ABSTRACT

Title of Dissertation: CLOSED AFFINE MANIFOLDS WITH AN INVARIANT LINE

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Radiant manifolds are affine manifolds whose holonomy preserves a point. Here we discuss certain properties of closed affine manifolds whose holonomy preserves an affine line. Particular attention is given to the case wherein the holonomy acts on the invariant line by translations and reflections. We show that in this case, the developing image must avoid the translation invariant line providing a generalization to the well known fact that closed radiant affine manifolds cannot have their fixed points inside the developing image. We conclude by generalizing this result to translation and reflection invariant proper subspaces of the holonomy.

CLOSED AFFINE MANIFOLDS WITH AN INVARIANT LINE

by

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Dedication

To my mom and dad, who have given me a life full of opportunity, love, and so much more.

Acknowledgments

I would not be writing this without the many people in my life who have looked after me. Professor Goldman, thank you for your constant patience and academic guidance throughout my career. To the Portmans and Marlows, I remember each holiday visit spent together enjoying each other's company. You've always been guiding examples of kindness in my life. To Andy, Brittany, Carl, Christophe, D-Benz, Fill, Jasonia, Kareem, Manderson, Matt, Philip, and each of your families, I cannot put to words the debt of gratitude I hold to each and every one of you. Lauren, I am so thankful for all the years we get to spend together and I cannot wait to see where life takes us. How lucky I am to spend it with you. O'Sharn and Anatoly, thank you for helping raise me, inspiring me with imagination, and always keeping me in your thoughts. What a joy it's been to be your little brother. Finally, to mom and dad, I am so indescribably proud and lucky to be your son. You have sacrificed so much to give me a life of opportunities, hope, happiness, and love. Thank you for every skateboard, music recommendation, guiding hand, life lesson, hug when I needed one, piece of advice, home cooked meal, and the unconditional love I have been blessed with in our family. Thank you all. I am so very lucky and I hope you all know just how deeply and truly I love you all.

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Chapter 1: Introduction

This thesis is concerned with *affine structures* on closed manifolds, in particular consequences of having proper invariant subspaces of the corresponding holonomy representation of the manifold in question. Stated in relatively elementary terms, affine manifolds are manifolds equipped with a preferential atlas of charts into affine space, $\mathbb{A}^n$, whose change of coordinates are the restriction of affine automorphisms of $\mathbb{A}^n$. Several examples of varying complexity are provided in Section 2.3.

Manifolds of this nature are of interest as one may conduct affine geometry upon them. Familiar concepts from affine geometry such as *convexity*, *parallelism*, and *geodesics* may all be defined on these manifolds through the use of a construction called a *developing pair*. This construction is defined for a much larger class of manifolds known as (G, X)-manifolds in Section 2.4. With this construction in mind, some basic properties and definitions are reviewed such as *completeness* of a (G, X)-structure and the correspondence between locally homogenous closed Riemannian manifolds and discrete co-compact subgroups of G acting freely on X.

Chapter 3 provides some background on a subclass of affine manifolds known as *radiant* manifolds. These are affine manifolds whose holonomy representation preserves a point in affine space. Examples are provided and in particular an example from Thurston and Sullivan is provided wherein the corresponding developing map is not a covering onto its image [ST83]. These pathological examples of when the developing map fails to be a covering map onto its image showcase some of the complexities that may arise even in the low-dimensional cases. While Nagano and Yagi showed that in the two dimensional case, the developing map of a closed affine manifold is a covering onto its image, this statement is no longer true in higher dimensions [NY74]. In John Smillie's thesis, he constructs an affine structure on the three-torus whose developing map fails to be a covering onto its image providing a counter example to a conjecture of Thurston which stipulated that the developing map of a closed affine manifold is a covering onto its image [Smi77].

The following theorem is a well-known fact regarding the incompleteness of closed radiant affine manifolds.

Theorem 1.1. *Let M be a closed radiant manifold. Then fixed points of the affine holonomy are not contained within the developing image. In particular this means the radiant structure on M is incomplete.*

We reprove this theorem with a new argument that makes use of the complete radiant flow associated to the vector field on $\mathbb{R}^n$ given by $R = -y^i\partial/\partial y^i$. Lemma 3.5 shows that if one can lift the radiant flow associated to R through a local diffeomorphism $F : N \longrightarrow \mathbb{R}^n$, then F may be promoted to a global diffeomorphism if $0 \in F(N)$. With this lemma, we show if a fixed point of the holonomy lies inside the developing image, the developing map must be a diffeomorphism and the holonomy acts trivially, contradicting the compactness hypothesis.

Chapter 4 investigates the consequences of having a fixed vector in the linear holonomy of a closed affine manifold. The condition of having a fixed vector in the linear holonomy provides an $\mathbb{R}$-action on the manifold M with *parallel flow*. In fact, this condition provides us with 'large' open subsets of the universal cover of M for which the developing map restricted to these open subsets is a diffeomorphism onto its image. These open subsets are invariant with respect to the induced $\mathbb{R}$-action on the universal cover of M and map to open cylinders under the developing map. This result is stated in formality below and its proof is largely taken from a preprint with some modifications and change in notation [Dal20].

Theorem 1.2. *Let M be a closed affine manifold whose linear holonomy fixes a vector in $\mathbb{R}^{n+1}$. Then there exists a complete parallel flow on M which lifts to the universal cover $\widetilde{M}$. In addition, for any point in the universal cover, we may find a neighborhood of the point saturated with respect to this flow such that the developing map restricted to these neighborhoods are diffeomorphisms onto their images.*

This theorem is further generalized to the situation wherein the linear holonomy admits a k-dimensional subspace upon which it acts by translations in Theorem 4.2.

Chapter 5 is concerned with a generalization of radiant manifolds. Whereas radiant manifolds preserve a single point, Chapter 5 investigates manifolds that preserve an affine line. Whereas the affine holonomy may only act trivially to preserve a point in the radiant case, the affine holonomy may act by translations, scalings, reflections, and their compositions to preserve an affine line. Fried, Goldman and

Hirsch proved that there are no complete closed affine manifolds with reducible holonomy [FGH81]. We prove the following mild generalization in Chapter 5. Let G be the group of affine transformations preserving some fixed affine line.

Theorem 1.3. *Let Ω be a connected open subset of $\mathbb{R} \times \mathbb{R}^n$ containing the line $\mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{R}^n$. There does not exist a subgroup $\Gamma \subset G$ with the discrete topology acting on Ω both properly and freely with a compact quotient.*

After this theorem is established, Section 5.3 returns to the general study of closed affine manifolds with an invariant line. Combining the results of Theorem 1.1 and Theorem 1.2, we prove the following theorem.

Theorem 1.4. *Let M be an $(n+1)$-dimensional closed affine manifold with $n \geq 1$ whose holonomy admits an invariant line. If the holonomy acts on the invariant line by translations, then the developing image cannot meet this invariant line.*

The corresponding result about when the affine holonomy admits an invariant k-plane upon which the holonomy acts by reflections and translations is addressed in Corollary 5.2. This result provides a partial affirmation of a conjecture of Goldman and Fried which states for closed affine manifolds, proper invariant subspaces that are acted upon unipotently by the holonomy cannot meet the developing image.

Chapter 2: Affine Manifolds and Developing Pairs

It is worth establishing for the purpose of clarity, this thesis works largely in the category of *smooth manifolds*. Unless otherwise stated, one may assume that all such topological spaces and maps are *smooth*. In addition, most of these manifolds are assumed to be connected unless otherwise stated. That said, throughout this thesis, the word *space* may be taken to mean *smooth connected manifold* and the word *map* may be taken to mean *smooth map*.

2.1 Affine Space

In this section we recall some basic definitions of *affine space*. Let n be some non-negative integer.

Definition 2.1. Affine n-space is a space equipped with a *free* and *transitive* $\mathbb{R}^n$-action. In other words, affine n-space is an $\mathbb{R}^n$-torsor.

Fix an affine space $\mathbb{A}^n$. For each point $p \in \mathbb{A}^n$ denote the action of the vector $v \in \mathbb{R}^n$ on p by $p + v$. By prospect of the fact that $\mathbb{R}^n$ acts both freely and transitively on $\mathbb{A}^n$, for each two points $p, q \in \mathbb{A}^n$, there is a unique vector $v \in \mathbb{R}^n$ so that $p + v = q$. This vector is occasionally denoted by qp. If one picks an *origin* $p \in \mathbb{A}^n$, then one may identify $\mathbb{A}^n$ with $\mathbb{R}^n$ via the map that takes each q to qp.

This identification in turn provides $\mathbb{A}^n$ with a vector space structure isomorphic to $\mathbb{R}^n$. The identification is non-canonical and depends on the choice of an origin. While each such identification only differs by choice of an origin, it nevertheless depends on picking a point. That said, there is no natural identity element of affine space. This is in contrast to the case of a vector space that has a naturally defined origin, namely the zero vector.

