right)\right)\right)}_{j_{1}} \oplus \underbrace{m_{2} \mapsto \operatorname{unit}\left(m_{2}\right)}_{j_{2}})
\end{aligned}
$$

where $m: \operatorname{Mem}[\operatorname{Var}], m_{1}: \operatorname{Mem}[\mathrm{FV}(e)]$, and $m_{2}: \operatorname{Mem}[\operatorname{Var} \backslash \mathrm{FV}(e) \backslash\{x\}]$. Note that even though the components of $j$ do not preserve input to output, $j$ itself does preserve input to output; $j_{1}$ and $j_{2}$ also evidently have this property. Now since $k \sqsubseteq j$ and $k_{1} \vDash P$, we have $j \vDash P$. Since $j_{1} \sqsubseteq j_{1} \oplus j_{2}$ and $j_{1} \vDash(\mathrm{FV}(e) \triangleright x=e)$, we have $j_{1} \oplus j_{2} \vDash(\mathrm{FV}(e) \triangleright x=e)$ as well. Thus, we conclude $f^{\prime} \vDash P \circ(\mathrm{FV}(e) \triangleright x=e)$.
Case: SAmp. By restriction (Theorem A.20), there exists $k_{1} \subseteq f$ such that $\mathrm{FV}(d) \subseteq S \mathrm{FV}(P) \subseteq \operatorname{range}\left(k_{1}\right) \subseteq \mathrm{FV}(P)$; let $K=\operatorname{range}\left(k_{1}\right)$. Since $f$ has empty domain, we have $f=k_{1} \odot k_{2}$ for some $k_{2}: \operatorname{Mem}[K] \rightarrow \mathcal{D}(\operatorname{Mem}[\operatorname{Var}])$. Let $f^{\prime}=f_{\llbracket x \leftarrow e \rrbracket \mu}$ be the lifted output. By Theorem A. 27 and associativity, we have:

$$
\begin{aligned}
f^{\prime} & =f \odot\left(m \mapsto \operatorname{unit}\left(m^{\operatorname{Var} \backslash\{x\rangle}\right)\right) \odot\left((\llbracket d \rrbracket \odot(v \mapsto[x \mapsto v])) \oplus\left(m_{1} \mapsto \operatorname{unit}\left(m_{1}\right)\right) \oplus\left(m_{2} \mapsto \operatorname{unit}\left(m_{2}\right)\right)\right) \\
& =\underbrace{k_{1} \odot k_{2} \odot\left(m \mapsto \operatorname{unit}\left(m^{\operatorname{Var} \backslash\{x\}}\right)\right)}_{j} \odot(\underbrace{(\llbracket d \rrbracket \odot(v \mapsto[x \mapsto v])) \oplus\left(m_{1} \mapsto \operatorname{unit}\left(m_{1}\right)\right)}_{j_{1}} \oplus \underbrace{m_{2} \mapsto \operatorname{unit}\left(m_{2}\right)}_{j_{2}})
\end{aligned}
$$

where $m: \operatorname{Mem}[\operatorname{Var}], \llbracket d \rrbracket: \operatorname{Mem}[\operatorname{FV}(d)] \rightarrow \mathcal{D}(\operatorname{Mem}[\operatorname{Val}]), m_{1}: \operatorname{Mem}[\mathrm{FV}(d)]$, and $m_{2}: \operatorname{Mem}[\operatorname{Var} \backslash \mathrm{FV}(d) \backslash\{x\}]$. Note that even though the components of $j$ do not preserve input to output, $j$ itself does preserve input to output; $j_{1}$ and $j_{2}$ also evidently have this property. Now since $k \sqsubseteq j$ and $k_{1} \vDash P$, we have $j \vDash P$. Since $j_{1} \sqsubseteq j_{1} \oplus j_{2}$ and $j_{1} \vDash(\operatorname{FV}(d) \triangleright x \sim d)$, we have $j_{1} \oplus j_{2} \vDash(\mathrm{FV}(d) \triangleright x \sim d)$ as well. Thus, we conclude $f^{\prime} \vDash P ;(\mathrm{FV}(d) \triangleright x \sim d)$.
Case: Skip. Trivial.
Case: Seqn. Trivial.
Case: DCond. Since all assertions are in RDIBI, we have $\mathrm{FV}_{\mathrm{D}}(P) \subseteq \mathrm{FV}_{\mathrm{R}}(\emptyset \triangleright[b])=\{b\}$. Since $f \vDash(\emptyset \triangleright[b])$, there exists $k_{1}, k_{2}$ such that $k_{1} \odot k_{2}=f$, with $k_{1} \vDash(\emptyset \triangleright[b])$ and $k_{2} \vDash P$.
By restriction (Theorem A.20), there exists $j_{1}$ such that $j_{1} \sqsubseteq k_{1}$ and

$$
\begin{aligned}
\operatorname{dom}\left(j_{1}\right) & \subseteq \mathrm{FV}_{\mathrm{D}}(\emptyset \triangleright[b])=\emptyset \\
\{b\} & =\mathrm{FV}_{\mathrm{R}}(\emptyset \triangleright[b]) \subseteq \operatorname{range}\left(j_{1}\right) \subseteq \mathrm{FV}(\emptyset \triangleright[b])=\{b\} .
\end{aligned}
$$

By restriction (Theorem A.20), there exists $j_{2}$ such that $j_{2} \sqsubseteq k_{2}$ and $j_{2} \vDash P$, and $\operatorname{dom}\left(j_{2}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(P) \subseteq \mathrm{FV}_{\mathrm{R}}(\emptyset \triangleright[b])=\{b\}$. Since $\operatorname{dom}\left(k_{2}\right)=\operatorname{range}\left(k_{1}\right) \supseteq\{b\}$, we may assume without loss of generality that $j_{2} \vDash P, j_{2} \sqsubseteq k_{2}$, and $\operatorname{dom}\left(j_{2}\right)=\{b\}$. Thus $j_{1} \odot j_{2}$ is defined, and so $j_{1} \odot j_{2} \sqsubseteq k_{1} \odot k_{2} \sqsubseteq f$ by Theorem A. 36 ,
By Theorem A.10 there exists $j: \operatorname{Mem}\left[\mathbf{r a n g e}\left(j_{2}\right)\right] \rightarrow \mathcal{D}(\operatorname{Mem}[\operatorname{Var}])$ such that $j_{1} \odot\left(j_{2} \odot j\right)=\left(j_{1} \odot j_{2}\right) \odot j=f$. Since $j_{2} \sqsubseteq j_{2} \odot j$, we have $j_{2} \odot j \vDash P$. Thus, we may assume without loss of generality that range $\left(j_{2}\right)=\operatorname{Var}$ and $j_{1} \odot j_{2}=f=\bar{\mu}$.
Let $l_{t t}, l_{f f}: \operatorname{Mem}[\emptyset] \rightarrow \mathcal{D}(\operatorname{Mem}[b])$ be defined by $l_{t t}(\langle \rangle)=$ unit $[b=t t]$ and $l_{f f}(\langle \rangle)=$ unit $[b=f f]$; evidently, $l_{t t} \vDash(\emptyset \triangleright b=t t)$ and $l_{f f} \vDash(\emptyset \triangleright b=f f)$. Now, we have:

$$
\begin{aligned}
& f_{\mu \mid \llbracket b=t t \rrbracket}=l_{t t} \odot j_{2} \\
& f_{\mu\lfloor\llbracket b=f f \rrbracket}=l_{f f} \odot j_{2}
\end{aligned}
$$

where each equality holds if the left side is defined. Regardless of whether the conditional distributions are defined, we always have:

$$
\begin{aligned}
& l_{t t} \odot j_{2} \vDash(\emptyset \triangleright b=t t) \stackrel{\circ}{9} \\
& l_{f f} \odot j_{2} \vDash(\emptyset \triangleright b=f f) \stackrel{ }{q} .
\end{aligned}
$$

Since both of these kernels have empty domain, we have $l_{t t} \odot j_{2}=\overline{v_{t t}}$ and $l_{f f} \odot j_{2}=\overline{v_{f f}}$ for two distributions $v_{t t}, v_{f f} \in \mathcal{D}(\operatorname{Mem}[\operatorname{Var}])$. By induction, we have:

$$
\begin{aligned}
& f_{\llbracket c \rrbracket v_{t}} \vDash(\emptyset \triangleright b=t t) \dot{ }\left(b: b=t t \triangleright Q_{1}\right) \\
& f_{\llbracket c \rrbracket v_{f f}} \vDash(\emptyset \triangleright b=f f) ;\left(b: b=f f \triangleright Q_{2}\right) .
\end{aligned}
$$

By similar reasoning as for the pre-conditions, there exists $k_{1}^{\prime}, k_{2}^{\prime}: \operatorname{Mem}[b] \rightarrow \mathcal{D}(\operatorname{Mem}[\mathrm{Var}])$ such that $k_{1}^{\prime} \vDash\left(b: b=t t \triangleright Q_{1}\right)$ and $k_{2}^{\prime} \vDash\left(b: b=f f \triangleright Q_{2}\right)$, and:

$$
f_{\llbracket c \rrbracket v_{t t}}=l_{t t} \odot k_{1}^{\prime} \quad f_{\llbracket c \rrbracket v_{f f}}=l_{f f} \odot k_{2}^{\prime}
$$

Let $k^{\prime}: \operatorname{Mem}[b] \rightarrow \mathcal{D}(\operatorname{Mem}[\operatorname{Var}])$ be the composite kernel defined as follows:

$$
k^{\prime}([b \mapsto v]) \triangleq \begin{cases}k_{1}^{\prime}([b \mapsto t t]) & : v=t t \\ k_{2}^{\prime}([b \mapsto f f]) & : v=f f\end{cases}
$$

By assumption, $k^{\prime} \vDash\left(\left(b: b=t t \triangleright Q_{1}\right) \wedge\left(b: b=f f \triangleright Q_{2}\right)\right)$. Now, let $p \triangleq \mu(\llbracket b=t t \rrbracket)$ be the probability of taking the first branch. Then we can conclude:

$$
\begin{aligned}
f_{\llbracket i f} b \text { then } c \text { else } c^{\prime} \rrbracket \mu & =f_{\llbracket c \rrbracket(\mu \| \llbracket b=t \rrbracket \rrbracket) \oplus_{p} \llbracket c^{\prime} \rrbracket(\mu \mu \llbracket b=t t \rrbracket)} \\
& =f_{\llbracket c \rrbracket v_{t} \oplus_{\oplus} \llbracket c \rrbracket v_{f f}} \\
& =f_{\llbracket c \rrbracket v_{t t}} \oplus_{p} f_{\llbracket c \rrbracket v_{f f}} \\
& =\left(l_{t t} \odot k_{1}^{\prime}\right) \oplus_{p}\left(l_{f f} \odot k_{2}^{\prime}\right) \\
& =\left(l_{t t} \odot k^{\prime}\right) \oplus_{p}\left(l_{f f} \odot k^{\prime}\right) \\
& =\left(l_{t t} \bar{\oplus}_{p} l_{f f}\right) \odot k^{\prime} \\
& \vDash(\emptyset \triangleright[b]) \circ\left(\left(b: b=t t \triangleright Q_{1}\right) \wedge\left(b: b=f f \triangleright Q_{2}\right)\right) .
\end{aligned}
$$

