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Extending Singh-Maddala Distribution

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A new distribution, the exponentiated transmuted Singh-Maddala distribution (ETSM), is presented, and three important special distributions are illustrated. Some mathematical properties are obtained, and parameters estimation method is applied using maximum likelihood. Illustrations based on random numbers and a real data set are given.

Keywords: Singh-Maddala distribution, moments, order statistics, quantile function, maximum likelihood estimation

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Introduction

The Burr distribution was first discussed by Burr (1942) as a two-parameter family; it is a very flexible distribution that can express a wide range of distributions shapes. The Singh-Maddala (SM) distribution introduced by Singh and Maddala (1976). It is known under various other names, such as the Burr XII distribution (Tadikamalla, 1980; Al-Khazaleh, 2016), the Pareto IV (Arnold, 1983) distribution, beta-P (Mielke & Johnson, 1974) distribution and generalized log-logistic (El-Saidi et al., 1990) distribution. The SM distribution includes, overlaps, or has as a limiting case many commonly-used distributions such as gamma, lognormal, log logistic, bell-shaped, and J-shaped beta distributions (but not U-shaped). The SM distribution is used in various fields such as finance, hydrology, and reliability to model a variety of data types.

Generally, the cumulative distribution function (CDF) of the transmuted function (Aryal & Tsokos, 2011) is given by

$$\mathrm{F}(x)=(1+\lambda)\mathrm{G}(x)-(\lambda\mathrm{G}(x))^2;|\lambda|\le 1,-\infty<x<\infty, \tag{1}$$

then, the CDF of the exponentiated transmuted function is defined by

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$$\mathrm{F}(x)=\mathrm{Z}^{v}(x)=\left[(1+\lambda)\mathrm{G}(x)-\lambda(\mathrm{G}(x))^{2}\right]^{v};-\infty<x<\infty. \tag{2}$$

The aim of this study is to present and study a new distribution called the ETSM distribution based on the exponentiated transmuted function.

The CDF and PDF of the ETSM Distribution

The CDF and the probability density function (PDF) of the Singh and Maddala (Singh & Maddala, 1976) are, respectively,

$$\mathrm{G}(x;a,b,p)=1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p};x\geq 0;a,b,p>0 \tag{3}$$

and

$$\mathrm{g}(x;a,b,p)=\left(\frac{ap}{b}\right)\left(\frac{x}{b}\right)^{a-1}\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-(p+1)},x\geq 0,a,b,p>0. \tag{4}$$

The exponentiated transmuted Singh-Maddala (ETSM) distribution can be derived easily by substituting equation (3) into equation (2); it yields the CDF of the ESTM(a, b, p, v, λ) distribution as follows:

$$\mathrm{F}(x)=\left\{(1+\lambda)\left[1-\left(1+\left(\frac{x}{b}\right)^{a}\right)^{-p}\right]-\lambda\left[1-\left(1+\left(\frac{x}{b}\right)^{a}\right)^{-p}\right]^{2}\right\}^{v}. \tag{5}$$

where v, a, and p are shape parameters and b and λ are scale parameters. Differentiating equation (5) yields the PDF of the ETSM distribution:

$$f(x) = \left(\frac{vap}{b}\right)\left(\frac{x}{b}\right)^{a-1}\left(1+\left(\frac{x}{b}\right)^{a}\right)^{-(p+1)}$$
$$\times\left\{(1+\lambda)\left[1-\left(1+\left(\frac{x}{b}\right)^{a}\right)^{-p}\right]-\lambda\left[1-\left(1+\left(\frac{x}{b}\right)^{a}\right)^{-p}\right]^{2}\right\}^{v-1} \quad (6)$$
$$\times\left\{(1+\lambda)-2\lambda\left[1-\left(1+\left(\frac{x}{b}\right)^{a}\right)^{-p}\right]\right\}$$

The ETSM distribution has several special cases as follows: setting $\lambda = 0$ gives the exponentiated Singh-Maddala (ESM) distribution, setting $v = 1$ gives the transmuted Singh-Maddala (TSM) distribution, and setting $\lambda = 0$ and $v = 1$ gives the Singh-Maddala (SM) distribution. Displayed in Figure 1 are plots of the ETSM density for some values of the parameters a, b, p, v, and λ.

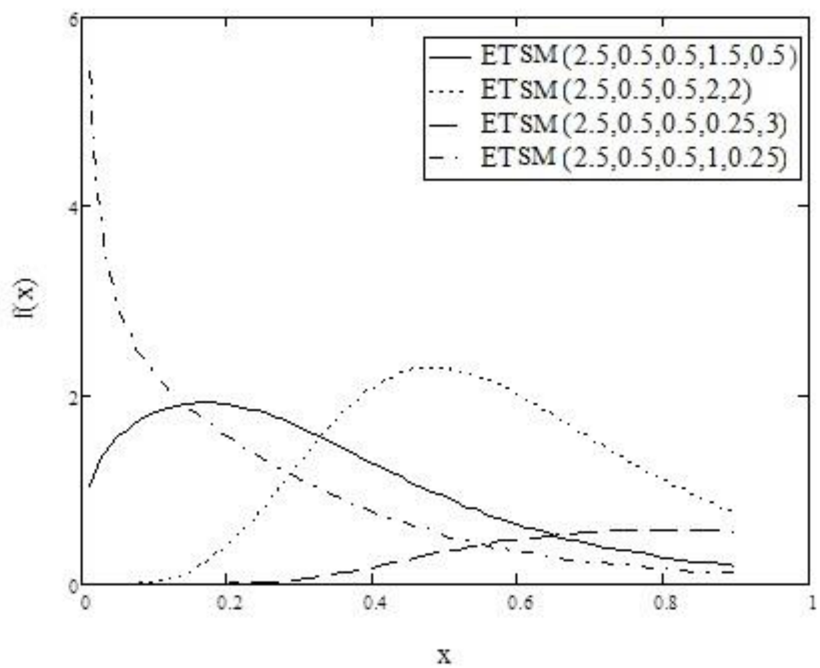

Figure 1. The PDF of the ETSM distribution with different parameters

Statistical Properties

The Quantile Function of the ETSM Distribution

The quantile function of the ETSM distribution is derived in the following Corollary