An *affine homomorphism* A is a map between affine spaces $A : \mathbb{A}^n \longrightarrow \mathbb{A}^m$ such that for each $p \in \mathbb{A}^n$ there exists a linear map $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ where for each $v \in \mathbb{R}^n$ we have $A(p + v) = A(p) + L(v)$. It is worth noting that this linear map is well defined independent of choice of p.

For if $q \in \mathbb{A}^n$ is another point in $\mathbb{A}^n$ and M is another linear map satisfying the above criterion, then for each $v \in \mathbb{R}^n$ we have that

$$\begin{aligned} M(v) &= A(q+v) - A(q) = A\left(p + (qp + v)\right) - A\left(p + qp\right) \\ &= L(qp+v) - L(qp) = L(v) \end{aligned} \tag{2.1}$$

Note that this means that for all $p, q \in \mathbb{A}^n$ we have the vector taking $A(q)$ to $A(p)$, $A(q)A(p)$, satisfies $A(q)A(p) = L(qp)$. In fact, this is yet another means of defining an affine homomorphism.

The group of *affine automorphisms* is defined to be the group of invertible affine homomorphisms on the space $\mathbb{A}^n$. This group is denoted by $\mathrm{Aff}(n)$. Some elements of $\mathrm{Aff}(2)$ acting on a square in $\mathbb{A}^2$ may be seen in Figure 2.1.

Up to isomorphism, this group is given by the semi-direct product of $\mathrm{GL}(n, \mathbb{R})$

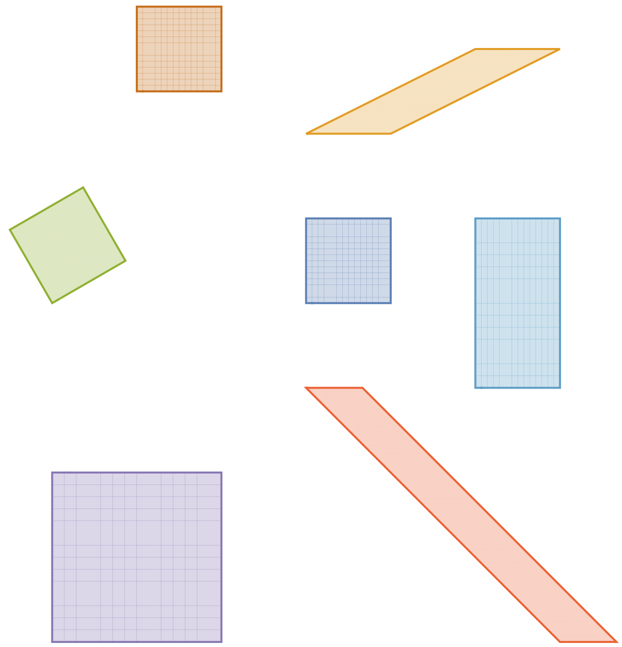

Figure 2.1: A figure of the affine group acting on the center dark blue square to produce several affinely equivalent shapes in the plane. All these shapes differ by elements of the affine group, namely through translation, reflection, rotation, sheering, scaling, or some composition of them. Note how even though lengths and areas are distorted, parallel lines are sent to parallel lines. This illustrates how the affine group does not preserve the standard Euclidean metric, but nevertheless preserves a notion of parallelism which is addressed in Section 2.5.

acting on $\mathbb{R}^n$ in the natural fashion. This may be seen by choosing an origin $p \in \mathbb{A}^n$ and identifying $\mathbb{A}^n$ with $\mathbb{R}^n$ via the map taking q to qp. From this identification, we associate to each affine automorphism $A : \mathbb{A}^n \longrightarrow \mathbb{A}^n$ its linear part $L_A \in \mathrm{GL}(n, \mathbb{R})$ and its translational part $T_A := A(p)p \in \mathbb{R}^n$, the unique vector taking p to $A(p)$.

Associate to the affine automorphisms $A, B \in \mathrm{Aff}(n)$ their linear parts $L_A, L_B \in GL(n, \mathbb{R})$ and their translational parts $T_A, T_B \in \mathbb{R}^n$. Composition of the two affine automorphisms obeys the equalities in Equation 2.2

$$\begin{aligned} BA\,(p+v) &= B\big(A(p)+L_A(v)\big) = B\big(p+T_A+L_A(v)\big) \\ &= B(p)+L_B\big(T_A+L_A(v)\big) = \big(B(p)+L_B(T_A)\big)+L_BL_A(v) \end{aligned} \tag{2.2}$$

Equation 2.2, the composition of BA has linear part $L_{BA} = L_B L_A$ and translational part $T_{BA} = \big(B(p) + L_B(T_A)\big)p = T_B + L_B(T_A)$ which is the product of (L_B, T_B) and (L_A, T_A) in the semi-direct product $\mathrm{GL}(n,\mathbb{R}) \ltimes \mathbb{R}^n$. Thus, we have the isomorphism between $\mathrm{Aff}(n)$ and $\mathrm{GL}(n,\mathbb{R}) \ltimes \mathbb{R}^n$ taking A to $(L_A, T_A) \in \mathrm{GL}(n,\mathbb{R})$. We frequently write the group $\mathrm{GL}(n,\mathbb{R}) \ltimes \mathbb{R}^n$ as $\mathrm{Aff}(n,\mathbb{R})$.

Affine geometry is the study of $\mathbb{A}^n$ and its invariants under the group of affine automorphisms, $\mathrm{Aff}(n)$. This is in contrast to *Euclidean* geometry which is the study of Euclidean space, $\mathbb{E}^n$, and its invariants under the group of isometries that preserve the Euclidean inner product $x \cdot y = \sum_i x^i y^i$. Whereas the group of Euclidean isometries preserves notions such as *distance*, *angle*, and *volume*, these invariants are lost in the affine group. In fact, perhaps the most important invariant of affine geometry is that of *parallelism* which is defined in Section 2.5. Figure 2.1 shows the affine group preserving the parallel sides of a square.

2.2 Affine Manifolds

In this section we provide some definitions and fundamentals regarding affine manifolds. A smooth manifold is a topological manifold whose coordinate charts $\phi_i : U_i \longrightarrow \mathbb{R}^n$ enjoy the property that their coordinate changes $\phi_i \circ \phi_j^{-1}$ are restrictions of smooth maps from $\mathbb{R}^n \longrightarrow \mathbb{R}^n$. A recurring theme of geometry is to insist these coordinate changes lie inside some group that preserve some structure in $\mathbb{R}^n$, such as the Euclidean inner product. In so doing, one may pullback the quantity via the coordinate charts to endow the manifold with such a structure. This for example is of natural interest in the case where the coordinate changes lie inside the group $\mathrm{Aff}(n, \mathbb{R})$ as the standard connection ∇ on $\mathbb{R}^n$ is invariant under this group. Due to this invariance, we may pullback the standard connection ∇ via the coordinate charts to endow M with a connection modeled on affine space. This is perhaps reasonable motivation for taking affine manifolds into consideration, as we know every manifold admits a connection, but the choice of this is highly arbitrarily and usually relies on some partition of unity [Lee91, Prop 4.5].

Definition 2.2. An affine structure on an n-dimensional manifold M is a collection of charts $\phi_i : U_i \subset M \longrightarrow \mathbb{A}^n$ so that for each connected component of $V \subset U_i \cap U_j$ there exists an affine automorphism $A_{i,j} : \mathbb{A}^n \longrightarrow \mathbb{A}^n$ so that $\phi_i \circ \phi_j^{-1} : \phi_j(V) \longrightarrow \phi_i(V)$ is equal to $A_{i,j}$ restricted to $\phi_j(V)$. In terms of diagrams, for each connected component $V \subset U_i \cap U_j$ we have Equation 2.3 below commutes.

$$\begin{tikzcd} & V \arrow[dl, "\phi_j"'] \arrow[dr, "\phi_i"] & \\ \phi_j(V) \arrow[rr, "A_{i,j}=\phi_i\circ\phi_j^{-1}"'] & & \phi_i(V) \end{tikzcd} \tag{2.3}$$