Above, $k_{1} \bar{\oplus}_{p} k_{2}$ lifts the convex combination operator from distributions to kernels from Mem[Ø]. We show the last equality in more detail. For any $r \in \operatorname{Mem}[\mathrm{Var}]$ :

$$
\begin{aligned}
& \left(l_{t t} \odot k^{\prime}\right) \bar{\oplus}_{p}\left(l_{f f} \odot k^{\prime}\right)(\langle \rangle)(r) \\
& =p \cdot\left(l_{t t} \odot k^{\prime}\right)(\langle \rangle)(r)+(1-p) \cdot\left(l_{f f} \odot k^{\prime}\right)(\langle \rangle)(r) \\
& =p \cdot\left(l_{t t} \odot k^{\prime}\right)(\langle \rangle)(r)+(1-p) \cdot\left(l_{f f} \odot k^{\prime}\right)(\langle \rangle)(r) \\
& =p \cdot l_{t t}(\langle \rangle)(b \mapsto t t) \cdot k^{\prime}(b \mapsto t t)(r)+(1-p) \cdot l_{f f}(\langle \rangle)(b \mapsto f f) \cdot k^{\prime}(b \mapsto f f)(r) \\
& =\left(\left(l_{t t} \oplus_{p} l_{f f}\right) \odot k^{\prime}\right)(\langle \rangle)(r) .
\end{aligned}
$$

where the penultimate equality holds because $l_{t t}$ and $l_{f f}$ are deterministic.
Case: Weak. Trivial.
Case: Frame. The proof for this case follows the argument for Frame rule in PSL, with a few minor changes.
There exists $k_{1}, k_{2}$ such that $k_{1} \oplus k_{2} \sqsubseteq f$, and $k_{1} \vDash P$ and $k_{2} \vDash R$; let $S_{1} \triangleq \operatorname{range}\left(k_{1}\right)$, and note that $\mathrm{RV}(c) \subseteq S_{1}$ by the last side-condition. By restriction (Theorem A.20), there exists $k_{2}^{\prime} \sqsubseteq k_{2}$ such that $k_{2}^{\prime} \models R$ and range $\left(k_{2}^{\prime}\right) \subseteq \mathrm{FV}(R)$; let $S_{2} \triangleq \operatorname{range}\left(k_{2}^{\prime}\right)$. Since $k_{1}$ and $k_{2}$ have empty domains, $S_{1}$ and $S_{2}$ must be disjoint. Let $S_{3}=\operatorname{Var} \backslash S_{2} \backslash S_{1}$. Since WV(c) is disjoint from $S_{2}$ by the first side-condition, we have $\mathrm{WV}(c) \subseteq S_{1} \cup S_{3}$.
Let $f^{\prime}=f_{\llbracket c \rrbracket \mu}$ be the lifted output. By induction, we have $f^{\prime} \vDash Q$; by restriction (Theorem A.20), there exists $k_{1}^{\prime} \sqsubseteq f^{\prime}$ such that range $\left(k_{1}^{\prime}\right) \subseteq \mathrm{FV}(Q)$ and $k_{1}^{\prime} \models Q$. By the third side condition, $\mathrm{RV}(c) \subseteq \mathrm{FV}_{\mathrm{R}}(P) \subseteq S_{1}$.
By soundness of RV and WV (Theorem A.28, all variables in WV(c) must be written before they are read and there is a function $F: \operatorname{Mem}\left[S_{1}\right] \rightarrow \mathcal{D}\left(\operatorname{Mem}\left[W V(c) \cup S_{1}\right]\right)$ such that:

$$
\pi_{\mathrm{WV}(c) \cup S_{1}} \llbracket c \rrbracket \mu=\operatorname{bind}\left(\mu, m \mapsto F\left(m^{S_{1}}\right)\right)
$$

Since $S_{2} \subseteq \mathrm{FV}(R)$, variables in $S_{2}$ are not in $\mathrm{MV}(c)$ by the first side-condition, and $S_{2}$ is disjoint from $\mathrm{WV}(c) \cup S_{1}$. By soundness of MV, we have:

$$
\pi_{\mathrm{WV}(c) \cup S_{1} \cup S_{2}} \llbracket c \rrbracket \mu=\operatorname{bind}\left(\pi_{\mathrm{WV}(c) \cup S_{1} \cup S_{2}} \mu, F \oplus \text { unit }\right)
$$

where unit : $\operatorname{Mem}\left[\mathrm{WV}(c) \cup S_{2}\right] \rightarrow \mathcal{D}\left(\operatorname{Mem}\left[\mathrm{WV}(c) \cup S_{2}\right]\right)$.
Since $S_{1}$ and $S_{2}$ are independent in $\mu$, we know that $S_{1} \cup \mathrm{WV}(c)$ and $S_{2}$ are independent in $\llbracket c \rrbracket \mu$. Hence:

$$
f_{\pi_{S_{1} \cup W V(c)} \llbracket c \rrbracket \mu} \oplus f_{\pi_{S_{2}} \llbracket c \rrbracket \mu} \sqsubseteq f^{\prime}
$$

By induction, $f^{\prime} \vDash Q$. Furthermore, $\mathrm{FV}(Q) \subseteq \mathrm{FV}_{\mathrm{R}}(P) \cup \mathrm{WV}(c) \subseteq S_{1} \cup \mathrm{WV}(c)$ by the second side-condition. By restriction (Theorem A.20), $f_{\pi_{S_{1} \cup W V(c)} \llbracket c \rrbracket \mu} \vDash Q$. Furthermore, $\pi_{S_{2}} \llbracket c \rrbracket \mu=\pi_{S_{2}} \mu$, so $\pi_{S_{2}} \llbracket c \rrbracket \mu \vDash R$ as well. Thus, $f^{\prime} \vDash Q * R$ as desired.

## O. Section $\square$ proving CI: omitted proofs

Proposition A.29. (Axioms for RDIBI) The following axioms are sound, assuming both precedent and antecedent are in Form ${ }_{\text {RDIBI }}$.

$$
\begin{align*}
& (P ; Q) \stackrel{\circ}{\circ} \mathrm{P} \circ(Q * R)  \tag{INDEP-1}\\
& P ; Q \rightarrow P * Q  \tag{Indep-2}\\
& P ; Q \rightarrow P \circ(Q *(S \triangleright[S]))  \tag{PAD}\\
& (P * Q) \stackrel{ }{\circ}(R * S) \rightarrow(P ; R) *(Q ; S)
\end{align*}
$$

(RestExch)
Proof. We prove them one by one.
Indep-1 We want to show that when $(P ; Q) \stackrel{\circ}{q} R, P \circ(Q * R)$ are both formula in $R D I B I, f \vDash(P \circ Q) \stackrel{q}{q}$ implies $f \vDash P ;(Q * R)$. By proof system of DIBI, $f \vDash(P \circ Q) \stackrel{ }{q} R$ implies that $f \vDash P \circ(Q \circ R)$. While $P \circ(Q \circ R)$ may not satisfy the restriction property, that is okay because we will only used conditions guaranteed by the fact that $(P \circ Q){ }_{9} R, P \circ(Q * R) \in$ Form $_{\mathrm{RDIBI}}$. In particular, we rely on $P, Q, R$ each satisfies restriction, and $\mathrm{FV}_{\mathrm{D}}(Q * R) \subseteq \mathrm{FV}_{\mathrm{R}}(P)$, which implies that

$$
\begin{equation*}
\mathrm{FV}_{\mathrm{D}}(R) \subseteq \mathrm{FV}_{\mathrm{D}}(Q * R) \subseteq \mathrm{FV}_{\mathrm{R}}(P) \tag{25}
\end{equation*}
$$

$f \vDash P \stackrel{\circ}{q}(Q \stackrel{\circ}{q})$ implies there exists $f_{p}, f_{q}, f_{r}$ such that $f_{p} \vDash P, f_{q} \vDash Q$, and $f_{r} \vDash R$, and $f_{p} \odot\left(f_{q} \odot f_{r}\right)=f$.
By restriction property Theorem A.20, $f_{q} \vDash Q$ implies that there exists $f_{q}^{\prime} \sqsubseteq f_{q}$ such that $\mathrm{FV}_{\mathrm{R}}(Q) \subseteq \operatorname{range}\left(f_{q}^{\prime}\right) \subseteq \mathrm{FV}(Q)$ and $\operatorname{dom}\left(f_{q}^{\prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(Q) . f_{q}^{\prime} \sqsubseteq f_{q}$ so there exists $v, T$ such that $f_{q}=\left(f_{q}^{\prime} \oplus_{k}\right.$ unit $\left._{T}\right) \odot v$.
Similarly, $f_{r} \vDash R$, by Theorem A.20, there exists $f_{r}^{\prime} \subseteq f_{r}$ such that $\mathrm{FV}(R) \subseteq \operatorname{range}\left(f_{r}^{\prime}\right) \subseteq \mathrm{FV}(R)$ and $\operatorname{dom}\left(f_{r}^{\prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(R)$. $f_{r}^{\prime} \sqsubseteq f_{r}$ so there exists $u, S$ such that $f_{r}=\left(f_{r}^{\prime} \oplus_{k}\right.$ unit $\left._{S}\right) \odot u$.
Now, we claim that $\mathrm{FV}_{\mathrm{D}}(R) \subseteq \operatorname{dom}\left(f_{q}^{\prime} \oplus\right.$ unit $\left._{T}\right)$ :
By Theorem A.20 $f_{p} \vDash P$ implies that there exists $f_{p}^{\prime} \subseteq f_{p}$ such that $\mathrm{FV}_{\mathrm{R}}(P) \subseteq \operatorname{range}\left(f_{p}^{\prime}\right) \subseteq \mathrm{FV}(P), \operatorname{dom}\left(f_{p}^{\prime}\right) \subseteq$ $F \mathrm{FV}(P)$, and $f_{p}^{\prime} \models P$. Thus, $\mathrm{FV}_{\mathrm{R}}(P) \subseteq \operatorname{range}\left(f_{p}\right)=\operatorname{dom}\left(f_{q}\right)$.
Recall that $\mathrm{FV}_{\mathrm{D}}(R) \subseteq \mathrm{FV}_{\mathrm{R}}(P)$, so $\mathrm{FV}_{\mathrm{D}}(R) \subseteq \operatorname{dom} f_{q}=\operatorname{dom} f_{q}^{\prime} \oplus$ unit $_{T}$.
As a corollary, we have $\operatorname{dom}\left(f_{r}^{\prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(R) \subseteq \operatorname{dom}\left(f_{q}^{\prime} \oplus \operatorname{unit}_{T}\right) \subseteq \operatorname{dom}(v)$, and $\boldsymbol{\operatorname { d o m }}\left(f_{r}^{\prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(R) \subseteq \operatorname{dom}\left(f_{q}^{\prime} \oplus\right.$ unit $)$. Then,