Corollary 1. The quantile function of the random variable X having the CDF of the ETSM distribution is given by the nonlinear equation

$$x_q = \left[\frac{q^{1/v}}{\left(\lambda b^{-2ap} - (1+\lambda) x_q^{1/2} b^{ap} \right)} \right]^{-\frac{1}{2ap}}. \tag{7}$$

Proof. Equating q to the CDF,

$$q = \mathrm{p}\left(X \le x_q \right) = \mathrm{F}\left(x_q \right) = q; x_q > 0, 0 < q < 1.$$

Then

$$q^{\frac{1}{v}} = (1+\lambda)\left(1 - \left[1 + (x)^a \right]^{-p} \right) - \lambda \left(1 - \left[1 + \left(\frac{x}{b} \right)^a \right]^{-p} \right)^2$$

and

$$x_q = \left[\frac{q^{1/v}}{\left(\frac{\lambda}{b^{-2ap}} - \frac{1+\lambda}{b^{-ap} x_q^{1/2}} \right)} \right]^{-\frac{1}{2ap}}, \tag{8}$$

where the last equation is a nonlinear quantile function and it needs a numerical solution to be solved.

The r^{th} Moment

The r^{th} moment of a random variable X of the ETSM distribution can be obtained from the following theorem:

Theorem 1. The r^{th} moment of the random variable X having the PDF of the ETSM distribution is given by

$$\mathrm{E}\left(X^r\right) = (-1)^{r/a} b^r - \left(\frac{rb^r}{ap}\right)\left(\sum_{i=0}^{v}\sum_{j=0}^{\infty}\binom{v}{i}\binom{r/a-1}{j}(-1)^{r/a-1-j}(\lambda)^i \mathrm{B}\left(\frac{i-j+1}{p}, v+1\right)\right) \quad (9)$$

Proof. The r^{th} moment of a random variable X can be obtained from

$$\mathrm{E}\left(x^r\right) = \int_x x^r \, \mathrm{f}(x) dx . \quad (10)$$

Then, substituting equation (6) into (10) yields

$$\mathrm{E}\left(X^r\right) = v\int_0^\infty x^r \left[(1+\lambda)\left(1-\left[1+\left(\frac{x}{b}\right)^a\right]^{-p}\right) - \lambda\left(1-\left[1+\left(\frac{x}{b}\right)^a\right]^{-p}\right)^2\right]^{v-1} \times\left(\frac{ap}{b}\right)\left(\frac{x}{b}\right)^{a-1}\left[1+\left(\frac{x}{b}\right)^a\right]^{-(p+1)}\left[(1+\lambda)-2\lambda\left(1-\left[1+\left(\frac{x}{b}\right)^a\right]^{-p}\right)\right]dx \quad (11)$$

Setting $y = 1 - [1 + (x / b)^a]^{-p}$ gives $x^r = b^r[(1 - y)^{-1/p} - 1]^{r/a}$. Substituting this into (11) yields

$$\mathrm{E}\left(Y^r\right) = \int_0^1 vb^r \left[(1-y)^{-1/p} - 1\right]^{\frac{r}{a}} \left(y + \lambda y - \lambda y^2\right)^{v-1} (1+\lambda-2\lambda y) dy . \quad (12)$$

Then, integration by parts and binomial expansion yield

$$\mathrm{E}\left(X^r\right)=$$
$$(-1)^{r/a}\, b^r-\left(\frac{rb^r}{ap}\right)\left(\sum_{i=0}^{v}\sum_{j=0}^{\infty}\binom{v}{i}\binom{r/a-1}{j}(-1)^{r/a-1-j}\,(\lambda)^i\,\mathrm{B}\left(\frac{i-j+1}{p},v+1\right)\right) \quad (13)$$

Setting r = 0 gives E(x^0) = 1. Setting r = 1 gives

$$\mathrm{E}\left(X\right)=$$
$$(-1)^{1/a}\, b-\left(\frac{b}{ap}\right)\left(\sum_{i=0}^{v}\sum_{j=0}^{\infty}\binom{v}{i}\binom{1/a-1}{j}(-1)^{1/a-1-j}\,(\lambda)^i\,\mathrm{B}\left(\frac{i-j+1}{p},v+1\right)\right) \quad (14)$$

Setting r = 2 gives

$$\mathrm{E}\left(X^2\right)=$$
$$(-1)^{2/a}\, b^2-\left(\frac{2b^2}{ap}\right)\left(\sum_{i=0}^{v}\sum_{j=0}^{\infty}\binom{v}{i}\binom{2/a-1}{j}(-1)^{2/a-1-j}\,(\lambda)^i\,\mathrm{B}\left(\frac{i-j+1}{p},v+1\right)\right) \quad (15)$$