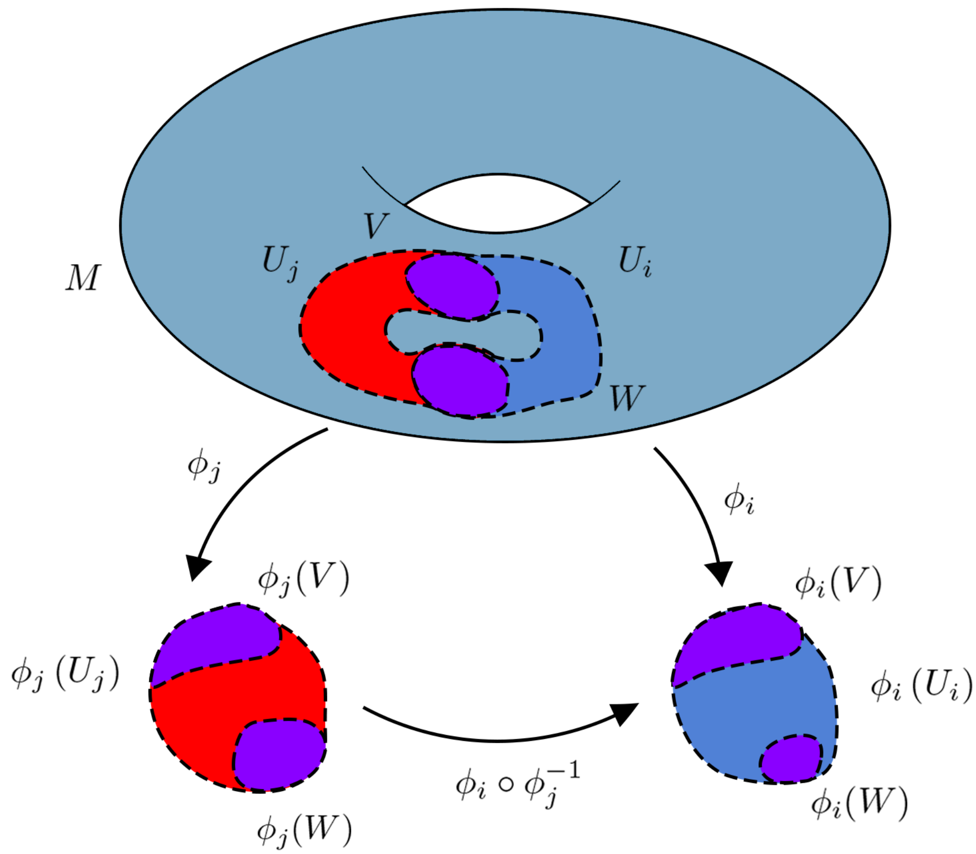

Figure 2.2: Here we have two overlapping coordinate charts $\phi_i : U_i \longrightarrow \mathbb{A}^2$ and $\phi_j : U_j :\longrightarrow \mathbb{A}^2$ on M with connected components $V, W \subset U_i \cap U_j$. As depicted, $\phi_j(V)$ and $\phi_i(V)$ differ by a translation in the affine plane, whereas $\phi_j(W)$ and $\phi_i(W)$ differ by a dilation and rotation by π-radians.

2.3 Examples of Affine Manifolds

In this section we provide several examples of affine structures on manifolds.

Example 2.1. Let Γ be a discrete subgroup of $\mathrm{Aff}(n, \mathbb{R})$ acting both properly and freely on $\mathbb{R}^n$. Since the action is both free and proper, one may form the quotient manifold $M := \mathbb{R}^n/\Gamma$ [Lee03, Thm 21.10]. As $\mathbb{R}^n$ is simply connected, the associated quotient map $p : \mathbb{R}^n \longrightarrow M$ is the universal covering for the manifold M.

This defines an affine structure on M as each point $m \in M$ is evenly covered by a neighborhood U so that $p^{-1}(U)$ is a collection of disjoint open subsets $\{U_i\}$ in $\mathbb{R}^n$ projecting diffeomorphically onto U. For each i, let p_i be the restriction of p to U_i onto U, $p_i : U_i \longrightarrow U$. Define a chart for U into $\mathbb{R}^n$ by $\phi_i : U \longrightarrow U_i \subset \mathbb{R}^n$ via $\phi_i = p_i^{-1}$.

Let $W \subset U \cap V$ be a connected component of the intersection of two charts $\phi_i : U \longrightarrow U_i$ and $\phi_j : V \longrightarrow V_j$ where U_i and V_j are subsets of $p^{-1}(U)$ and $p^{-1}(V)$ respectively that are projected diffeomorphically onto U and V respectively. Then for each $x \in \phi_j(W)$, the coordinate changes are given by

$$\left(\phi_i \circ \phi_j^{-1}\right)(x) = \left(p_i^{-1} \circ p_j\right)(x) = \gamma(x) \tag{2.4}$$

where γ is some element of Γ taking U_j to U_i. This shows the coordinate changes are indeed locally affine.

A standard example of an affine structure arises by taking $\Gamma \subset \mathrm{Aff}(2, \mathbb{R})$ to be a lattice of translations and forming the quotient $\mathbb{R}^2/\Gamma$. This quotient is diffeomorphic to the two-torus. As the group of translations belongs to the isometries of Euclidean space, the Euclidean metric passes down to the quotient providing the two-torus with a flat *Euclidean structure* by which we mean the coordinate changes are locally the restrictions of Euclidean isometries. Figure 2.3 below illustrates this construction.

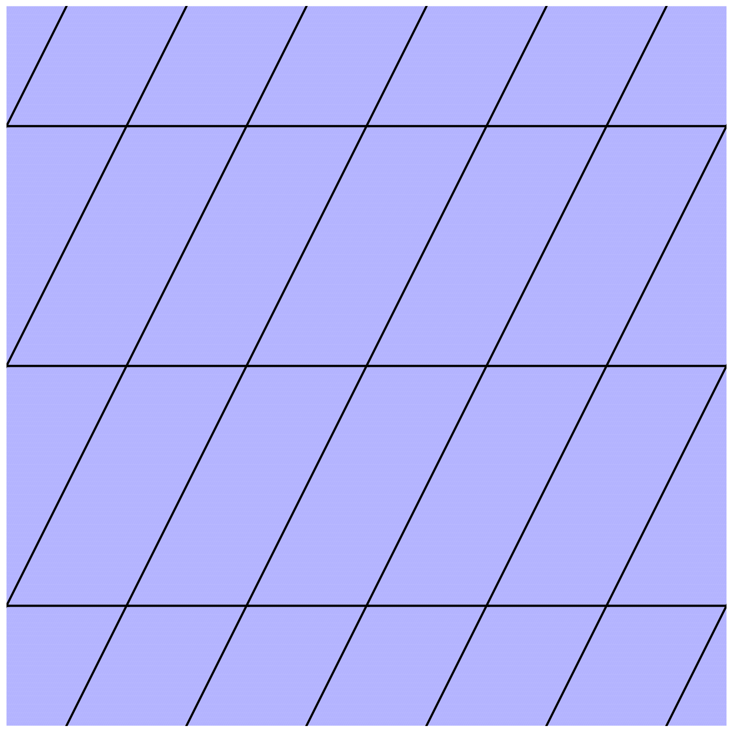

Figure 2.3: Fundamental domains of a lattice of translations generated by the vectors $(1, 0)$ and $(1, 2)$ acting on $\mathbb{R}^2$. The quotient of $\mathbb{R}^2$ by this lattice of translations provides the two-torus with a Euclidean structure.

This should be contrasted with Example 3.1 which is an example of a *similarity* structure. In fact the Hopf-manifolds defined in Example 3.2 all provide similarity structures whose coordinate changes lie inside the similarity group.

Before continuing it is useful to have the notion of an *affine map* between affine manifolds.

Definition 2.3. Let M and N be n-dimensional affine manifolds. Let $F : M \longrightarrow N$ be a local diffeomorphism. F is called an affine map if for each pair of charts $\phi : U \subset M \longrightarrow \mathbb{A}^n$ and $\psi : V \subset N \longrightarrow \mathbb{A}^n$, we have that F is locally affine in the sense that the restriction of $\psi \circ F \circ \phi^{-1}$ to each connected component of $\phi\left(U \cap F^{-1}(V)\right)$ is the restriction of an affine automorphism on $\mathbb{A}^n$.

Example 2.2. Let M have an affine structure. Then each covering space $p : C \longrightarrow M$ is naturally endowed with an affine structure such that the covering map is an affine map. For each point $c \in C$, choose an evenly covered neighborhood U of $p(c)$. Shrink this neighborhood U sufficiently small so that there's a chart $\phi : U \longrightarrow \mathbb{A}^n$. Since c is contained in a unique connected component of $V \subset p^{-1}(U)$ so that $p|_V$ is a diffeomorphism onto U, we may compose p with the chart ϕ to obtain a chart for C, $\psi := \phi \circ p : V \longrightarrow \mathbb{A}^n$.

This provides an affine atlas for C. Let $\psi_i : V_i \longrightarrow X$ and $\psi_j : V_j \longrightarrow X$ be two patches in C and $W \subset V_i \cap V_j$ be a connected component of their intersection. By construction $\psi_i = \phi_i \circ p$ and $\psi_j = \phi_j \circ p$ for some patches $\phi_i : U_i \longrightarrow X$ and $\phi_j : U_j \longrightarrow X$ in M.

Let $p_i : V_i \longrightarrow U_i$ denote the restriction of p to V_i. The coordinate changes of the patches ψ_i and ψ_j for any $x \in \psi_j(W)$ are given by Equation 2.5 below.

$$
\begin{aligned}
(\psi_i \circ \psi_j^{-1})(x) &= (\phi_i \circ p_i) \circ (\phi_j \circ p_j)^{-1}(x) = \phi_i\left(p_i p_j^{-1}\left(\phi_j^{-1}(x)\right)\right) \\
&= \left(\phi_i \circ \phi_j^{-1}\right)(x) \qquad (2.5)
\end{aligned}
$$

By hypothesis, since M is affine, $\phi_i \circ \phi_j^{-1}$ is locally the restriction of some affine automorphism, and thus so too is $\psi_i \circ \psi_j^{-1}$.