$$
\begin{align*}
f_{q} \odot f_{r} & =\left(\left(f_{q}^{\prime} \oplus \text { unit }_{T}\right) \odot v\right) \odot\left(\left(f_{r}^{\prime} \oplus \text { unit }_{S}\right) \odot u\right) \\
& \left.=\left(f_{q}^{\prime} \oplus \text { unit }_{T}\right) \odot\left(v \odot\left(f_{r}^{\prime} \oplus \text { unit }_{S}\right)\right) \odot u \quad \quad \text { (By standard associativity of } \odot\right) \\
& \left.=\left(f_{q}^{\prime} \oplus \text { unit }_{T}\right) \odot\left(f_{r}^{\prime} \oplus v\right) \odot u \quad \quad \text { By Theorem A.34 and dom }\left(f_{r}^{\prime}\right) \subseteq \operatorname{dom}(v)\right) \\
& =\left(f_{q}^{\prime} \oplus \text { unit }_{T}\right) \odot\left(\left(f_{r}^{\prime} \odot \text { unit }_{\text {range }\left(f_{r}^{\prime}\right)}\right) \oplus\left(\text { unit }_{\text {dom }^{\prime}(v)} \odot v\right) \odot u\right. \\
& \left.=\left(f_{q}^{\prime} \oplus \text { unit }_{T}\right) \odot\left(f_{r}^{\prime} \oplus \text { unit }_{\text {dom }(v)}\right) \odot\left(\text { unit }_{\text {range }\left(f_{r}^{\prime}\right)}\right) \oplus v\right) \odot u \\
& =\left(\left(f_{q}^{\prime} \oplus \text { unit }_{T}\right) \oplus f_{r}^{\prime}\right) \odot\left(v \oplus \text { unit }_{\text {range }\left(f_{f}^{\prime}\right)}^{\prime}\right) \odot u \\
& =\left(\left(f_{q}^{\prime} \oplus \text { unit }_{T}\right) \odot v\right) \oplus\left(f_{r}^{\prime} \odot \text { unit }_{\text {range }\left(f_{r}^{\prime}\right)}^{\prime}\right) \odot u \\
& =f_{q} \oplus f_{r}
\end{align*}
$$

where $\dagger$ follows from Theorem A.34 $\operatorname{dom}\left(f_{r}^{\prime}\right) \subseteq \operatorname{dom}\left(f_{q}^{\prime} \oplus\right.$ unit $\left._{T}\right)$ and exact commutativity, $\odot$ follows from Eq. Exchange equality and Theorem A.33
Thus, $f_{q} \odot f_{r} \vDash Q * R$. And by satisfaction rules,

$$
f \vDash P \stackrel{\circ}{g}(Q * R)
$$

Inder-2 We want to show that under the special condition $\mathrm{FV}_{\mathrm{D}}(Q)=\emptyset, f \vDash P ; Q$ implies that $f \vDash P * Q$.
If $f \vDash P ; Q$, then there exists $f_{p}, f_{q}$ such that $f_{p} \odot f_{q}=f$ and $f_{p} \vDash P, f_{q} \vDash Q$.
By restriction property Theorem A.20 $f_{q} \vDash Q$ implies that there exists $f_{q}^{\prime} \sqsubseteq f_{q}$ such that $\mathrm{FV}_{\mathrm{R}}(Q) \subseteq \operatorname{range}\left(f_{q}^{\prime}\right) \subseteq \mathrm{FV}(Q)$ and $\operatorname{dom}\left(f_{q}^{\prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(Q) . f_{q}^{\prime} \sqsubseteq f_{q}$ so there exists $v, T$ such that $f_{q}=\left(f_{q}^{\prime} \oplus_{k}\right.$ unit $\left._{T}\right) \odot v$.

Since $\operatorname{dom}\left(f_{q}^{\prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(Q)$ and $\mathrm{FV}(Q)=\emptyset$, it must $\operatorname{dom}\left(f_{q}^{\prime}\right)=\emptyset$, and thus no matter what the domain of $f_{p}$ is, $\operatorname{dom}\left(f_{q}^{\prime}\right) \subseteq$ $\operatorname{dom}\left(f_{p}\right)$. Thus,

$$
f_{p} \odot f_{q}=f_{p} \odot\left(f_{q}^{\prime} \oplus \mathrm{unit}_{T}\right) \odot v
$$

$$
=\left(f_{p} \oplus f_{q}^{\prime}\right) \oplus v \quad\left(\text { By Theorem A.34 and } \operatorname{dom}\left(f_{q}^{\prime}\right) \subseteq \operatorname{dom}\left(f_{p}\right)\right)
$$

Thus, $f_{p} \oplus f_{q}^{\prime} \sqsubseteq f_{p} \odot f_{q}=f$. By satisfaction rules, $f_{p} \vDash P$ and $f_{q}^{\prime} \vDash Q$ implies that $f_{p} \oplus f_{q}^{\prime} \vDash P * Q$. Thus, by persistence, $f \vDash P * Q$
Pad We want to show that when $P ; Q, P ;(Q *(S \triangleright[S]))$ are both in Form ${ }_{\text {RDIBI }}, f \vDash P ; Q$ implies $f \vDash P ;(Q *(S \triangleright[S])$.
One key guarantee we rely on from the grammar of Form ${ }_{\text {RDIBI }}$ is that

$$
\mathrm{FV}_{\mathrm{D}}(Q) \cup S=\mathrm{FV}_{\mathrm{D}}(Q *(S \triangleright[S])) \subseteq \mathrm{FV}_{\mathrm{R}}(P)
$$

When $f \vDash P \circ Q$, there exists $f_{p}, f_{q}$ such that $f_{p} \odot f_{q}=f$ and $f_{p} \vDash P, f_{q} \vDash Q$,
By Theorem A.20, $f_{p} \vDash P$ implies that there exists $f_{p}^{\prime} \sqsubseteq f_{p}$ such that $\mathrm{FV}_{\mathrm{R}}(P) \subseteq \operatorname{range}\left(f_{p}^{\prime}\right) \subseteq \mathrm{FV}(P), \operatorname{dom}\left(f_{p}^{\prime}\right) \subseteq$ $F \mathrm{FV}(P)$, and $f_{p}^{\prime} \vDash P$. By the fact that $f_{p} \odot f_{q}$ is defined, and that the definition of preorder in our concrete models, $f_{p}^{\prime} \sqsubseteq f_{p}$ implies

$$
\operatorname{dom}\left(f_{q}\right)=\operatorname{range}\left(f_{p}\right) \supseteq \operatorname{range}\left(f_{p}^{\prime}\right) \supseteq \mathrm{FV}_{\mathrm{R}}(P) \supseteq S
$$

Since $f_{q}$ preserves input, $S \subseteq \operatorname{dom}\left(f_{q}\right)$ implies that $f_{q}=f_{q} \oplus$ unit $_{s}$, and thus $f_{p} \odot f_{q}=f_{p} \odot\left(f_{q} \oplus\right.$ unit $\left._{S}\right)$.
Note that units $\vDash(S \triangleright[S])$, and $f_{q} \vDash Q$. Thus, $f_{q} \oplus$ unit $_{S} \vDash Q *(S \triangleright[S])$. Since $f_{p} \vDash P$, it follows that

$$
f_{p} \odot\left(f_{q} \oplus \text { unit }_{S}\right) \vDash P \circ(Q *(S \triangleright[S]))
$$

Since $f=f_{p} \odot f_{q}=f_{p} \odot\left(f_{q} \oplus\right.$ unit $\left._{S}\right)$,

$$
f \vDash P \circ(Q *(S \triangleright[S]))
$$

RESTExCH We want to show that when $(P * Q) \stackrel{ }{g}(R * S)$ and $(P \circ R) *\left(Q{ }_{g} S\right)$ are both formula in Form ${ }_{\text {RDIbI }}, f \vDash(P * Q) \stackrel{\circ}{g}(R * S)$ implies $f \vDash(P * R) *(Q * S)$.
The key properties that being in Form ${ }_{\text {RDIBI }}$ guarantees us are that

$$
\begin{aligned}
& \mathrm{FV}_{\mathrm{D}}(R) \subseteq \mathrm{FV}_{\mathrm{R}}(P) \quad \mathrm{FV}_{\mathrm{D}}(S) \subseteq \mathrm{FV}_{\mathrm{R}}(Q) \\
& \mathrm{FV}_{\mathrm{D}}(R * S)=\mathrm{FV}_{\mathrm{D}}(R) \cup \mathrm{FV}_{\mathrm{D}}(S) \subseteq \mathrm{FV}_{\mathrm{R}}(P * Q)=\mathrm{FV}_{\mathrm{R}}(P) \cup \mathrm{FV}_{\mathrm{R}}(Q)
\end{aligned}
$$

If $f \vDash(P * Q) \stackrel{\circ}{g}(R * S)$, then there exists $f_{1}, f_{2}$ such that $f_{1} \odot f_{2}=f, f_{1} \vDash P * Q, f_{2} \vDash R * S$. That is, there exist $u_{1}$, $v_{1}$ such that $u_{1} \oplus v_{1} \sqsubseteq f_{1}, u_{1} \vDash P$, and $v_{1} \vDash Q$; there exist $u_{2}, v_{2}$ such that $u_{2} \oplus v_{2} \sqsubseteq f_{2}, u_{2} \vDash R, v_{2} \vDash S$.
By Theorem A. 20

- $u_{1} \vDash P$ implies there exists $u_{1}^{\prime} \sqsubseteq u_{1}$ such that $\mathrm{FV}_{\mathrm{R}}(P) \subseteq \operatorname{range}\left(u_{1}^{\prime}\right) \subseteq \mathrm{FV}(P)$, $\boldsymbol{\operatorname { d o m }}\left(u_{1}^{\prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(P)$, and $u_{1}^{\prime} \vDash P$.
- $v_{1} \models Q$ implies there exists $v_{1}^{\prime} \sqsubseteq v_{1}$ such that $\mathrm{FV}_{\mathrm{R}}(Q) \subseteq \operatorname{range}\left(v_{1}^{\prime}\right) \subseteq \mathrm{FV}(Q)$, $\boldsymbol{\operatorname { d o m }}\left(v_{1}^{\prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(Q)$, and $v_{1}^{\prime} \vDash Q$.
- $u_{2} \vDash R$ implies there exists $u_{2}^{\prime} \sqsubseteq u_{2}$ such that $\mathrm{FV}_{\mathrm{R}}(R) \subseteq \operatorname{range}\left(u_{2}^{\prime}\right) \subseteq \mathrm{FV}(R), \operatorname{dom}\left(u_{2}^{\prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(R)$, and $u_{2}^{\prime} \vDash R$.
- $v_{2} \models S$ implies there exists $v_{2}^{\prime} \sqsubseteq v_{2}$ such that $\mathrm{FV}(S) \subseteq \operatorname{range}\left(v_{2}^{\prime}\right) \subseteq \mathrm{FV}(S), \operatorname{dom}\left(v_{2}^{\prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(S)$, and $v_{2}^{\prime} \models S$.