Similarly, E(X^3) and E(X^4) can be calculated. The variance can be given by the fact that Var(X) = E(X^2) – [E(X)]^2. Therefore, Skewness and Kurtosis can be given, respectively, by

$$\mathrm{Skewness}(X)=\frac{\mathrm{E}\left(X^3\right)-3\,\mathrm{E}(X)\,\mathrm{E}\left(X^2\right)+2\,\mathrm{E}^3(X)}{\mathrm{Var}^{3/2}(X)},$$
$$\mathrm{Kurtosis}(X)=\frac{\mathrm{E}\left(X^4\right)-4\,\mathrm{E}(X)\,\mathrm{E}\left(X^3\right)+6\,\mathrm{E}\left(X^2\right)\mathrm{E}^2(X)-3\,\mathrm{E}^4(X)}{\mathrm{Var}^2(X)}$$

The Moment Generating Function

The moment generating function of the ETSM distribution is obtained in the following theorem:

Theorem 2. The moment generating function of the random variable X which has the PDF of the ETSM distribution is given by

$$\mathrm{M}_x(t)=\sum_{r=0}^{\infty}\frac{(t)^r}{r!}\mu_x^r\ .$$

Proof. Clearly, from the following fact

$$\mathrm{M}_x(t)=\mathrm{E}(\exp(xt)),$$

using the expansion of exp(*xt*) yields

$$\mathrm{M}_x(t)=\mathrm{E}\left(\sum_{r=0}^{\infty}\frac{(xt)^r}{r!}\right).$$

Then

$$\mathrm{M}_x(t)=\sum_{r=0}^{\infty}\frac{(t)^r}{r!}\mathrm{E}(x^r)$$

and hence

$$\mathrm{M}_x(t)=\sum_{r=0}^{\infty}\frac{(t)^r}{r!}\mu_x^r\ . \qquad (16)$$

The Mode

The log function of the PDF is

$$\log f(x)=\log v+(v-1)\log\left[(1+\lambda)\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)\right.$$

$$\left.-\lambda\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)^{2}\right]+\log\left(\frac{ap}{b}\right)+(a-1)\log\left(\frac{x}{b}\right) \tag{17}$$

$$-(p+1)\log\left[1+\left(\frac{x}{b}\right)^{a}\right]+\log\left[(1+\lambda)-2\lambda\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)\right]$$

Then

$$\frac{d}{dx}\log f(x)=(v-1)\left\{\frac{(1+\lambda)\left[1+\frac{x}{b}^{a}\right]^{-(p+1)}\left(\frac{ap}{b}\right)\left(\frac{x}{b}\right)^{a-1}}{\left[(1+\lambda)\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)-\lambda\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)^{2}\right]}\right.$$

$$\left.-\frac{2\lambda p\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-(p+1)}\left(\frac{a}{b}\right)\left(\frac{x}{b}\right)^{a-1}}{\left[(1+\lambda)\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)-\lambda\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)^{2}\right]}\right\}$$

$$+\frac{(a-1)}{x}-(p+1)\frac{\left(\frac{a}{b}\right)\left(\frac{x}{b}\right)^{a-1}}{\left[1+\left(\frac{x}{b}\right)^{a}\right]}-\frac{2\lambda p\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-(p+1)}\left(\frac{a}{b}\right)\left(\frac{x}{b}\right)^{a-1}}{\left[(1+\lambda)-2\lambda\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)\right]}$$

The mode can be determined by equating the previous equation to zero where it is a nonlinear equation and needs a numerical solution, to be solved with respect to x on one condition: that the value of x that satisfies the following equation must be less than zero.

$$\frac{d^2}{dx^2}\log \mathrm{f}\left(x\right)=\left(v-1\right)\left\{\frac{\left[\left(1+\lambda\right)\left(1-B_i\right)-\lambda A_i\right]\left[\left(1+\lambda\right)\left(\frac{a^2 p}{b^2}\right)D_i^{2(a-1)}\left(p+1\right)F_i^{(p+2)}\right]}{\left[\left(1+\lambda\right)\left(1-B_i\right)-\lambda A_i\right]^2}\right.$$

$$+\frac{\left[\left(1+\lambda\right)\left(1-B_i\right)-\lambda A_i\right]\left[\left(1+\lambda\right)p\left(a-1\right)D_i^{(a-2)}E_i\right]}{\left[\left(1+\lambda\right)\left(1-B_i\right)-\lambda A_i\right]^2}$$

$$-\frac{\left[\left(1+\lambda\right)\left(1-B_i\right)-\lambda A_i\right]\left[2\lambda pF_i^{(p+2)}\left(\frac{a^2}{b^2}\right)D_i^{2(a-1)}\left(p+1\right)\left(1-B_i\right)\right]}{\left[\left(1+\lambda\right)\left(1-B_i\right)-\lambda A_i\right]^2}$$

$$+\frac{\left[\left(1+\lambda\right)\left(1-B_i\right)-\lambda A_i\right]\left[2\lambda p\left(\frac{a}{b^3}\right)\left(a-1\right)D_i^{(a-2)}\left(1-B_i\right)E_i\right]}{\left[\left(1+\lambda\right)\left(1-B_i\right)-\lambda A_i\right]^2}$$

$$\left.-\frac{\left[\left(1+\lambda\right)\left(\frac{ap}{b}\right)D_i^{(a-1)}E_i-2\lambda\left(\frac{ap}{b}\right)D_i^{(a-1)}E_i\left(1-B_i\right)\right]}{\left[\left(1+\lambda\right)\left(1-B_i\right)-\lambda A_i\right]^2}\right\}$$

$$-\frac{\left(a-1\right)}{x^2}-\left(p+1\right)\frac{\left(1+C_i\right)\left(a-1\right)\left(\frac{a}{b^3}\right)D_i^{(a-2)}-\left(\frac{a}{b^3}\right)D_i^{(a-1)}}{\left(1+C_i\right)^2}$$

$$+\frac{\left(\left(1+\lambda\right)-2\lambda\left(1-B_i\right)\right)\left(2\lambda p\left(p+1\right)\left(\frac{a^2}{b^2}\right)D_i^{2(a-1)}F_i^{(p+2)}-2\lambda p\left(a-1\right)\left(\frac{a}{b^3}\right)D_i^{(a-2)}E_i\right)}{\left(\left(1+\lambda\right)-2\lambda\left(1-B_i\right)\right)^2}$$

$$-\frac{\left(-\left(\frac{2\lambda pa}{b}\right)D_i^{(a-1)}E_i\right)^2}{\left(\left(1+\lambda\right)-2\lambda\left(1-B_i\right)\right)^2}$$

Reliability Properties

Properties of reliability (Meeker & Escobar, 1998) will be obtained.