This provides our covering space C with an affine structure induced by M. Now choose coordinate charts $\psi : V \longrightarrow X$ in C and $\phi : U \longrightarrow X$ in M where $\psi = \phi \circ p$. Let W be a connected component of $\psi\left(V \cap p^{-1}(U)\right)$. For each $x \in \psi(W)$ we have that

$$\left(\phi \circ p \circ \psi^{-1}\right)(x) = \phi \circ p \circ (\phi \circ p)^{-1}(x) = x \tag{2.6}$$

Thus, locally, in these charts, the coordinate expression of p is the restriction of an affine map, namely the identity. Since in these charts the coordinate representation is affine, the same holds true for any such charts, as coordinate changes are also affine. Thus $p : C \longrightarrow M$ is indeed an affine map in the sense of Definition 2.3.

Example 2.3. In a similar spirit to Example 2.1, consider an n-dimensional affine manifold M and let Γ be a group of affine automorphisms acting on M both properly and freely. The quotient space M/Γ inherits a natural affine structure such that the projection map $p : M \longrightarrow M/\Gamma$ is an affine map.

As the projection is a covering space, for each point $x \in M/\Gamma$, we may choose an evenly covered neighborhood $x \in U$ with $p^{-1}(U)$ equal to the disjoint union of $\{U_i\}$ where p restricted to each U_i is a diffeomorphism onto U. Shrink U_i if necessary to assume we may find a chart $\phi_i : U_i \longrightarrow \mathbb{A}^n$. The composition $\Phi_i := \phi_i \circ p_i^{-1} : U \longrightarrow \mathbb{A}^n$ serves as a chart about $x \in U$ where $p_i^{-1} : U \longrightarrow U_i$ denotes the inverse of p onto U_i.

Given two charts $\Phi_i : U_i \longrightarrow \mathbb{A}^n$ and $\Phi_j : U_j \longrightarrow \mathbb{A}^n$, let $W \subset U_i \cap U_j$ be a connected component of their intersection. By construction, for each $x \in \Phi_j(W)$ we have that

$$\left(\Phi_i \circ \Phi_j^{-1}\right)(x) = \left(\phi_i \circ p_i^{-1}\right) \circ \left(\phi_j \circ p_j^{-1}\right)^{-1}(x) = \phi_i\left(p_i^{-1} \circ p_j\left(\phi_j^{-1}(x)\right)\right)$$
$$= \phi_i\left(\gamma \phi_j^{-1}(x)\right) \tag{2.7}$$

where $\gamma \in \Gamma$ is some element of Γ taking U_j to U_i. Equation 2.7 shows that the change of coordinates $\Phi_i \circ \Phi_j^{-1}$ is locally given by the composition of $\phi_j \gamma \phi_i^{-1}$. As $\gamma : M \longrightarrow M$ is an affine map as defined by Definition 2.3, this means that the expression $\phi_i \circ \gamma \circ \phi_j^{-1} : \mathbb{A}^n \longrightarrow \mathbb{A}^n$, in appropriately chosen charts, is the restriction of an affine automorphism of $\mathbb{A}^n$ as desired. Hence M/Γ supports an affine structure.

To show $p : M \longrightarrow M/\Gamma$ is an affine map choose charts $\phi_i : U_i \longrightarrow \mathbb{A}^n$ for M and $\Phi_i : U \longrightarrow \mathbb{A}^n$ for M/Γ. For each $x \in \phi_i\left(U_i \cap p^{-1}(U)\right)$, the coordinate representation of p is given by

$$\left(\Phi_i \circ p \circ \phi_i^{-1}\right)(x) = \left(\phi_i \circ p_i^{-1}\right) \circ \left(\phi_i \circ p_i^{-1}\right)^{-1}(x) = x \tag{2.8}$$

which is affine. Since the coordinate expression of p is locally affine in these charts, it is locally affine in all such charts, and thus p is an affine map in the sense of Definition 2.3 as claimed.

Example 2.4. Let M and N be affine manifolds of dimension m and n respectively. The product manifold $M \times N$ supports an affine structure. This may be seen in the following fashion. Note that the product of two affine space, $\mathbb{A}^m \times \mathbb{A}^n$ is isomorphic to the affine space $\mathbb{A}^{m+n}$. Given two affine automorphisms F of $\mathbb{A}^m$ and G of $\mathbb{A}^n$, the product of these two maps is also an affine automorphism on $\mathbb{A}^{m+n}$. Another way of putting this is that $\mathrm{Aff}(m,\mathbb{R}) \times \mathrm{Aff}(n,\mathbb{R})$ is a subgroup of $\mathrm{Aff}(m+n,\mathbb{R})$. Let $(A,v) \in \mathrm{Aff}(m,\mathbb{R})$ and $(B,w) \in \mathrm{Aff}(n,\mathbb{R})$.

The map taking this pair of affine automorphisms to the affine automorphism of $\mathbb{A}^{m+n}$ given by Equation 2.9 below

$$(A \times B, v \times w) := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} \tag{2.9}$$

is easily verified to be an injective group homomorphism.

Thus, given any two coordinate charts $\phi : U \subset M \longrightarrow \mathbb{A}^m$ and $\psi : V \subset N \longrightarrow \mathbb{A}^n$ one may form the product chart $\phi \times \psi : U \times V \subset M \times N \longrightarrow \mathbb{A}^{m+n}$. This defines an atlas of charts $U \times V \longrightarrow \mathbb{A}^{m+n}$ on $M \times N$. By construction, the coordinate changes are given by product of two affine maps which is affine as shown by Equation 2.9.

Example 2.5. Let M be an affine manifold and $A : M \longrightarrow M$ be an affine automorphism of the affine structure on M. Consider the affine product structure on $\mathbb{R} \times M$ where $\mathbb{R}$ has the standard affine structure induced by an identification of $\mathbb{A}^1$ and $\mathbb{R}$. Let Γ be the cyclic group generated by the affine automorphism of $\mathbb{R} \times M$ taking $(t,m) \in \mathbb{R} \times M$ to $(t+1, Am)$. This action is easily seen to be both free and proper on $\mathbb{R} \times M$, and thus by means of Example 2.3, $(\mathbb{R} \times M)/\Gamma$ has an affine structure. Thus the mapping torus of an affine automorphism supports an affine structure. This construction is known as the *parallel suspension* of an affine automorphism $A : M \longrightarrow M$. More details on this construction may be found in

Goldman's text [Gol21, p. 118].

Before proceeding we introduce the notion of a *projective structure*. Projective manifolds are a larger class of manifolds which include all affine ones. Intuitively this comes from the fact that affine space, $\mathbb{A}^n$, sits inside projective space, $\mathbb{R}P^n$, and every affine automorphism induces a projective automorphism. That is to say that projective geometry is an extension of affine geometry, much like how affine geometry is an extension of Euclidean geometry. This extension is illustrated in Figure 2.4.

Definition 2.4. A projective structure on an n-dimensional manifold M is a collection of charts $\phi_i : U_i \subset M \longrightarrow \mathbb{R}P^n$ so that for each connected component of $V \subset U_i \cap U_j$ there exists a projective automorphism $P_{i,j} : \mathbb{R}P^n \longrightarrow \mathbb{R}P^n$ so that $\phi_i \circ \phi_j^{-1} : \phi_j(V) \longrightarrow \phi_i(V)$ is equal to $P_{i,j}$ restricted to $\phi_j(V)$. In terms of diagrams, for each connected component $V \subset U_i \cap U_j$ we have Equation 2.10 below commutes.

$$\begin{tikzcd} & V \arrow[dl, "\phi_j"'] \arrow[dr, "\phi_i"] & \\ \phi_j(V) \arrow[rr, "P_{i,j}=\phi_i\circ\phi_j^{-1}"'] & & \phi_i(V) \end{tikzcd} \tag{2.10}$$

As mentioned above if one picks an origin and identifies $\mathbb{A}^n$ with $\mathbb{R}^n$, then $\mathbb{R}^n \subset \mathbb{R}P^n$ via the inclusion sending $v \in \mathbb{R}^n$ to the point $[v : 1] \in \mathbb{R}P^n$. Moreover, each affine automorphism $(A, v) \in \mathrm{Aff}(n, \mathbb{R})$ induces a projective automorphism in $\mathrm{PGL}(n+1, \mathbb{R})$ via the $(n+1) \times (n+1)$ matrix acting on $\mathbb{R}P^n$ as shown in Equation 2.11.