By Downwards closure property of $\oplus, u_{2}^{\prime} \oplus v_{2}^{\prime}$ is defined and $u_{2}^{\prime} \oplus v_{2}^{\prime} \sqsubseteq u_{2} \oplus v_{2} \sqsubseteq f_{2}$. Say that $f_{1}=\left(u_{1} \oplus v_{1} \oplus\right.$ unit $\left.S_{1}\right) \odot h_{1}$, $f_{2}=\left(u_{2}^{\prime} \oplus v_{2}^{\prime} \oplus\right.$ unit $\left._{S_{2}}\right) \odot h_{2}$. Also,

$$
\begin{aligned}
\operatorname{dom}\left(u_{2}^{\prime} \oplus v_{2}^{\prime}\right) & =\operatorname{dom}\left(u_{2}^{\prime}\right) \cup \operatorname{dom}\left(v_{2}^{\prime}\right) \subseteq \mathrm{FV}_{\mathrm{D}}(R) \cup \mathrm{FV}_{\mathrm{D}}(S) \subseteq \mathrm{FV}_{\mathrm{R}}(P) \cup \mathrm{FV} \mathrm{~V}_{\mathrm{D}}(Q) \\
& \subseteq \operatorname{range}\left(u_{1}^{\prime}\right) \cup \operatorname{range}\left(v_{1}^{\prime}\right) \subseteq \operatorname{range}\left(u_{1}\right) \cup \operatorname{range}\left(v_{1}\right)=\operatorname{range}\left(u_{1} \oplus v_{1}\right)
\end{aligned}
$$

Then

$$
\begin{align*}
f_{1} \odot f_{2} & =\left(u_{1} \oplus v_{1} \oplus \text { unit }_{S_{1}}\right) \odot h_{1} \odot\left(u_{2}^{\prime} \oplus v_{2}^{\prime} \oplus \text { unit }_{S_{2}}\right) \odot h_{2} \\
& =\left(u_{1} \oplus v_{1} \oplus \text { unit }_{S_{1}}\right) \odot\left(\left(u_{2}^{\prime} \oplus v_{2}^{\prime}\right) \oplus h_{1}\right) \odot h_{2} \\
& \left.=\left(u_{1} \oplus v_{1} \oplus \text { unit }_{S_{1}}\right) \odot\left(\left(u_{2}^{\prime} \oplus v_{2}^{\prime}\right) \odot \text { unit }_{\text {range }}\left(u_{2}^{\prime} \oplus v_{2}^{\prime}\right)\right) \oplus\left(\text { unit }_{\text {dom }\left(h_{1}\right)}\right) \odot h_{1}\right) \odot h_{2} \\
& \left.=\left(u_{1} \oplus v_{1} \oplus \text { unit }_{S_{1}}\right) \odot\left(u_{2}^{\prime} \oplus v_{2}^{\prime} \oplus \text { unit }_{\text {dom }\left(h_{1}\right)}\right) \odot\left(\text { unit }_{\text {range }\left(u_{2}^{\prime} \oplus v_{2}^{\prime}\right)}\right) h_{1}\right) \odot h_{2} \\
& =\left(u_{1} \oplus v_{1} \oplus \text { unit }_{S_{1}}\right) \odot\left(u_{2}^{\prime} \oplus v_{2}^{\prime} \oplus \text { unit }_{\text {range }}\left(u_{1} \oplus v_{1}\right) \oplus \text { unit }_{S_{1}}\right) \odot\left(\text { unit }_{\text {range }}\left(u_{2}^{\prime} \oplus v_{2}^{\prime}\right) \oplus h_{1}\right) \odot h_{2} \\
& =\left(\left(\left(u_{1} \oplus v_{1}\right) \odot\left(u_{2}^{\prime} \oplus v_{2}^{\prime} \oplus \text { unit }_{\text {range }\left(u_{1} \oplus v_{1}\right)}\right)\right) \oplus \text { unit }_{S_{1}}\right) \odot\left(\text { unit }_{\text {range }}\left(u_{\left.u_{2}^{\prime} \oplus v_{2}^{\prime}\right)} \oplus h_{1}\right) \odot h_{2}\right. \\
& =\left(\left(u_{1} \odot\left(u_{2}^{\prime} \oplus \text { unit }_{\text {range }}\left(u_{1}\right)\right)\right) \oplus\left(v_{1} \odot\left(v_{2}^{\prime} \oplus \text { unit }_{\text {range }\left(v_{1}\right)}\right)\right) \oplus \text { unit }_{S_{1}}\right) \\
& \odot\left(\text { unit }_{\text {range }}\left(u_{2}^{\prime} \oplus v_{2}^{\prime}\right) \oplus h_{1}\right) \odot h_{2} \quad \quad(\dagger \text { and exact commutativity, associativity })
\end{align*}
$$

where $\diamond$ follows from Theorem A.34, $\operatorname{dom}\left(u_{2}^{\prime} \oplus v_{2}^{\prime}\right) \subseteq \operatorname{range}\left(u_{1} \oplus v_{1}\right) \subseteq \operatorname{dom}\left(h_{1}\right)$, and $\dagger$ follows from Eq. Exchange equality) and Theorem A. 33 .
Thus, $\left(u_{1} \odot\left(u_{2}^{\prime} \oplus\right.\right.$ unit $\left.\left._{\text {range }\left(u_{1}\right)}\right)\right) \oplus\left(v_{1} \odot\left(v_{2}^{\prime} \oplus\right.\right.$ unit $\left.\left._{\text {range }\left(v_{1}\right)}\right)\right) \sqsubseteq f_{1} \odot f_{2}$. Recall that $u_{2}^{\prime} \vDash R$. By persistence, $u_{2}^{\prime} \oplus$ unit $_{\text {range }\left(u_{1}\right)} \vDash R$. Similarly, $v_{2}^{\prime} \vDash S$, so by persistence, $v_{2}^{\prime} \oplus$ unit $_{\text {range }\left(v_{1}\right)} \vDash S$. Therefore,

$$
\left(u_{1} \odot\left(u_{2}^{\prime} \oplus \operatorname{unit}_{\text {range }\left(u_{1}\right)}\right)\right) \oplus\left(v_{1} \odot\left(v_{2}^{\prime} \oplus \operatorname{unit}_{\text {range }\left(v_{1}\right)}\right)\right) \vDash(P \circ R) *(Q \circ S)
$$

Then, by persistence, $f \vDash(P \circ R) *(Q \circ S)$.

Proposition A.30. (Axioms for atomic propositions) The following axioms are sound.

$$
\begin{aligned}
& (S \triangleright[A] *[B]) \rightarrow(S \triangleright[A]) *(S \triangleright[B]) \quad \text { if } A \cap B \subseteq S \\
& (S \triangleright[A] *[B]) \rightarrow(S \triangleright[A \cup B]) \\
& (A \triangleright[B]) \stackrel{\circ}{(B \triangleright[C]) \rightarrow(A \triangleright[C])} \\
& (A \triangleright[B]) \rightarrow(A \triangleright[A]) \stackrel{q}{(A \triangleright[B])} \\
& (A \triangleright[B]) \rightarrow(A \triangleright[B]) \stackrel{\circ}{(B \triangleright[B])}
\end{aligned}
$$

Proof. We prove it one by one.
RevPar Given any $f \vDash(S \triangleright[A] *[B])$, by satisfaction rules and semantic of atomic propositions, there exists $f^{\prime} \sqsubseteq f$ such that for all $m \in M_{d}$ such that $m \vDash_{d} S, f^{\prime}(m) \models_{r}[A] *[B]$.
Since $f^{\prime}(m)$ is defined and $f^{\prime}(m) \vDash_{r}[A] *[B]$, it follows that $\operatorname{dom}\left(f^{\prime}\right)=S$ and range $\left(f^{\prime}\right) \supseteq S \cup A \cup B$. Thus, we can define $f_{1}=\pi_{S \cup A} f^{\prime}, f_{2}=\pi_{S \cup B} f^{\prime}$. Note that $f_{1} \vDash(S \triangleright[A]), f_{2} \vDash(S \triangleright[B])$. Also, because $A \cap B \subseteq S$,

$$
\operatorname{range}\left(f_{1}\right) \cap \operatorname{range}\left(f_{2}\right)=(S \cup A) \cap(S \cup B)=S
$$

and thus $f_{1} \oplus f_{2}$ is defined. We now want to show that $f_{1} \oplus f_{2} \sqsubseteq f$.
Note $f^{\prime}(m) \models_{r}[A] *[B]$ implies that there exists $\mu_{1}, \mu_{2}$ such that $\mu_{1} \oplus_{r} \mu_{2} \sqsubseteq f^{\prime}(m)$, and $\operatorname{dom}\left(\mu_{1}\right) \supseteq A$, dom $\left(\mu_{2}\right) \supseteq B$. Since $f^{\prime}$ preserves input on its domain $S, \pi_{S} f^{\prime}(m)=\operatorname{unit}(m)$, so $\left(\mu_{1} \oplus_{r} \operatorname{unit}(m)\right) \oplus_{r}\left(\mu_{2} \oplus_{r} \operatorname{unit}(m) \sqsubseteq f^{\prime}(m) \oplus_{r} \operatorname{unit}(m) \oplus_{r} \operatorname{unit}(m)=f^{\prime}(m)\right.$ too. Let $\mu_{1}^{\prime}=\pi_{A \cup S}\left(\mu_{1} \oplus_{r} \operatorname{unit}(m)\right)$ and $\mu_{2}^{\prime}=\pi_{B \cup S}\left(\mu_{2} \oplus_{r} \operatorname{unit}(m)\right)$. Then due to Downwards closure in $M_{d}, \mu_{1}^{\prime} \oplus_{r} \mu_{2}^{\prime}$ will also be defined, and

$$
\mu_{1}^{\prime} \oplus_{r} \mu_{2}^{\prime} \sqsubseteq\left(\mu_{1} \oplus_{r} \operatorname{unit}(m)\right) \oplus_{r}\left(\mu_{2} \oplus_{r} \operatorname{unit}(m)\right) \sqsubseteq f^{\prime}(m),
$$

which implies that $\mu_{1}^{\prime} \oplus_{r} \mu_{2}^{\prime}=\pi_{S \cup A \cup B} f^{\prime}(m)$. In the range model, this means that $\mu_{1}^{\prime}=\pi_{S \cup A} f^{\prime}(m), \mu_{2}^{\prime}=\pi_{S \cup B} f^{\prime}(m)$.
Then for any $m^{\prime} \in \operatorname{Mem}[S]$, any $r \in \operatorname{Mem}[A \cup B \cup S]$,

$$
\begin{aligned}
\left(\pi_{S \cup A \cup B} f^{\prime}\right)\left(m^{\prime}\right)(r) & =\left(\pi_{S \cup A \cup B} f^{\prime}\left(m^{\prime}\right)\right)(r)=\mu_{1}^{\prime} \oplus_{r} \mu_{2}^{\prime}(r)=\mu_{1}^{\prime}\left(r^{S \cup A}\right) \cdot \mu_{2}^{\prime}\left(r^{S \cup B}\right) \\
\left(f_{1} \oplus f_{2}\right)\left(m^{\prime}\right)(r) & =f_{1}\left(m^{\prime}\right)\left(r^{S \cup A}\right) \cdot f_{2}\left(m^{\prime}\right)\left(r^{\cup \cup B}\right) \\
& =\left(\pi_{S \cup A} f^{\prime}\right)\left(m^{\prime}\right)\left(r^{S \cup A}\right) \cdot\left(\pi_{S \cup B} f^{\prime}\left(m^{\prime}\right)\left(r^{S \cup B}\right)\right. \\
& =\mu_{1}^{\prime}\left(r^{S \cup A}\right) \cdot \mu_{2}^{\prime}\left(r^{S \cup B}\right)
\end{aligned}
$$

Thus, $f_{1} \oplus f_{2}=\pi_{S \cup A \cup B} f^{\prime}$, which implies that $f_{1} \oplus f_{2} \sqsubseteq f$. By their types, $f_{1} \oplus f_{2} \vDash(S \triangleright[A]) *(S \triangleright[B])$.
By persistence, $f \vDash(S \triangleright[A]) *(S \triangleright[B])$.
UnIONRAN Obvious from the semantics of atomic proposition and the range logic.
AtomSeq Given any $f \vDash(A \triangleright[B]) \stackrel{\circ}{ }(B \triangleright[C])$, by satisfaction rules and semantic of atomic propositions, there exists

- $f_{1}, f_{2}$ such that $f_{1} \odot f_{2}=f$;
- $f_{1}^{\prime} \sqsubseteq f_{1}$ such that for any $m \in M_{d}$ such that $m \vDash_{d} A, f_{1}^{\prime}(m) \models_{r}[B]$.
- $f_{2}^{\prime} \sqsubseteq f_{2}$ such that for any $m \in M_{d}$ such that $m \vDash_{d} B, f_{2}^{\prime}(m) \vDash_{r}[C]$.