The Survival Function

Because

$$\bar{\mathrm{F}}(x)=1-\mathrm{F}(x),$$

the survival function is

$$\bar{\mathrm{F}}(x)=1-\left[(1+\lambda)\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)-\lambda\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)^{2}\right]^{v}. \qquad (18)$$

The Hazard Rate Function

The hazard rate function of the ETSM distribution is derived in the following Corollary:

Corollary 2. The hazard function of the random variable X having CDF and PDF of the ETSM Distribution is given by

$$\mathrm{h}(x)=\frac{v\left[(1+\lambda)\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)-\lambda\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)^{2}\right]^{v-1}}{1-\left[(1+\lambda)\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)-\lambda\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)^{2}\right]^{v}}$$
$$\times\left(\frac{ap}{b}\right)\left(\frac{x}{b}\right)^{a-1}\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-(p+1)}\left[(1+\lambda)-2\lambda\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)\right]$$

Proof. Generally, the hazard function of the random variable X is given by

$$\mathrm{h}(x) = \frac{\mathrm{f}(x)}{1-\mathrm{F}(x)}.$$

Substituting equations (5) and (6) into the previous equation yields

$$\mathrm{h}(x) = \frac{v\left[(1+\lambda)\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)-\lambda\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)^{2}\right]^{v-1}}{1-\left[(1+\lambda)\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)-\lambda\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)^{2}\right]^{v}} \times\left(\frac{ap}{b}\right)\left(\frac{x}{b}\right)^{a-1}\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-(p+1)}\left[(1+\lambda)-2\lambda\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)\right] \qquad (19)$$

Figure 2 illustrates the Hazard function of the ETSM distribution with different parameters. One can see, in Figure 2, two types of Hazard functions curves of the ETSM distribution are described as follows: An increasing then constant then decreasing Hazard curve and an increasing then decreasing Hazard curve.

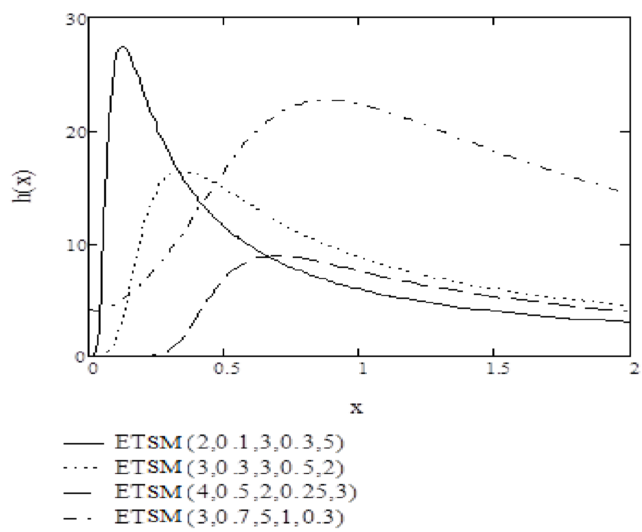

Figure 2. The hazard function of the ESTM distribution with different parameters

The Cumulative Hazard Rate Function

Based on

$$\mathrm{H}(x)=\int_0^x \mathrm{h}(x)\,dx$$

and substituting equation (8) into the previous equation yields

$$\begin{aligned}\mathrm{H}(x)&=\int_0^x \mathrm{h}(x)\,dx\\&=-\ln\left[1-\lambda\left(\frac{x}{b}\right)^{-2ap}-(1+\lambda)\left(\frac{x}{b}\right)^{-ap}\right]^{v}\end{aligned}\qquad(20)$$

Order Statistics of the ESTM Distribution

The r^{th} moment of order statistics of the ETSM distribution (Arnold et al., 1992) is derived in the following theorem:

Theorem 3. The density $\mathrm{f}_{u:n}(x_u)$ of the u^{th} order statistic, for $u = 1, 2,\ldots, n$, from iid random variables $X_1, X_2,\ldots, X_n$ following the ETSM distribution (Arnold et al., 1992) is given by

$$\begin{aligned}\mathrm{f}_{u:n}(x_u)&=\frac{1}{\beta(u,n-u+1)}\\&\times\sum_{w=0}^{n-u}(-1)^{w}\binom{n-u}{w}v\left[(1+\lambda)\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)\right.\\&\qquad\left.-\lambda\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)^{2}\right]^{v-1+vu+vw-v}\\&\times\left(\frac{ap}{b}\right)\left(\frac{x}{b}\right)^{a-1}\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-(p+1)}\left[(1+\lambda)-2\lambda\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)\right]\end{aligned}\qquad(21)$$

Proof. Generally, the density $f_{u:n}(x_u)$ of the u^{th} order statistic, for $u = 1, 2, \ldots, n$, from iid random variables $X_1, X_2, \ldots, X_n$ (Arnold et al., 1992) is given by

$$f_{u:n}(x_u) = \frac{f(x_u)}{\beta(u, n-u+1)} F(x_u)^{u-1} \{1 - F(x_u)\}^{n-u}.$$