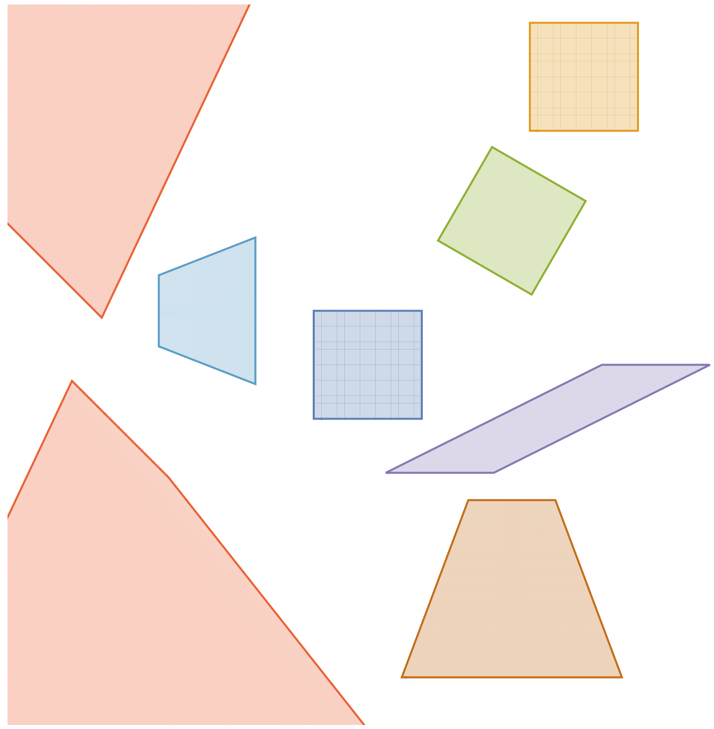

Figure 2.4: A figure of the projective group acting on the central dark blue square to produce several projectively equivalent shapes. This figure should be contrast with Figure 2.1. Technically this is only an affine patch of $\mathbb{R}P^2$, but nevertheless conveys how the projective group contains the affine transformations, and more. The red, tan, and light blue shapes are projectively equivalent to the dark blue center square, but are not affinely equivalent it. Note that unlike the affine group, the projective group fails to preserve parallelism as illustrated by the fact that the parallel sides of the dark blue square are not sent to parallel sides of the light blue square. Nevertheless the projective group still takes lines to lines.

$$\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} [x : t] = [Ax + vt : t] \tag{2.11}$$

It is for this reason we say that projective geometry is an extension of affine geometry as both affine space and its group of transformations sit inside projective space and its group of projective transformations.

Example 2.6. Let $\mathbb{R}^3$ have the Minkowski metric given by $(x, y) = x^1y^1 - x^2y^2 - x^3y^3$. Define P as the locus of points $(x, x) = 1$ where $x > 0$ in $\mathbb{R}^3$ with the pullback metric induced by the inclusion map $P \longrightarrow \mathbb{R}^3$. This is the hyperboloid model of the hyperbolic plane whose orientation preserving isometry group is $\mathrm{SO}^+(2, 1)$.

The map $P \longrightarrow \mathbb{R}P^2$ sending each point (x, y, z) in P to its homogenous coordinates $[x : y : z] \in \mathbb{R}P^2$ is easily seen to be an embedding. This inclusion is equivariant with respect to the homomorphsim $\mathrm{SO}^+(2, 1) \longrightarrow \mathrm{PSO}(2, 1)$ and shows that every hyperbolic two-manifold has a projective structure. This may be easily generalized to higher dimensions, but for familiarity's sake has been restricted to the two dimensional case.

This extension of geometries means that one may construct hyperbolic structures on say for example a closed orientable genus two surface to obtain non-affine projective structures. The fact that these are not affine follows by Benzécri's Theorem which states that the Euler characteristic of a two-dimensional closed affine manifold vanishes [Ben59]. Samual Ballas, Jeffrey Danciger, and Gye-Seon Lee have done work to provide examples of projective structures that are not hyperbolic on certain three manifolds [BDGS18].

2.4 (G, X)-manifolds and Developing Pairs

As one can see Definition 2.2 and Definition 2.4 are very similar in nature. In fact, they fall under the much larger class of manifolds known as (G, X)-manifolds. Here G is a Lie group that acts on a *model space* X. Here we assume the action of G is *strongly effective* in the sense that if $g \in G$ such that there exists an open subset $U \subset X$ for which $g|_U = \mathrm{id}|_U$, then $g = 1$.

With this context in mind, we define a (G, X)-structure on a manifold in the following fashion.

Definition 2.5. Let G be a Lie group acting strongly effectively on a connected manifold X. A (G, X)-structure on an n-dimensional manifold M is a collection of charts $\phi_i : U_i \subset M \longrightarrow X$ so that for each connected component of $V \subset U_i \cap U_j$ there exists a transformation $g_{i,j} : X \longrightarrow X$ so that $\phi_i \circ \phi_j^{-1} : \phi_j(V) \longrightarrow \phi_i(V)$ is equal to $g_{i,j}$ restricted to $\phi_j(V)$. In terms of diagrams, for each connected component $V \subset U_i \cap U_j$ we have Equation 2.12 below commutes.

$$\begin{tikzcd} & V \arrow[dl, "\phi_j"'] \arrow[dr, "\phi_i"] & \\ \phi_j(V) \arrow[rr, "g_{i,j}=\phi_i\circ\phi_j^{-1}"'] & & \phi_i(V) \end{tikzcd} \tag{2.12}$$

In this language, an affine structure is a $(\mathrm{Aff}(n), \mathbb{A}^n)$-structure, and a Euclidean structure is an $(\mathrm{Isom}(n), \mathbb{E}^n)$-structure, where $\mathrm{Isom}(n)$ is the isometry group preserving the Euclidean metric. A similarity structure is a $(\mathrm{Sim}(n), \mathbb{A}^n)$-structure, where $\mathrm{Sim}(n)$ is the group of similarity preserving transformations acting on affine space. Finally a projective structure is a $(\mathrm{PGL}(n+1), \mathbb{R}P^n)$-structure.

If we pick an origin in affine space and identify it with $\mathbb{R}^n$ we have the following inclusions of the groups below.

$$\mathrm{Isom}(n, \mathbb{R}) < \mathrm{Sim}(n, \mathbb{R}) < \mathrm{Aff}(n, \mathbb{R}) < \mathrm{PGL}(n+1, \mathbb{R}) \tag{2.13}$$

with $\mathbb{R}^n \subset \mathbb{R}P^n$ via the inclusion as defined in the paragraph after Equation 2.10. Due to these inclusions, we say that similarity structures are a generalization of Euclidean structures and affine structures are a generalization of similarity structures. Projective structures contain all of these and even hyperbolic structures as seen in Example 2.6.

A result of Bieberbach classified Euclidean structures on closed manifolds [Wol11]. As it so happens, these manifolds are entirely classified by an associated representation of their fundamental group into the group of isometries of Euclidean space. Remarkably there are only finitely many of them and also the manifolds are finitely covered by Euclidean tori. On the other hand, David Fried classified similarity structures on closed manifolds [Fri80]. These manifolds include the Euclidean structures, but in addition, have non-Euclidean structures. Example 3.1 provides an example of this sort of possibility. In fact, in his work, Fried showed that these are essentially the only other types of similarity structures one obtains, namely finite quotients of Hopf-manifolds.

Both of these results rely heavily on the use an indispensable tool associated to (G, X)-structures called a *developing pair*. We now endeavor to describe this construction.

Let M be a (G, X)-manifold. Pick a point $p \in M$ and base the fundamental group there. As M is path-connected, we frequently suppress the base point and simply write $\pi_1(M)$ for the fundamental group. Let $p \in U$ be a coordinate patch, $\phi : U \longrightarrow X$ and $\gamma : [0, 1] \longrightarrow M$ be any path beginning at p. We assign to this path a point in X in the following fashion.

Cover the image of the path γ by $(k+1)$-coordinates patches $\phi_i : U \longrightarrow X$ for $0 \leq i \leq k$ beginning with $U_0 = U$ and so that consecutive coordinate patches overlap, meaning $U_i \cap U_{i+1} \neq \emptyset$.

Pick a mesh of times $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = 1$ in $[0,1]$ so that $\gamma(t') \in U_{i-1}$ for all $t_{i-1} \leq t' \leq t_i$ for $i = 1 \ldots k$. Note this necessitates that $\gamma(t_i) \in U_{i-1} \cap U_i$ for each $1 \leq i \leq k$. Define $\gamma_i = \gamma|_{[t_i, t_{i+1}]}$ for each $i = 0, \ldots, k$.

While describing this process in full detail, it is worth following along to Figure 2.5 to get a feel for the construction. We proceed by inductively defining paths α_i in X.

Define $\alpha_0 := \phi_0 \gamma_0$.

Let $V_1 \subset U_0 \cap U_1$ be the connected component of $U_0 \cap U_1$ containing $\gamma(t_1)$ and $g_{0,1}$ be the element of G so that $g_{0,1}|_{\phi_1(V_1)} = \phi_0 \circ \phi_1^{-1} : \phi_1(V_1) \longrightarrow \phi_0(V_1)$. This element exists by definition of being a (G, X)-manifold, and moreover is unique by prospect of G acting strongly effectively on X.