Note that $f_{1}^{\prime}(m) \models_{r}[B]$ implies that $B \subseteq \operatorname{range}\left(f_{1}^{\prime}\right)$, so $\pi_{B} f_{1}^{\prime}$ is defined. Let $f_{1}^{\prime \prime}=\pi_{B} f_{1}^{\prime}$.
Note that for any $m \in M_{d}$ such that $m \models_{d} A, f_{1}^{\prime \prime}(m) \models_{r}[B]$ too, so $f^{\prime \prime} \vDash(A \triangleright[B])$ too. Also, by transitivity, $f_{1}^{\prime \prime} \sqsubseteq f_{1}^{\prime} \sqsubseteq f_{1}$.
Say $f_{1}=\left(f_{1}^{\prime \prime} \oplus \eta_{S_{1}}\right) \odot v_{1}, f_{2}=\left(f_{2}^{\prime} \oplus \eta_{S_{2}}\right) \odot v_{2}$, then since range $\left(f_{1}^{\prime \prime}\right)=B=\operatorname{dom}\left(f_{2}^{\prime}\right)$,

$$
\left.\begin{array}{rl}
f_{1} \odot f_{2} & =\left(f_{1}^{\prime \prime} \oplus \eta_{S_{1}}\right) \odot v_{1} \odot\left(f_{2}^{\prime} \oplus \eta_{S_{2}}\right) \odot v_{2} \\
& =\left(f_{1}^{\prime \prime} \oplus \eta_{S_{1}}\right) \odot\left(f_{2}^{\prime} \oplus v_{1}\right) \odot v_{2} \quad \quad\left(\text { By Theorem A.34 and dom }\left(f_{2}^{\prime}\right)=B=\operatorname{range}\left(f_{1}^{\prime \prime}\right) \subseteq \operatorname{dom}\left(v_{1}\right)\right) \\
& =\left(f_{1}^{\prime \prime} \oplus \eta_{S_{1}}\right) \odot\left(f_{2}^{\prime} \oplus \eta_{\operatorname{dom}\left(v_{1}\right)}\right) \odot\left(v_{1} \oplus \eta_{\text {range }\left(f_{1}\right)}\right) \odot v_{2} \\
& =\left(f_{1}^{\prime \prime} \oplus \eta_{S_{1}}\right) \odot\left(f_{2}^{\prime} \oplus \eta_{S}\right) \odot\left(v_{1} \oplus \eta_{\text {range }\left(f_{1}\right)}\right) \odot v_{2} \\
& =\left(\left(f_{1}^{\prime \prime} \odot f_{2}^{\prime}\right) \oplus \eta_{S_{1}}\right) \odot\left(v_{1} \oplus \eta_{\text {range }\left(f_{1}\right)}\right) \odot v_{2}
\end{array} \quad \text { (By Theorem A.35) }\right)
$$

So $f_{1}^{\prime \prime} \odot f_{2}^{\prime} \sqsubseteq f_{1} \odot f_{2}=f$.
$f_{1}^{\prime \prime}: \operatorname{Mem}[A] \rightarrow \mathcal{D}(\operatorname{Mem}[B]), f_{2}^{\prime}: \operatorname{Mem}[B] \rightarrow \mathcal{D}\left(\operatorname{Mem}\left[\operatorname{range}\left(f_{2}^{\prime}\right)\right]\right) A$, so $f_{1}^{\prime \prime} \odot f_{2}^{\prime}: \operatorname{Mem}[A] \rightarrow \mathcal{D}\left(\operatorname{Mem}\left[\operatorname{range}\left(f_{2}^{\prime}\right)\right]\right)$. Since
range $\left(f_{2}^{\prime}\right) \supseteq C$, it follows that $f_{1}^{\prime \prime} \odot f_{2}^{\prime} \vDash(A \triangleright[C])$, and thus $f \vDash(A \triangleright[C])$ by persistence.
UnITL If $f \vDash(A \triangleright[B])$, then there must exists $f^{\prime} \sqsubseteq f$ such that for all $m \in M_{d}$ such that $m \vDash A, f^{\prime}(m) \models_{r}[B]$.
Given any witness $f^{\prime}, f^{\prime}=$ unit $_{\operatorname{Mem}[A]} \odot f^{\prime}$, and also $f^{\prime} \vDash_{r}(A \triangleright[B])$.
Note that unit ${ }_{\text {Mem }[A]} \models_{r}(A \triangleright[A])$, so $f^{\prime}=$ unit $_{\text {Mem }[A]} \odot f^{\prime} \vDash(A \triangleright[A]) \stackrel{ }{q}(A \triangleright[B])$.
UNITR Analogous as the UnITL case, except that now using the fact $f^{\prime}=f^{\prime} \odot$ unit $_{\operatorname{Mem}[B]}$ for any $f^{\prime}: \operatorname{Mem}[A] \rightarrow \mathcal{D}(\operatorname{Mem}[B])$.

## P. Common properties of models $M^{D}$ and $M^{P}$

We define a more general class of models, parametric on a monad $\mathcal{T}$, which encompasses both our concrete models $M^{P}$ and $M^{D}$. We will call them $\mathcal{T}$-models and use their properties to simplify proofs of certain properties of $M^{D}$ and $M^{P}$.
Definition A. 9 ( $\mathcal{T}$-models). We say that $(M, \sqsubseteq, \oplus, \odot, M)$ is a $\mathcal{T}$-model if it satisfies the following conditions.

1) $M$ consists of all maps of the type $\operatorname{Mem}[S] \rightarrow \mathcal{T}(\operatorname{Mem}[S \cup U])$, where $S, U$ are finite subsets of Var.
2) All $m \in M$ preserve the input $m: \operatorname{Mem}[S] \rightarrow \mathcal{T}(\operatorname{Mem}[S \cup U])$ is in $M$ only if $\pi_{S} m=$ unit ;
3) $\odot$ is defined to be the Kleisli composition associated with $\mathcal{T}$;
4) $\oplus$ is deterministic and partial: $f \oplus g$ is defined when range $(f) \cap \operatorname{range}(g)=\operatorname{dom}(f) \cap \operatorname{dom}(g)$;
5) $\oplus$ satisfies standard associativity: when both $(f \oplus g) \oplus h$ and $f \oplus(g \oplus h)$ are defined, $(f \oplus g) \oplus h=f \oplus(g \oplus h)$;
6) When $f \oplus g$ are $g \oplus f$ are both defined, $f \oplus g=g \oplus f$.
7) For any $f: \operatorname{Mem}[A] \rightarrow \mathcal{T}(\operatorname{Mem}[A \cup X]) \in M$, and any $S \subseteq A$,

$$
f \oplus \mathrm{unit}_{S}=f
$$

(Padding equality)
8) When both $\left(f_{1} \oplus f_{2}\right) \odot\left(f_{3} \oplus f_{4}\right)$ and $\left(f_{1} \odot f_{3}\right) \oplus\left(f_{2} \odot f_{4}\right)$ are defined,

$$
\left(f_{1} \oplus f_{2}\right) \odot\left(f_{3} \oplus f_{4}\right)=\left(f_{1} \odot f_{3}\right) \oplus\left(f_{2} \odot f_{4}\right)
$$

(Exchange equality)
9) $M$ is closed under $\oplus$ and $\odot$;
10) For $f, g \in M, f \sqsubseteq g$ if and only if there exist $v \in M$ and some finite set $S$ such that,

$$
\begin{equation*}
g=\left(f \oplus \text { unit }_{S}\right) \odot v \tag{26}
\end{equation*}
$$

Below, we prove properties $\mathcal{T}$-models, which would be common properties of $M^{D}$ and $M^{P}$. Two main results are that all $\mathcal{T}$-models are DIBI frames (Theorem A.37).

Lemma A. 31 (Standard associativity of $\oplus)$. For any $f_{1}, f_{2}, f_{3} \in M$, $\left(f_{1} \oplus f_{2}\right) \oplus f_{3}$ is defined if and only if $f_{1} \oplus\left(f_{2} \oplus f_{3}\right)$ is defined and they are equal.

Proof. $\left(f_{1} \oplus f_{2}\right) \oplus f_{3}$ is defined if and only if $R_{1} \cap R_{2}=D_{1} \cap D_{2}$ and $\left(R_{1} \cup R_{2}\right) \cap R_{3}=\left(D_{1} \cup D_{2}\right) \cap D_{3}$.
$f_{1} \oplus\left(f_{2} \oplus f_{3}\right)$ is defined if and only if $R_{2} \cap R_{3}=D_{2} \cap D_{3}$ and $R_{1} \cap\left(R_{2} \cup R_{3}\right)=D_{1} \cap\left(D_{2} \cup D_{3}\right)$. Thus, to show that $\left(f_{1} \oplus f_{2}\right) \oplus f_{3}$ is defined if and only if $f_{1} \oplus\left(f_{2} \oplus f_{3}\right)$ is defined, it suffices to show that

$$
\begin{align*}
R_{1} \cap R_{2} & =D_{1} \cap D_{2}  \tag{27}\\
\left(R_{1} \cup R_{2}\right) \cap R_{3} & =\left(D_{1} \cup D_{2}\right) \cap D_{3} \tag{28}
\end{align*}
$$

if and only if

$$
\begin{align*}
R_{2} \cap R_{3} & =D_{2} \cap D_{3}  \tag{29}\\
R_{1} \cap\left(R_{2} \cup R_{3}\right) & =D_{1} \cap\left(D_{2} \cup D_{3}\right) \tag{30}
\end{align*}
$$

We show that Eq. (29) and Eq. (30) follows from Eq. (27) and Eq. (28):
Recall that $D_{1} \subseteq R_{1}, D_{2} \subseteq R_{2}, D_{3} \subseteq R_{3}$, so

- Eq. (29) follows from $D_{2} \cap D_{3} \subseteq R_{2} \cap R_{3}$ and $D_{2} \cap D_{3} \supseteq R_{2} \cap R_{3}$, which holds because

$$
\begin{align*}
R_{2} \cap R_{3} & =R_{2} \cap\left(R_{2} \cap R_{3}\right) \subseteq R_{2} \cap\left(\left(R_{1} \cup R_{2}\right) \cap R_{3}\right) \\
& =R_{2} \cap\left(\left(D_{1} \cup D_{2}\right) \cap D_{3}\right)=R_{2} \cap\left(D_{1} \cap D_{3}\right)  \tag{28}\\
& \subseteq\left(R_{2} \cap D_{1}\right) \cap D_{3} \subseteq\left(R_{2} \cap R_{1}\right) \cap D_{3} \\
& =\left(D_{2} \cap D_{1}\right) \cap D_{3} \subseteq D_{2} \cap D_{3}
\end{align*}
$$ (By $D_{1} \subseteq R_{1}$ )

(By Eq. (27))