Using binomial expansion yields

$$f_{u:n}(x_u) = \frac{f(x_u)}{\beta(u, n-u+1)} \sum_{w=0}^{n-u} (-1)^w \binom{n-u}{w} F(x_u)^{u+w-1}.$$

Then,

$$\begin{aligned} f_{u:n}(x_u) = {} & \frac{1}{\beta(u, n-u+1)} \\ & \times \sum_{w=0}^{n-u} (-1)^w \binom{n-u}{w} v \left[(1+\lambda)\left(1 - \left[1 + \left(\frac{x}{b}\right)^a\right]^{-p}\right) \right. \\ & \left. - \lambda\left(1 - \left[1 + \left(\frac{x}{b}\right)^a\right]^{-p}\right)^2 \right]^{v-1+vu+vw-v} \qquad (22) \\ & \times \left(\frac{ap}{b}\right)\left(\frac{x}{b}\right)^{a-1}\left[1 + \left(\frac{x}{b}\right)^a\right]^{-(p+1)} \left[(1+\lambda) - 2\lambda\left(1 - \left[1 + \left(\frac{x}{b}\right)^a\right]^{-p}\right)\right] \end{aligned}$$

Moments of order statistics can be given by

$$E\left(X_{u:n}^r\right) = \int_0^\infty x_u^r \, f_{u:n}(x_u)\, dx_u.$$

Substituting equation (6) into the previous equation yields

$$\mathrm{f}_{u:n}\left(x_u\right)=\frac{1}{\beta\left(u,n-u+1\right)}$$
$$\times\sum_{w=0}^{n-u}\left(-1\right)^{w}\binom{n-u}{w}v\int_{0}^{\infty}x_u^r\left[\left(1+\lambda\right)\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)\right.$$
$$\left.-\lambda\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)^{2}\right]^{v-1+vu+vw-v} \quad (23)$$
$$\times\left(\frac{ap}{b}\right)\left(\frac{x}{b}\right)^{a-1}\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-(p+1)}\left[\left(1+\lambda\right)-2\lambda\left(1-\left[1+\left(\frac{x}{b}\right)^{a}\right]^{-p}\right)\right]dx_u$$

Setting $y = 1 - [1 + (x / b)^a]^{-p}$ gives $x^r = b^r[(1 - y)^{-1/p} - 1]^{r/a}$. Substituting this into (23) yields

$$\mathrm{E}\left(y^r\right)=\int vb^r\left[\left(1-y\right)^{(-1/p)}-1\right]^{(r/a)}\left(y+\lambda y-\lambda y^2\right)^{vu+vw}\left(1+\lambda-2\lambda y\right)dy\,. \quad (24)$$

Using integration by parts and binomial expansion yields

$$\mathrm{E}\left(X^r\right)=\frac{v}{vu+vw}\left(-1\right)^{r/a}b^r-\frac{v}{vu+vw}\left(\frac{rb^r}{ap}\right)$$
$$\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{vu+vw}{i}\binom{r/a-1}{j}\left(-1\right)^{r/a-1-j}\left(\lambda\right)^{i}\mathrm{B}\left(\frac{i-j+1}{p},vu+vw+1\right) \quad (25)$$

Maximum Likelihood Estimation

Let X_1, X_2,…, X_n be iid random variables following the ETSM(a, b, p, λ, v) distribution; then the likelihood function (Garthwait et al., 1995) is given by

$$L = v^n \prod_{i=1}^{n} \left[(1+\lambda)\left(1-\left[1+\left(\frac{x_i}{b}\right)^a\right]^{-p}\right) - \lambda\left(1-\left[1+\left(\frac{x_i}{b}\right)^a\right]^{-p}\right)^2 \right]^{v-1}$$

$$\times\left(\frac{ap}{b}\right)\left(\frac{x_i}{b}\right)^{a-1}\left[1+\left(\frac{x_i}{b}\right)^a\right]^{-(p+1)}\left[(1+\lambda)-2\lambda\left(1-\left[1+\left(\frac{x_i}{b}\right)^a\right]^{-p}\right)\right]$$

Hence, the log likelihood function is

$$\ell(xi;B) = n\log v + \log\prod_{i=1}^{n}\left[(1+\lambda)\left(1-\left[1+\left(\frac{x_i}{b}\right)^a\right]^{-p}\right) - \lambda\left(1-\left[1+\left(\frac{x_i}{b}\right)^a\right]^{-p}\right)^2\right]^{v-1}$$

$$+\log\prod_{i=1}^{n}(ap/b)(x_i/\mathrm{b})^{a-1}\left[1+(x_i/\mathrm{b})^a\right]^{-(p+1)}$$

$$+\log\prod_{i=1}^{n}\left[(1+\lambda)-2\lambda\left(1-\left[1+\left(\frac{x_i}{b}\right)^a\right]^{-p}\right)\right]$$

Then

$$\ell(xi;B) = n\log v + (v-1)\log\prod_{i=1}^{n}\left[(1+\lambda)\left(1-\left[1+\left(\frac{x_i}{b}\right)^a\right]^{-p}\right)\right]$$

$$-\prod_{i=1}^{n}\left[\lambda\left(1-\left[1+\left(\frac{x_i}{b}\right)^a\right]^{-p}\right)^2\right] + \log\prod_{i=1}^{n}\left(\frac{ap}{b}\right)\left(\frac{x_i}{b}\right)^{a-1}\left[1+\left(\frac{x_i}{b}\right)^a\right]^{-(p+1)}$$

$$+\log\prod_{i=1}^{n}\left[(1+\lambda)-2\lambda\left(1-\left[1+\left(\frac{x_i}{b}\right)^a\right]^{-p}\right)\right]$$