Define $\alpha_1 := g_{0,1}(\phi_1 \gamma_1)$. The initial point of α_1 is the terminal point of α_0. To see this note that by definition, $\alpha_1(t_1) = g_{0,1}(\phi_1 \gamma_1(t_1))$. Additionally, $\phi_1 \gamma_1(t_1) \in \phi_1(V_1)$, as $\gamma_1(t_1) \in V_1$. Since $g_{0,1}|_{\phi_1(V_1)} = \phi_0 \circ \phi_1^{-1}$ and $\phi_1(\gamma_1(t_1)) \in \phi_1(V_1)$, this means,

$$\begin{aligned} \alpha_1(t_1) &= g_{0,1}(\phi_1 \gamma_1(t_1)) = \left(\phi_0 \circ \phi_1^{-1}\right)(\phi_1 \gamma_1(t_1)) = \phi_0 \gamma_1(t_1) = \phi_0 \gamma_0(t_1) \\ &= \alpha_0(t_1) \end{aligned} \tag{2.14}$$

Equation 2.14 shows the initial point of α_1 equals the terminal point of α_0 as claimed.

Let V_2 be the connected component of $U_1 \cap U_2$ containing $\gamma(t_2)$ and $g_{1,2}$ be the unique element of G so that $g_{1,2}|_{\phi_2(V_2)} = \phi_1 \circ \phi_2^{-1} : \phi_2(V_2) \longrightarrow \phi_1(V_2)$.

Define $\alpha_2 := g_{0,1}g_{1,2}(\phi_2\gamma_2)$. The initial point of α_2 is the terminal point of α_1. To see this note that by definition $\alpha_2(t_2) = g_{0,1}g_{1,2}(\phi_2\gamma_2(t_2))$. Additionally, $\phi_2\gamma_2(t_2) \in \phi_2(V_2)$, as $\gamma_2(t_2) \in V_2$. Since $g_{1,2}|_{\phi_2(V_2)} = \phi_1 \circ \phi_2^{-1}$ and $\phi_2\gamma_2(t_2) \in \phi_2(V_2)$, we have

$$\begin{aligned} \alpha_2(t_2) = g_{0,1}g_{1,2}(\phi_2\gamma_2(t_2)) &= g_{0,1}\left(\phi_1 \circ \phi_2^{-1}\right)(\phi_2\gamma_2(t_2)) = g_{0,1}(\phi_1\gamma_2(t_2)) \\ &= g_{0,1}(\phi_1\gamma_1(t_2)) = \alpha_1(t_2) \qquad (2.15) \end{aligned}$$

Equation 2.15 shows the initial point of α_2 equals the terminal point of α_1 as claimed.

Inductively define and construct $(k+1)$-paths in X, $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_k$ in the fashion above with $\alpha_i = g_{0,1}g_{1,2}\ldots g_{i-1,i}(\phi_i\gamma_i)$ for $1 \leq i \leq k$ each with the property that the terminal point of α_{i-1} is the initial point of α_i. We may concatenate these paths together to obtain the path

$$\begin{aligned} \alpha_0\alpha_1\ldots\alpha_k = (\phi_0\gamma_0)(g_{0,1}(\phi_1\gamma_1))(g_{0,1}g_{1,2}(\phi_2\gamma_2))\ldots \\ (g_{0,1}g_{1,2}\ldots g_{k-1,k}(\phi_k\gamma_k)) \qquad (2.16) \end{aligned}$$

This path begins at $\alpha_0(0) = \phi_0\gamma_0(0) = \phi(p)$ where $\phi : U \longrightarrow X$ is the original coordinate patch. We assign to the path γ in M based at p the terminal point of the path as defined in Equation 2.16. Figure 2.5 illustrates this construction with three charts.

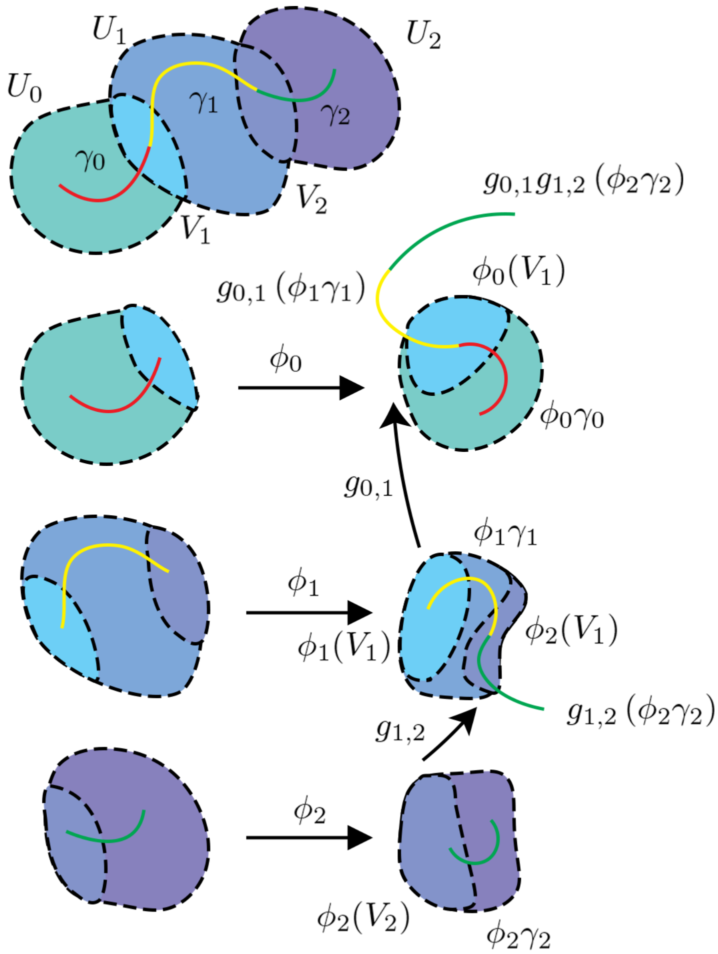

Figure 2.5: Here the path γ is covered by three coordinate charts, U_0, U_1, and U_2. The path γ is broken up into γ_0, γ_1, and γ_2 in red, yellow, and green respectively. The U_i's are illustrated below the covering of γ along with their corresponding images under each chart ϕ_i. This figure shows how to construct the path in X given by Equation 2.16. Starting with the image of γ_2 in $\phi_2(U_2)$, one maps $\phi_2\gamma_2$ back under the unique $g_{1,2}$ taking $\phi_2(V_2)$ to $\phi_1(V_2)$ where V_2 is the component containing γ_1's end point. This yields the yellow and green curve in the second row of charts. We pull this path back by $g_{0,1}$, the unique element taking $\phi_1(V_1)$ to $\phi_0(V_1)$ where V_1 is the component containing γ_0's end point. This yields the red, yellow, and green path starting from the original chart $\phi_0 : U_0 \longrightarrow X$. The end point of this path is the developing map applied to γ.

At this moment we have a map from the space of smooth paths based at p in M to the model space X. Denote this map applied to the path γ by $\text{dev}(\gamma)$. This construction appears to depend on several choices including for example, the path γ in question, and both the choice and number of coordinate patches used along with their corresponding meshes of times chosen.

In regards to the covering of γ by coordinates patches, we show that this construction depends only on the choice of the original coordinate patch $\phi : U \longrightarrow X$ and not subsequent patches chosen in the covering.

To this end, let $\{U_i\}$ be a covering of γ by $(k+1)$-coordinate patches into X such that $U_i \cap U_{i+1} \neq \emptyset$ with $U_0 = U$ as before. Insert an additional covering patch $\psi : W \longrightarrow X$ somewhere after the original coordinate patch $\phi : U \longrightarrow X$ so that the set $\{U_0, \ldots, U_i, W, U_{i+1}, \ldots, U_k\}$ covers γ. Insert an additional s so that $0 = t_0 < \ldots < t_i < s < t_{i+1} < \ldots t_{k+1} = 1$ and satisfies the property that the image of γ on each interval is contained in the corresponding coordinate patch covering γ. In particular this means $\gamma(t') \in W$ for all $s \leq t' \leq t_{i+1}$.

At the $(i+1)$-th step of the construction applied to our refined cover and mesh, let W_{i+1} be the connected component of $U_i \cap W$ containing $\gamma(s)$ and h_{i+1} be the unique element of G so that $h|_{\psi(W_{i+1})} = \phi_i \circ \psi^{-1} : \psi(W_{i+1}) \longrightarrow \phi_i(W_{i+1})$.

Similarly at the $(i+2)$-th step, let W_{i+2} be the connected component of $W \cap U_{i+1}$ containing $\gamma(t_{i+1})$ and j being the unique element of G so that $j|_{\phi_{i+1}(W_{i+2})} = \psi \circ \phi_{i+1}^{-1} : \phi_{i+1}(W_{i+2}) \longrightarrow \psi(W_{i+2})$.

At the $(i+1)$-th step of the original construction with the unrefined cover and mesh, V_{i+1} is the connected component of $U_i \cap U_{i+1}$ containing $\gamma(t_{i+1})$ and $g_{i,i+1}$ is the unique element of G so that $g_{i,i+1}|_{\phi_{i+1}(V_{i+1})} = \phi_i \circ \phi_{i+1}^{-1} : \phi_{i+1}(V_{i+1}) \longrightarrow \phi_i(V_{i+1})$.