- Eq. (30) follows from $\left(D_{1} \cup D_{2}\right) \cap D_{3} \subseteq\left(R_{1} \cup R_{2}\right) \cap R_{3}$ and $\left(D_{1} \cup D_{2}\right) \cap D_{3} \supseteq\left(R_{1} \cup R_{2}\right) \cap R_{3}$, which holds because

$$
\begin{array}{rlrl}
R_{1} \cap\left(R_{2} \cup R_{3}\right) & =\left(R_{1} \cap R_{2}\right) \cup\left(R_{1} \cap R_{3}\right) \subseteq\left(R_{1} \cap R_{2}\right) \cup\left(R_{1} \cap\left(R_{1} \cup R_{2}\right) \cap R_{3}\right) & & \\
& =\left(D_{1} \cap D_{2}\right) \cup\left(R_{1} \cap\left(D_{1} \cup D_{2}\right) \cap D_{3}\right) & &  \tag{27}\\
& =\left(D_{1} \cap D_{2}\right) \cup\left(\left(R_{1} \cap D_{1} \cap D_{3}\right) \cup\left(R_{1} \cap D_{2} \cap D_{3}\right)\right) & & \\
& \subseteq\left(D_{1} \cap D_{2}\right) \cup\left(\left(D_{1} \cap D_{3}\right) \cup\left(R_{1} \cap R_{2} \cap D_{3}\right)\right) & \text { (By } \left.D_{2} \subseteq R_{2}\right) \\
& \subseteq\left(D_{1} \cap D_{2}\right) \cup\left(\left(D_{1} \cap D_{3}\right) \cup\left(D_{1} \cap D_{2} \cap D_{3}\right)\right) & \text { (By Eq. (28) } \\
& \subseteq\left(D_{1} \cap D_{2}\right) \cup\left(D_{1} \cap D_{3}\right)=D_{1} \cap\left(D_{2} \cup D_{3}\right) & &
\end{array}
$$

We show that Eq. (27) and Eq. (28) follows from Eq. (29) and Eq. (30):

- Eq. (27) follows from $D_{1} \cap D_{2} \subseteq R_{1} \cap R_{2}$ and $D_{1} \cap D_{2} \supseteq R_{1} \cap R_{2}$, which holds because

$$
\begin{align*}
R_{1} \cap R_{2} & =R_{1} \cap\left(R_{2} \cup R_{3}\right) \cap R_{2}=D_{1} \cap\left(D_{2} \cup D_{3}\right) \cap R_{2}  \tag{29}\\
& =D_{1} \cap\left(\left(D_{2} \cap R_{2}\right) \cup\left(D_{3} \cap R_{2}\right)\right)=D_{1} \cap\left(D_{2} \cup\left(D_{3} \cap R_{2}\right)\right) \\
& \subseteq D_{1} \cap\left(D_{2} \cup\left(R_{1} \cap R_{2}\right)\right) \\
& =D_{1} \cap\left(D_{2} \cup\left(D_{1} \cap D_{2}\right)\right) \\
& =D_{1} \cap D_{2}
\end{align*}
$$

(By $D_{2} \subseteq R_{1}$ )
(By Eq. (29))

- Eq. (28) follows from $\left(D_{1} \cup D_{2}\right) \cap D_{3} \subseteq\left(R_{1} \cup R_{2}\right) \cap R_{3}$ and $\left(D_{1} \cup D_{2}\right) \cap D_{3} \supseteq\left(R_{1} \cup R_{2}\right) \cap R_{3}$, which holds because

$$
\begin{aligned}
\left(R_{1} \cup R_{2}\right) \cap R_{3} & =\left(R_{1} \cap R_{3}\right) \cup\left(R_{2} \cap R_{3}\right) \\
& =\left(R_{1} \cap\left(R_{2} \cup R_{3}\right) \cap R_{3}\right) \cup\left(R_{2} \cap R_{3}\right) \\
& =\left(D_{1} \cap\left(D_{2} \cup D_{3}\right) \cap R_{3}\right) \cup\left(D_{2} \cap D_{3}\right) \\
& =\left(D_{1} \cap\left(\left(D_{2} \cap R_{3}\right) \cup\left(D_{3} \cap R_{3}\right)\right)\right) \cup\left(D_{2} \cap D_{3}\right) \\
& \subseteq\left(D_{1} \cap\left(\left(R_{2} \cap R_{3}\right) \cup D_{3}\right)\right) \cup\left(D_{2} \cap D_{3}\right) \\
& =\left(D_{1} \cap\left(\left(D_{2} \cap D_{3}\right) \cup D_{3}\right)\right) \cup\left(D_{2} \cap D_{3}\right) \\
& =\left(D_{1} \cap D_{3}\right) \cup\left(D_{2} \cap D_{3}\right)=\left(D_{1} \cup D_{2}\right) \cap D_{3}
\end{aligned}
$$

(By Eq. (30)

$$
\subseteq\left(D_{1} \cap\left(\left(R_{2} \cap R_{3}\right) \cup D_{3}\right)\right) \cup\left(D_{2} \cap D_{3}\right) \quad\left(\text { By } D_{2} \subseteq R_{2}, D_{3} \subseteq R_{3}\right)
$$

$$
\left.=\left(D_{1} \cap\left(\left(D_{2} \cap D_{3}\right) \cup D_{3}\right)\right) \cup\left(D_{2} \cap D_{3}\right) \quad \text { (By Eq. (29) }\right)
$$

Thus, Eq. (27) and Eq. (28) hold if and only if Eq. (29) and Eq. (30) hold. Therefore, $\left(f_{1} \oplus f_{2}\right) \oplus f_{3}$ is defined if and only if $f_{1} \oplus\left(f_{2} \oplus f_{3}\right)$ is defined and by Definition A.9(5) they are equal.

Lemma A. 32 (Reflexivity and transitivity of order). For any $\mathcal{T}$-model $M$, the order $\sqsubseteq$ defined in $M$ is transitive and reflexive.
Proof. Let $x: \operatorname{Mem}[A] \rightarrow \mathcal{T}(\operatorname{Mem}[X]) \in M, S=\emptyset, v=$ unit $_{X}$. Then

$$
\begin{aligned}
\left(x \oplus \text { unit }_{S}\right) \odot v & =\left(x \oplus \text { unit }_{\theta}\right) \odot \text { unit }_{X} \\
& =x \odot \text { unit }_{X} \\
& =x
\end{aligned}
$$

(By Eq. Padding equality)
(By Definition A.9(3))
Thus, by Equation (26) we have $x \sqsubseteq x$, and the order is reflexive.
For any $x, y, z \in M$, if $x \sqsubseteq y$ and $y \sqsubseteq z$, then by definition of $\sqsubseteq$, there exist $S_{1}$ and $v_{1}$ such that $y=\left(x \oplus\right.$ unit $\left._{S_{1}}\right) \odot v_{1}$, and there exist $S_{2}$ and $v_{2}$ such that $z=\left(y \oplus\right.$ unit $\left._{S_{2}}\right) \odot v_{2}$.

We can now calculate:

$$
\begin{aligned}
z & =\left(y \oplus \text { unit }_{S_{2}}\right) \odot v_{2} \\
& =\left(\left(\left(x \oplus \text { unit }_{S_{1}}\right) \odot v_{1}\right) \oplus \text { unit }_{S_{2}}\right) \odot v_{2} \\
& =\left(\left(\left(x \oplus \text { unit }_{S_{1}}\right) \odot v_{1}\right) \oplus\left(\text { unit }_{S_{2}} \odot \text { unit }_{S_{2}}\right)\right) \odot v_{2} \\
& =\left(x \oplus \text { unit }_{S_{1}} \oplus \text { unit }_{S_{2}}\right) \odot\left(v_{1} \oplus \text { unit }_{S_{2}}\right) \odot v_{2} \\
& =\left(x \oplus \text { unit }_{S_{1} \cup S_{2}}\right) \odot\left(\left(v_{1} \oplus \text { unit }_{S_{2}}\right) \odot v_{2}\right)
\end{aligned}
$$

(By Exchange equality and Theorem A.33)
$M$ is closed under $\oplus, \odot$, so $\left(v_{1} \oplus\right.$ unit $\left._{S_{2}}\right) \odot v_{2} \in M$. Thus, we can instantiate Equation (26) with $S=S_{1} \cup S_{2}$ and $v=\left(v_{1} \oplus\right.$ unit $\left._{S_{2}}\right) \odot v_{2}$ obtaining $x \sqsubseteq z$. So the order is transitive.
Proposition A.33. For any $\mathcal{T}$-model $M$, states $f_{1}, f_{2}, f_{3}, f_{4}$ in $M,\left(f_{1} \odot f_{3}\right) \oplus\left(f_{2} \odot f_{4}\right)$ is defined implies $\left(f_{1} \oplus f_{2}\right) \odot\left(f_{3} \oplus f_{4}\right)$ is also defined. The converse does not always hold, but if $f_{1} \odot f_{3}$ and $f_{2} \odot f_{4}$ are defined, then $\left(f_{1} \oplus f_{2}\right) \odot\left(f_{3} \oplus f_{4}\right)$ is defined implies $\left(f_{1} \odot f_{3}\right) \oplus\left(f_{2} \odot f_{4}\right)$ is defined too.

Proof. We prove each direction individually:

- Given $\left(f_{1} \odot f_{3}\right) \oplus\left(f_{2} \odot f_{4}\right)$ is defined, it must that $R_{1}=D_{3}, R_{2}=D_{4}$, and $R_{3} \cap R_{4}=D_{1} \cap D_{2}$. Thus, $R_{1} \cap R_{2}=D_{3} \cap D_{4} \subseteq$ $R_{3} \cap R_{4}=D_{1} \cap D_{2}$, ensuring that $f_{1} \oplus f_{2}$ is defined; $R_{3} \cap R_{4}=D_{1} \cap D_{2} \subseteq R_{1} \cap R_{2}=D_{3} \cap D_{4}$, ensuring that $f_{3} \oplus f_{4}$ is defined; range $\left(f_{1} \oplus f_{2}\right)=R_{1} \cup R_{2}=D_{3} \cup D_{4}=\boldsymbol{\operatorname { d o m }}\left(f_{3} \oplus f_{4}\right)$, ensuring $\left(f_{1} \oplus f_{2}\right) \odot\left(f_{3} \oplus f_{4}\right)$ is defined.
- Given $f_{1} \odot f_{3}$ and $f_{2} \odot f_{4}$ are defined, $\left(f_{1} \odot f_{3}\right) \oplus\left(f_{2} \odot f_{4}\right)$ is defined if $R_{3} \cap R_{4}=D_{1} \cap D_{2}$. When $\left(f_{1} \oplus f_{2}\right) \odot\left(f_{3} \oplus f_{4}\right)$ is defined,

$$
\begin{aligned}
R_{3} \cap R_{4} & =D_{3} \cap D_{4} \\
& =R_{1} \cap R_{2} \\
& =D_{1} \cap D_{2}
\end{aligned}
$$

(Because $f_{3} \oplus f_{4}$ is defined)
(Because $f_{1} \odot f_{3}$ and $f_{2} \odot f_{4}$ are defined)
(Because $f_{1} \oplus f_{2}$ is defined)
So $\left(f_{1} \odot f_{3}\right) \oplus\left(f_{2} \odot f_{4}\right)$ is also defined.
Lemma A. 34 ( $\odot$ elimination). For any $\mathcal{T}$-model $M$, and $f, g \in M$, if $f \odot\left(g \oplus\right.$ unit $\left._{X}\right)$ is defined and $\operatorname{dom}(g) \subseteq \operatorname{dom}(f)$, then $f \odot\left(g \oplus u^{\prime} t_{X}\right)=g \oplus f$.
Proof. Let $f: \operatorname{Mem}[S] \rightarrow \mathcal{T}(\operatorname{Mem}[S \cup T])$ and $g: \operatorname{Mem}[U] \rightarrow \mathcal{T}(\operatorname{Mem}[U \cup V])$ be in $M$. When $U \subseteq S$,

$$
\left.\begin{array}{l}
f \odot\left(g \oplus \text { unit }_{X}\right) \\
=\left(f \oplus \text { unit }_{U}\right) \odot\left(g \oplus \text { unit }_{X} \oplus \text { unit }_{S \cup T}\right) \\
=\left(\text { unit }_{U} \oplus f\right) \odot\left(g \oplus \text { unit }_{X} \oplus \text { unit }_{S \cup T}\right) \\
=\left(\text { unit }_{U} \oplus f\right) \odot\left(g \oplus \text { unit }_{S \cup T}\right) \\
=\left(\text { unit }_{U} \odot g\right) \oplus\left(f \odot \text { unit }_{S \cup T}\right) \\
=g \oplus f
\end{array} \quad \text { (By Padding equality) }\right) \text { (By commutativity) }
$$

where $\dagger$ follows from $X \subseteq S \cup T$, which holds as $f \odot\left(g \oplus\right.$ unit $\left._{X}\right)$ defined implies $S \cup T=X \cup U$.
Lemma A. 35 (Converting $\oplus$ to $\odot$ ). For any $\mathcal{T}$-model $M$, let $f: \operatorname{Mem}[S] \rightarrow \mathcal{T}(\operatorname{Mem}[S \cup T])$ and $g: \operatorname{Mem}[U] \rightarrow \mathcal{T}(\operatorname{Mem}[U \cup V])$ be in M. If $f \oplus g$ is defined, then $f \oplus g=\left(f \oplus\right.$ unit $\left._{U}\right) \odot\left(\right.$ unit $\left._{S \cup T} \oplus g\right)$.