Let

$$A_i = \left(1 - \left[1 + \left(\frac{x_i}{b}\right)^a\right]^{-p}\right)^2, B_i = \left[1 + \left(\frac{x_i}{b}\right)^a\right]^{-p}, C_i = \left(\frac{x_i}{b}\right)^a, D = \frac{x_i}{b},$$

$$E_i = \left[1 + \left(\frac{x_i}{b}\right)^a\right]^{-(p+1)}, F_i = \left[1 + \left(\frac{x_i}{b}\right)^a\right]^{-1}, \text{ and } G_i = \left[1 + \left(\frac{x_i}{b}\right)^a\right]$$

Then, differentiating with respect to a yields

$$\frac{\partial \ell(x;B)}{\partial a} = (v-1)\sum_{i=1}^{n} \frac{\left(pE_iC_i(\ln D_i)\right)\left[(1+\lambda) - 2\lambda(1-B_i)\right]}{\left[(1+\lambda)(1-B_i) - \lambda A_i\right]} + \sum_{i=1}^{n} \frac{\left(\left(\frac{ap}{b}\right)D_i^{a-1}(\ln D_i)E_i\right)\left[1-(p+1)F_iC_i\right]}{\left[\left(\frac{ap}{b}\right)D_i^{a-1}E_i\right]} + \sum_{i=1}^{n} \frac{\left[(-2p\lambda)E_iC_i(\ln D_i)\right]}{\left[(1+\lambda) - 2\lambda(1-B_i)\right]} \tag{26}$$

differentiating with respect to b yields

$$\frac{\partial \ell(x;B)}{\partial b} = (v-1)\sum_{i=1}^{n} \frac{\left(E_i\left(\frac{ap}{b}\right)C_i\right)\left[-(1+\lambda) + 2\lambda(1-B_i)\right]}{\left[(1+\lambda)(1-B_i) - \lambda A_i\right]} - \sum_{i=1}^{n} \frac{\left(\frac{ap}{b^2}\right)D_i^{a-1}E_i\left[-1 + (1-a) + aC_i(p+1)F_i\right]}{\left[\left(\frac{ap}{b}\right)D_i^{a-1}E_i\right]} + \sum_{i=1}^{n} \frac{\left[(2p\lambda)E_iC_i\left(\frac{a}{b}\right)\right]}{\left[(1+\lambda) - 2\lambda(1-B_i)\right]} \tag{27}$$

differentiating with respect to p yields

$$\frac{\partial \ell(x;B)}{\partial p} = (v-1)\sum_{i=1}^{n} \frac{(B_i \ln G_i)\left[\left((1+\lambda)\left(\frac{ap}{b}\right)\right)+2\lambda(1-B_i)\right]}{\left[(1+\lambda)(1-B_i)-\lambda A_i\right]} + \sum_{i=1}^{n} \frac{\left(\left(\frac{a}{b}\right)D_i^{a-1}E_i\right)\left[1+p\ln G_i\right]}{\left[\left(\frac{ap}{b}\right)D_i^{a-1}E_i\right]} - \sum_{i=1}^{n} \frac{\left[(2\lambda)(1-B_i)\ln G_i\right]}{\left[(1+\lambda)-2\lambda(1-B_i)\right]} \tag{28}$$

differentiating with respect to v yields

$$\frac{\partial \ell(x;B)}{\partial v} = \left(\frac{n}{v}\right) + \sum_{i=1}^{n} \log\left[(1+\lambda)(1-B_i)-\lambda A_i\right], \tag{29}$$

and differentiating with respect to λ yields

$$\frac{\partial \ell(x;B)}{\partial \lambda} = (v-1)\sum_{i=1}^{n} \frac{(1-B_i)-A_i}{\left[(1+\lambda)(1-B_i)-\lambda A_i\right]} + \sum_{i=1}^{n} \frac{\left[1-2(1-B_i)\right]}{\left[(1+\lambda)-2\lambda(1-B_i)\right]}. \tag{30}$$

Let $\boldsymbol{\theta}$ be the vector of the unknown parameters (a, b, p, λ, v); then elements of the 5×5 information matrix $\mathbf{I}(a, b, p, \lambda, v)$ can be obtained by

$$\mathbf{I}_{ij}\left(\hat{\boldsymbol{\theta}}\right) = \mathrm{E}\left[-\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_i \partial \boldsymbol{\theta}_j}\bigg|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}} \right].$$

Then $\mathbf{I}^{-1}(a, b, p, \lambda, v)$ is the variance covariance matrix of the unknown parameters (a, b, p, λ, v) and the asymptotic distributions of the maximum likelihood estimators (MLE) parameters are

$$\sqrt{n}\left(\hat{\boldsymbol{\theta}}_i - \boldsymbol{\theta}_i\right) \approx N_5\left(0, \mathbf{I}^{-1}\left(\hat{\boldsymbol{\theta}}_i\right)\right), \quad i = 1, \ldots, 5.$$

The approximation of the $100(1-\alpha)\%$ confidence intervals for the unknown parameters based on the asymptotic distribution of the ETSM(a, b, p, λ, v) are determined as

$$\hat{\boldsymbol{\theta}}_i \pm z_{\alpha/2}\sqrt{\mathbf{I}^{-1}\left(\hat{\boldsymbol{\theta}}_i\right)}, \quad i = 1, \ldots, 5,$$

where $z_{\alpha/2}$ is the upper $(\alpha / 2)^{\text{th}}$ percentile of a standard normal distribution.