We claim $W_{i+1} \cap W_{i+2} \neq \emptyset$. By the old construction, $\gamma(t') \in U_i$ for all $t_i \leq t' \leq t_{i+1}$. By the new construction, $\gamma(t') \in W$ for all $s \leq t' \leq t_{i+1}$. Thus $\gamma(t') \in U_i \cap W$ for all $s \leq t' \leq t_{i+1}$. Thus there is a path entirely contained in $U_i \cap W$ containing both $\gamma(s)$ and $\gamma(t_{i+1})$, so they are in the same connected component of $U_i \cap W$. Since $\gamma(s) \in W_{i+1} \subset U_i \cap W$, this means $\gamma(t_{i+1}) \in W_{i+1}$, thus $\gamma(t_{i+1})$ is in both W_{i+1} and W_{i+2} showing their intersection is non-empty.

Since $\gamma(t_{i+1}) \in V_{i+1} \cap W_{i+2}$, and $\gamma(t_{i+1}) \in W_{i+1} \cap W_{i+2}$, we may choose a sufficiently small connected open subset in their intersection, call it C.

The composition of hj is defined on all of X, but we may restrict it to the domain $\phi_{i+1}(C) \subset \phi_{i+1}(W_{i+2})$. Because $C \subset W_{i+1} \cap W_{i+2}$, this means hj restricted to $\phi_{i+1}(C)$ is given by

$$hj = \left(\phi_i \circ \psi^{-1}\right) \circ \left(\psi \circ \phi_{i+1}^{-1}\right) = \phi_i \circ \phi_{i+1}^{-1} \tag{2.17}$$

Similarly since $g_{i,i+1}|_{\phi_{i+1}(V_{i+1})} = \phi_i \circ \phi_{i+1}^{-1}$, and $\phi_{i+1}(C) \subset \phi_{i+1}(V_{i+1})$, this means $g_{i,i+1}|_{\phi_{i+1}(C)} = hj_{\phi_{i+1}(C)}$. By prospect of the fact that G acts strongly effectively on X, and both hj and $g_{i,i+1}$ agree on the open subset C, this means $hj = g_{i,i+1}$.

In the old construction, the developing map applied to the path γ with the cover $\{U_i\}$ and mesh of $\{t_i\}$ is given by the terminal point of Equation 2.16. Taking the terminal point of the path in Equation 2.16 yields the expression in Equation 2.18.

$$\operatorname{dev}(\gamma) = g_{0,1}g_{1,2}\ldots g_{i-1,i}g_{i,i+1}g_{i+1,i+2}\ldots g_{k-1,k}\left(\phi_k\gamma_k(1)\right) \tag{2.18}$$

On the other hand, in the new construction, the developing map applied to the path γ with the refined cover and refined mesh provides a new path. Taking the terminal point of this path yields the expression in Equation 2.19

$$\begin{aligned}\operatorname{dev}(\gamma) &= g_{0,1}g_{1,2}\dots g_{i-1,i}hjg_{i+1,i+2}\dots g_{k-1,k}\left(\phi_k\gamma_k(1)\right)\\ &= g_{0,1}g_{1,2}\dots g_{i-1,i}g_{i,i+1}g_{i+1,i+2}\dots g_{k-1,k}\left(\phi_k\gamma_k(1)\right) \qquad (2.19)\end{aligned}$$

The second equality follows by Equation 2.17. Thus the developing map applied to this refined cover is left unchanged. Consequently, given any two covers $\{U_i\}$ and $\{U'_j\}$ of γ with meshes $\{t_i\}$ and $\{s_j\}$, we may take the union of these two covers and meshes to obtain a refined cover and mesh of both.

Since we showed the developing map is left unchanged on a refinement by an additional chart and additional time in the mesh, induction shows the developing map is left unchanged on the refinement on their unions. Thus the construction applied to $\{U_i\}$, $\{t_i\}$ and $\{U'_j\}$, $\{s_j\}$ on γ provide the same terminal point in X, so this construction is well-defined for any choice of cover or mesh of times.

That said, one can also show that if γ and β are two paths in M that are homotopic relative endpoints, then $\operatorname{dev}(\gamma) = \operatorname{dev}(\beta)$. The technicalities of this proof are somewhat involved, the basic idea is as follows and taken from Goldman's text [Gol21, p. 99]. If γ and β are paths homotopic relative endpoints in M and contained in the original coordinate chart $\phi : U \longrightarrow X$, then clearly both γ and β develop to the same point in X.

Now let $h : [0,1]^2 \longrightarrow X$ be a homotopy between γ and β relative endpoints so that $h(t, 0) = \gamma(t)$ and $h(t, 1) = \beta(t)$. This homotopy can be broken up into 'small' homotopies, namely homotopies such that we may find meshes of times $0 = s_0 < s_1 < \ldots < s_m = 1$ so that γ_{s_j} and $\gamma_{s_{j+1}}$ are homotopic relative endpoints so that one may find a cover and mesh of both paths $\{U_i\}$ and $\{t_i\}$ so that both $\gamma_{s_j}(t_i)$ and $\gamma_{s_{j+1}}(t_i)$ are in the same connected component of $U_{i-1} \cap U_i$. Figure 2.6 provides an example of this.

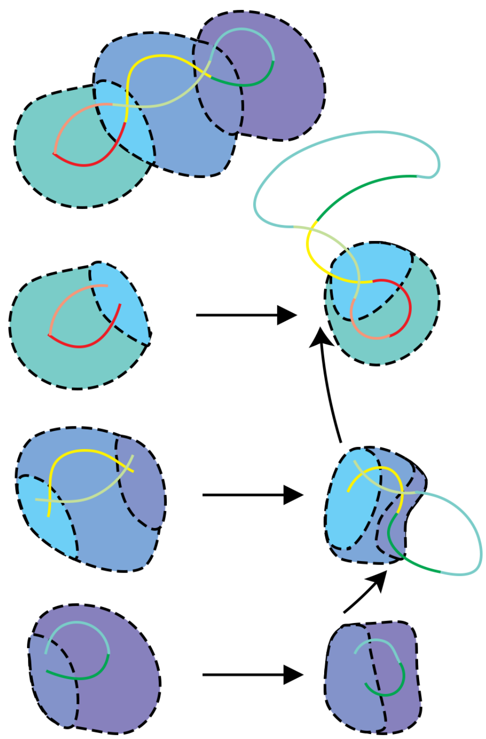

Figure 2.6: Here two homotopic paths, relative endpoints, are provided where each endpoint of each segments is contained in the same connected component. If we denote the new path as β, then both $\gamma(t_1), \beta(t_1) \in V_1$ and $\gamma(t_2), \beta(t_2) \in V_2$. As both β and γ are covered by the same charts, and each $\gamma(t_i)$ and $\beta(t_i)$ are contained in the same connected component of $U_{i-1} \cap U_i$, this guarantees the change of coordinate elements $g_{i-1,i}$ are the same for all i in Equation 2.18, thus yielding the same developing image.

Given such a cover one can then show both these paths develop to the same point, and by induction, γ and β both develop to the same point, so the developing map is well defined on the homotopy classes of paths based at $p \in M$. In fact, this construction is provided in full detail for a similar process regarding the Baker-Campbell-Hausdorff formula [Hal04, p. 76].

Consequently the developing map is well defined on the universal cover of M. We denote this map by $\mathrm{dev} : \widetilde{M} \longrightarrow X$. As this map is defined on the space of homotopy classes of paths based at $p \in M$, it is natural to ask how the developing map changes if one considers the concatenation of a path γ beginning at p with a loop β based at p.

To this end, let $\{U_i\}$ and $\{t_i\}$ be a cover and mesh for γ and $\{V_j\}$ and $\{s_j\}$ be a cover and mesh for β. One may form the concatenation $\beta\gamma$ and combine the covers and meshes appropriately to obtain the cover $\{U_0 = V_0, \dots, V_m, U_0, \dots, U_k\}$ and mesh $0 = s_0/2 < s_1/2 < \dots < s_{m+1}/2 = 1/2 = (t_0+1)/2 < (t_1+1)/2 < \dots < (t_{k+1}+1)/2 = 1$. Following Equation 2.18, the developing map applied to the path β will yield

$$\mathrm{dev}(\beta) = h_{0,1}h_{1,2}\dots h_{m-1,m}\left(\psi_m\gamma_m(1)\right) \tag{2.20}$$

where $\psi_j : V_j \longrightarrow X$ are the charts covering β. Thus, when the developing map is applied to the concatenation $\beta\gamma$ with the covering and mesh as described above, one has the result as in Equation 2.21.

$$\mathrm{dev}(\beta\gamma) = \left(h_{0,1}h_{1,2}\dots h_{m-1,m}\right)\left(g_{0,1}g_{1,2}\dots g_{k-1,k}\left(\phi_k\gamma_k(1)\right)\right) = h\mathrm{dev}(\gamma) \tag{2.21}$$

where $\phi_i : U_i \longrightarrow X$ are the charts covering γ. The expression h is the fixed element in G corresponding to the product $h_{0,1} \ldots h_{m-1,m}$ which depends only on the choice of β.