Proof.

$$
f \oplus g=\left(f \odot \text { unit }_{S \cup T}\right) \oplus\left(\text { unit }_{U} \odot g\right)
$$

$$
=\left(f \oplus \text { unit }_{U}\right) \odot\left(\text { unit }_{S \cup T} \oplus g\right) \quad \text { (By Theorem A.33 and Exchange equality) }
$$

Lemma A. 36 (Quasi-Downwards-closure of $\odot$ ). For any $\mathcal{T}$-model M, and $f, g, h, i \in M$, if $f \sqsubseteq h, g \sqsubseteq i$, and $f \odot g, h \odot i$ are all defined, then $f \odot g \sqsubseteq h \odot i$.

Proof. Since $f \sqsubseteq h, g \sqsubseteq i$, there must exist sets $S_{1}, S_{2}$ and $v_{1}, v_{2} \in M$ such that $h=\left(f \oplus\right.$ unit $\left._{S_{1}}\right) \odot v_{1}, i=\left(g \oplus\right.$ unit $\left._{S_{2}}\right) \odot v_{2} . f \odot g$ is defined, so $\operatorname{dom}(g)=\operatorname{range}(f) \subseteq \operatorname{range}\left(f \oplus \operatorname{unit}_{S_{1}}\right)=\operatorname{dom}\left(v_{1}\right)$. Thus,

$$
\begin{align*}
h \odot i & =\left(f \oplus \text { unit }_{S_{1}}\right) \odot v_{1} \odot\left(g \oplus \text { unit }_{S_{2}}\right) \odot v_{2} \\
& \left.=\left(f \oplus \text { unit }_{S_{1}}\right) \odot\left(g \oplus v_{1}\right) \odot v_{2} \quad \text { (By Theorem A.34 and dom }(g) \subseteq \operatorname{dom}\left(v_{1}\right)\right) \\
& =\left(f \oplus \text { unit }_{S_{1}}\right) \odot\left(g \oplus \text { unit }_{\text {dom }\left(v_{1}\right)}\right) \odot\left(\text { unit }_{\text {range }(g)} \oplus v_{1}\right) \odot v_{2} \quad \text { (By Theorem A.35) } \\
& =\left(f \oplus \text { unit }_{S_{1}}\right) \odot\left(g \oplus \text { unit }_{S_{1}}\right) \odot\left(\text { unit }_{\text {range }(g)} \oplus v_{1}\right) \odot v_{2} \\
& =\left((f \odot g) \oplus\left(\text { unit }_{S_{1}} \odot \text { unit }_{S_{1}}\right)\right) \odot\left(\text { unit }_{\text {range }(g)} \oplus v_{1}\right) \odot v_{2} \\
& \left.=\left((f \odot g) \oplus \text { unit }_{S_{1}}\right) \odot \text { unit }_{\text {range }(g)} \oplus v_{1}\right) \odot v_{2}
\end{align*}
$$

where $\dagger$ follows from $\operatorname{dom}(g)=\operatorname{range}(f)$ and Eq. Padding equality, and $\odot$ follows from Theorem A. 33 and Exchange equality.
Therefore, $f \odot g \sqsubseteq h \odot i$.
Lemma A.37. Any $\mathcal{T}$-model $M$ is in DIBI.
Proof. The axioms that we need to check are the follows.
$\oplus$ Down-Closed We want to show that for any $x^{\prime}, x, y^{\prime}, y \in M$, if $x^{\prime} \sqsubseteq x$ and $y^{\prime} \sqsubseteq y$ and $x \oplus y=z$, then $x^{\prime} \oplus y^{\prime}$ is defined, and $x^{\prime} \oplus y^{\prime}=z^{\prime} \sqsubseteq z$.

Since $x^{\prime} \sqsubseteq x$ and $y^{\prime} \sqsubseteq y$, there exist sets $S_{1}, S_{2}$, and $v_{1}, v_{2} \in M$ such that $x=\left(x^{\prime} \oplus\right.$ unit $\left._{S_{1}}\right) \odot v_{1}$, and $y=\left(y^{\prime} \oplus\right.$ unit $\left._{S_{2}}\right) \odot v_{2}$. Thus,

$$
\begin{aligned}
x \oplus y & =\left(\left(x^{\prime} \oplus \text { unit }_{S_{1}}\right) \odot v_{1}\right) \oplus\left(\left(y^{\prime} \oplus \text { unit }_{S_{2}}\right) \odot v_{2}\right) \\
& =\left(\left(x^{\prime} \oplus \text { unit }_{S_{1}}\right) \oplus\left(y^{\prime} \oplus \text { unit }_{S_{2}}\right)\right) \odot\left(v_{1} \oplus v_{2}\right) \\
& =\left(\left(x^{\prime} \oplus y^{\prime}\right) \oplus\left(\text { unit }_{S_{1}} \oplus \text { unit }_{S_{2}}\right)\right) \odot\left(v_{1} \oplus v_{2}\right) \\
& =\left(\left(x^{\prime} \oplus y^{\prime}\right) \oplus\left(\text { unit }_{S_{1} \cup S_{2}}\right)\right) \odot\left(v_{1} \oplus v_{2}\right)
\end{aligned}
$$

(By Theorem A. 33 and Exchange equality)
(By commutativity and associativity)

This derivation proved that $x^{\prime} \oplus y^{\prime}$ is defined, and $x^{\prime} \oplus y^{\prime} \sqsubseteq x \oplus y=z$.
( $\odot$ Up-Closed) We want to show that for any $z^{\prime}, z, x, y \in M$, if $z=x \odot y$ and $z^{\prime} \sqsupseteq z$, then there exists $x^{\prime}, y^{\prime}$ such that $x^{\prime} \sqsupseteq x$, $y^{\prime} \sqsupseteq y$, and $z^{\prime}=x^{\prime} \odot y^{\prime}$.
Since $z^{\prime} \sqsupseteq z$, there exist set $S$, and $v \in M$ such that $z^{\prime}=\left(z \oplus\right.$ unit $\left._{S}\right) \odot v$. Thus,

$$
\begin{aligned}
z^{\prime} & =\left(z \oplus \text { unit }_{S}\right) \odot v \\
& =\left((x \odot y) \oplus \text { unit }_{S}\right) \odot v \\
& =\left((x \odot y) \oplus\left(\text { unit }_{S} \odot \text { unit }_{S}\right)\right) \odot v \\
& =\left(\left(x \oplus \text { unit }_{S}\right) \odot\left(y \oplus \text { unit }_{S}\right)\right) \odot v \\
& =\left(x \oplus \text { unit }_{S}\right) \odot\left(\left(y \oplus \text { unit }_{S}\right) \odot v\right)
\end{aligned}
$$

$$
=\left(\left(x \oplus \text { unit }_{S}\right) \odot\left(y \oplus \text { unit }_{S}\right)\right) \odot v \quad \text { (By Theorem A. } 33 \text { and Exchange equality }
$$

(By standard associativity of $\odot$ )
Thus, for $x^{\prime}=x \oplus$ unit $_{S}$ and $y^{\prime}=\left(y \oplus\right.$ unit $\left._{S}\right) \odot v, z^{\prime}=x^{\prime} \odot y^{\prime}$.
( $\oplus$ Commutativity) We want to show that $z=x \oplus y$ implies that $z=y \oplus x$. By definition of $T$-models: first, $x \oplus y$ is defined iff range $(x) \cap \operatorname{range}(y)=\boldsymbol{\operatorname { d o m }}(x) \cap \operatorname{dom}(y)$ iff $y \oplus x$ is defined; second, when $x \oplus y$ and $y \oplus x$ are both defined, they are equal. Thus, $\oplus$ commutativity frame condition is satisfied.
( $\oplus$ Associativity) Since $\oplus$ is deterministic and partial,the associativity of $\oplus$ frame axiom reduces to Theorem A. 31
$(\oplus$ Unit existence) We want to show that for any $x \in M$, there exists $e \in E$ such that $x=e \oplus x$. For any $x: \operatorname{Mem}[A] \rightarrow$ $\mathcal{D}(\operatorname{Mem}[B]), x \oplus$ unit $_{\text {Mem }[\emptyset]}$ is defined because $B \cap \emptyset=\emptyset=A \cap \emptyset$, and by Eq. (Padding equality), $\left(x \oplus\right.$ unit $\left._{\text {Mem }[\emptyset]}\right)=x$. Also, unit ${ }_{\text {Mem }[0]} \in E=M$. So $e=$ unit $_{\text {Mem }[0]}$ serves as the unit under $\oplus$ for any $x$.
( $\oplus$ Unit Coherence) We want to show that for any $y \in M, e \in E=M$, if $x=y \oplus e$, then $x \sqsupseteq y$.

$$
\begin{aligned}
x=y \oplus e & =\left(y \odot \text { unit }_{\text {range }(y)}\right) \oplus\left(\text { unit }_{\text {dom }(e)} \odot e\right) \\
& =\left(y \oplus \text { unit }_{\boldsymbol{d o m}(e)}\right) \odot\left(\text { unit }_{\text {range }(y)} \oplus e\right) \\
& =\left(y \oplus \text { unit }_{\operatorname{dom}(e)}\right) \odot\left(e \oplus \text { unit }_{\text {range }(y)}\right)
\end{aligned}
$$

(By Eq. (Exchange equality))
( $\oplus$ Commutativity)

Thus, $x \sqsupseteq y$.
( $\odot$ Associativity) Since $\odot$ is deterministic and partial, the associativity of $\odot$ frame axiom reduces to the standard associativity. Kleisli composition satisfies standard associativity, so $\odot$ also satisfies standard associativity.
$\left(\odot\right.$ Unit $^{\left.\text {Existence }_{\mathbf{L}} \text { and } \mathbf{R}\right)}$ Since $\odot$ is the Kleisli composition, for any morphism $x: \operatorname{Mem}[A] \rightarrow \mathcal{D}(\operatorname{Mem}[B])$, unit $\mathbf{M e m}_{[A]}$ is the left unit, and unit $\operatorname{Mem}[B]$ is the right unit. For all $S$, unit $\operatorname{Mem}[S] \in M=E$. Thus, for any $x \in M$, there exists $e \in E$ such that $e \odot x=x$, and there exists $e^{\prime} \in E$ such that $x \odot e^{\prime}=x$.
( $\odot$ Coherence $_{R}$ ) For any $y \in M, e \in E=M$ such that $x=y \odot e$, we want to show that $x \sqsupseteq y$. We just proved that $\left(y \oplus\right.$ unit $\left._{\text {Mem }[0]}\right)=y$ for any $y$, so $x=y \odot e=\left(y \oplus\right.$ unit $\left._{\operatorname{Mem}[0]}\right) \odot e$, and $x \sqsubseteq y$ as desired.
(Unit closure) We want to show that for any $e \in E$ and $e^{\prime} \sqsupseteq e, e^{\prime} \in E$. This is evident because $E=M$ and $M$ is closed under $\oplus$ and $\odot$.
(Reverse exchange) Given $x=y \oplus z$ and $y=y_{1} \odot y_{2}, z=z_{1} \odot z_{2}$, we want to show that there exists $u=y_{1} \oplus z_{1}, v=y_{2} \oplus z_{2}$, and $x=u \odot v$.
After substitution, we get $\left(y_{1} \odot y_{2}\right) \oplus\left(z_{1} \odot z_{2}\right)=y \oplus z=x$. By Exchange equality and Theorem A.33, when $\left(y_{1} \odot y_{2}\right) \oplus\left(z_{1} \odot z_{2}\right)$ is defined, $\left(y_{1} \oplus z_{1}\right) \odot\left(y_{2} \odot z_{2}\right)$ is also defined, and $\left(y_{1} \odot y_{2}\right) \oplus\left(z_{1} \odot z_{2}\right)=\left(y_{1} \oplus z_{1}\right) \odot\left(y_{2} \oplus z_{2}\right)$. Thus $\left(y_{1} \oplus z_{1}\right) \odot\left(y_{2} \oplus z_{2}\right)=y \oplus z=x$, and thus $u=y_{1} \oplus z_{1}, v=y_{2} \oplus z_{2}$ completes the proof.