Illustration

This purpose is to obtain MLEs of the ETSM distribution parameters using random numbers to study the MLEs sample behavior. Obtaining parameters estimates is described as follows:

Step (1): Generating a random sample $x_1, x_2, \ldots, x_n$ of sizes n = 10, 30, 50, and 100 using the ETSM distribution.
Step (2): Selecting parameters values: a = 0.7, b = 2, p = 2, λ = 0.3, v = 1.5
Step (3): Solving (26) to (30) by iteration to get MLEs, biases, root of mean squared error (RMSE), and the Pearson type of parameter estimators (Pearson, 1895) of the ETSM distribution.
Step (4): Repeating steps from 1 to 3 10,000 times.

In this study, random numbers samples are generated with Mathcad using conjugate gradient iteration method. All results are illustrated in Table 1.

The more sample size increases the more biases and RMSE decrease. In addition, the sampling distribution of a is a Pearson Type I distribution at all times, the sampling distribution of p is a Pearson Type IV distribution at all times, the sampling distribution of λ is a Pearson Type I distribution at all times, the sampling distribution of v is a Pearson Type I distribution at all times, and the sampling distribution of b differs according to sample size. The estimators can be consistent, specially, when sample size increases.

Table 1. Biases and RMSE of parameters estimation within small, medium, and large samples

Sample Size	Parameter	Mean for 1000 times	Biases		RMSE		Pearson system coefficient	Pearson type
			Each	Total	Each	Total		
10	a = 0.7	1.796	1.096	4.444	1.982	9.460	-0.5520	I
	b = 2	5.436	3.436		7.469		0.2650	IV
	p = 2	4.126	2.126		4.125		0.2370	IV
	λ = 0.3	0.151	-0.149		0.425		-0.2420	I
	v = 1.5	2.986	1.486		3.547		-0.4150	I

Table 1 (continuous).

Sample Size	Parameter	Mean for 1000 times	Biases Each	Biases Total	RMSE Each	RMSE Total	Pearson system coefficient	Pearson type
30	$a = 0.7$	1.098	0.398	3.507	1.043	7.523	-0.5290	I
	$b = 2$	5.071	3.071		6.331		0.2960	IV
	$p = 2$	3.207	1.207		2.719		0.1980	IV
	$\lambda = 0.3$	0.237	-0.063		0.262		-0.1510	I
	$v = 1.5$	2.620	1.120		2.823		-0.8460	I
50	$a = 0.7$	0.890	0.190	2.771	0.678	5.787	-5.2560	I
	$b = 2$	1.796	2.392		4.678		-0.6430	I
	$p = 2$	5.436	0.965		2.160		0.0098	IV
	$\lambda = 0.3$	4.126	-0.049		0.231		-0.1360	I
	$v = 1.5$	0.151	0.995		2.537		-0.4810	I
100	$a = 0.7$	0.829	0.129	2.108	0.510	4.865	-3.4980	I
	$b = 2$	3.865	1.865		4.039		-0.3960	I
	$p = 2$	2.704	0.704		1.841		0.0022	IV
	$\lambda = 0.3$	0.255	-0.045		0.212		-0.1150	I
	$v = 1.5$	2.173	0.673		1.914		-0.2980	I
300	$a = 0.7$	0.725	0.025	0.799	0.201	1.580	-3.0129	I
	$b = 2$	2.730	0.730		1.374		-0.3210	I
	$p = 2$	2.301	0.301		0.594		0.0010	IV
	$\lambda = 0.3$	0.281	-0.019		0.104		-0.1010	I
	$v = 1.5$	1.620	0.120		0.455		-0.2130	I

Application

A practical example using a real data set is given to see how the empirical model works. In our example, the different distributions used are the ETSM, ESM, TSM, and SM distributions. The following data represents the lifetime (hours) of candle lamps for 50 devices (https://www.npl.co.uk/)

0.172, 0.173 0.270, 0.200, 0.260, 0.186, 0.186, 0.191, 0.192, 0.196,
0.202, 0.212, 0.216, 0.217, 0.218, 0.219, 0.224, 0.226, 0.227, 0.227,
0.233, 0.234, 0.241, 0.244, 0.244, 0.245, 0.247, 0.250, 0.250, 0.252,
0.253, 0.234, 0.256, 0.235, 0.265, 0.265, 0.265, 0.269, 0.275, 0.276,
0.278, 0.285, 0.288, 0.290, 0.294, 0.216, 0.234, 0.217, 0.238, 0.204

The results of some goodness-of-fit measures and likelihood ratio tests are computed using Mathcad (version 15) and are included in Table 2 and Table 3, respectively. Figure 3 illustrates probability density functions for different distributions which fit the data.

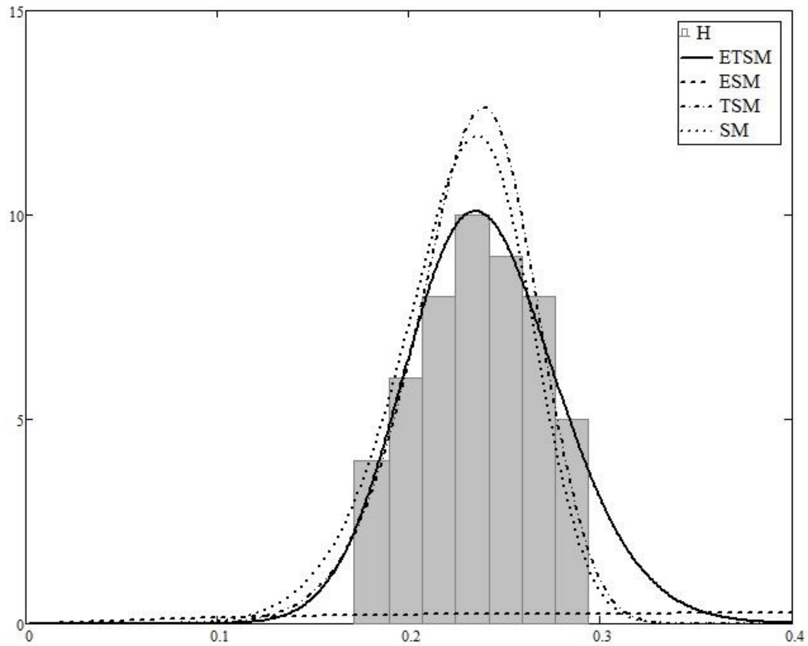

Figure 3. Estimated probability density functions for different distributions

Table 2. The MLE of the parameter(s) and the associated AIC and BIC values

	MLE of parameters									
Model	***a***	***b***	***p***	***λ***	***v***	**KS**	***p*-value**	**AIC**	**CAIC**	**BIC**
ETSM	6.995	0.365	19.953	0.218	1.698	0.073	0.951	-189.188	-187.824	-179.628
	(0.107)	(0.024)	(0.171)	(0.056)	(1.331)					
ESM	6.625	0.360	18.121	0.000	1.746	0.932	0.012	-114.744	-113.855	-107.096
	(2.157)	(0.029)	(6.339)	–	(1.283)					
TSM	9.368	0.334	13.057	0.336	1.000	0.850	0.034	-123.690	-123.168	-117.954
	(0.632)	(0.055)	(1.275)	(0.068)	–					
SM	8.882	0.335	14.526	0.000	1.000	0.128	0.028	-183.330	-182.441	-175.682
	(0.118)	(0.054)	(2.744)	–	–					

Table 3. The log-likelihood function, likelihood ratio tests statistic, and *p*-values

Model	**H_0**	**ℓ (log likelihood)**	**Λ (likelihood ratio test statistic)**	**df (degrees of freedom)**	***p*-value**
ESM	$\lambda = 0$	95.665	7.858	1	5.06E-03
TSM	$v = 0$	61.372	76.444	1	0.00
SM	$v = 0, \lambda = 0$	64.845	69.498	2	0.00

Note: The log likelihood of the ETSM = 99.594

In Table 2, the MLEs of distributions parameters, the corresponding RMSE (given in parentheses), Kolmogorov-Smirnov (KS) test statistic, AIC (Akaike Information Criterion), CAIC (consistent Akaike Information Criterion), and BIC (Bayesian information criterion) are computed for every distribution. The null hypothesis that the data follows the ETSM distribution, only, can be accepted at significance level $\alpha = 0.05$ and it is clear that the ETSM distribution has the smallest KS, AIC, CAIC, and BIC, so ETSM distribution can be the best fitted distribution to the data compared with other distributions.

In Table 3, based on the likelihood ratio test, the null hypothesis is that the data follow the nested model and the alternative is the data follow the full model, where the ESM, TSM, and SM distributions are nested by the ETSM distribution. Obviously (from the *p*-values) all null hypotheses can be rejected at the level of significance $\alpha = 0.05$, so ETSM distribution can fit the data better than the nested distributions as was illustrated before.

Conclusion

The ETSM distribution is a useful distribution having flexible statistical properties, wide applications, and generalizes some important distributions. The ETSM distribution can be used quite effectively to provide better fits compared to other distributions.

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References

Al-Khazaleh, A. M. H. (2016). Transmuted Burr type XII distribution: A generalization of the Burr type XII distribution. International Mathematical Forum, 11(12), 547-556. doi: 10.12988/imf.2016.6443

Arnold, C. B. (1983). *Pareto distributions*. Fairland, MD: International Cooperative Publishing House.

Arnold, C. B., Balakrishnan, N., & Nagaraja, H. N. (1992). *A first course in order statistics*. New York: John Wiley and Sons, Inc.

Aryal, G. R., & Tsokos, C. P. (2011). Transmuted Weibull distribution: A generalization of the Weibull probability distribution. *European Journal of Pure and Applied Mathematics, 4*(2), 89-102. Retrieved from https://ejpam.com/index.php/ejpam/article/view/1170

Burr, I. W. (1942). Cumulative frequency functions. *Annals of Mathematical Statistics, 13*(2), 215-232. doi: 10.1214/aoms/1177731607

El-Saidi, M. A., & George, E. O. (1990). A generalized logistic model for quantal response. *Biometrical Journal, 32*(8), 943-954. doi: 10.1002/bimj.4710320807

Garthwait, P. H., Jolliffe, I. P., & Jones, B. (1995). *Statistical inference*. London: Prentice Hall International.

Meeker, W. Q., & Escobar, L. A. (1998). *Statistical methods for reliability data*. New York: John Wiley.

Mielke, W., Jr., & Johnson, E. S. (1974). Some generalized beta distributions of the second kind having desirable application features in hydrology and meteorology. *Water Resources Research*, *10*(2), 223-226. doi: 10.1029/WR010i002p00223

Pearson, K. (1895). X. Contributions to the mathematical theory of evolution.–II. Skew variations in homogeneous material. *Philosophical Transactions of The Royal Society A*, *186*, 343-414. doi: 10.1098/rsta.1895.0010

Singh, S. K., & Maddala, G. S. (1976). A function for the size distribution of incomes. *Econometrica, 44*(5), 963-970. doi: 10.1007/978-0-387-72796-7_2

Tadikamalla, P. R. (1980). A look at the Burr and related distributions. *International Statistical Review, 48*(3), 337-344. doi: 10.2307/1402945