Lemma A. 38 (Classical flavor in intuitionistic model). For any $\mathcal{T}$-model M such that Disintegration holds (see Theorem A. 10 and Theorem A.13), and $f \in M$,

$$
f \vDash(\emptyset \triangleright[Z]) \stackrel{q}{ }((Z \triangleright[X]) *(Z \triangleright[Y]))
$$

if and only if there exist $g, h, i \in M$, such that $g: \operatorname{Mem}[\emptyset] \rightarrow \mathcal{T}(\operatorname{Mem}[Z]), h: \operatorname{Mem}[Z] \rightarrow \mathcal{T}(\operatorname{Mem}[Z \cup X]), i: \operatorname{Mem}[Z] \rightarrow$ $\mathcal{T}(\operatorname{Mem}[Z \cup Y])$, and $g \odot(h \oplus i) \sqsubseteq f$.

Proof. The backwards direction trivially follows from persistence. We detail the proof for the forward direction here. Suppose $f \vDash(\emptyset \triangleright[Z]) \stackrel{\circ}{( }(Z \triangleright[X]) *(Z \triangleright[Y]))$. Then, there exist $f_{1}, f_{2}, f_{3}, f_{4}$ such that $f_{1} \odot f_{2}=f, f_{3} \oplus f_{4} \sqsubseteq f_{2}, f_{1} \vDash(\emptyset \triangleright[Z])$, $f_{3} \vDash(Z \triangleright[X])$ and $f_{4} \vDash(Z \triangleright[Y])$.

- $f_{1} \vDash(\emptyset \triangleright[Z])$ implies that there exists $f_{1}^{\prime \prime} \sqsubseteq f_{1}$ such that $\operatorname{dom}\left(f_{1}^{\prime \prime}\right)=\emptyset$, and range $\left(f_{1}^{\prime \prime}\right) \supseteq Z$. Let $f_{1}^{\prime}=\pi_{Z} f_{1}^{\prime \prime}$. Note that $f_{1}^{\prime}: \operatorname{Mem}[\emptyset] \rightarrow \mathcal{T}(\operatorname{Mem}[Z])$ and $f_{1}^{\prime} \sqsubseteq f_{1}^{\prime \prime} \sqsubseteq f_{1}$. Hence, there exists some set $S_{1}$ and $v_{1} \in M$ such that $f_{1}=\left(f_{1}^{\prime} \oplus\right.$ unit $\left._{S_{1}}\right) \odot v_{1}$.
- $f_{3} \vDash(Z \triangleright[X])$ implies that there exists $f_{3}^{\prime \prime} \sqsubseteq f_{3}$ such that $\operatorname{dom}\left(f_{3}^{\prime \prime}\right)=Z$, and range $\left(f_{3}^{\prime \prime}\right) \supseteq X$. Define $f_{3}^{\prime}=\pi_{Z \cup X} f_{3}^{\prime \prime}$. Then $f_{3}^{\prime} \sqsubseteq f_{3}^{\prime \prime} \sqsubseteq f_{3}$, and $f_{3}^{\prime}: \operatorname{Mem}[Z] \rightarrow \mathcal{T}(\operatorname{Mem}[X \cup Z])$.
- $f_{4} \vDash(Z \triangleright[Y])$ implies that there exists $f_{4}^{\prime \prime} \sqsubseteq f_{4}$ such that $\operatorname{dom}\left(f_{4}^{\prime \prime}\right)=Z$, and range $\left(f_{4}^{\prime \prime}\right) \supseteq Y$. Define $f_{4}^{\prime}=\pi_{Z \cup Y} f_{4}^{\prime \prime}$ and note that $f_{4}^{\prime}: \operatorname{Mem}[Z] \rightarrow \mathcal{T}(\operatorname{Mem}[Y \cup Z])$.
- By Downwards closure of $\oplus$ (Appendix $\mathbb{B})$, having $f_{3} \oplus f_{4}$ defined implies that $f_{3}^{\prime} \oplus f_{4}^{\prime}$ is also defined and $f_{3}^{\prime} \oplus f_{4}^{\prime} \sqsubseteq f_{3} \oplus f_{4} \sqsubseteq f_{2}$. Thus, there exists some $v_{2} \in M$ and finite set $S_{2}$ such that $f_{2}=\left(f_{3}^{\prime} \oplus f_{4}^{\prime} \oplus\right.$ unit $\left._{S_{2}}\right) \odot v_{2}$.
Using these observations, we can now calculate and show that $f_{1}^{\prime} \odot\left(f_{3}^{\prime} \oplus f_{4}^{\prime} \oplus\right.$ unit $\left._{Z}\right) \sqsubseteq f_{1} \oplus f_{2}$ :

$$
\begin{aligned}
& f_{1} \odot f_{2} \\
& =\left(f_{1}^{\prime} \oplus \text { unit }_{S_{1}}\right) \odot v_{1} \odot\left(f_{3}^{\prime} \oplus f_{4}^{\prime} \oplus \text { unit }_{S_{2}}\right) \odot v_{2} \\
& \left.=\left(f_{1}^{\prime} \oplus \text { unit }_{s_{1}}\right) \odot\left(f_{3}^{\prime} \oplus f_{4}^{\prime} \oplus v_{1}\right) \odot v_{2} \quad \text { (By Theorem A. } 34 \text { and } \operatorname{dom}\left(f_{3}^{\prime} \oplus f_{4}^{\prime}\right)=Z \subseteq \operatorname{range}\left(f_{1}^{\prime} \oplus \text { unit }_{S_{1}}\right)\right) \\
& =\left(f_{1}^{\prime} \oplus \text { unit }_{S_{1}}\right) \odot\left(\left(f_{3}^{\prime} \oplus f_{4}^{\prime} \oplus \text { unit }_{\text {dom } \left._{\left(v_{1}\right)}\right)}\right) \odot\left(\text { unit }_{X \cup Y \cup Z} \oplus v_{1}\right)\right) \odot v_{2} \\
& =\left(f_{1}^{\prime} \oplus \text { unit }_{S_{1}}\right) \odot\left(f_{3}^{\prime} \oplus f_{4}^{\prime} \oplus \text { unit }_{Z} \oplus \text { unit }_{S_{1}}\right) \odot\left(\text { unit }_{X \cup Y \cup Z} \oplus v_{1}\right) \odot v_{2} \\
& =\left(\left(f_{1}^{\prime} \odot\left(f_{3}^{\prime} \oplus f_{4}^{\prime} \oplus \text { unit }_{z}\right)\right) \oplus\left(\text { unit }_{S_{1}} \odot \text { unit }_{S_{1}}\right)\right) \odot\left(\text { unit }_{X \cup Y \cup Z} \oplus v_{1}\right) \odot v_{2} \\
& =\left(\left(f_{1}^{\prime} \odot\left(f_{3}^{\prime} \oplus f_{4}^{\prime} \oplus \text { unit }_{z}\right)\right) \oplus \text { unit }_{S_{1}}\right) \odot\left(\text { unit }_{X \cup Y \cup Z} \oplus v_{1}\right) \odot v_{2} \\
& \left.=\left(\left(f_{1}^{\prime} \odot\left(f_{3}^{\prime} \oplus f_{4}^{\prime}\right)\right) \oplus \text { unit }_{S_{1}}\right) \odot\left(\text { unit }_{X \cup Y \cup Z} \oplus v_{1}\right) \odot v_{2} \quad \text { (Because } f_{3}^{\prime}, f_{4}^{\prime} \text { preserves input on } Z\right)
\end{aligned}
$$

To finish, take $g=f_{1}^{\prime}: \operatorname{Mem}[\emptyset] \rightarrow \mathcal{T}(\operatorname{Mem}[Z]), h=f_{3}^{\prime}: \operatorname{Mem}[Z] \rightarrow \mathcal{T}(\operatorname{Mem}[Z \cup X]), i=f_{4}^{\prime}: \operatorname{Mem}[Z] \rightarrow \mathcal{T}(\operatorname{Mem}[Z \cup Y])$, and note that $g \odot(h \oplus i)=f_{1}^{\prime} \odot\left(f_{3}^{\prime} \oplus f_{4}^{\prime}\right) \sqsubseteq f_{1} \oplus f_{2} \sqsubseteq f$.

Lemma A. 39 (Uniqueness). For any $\mathcal{T}$-model $M, f, g: \operatorname{Mem}[X] \rightarrow \mathcal{T}(\operatorname{Mem}[X \cup Y])$ in $M$, and arbitrary $h \in M$, if $f \sqsubseteq h$ and $g \sqsubseteq h$, then $f=g$.

Proof. $f \sqsubseteq h$ implies that there exists $v_{1}, S_{1}$ such that $\left(f \oplus\right.$ unit $\left._{S_{1}}\right) \odot v_{1}=h ; g \sqsubseteq h$ implies that there exists $v_{2}, S_{2}$ such that $\left(g \oplus\right.$ unit $\left._{S_{2}}\right) \odot v_{2}=h$. Take $h: \operatorname{Mem}[W] \rightarrow \mathcal{T}(\operatorname{Mem}[Z \cup W])$, and then

$$
\begin{aligned}
& f \oplus \text { unit }_{S_{1}}=\pi_{\text {range }\left(f \oplus \text { unit }_{S_{1}}\right)} h=\pi_{X \cup Y \cup \operatorname{dom}(h)} h \\
& g \oplus \text { unit }_{S_{2}}=\pi_{\text {range }\left(g \oplus \operatorname{unit}_{S_{2}}\right)} h=\pi_{X \cup Y \cup \operatorname{dom}(h)} h
\end{aligned}
$$

Thus, $f \oplus$ unit $_{S_{1}}=g \oplus$ unit $_{S_{2}}$. Now, suppose $f \neq g$. This would imply $f \oplus$ unit $_{S_{1}} \neq g \oplus$ unit $_{S_{2}}$ which is a contradiction. Thus, $f=g$.