In fact, since we showed the developing map is well defined on homotopy classes of paths, this yields a group homomorphism $\mathrm{hol} : \pi_1(M) \longrightarrow G$ that takes the homotopy class of β to the element h as defined in Equation 2.21. Given how the fundamental group acts on the universal covering space of M via deck transformations, we obtain the fundamental relation between the developing map and holonomy below in Equation 2.22. For each $[\gamma] \in \pi_1(M)$ which acts on $\widetilde{M}$ by the corresponding deck transformation, the diagram below commutes for any such $[\gamma]$.

$$\begin{tikzcd} \widetilde{M} \arrow[r, "{[\gamma]}"] \arrow[d, "\mathrm{dev}"'] & \widetilde{M} \arrow[d, "\mathrm{dev}"] \\ X \arrow[r, "{\mathrm{hol}\,[\gamma]}"'] & X \end{tikzcd} \tag{2.22}$$

The developing map is easily seen to be a local diffeomorphism, as locally it is given by the original coordinate patch $\phi : U \longrightarrow X$ or some translate of it by an element of G as seen by Equation 2.18. The corresponding pair $(\mathrm{dev}, \mathrm{hol})$ is frequently referred to as a *developing pair* for the (G, X)-structure on M.

Thus to each (G, X)-structure on M, we have a means of assigning a local diffeomorphism from $\widetilde{M}$ into X and a group homomorphism from the fundamental group into the group G of transformations of X that obey the commutative diagram in Equation 2.22 above.

On the other hand, one may start off with a simply connected space N and a discrete group Γ acting both properly and freely on N to form the quotient manifold $M := N/\Gamma$. Given a group homomorphism $\mathrm{hol} : \Gamma \longrightarrow G$, and a hol-equivariant local diffeomorphism $\mathrm{dev} : N \longrightarrow X$, this defines a (G, X)-structure on M.

Coordinate patches $\phi : U \longrightarrow X$ may be constructed in the following fashion. Take an evenly covered open neighborhood $U \subset M$ and the disjoint collection $\{U_i\}$ of its inverse image under the projection $p : N \longrightarrow M$. Define $\phi_i : U \longrightarrow X$ via $\mathrm{dev} \circ p_i^{-1}$ where $p_i := p|_{U_i} : U_i \longrightarrow U$ similar to Example 2.3. Given a connected component $W \subset U \cap V$ of overlapping charts $\phi_i : U \longrightarrow X$ and $\phi_j : V \longrightarrow X$, note that for any $x \in \phi_j(W)$, we have

$$\begin{aligned}(\phi_i \circ \phi_j^{-1})(x) &= (\mathrm{dev} \circ p_i^{-1}) \circ \left(\mathrm{dev} \circ p_j^{-1}\right)^{-1}(x) \\ &= \mathrm{dev} \circ p_i^{-1} p_j(n) = \mathrm{dev}(\gamma n) = \mathrm{hol}\,\gamma\,\mathrm{dev}(n) = (\mathrm{hol}\,\gamma)\,(x) \qquad (2.23)\end{aligned}$$

where n is the unique element in $U_j \subset N$ so that $\mathrm{dev}(n) = x$ and γ is the element in Γ taking U_j to U_i. Thus Equation 2.23 shows that locally the coordinate changes are given by elements of G, as $\mathrm{hol}\,\gamma \in G$, and consequently, M supports a (G, X)-structure as claimed.

The observation above provides us with a means of constructing developing maps via passing to the universal cover of an open subset of X equipped with a free and proper discrete subgroup of G acting on it.

Lemma 2.1. *Let $\Gamma \leq G$ be a discrete subgroup acting on a connected open subset $\Omega \subset X$ both properly and freely. The quotient manifold $M := \Omega/\Gamma$ inherits a (G, X)-structure whose developing map* $\mathrm{dev} : \widetilde{\Omega} \longrightarrow X$ *has Ω as its developing image and*

Γ *as its holonomy image. In this case, the developing map is a covering onto its image.*

Proof. As Ω is an open subset of X, it is easily seen to have a natural (G, X)-structure. One can easily generalize Example 2.3 to show the quotient manifold $M := \Omega/\Gamma$ inherits a (G, X)-structure such that the projection $p : \Omega \longrightarrow M$ is an (G, X)-map.

Let U be an evenly covered neighborhood of M and let $U_i \subset \Omega \subset X$ be a corresponding chart associated to the inverse image of U under p^{-1}. If we denote p_i as the restriction of p to each chart U_i, then the collection $\Phi_i := p_i^{-1} : U \longrightarrow X$ serve as charts for U. Here, as in Example 2.1, the charts $\phi_i : U_i \longrightarrow X$ are simply given by the identity restricted to U_i. Let $W \subset U \cap V$ be a connected component for two charts $\Phi_i : U \longrightarrow X$ and $\Phi_j : V \longrightarrow X$. For each $x \in \Phi_j(W)$, the change of coordinates is given by

$$\left(\Phi_i \circ \Phi_j^{-1}\right)(x) = \left(p_i^{-1} \circ p_j\right)(x) = \gamma(x) \tag{2.24}$$

where $\gamma \in \Gamma \leq G$ is some element of Γ taking U_j to U_i.

Pick a base point $p \in M$ and a point $q \in \Omega$ above it. Let $y \in \Omega$. We claim that y is in the developing image of M. Pick a path α based at q to y contained in Ω. Cover α with $(k+1)$-charts $\{V_i\} \subset \Omega$ so that $p : \Omega \longrightarrow M$ restricted to each V_i is a diffeomorphism onto a chart for M. Choose times $\{t_i\}$ so that $\alpha(t_i) \in V_{i-1} \cap V_i$ for each $1 \leq i \leq k$. Denote $\alpha_i = \alpha|_{[t_i, t_{i+1}]}$. The charts $U_i := p(V_i)$ cover the path $\beta := p(\alpha)$ in M which is based at $p \in M$ and $\beta(t_i) \in U_{i-1} \cap U_i$.

If one applies the developing map construction as in Equation 2.18, one can show that the curve β based at p develops to the curve α based at q. This follows because the original chart U_0 containing p takes p to q via $p|_{V_0}^{-1} : U_0 \longrightarrow V_0 \subset X$. Since each $V_{i-1} \cap V_i \neq \emptyset$, the change of coordinates $g_{i-1,i}$ in Equation 2.16 or γ in Equation 2.24 are all trivial. Hence β develops to the curve α. In particular, this means $\mathrm{dev}(\beta)$ is the terminal point of α, which is by construction y and is therefore part of the developing image. On the other hand, every path in M based at p lifts to a unique path based at q in Ω. Thus every path in M develops to a point in Ω, so the developing image of M equals Ω.

That the holonomy is Γ as claimed may be seen in the following fashion. Equation 2.24 shows that every element of the holonomy is some element of Γ. That every element of Γ is realized as a holonomy element may be seen as follows. A standard fact of algebraic topology yields that Γ is isomorphic to $\pi_1(M)/p_*\pi_1(\Omega)$ where $p : \Omega \longrightarrow M$ is the covering map projection [Hat01, Prop 1.40]. The isomorphism is defined by sending homotopy classes of loops $[\beta] \in \pi_1(M,p)$ to the unique deck transformation taking $q \in \Omega$ to the terminal point of the unique lift of β to Ω based at q. That said, let $[\beta] \in \pi_1(M,p)$ be a class of loops based at p whose image under the isomorphism is given by $\gamma \in \Gamma$.

Now let α be a path based at $p \in M$ and $\mathrm{dev}(\alpha)$ be its developing image which we know by the above arguments is the terminal point of the unique lift of α to Ω based at $q \in \Omega$. Denote this lift by $\widetilde{\alpha}$ so $\mathrm{dev}(\alpha) = \widetilde{\alpha}(1)$.

Consider the concatenation $\beta\alpha$. Again, we know that $\mathrm{dev}(\beta\alpha)$ is the terminal point of the unique lift of $\beta\alpha$ to Ω based at $q \in \Omega$. In particular, this means the latter half of the lift of $\beta\alpha$ is a lift of α based at the terminal point of the lift of β based at q, which is by construction, γq. By the uniqueness of paths, this means the latter half of the lift of $\beta\alpha$ is the unique lift of α based at γq. Thus the latter half of the lift of $\beta\alpha$ at q is equal to $\gamma\widetilde{\alpha}$. In particular, their endpoints agree, so $\mathrm{dev}(\beta\alpha) = \gamma\mathrm{dev}(\alpha)$. Hence γ is an element of the holonomy as claimed.

That the developing map is a covering onto its image follows as the developing map was shown to take paths in M based at p to the terminal point of its lift to Ω based at q. Passing to the universal cover, we get $\mathrm{dev} : \widetilde{M} \longrightarrow X$ that takes homotopy classes of paths in M based at p to the terminal point of their lifts to Ω based at q which, up to identification of $\widetilde{M}$ and $\widetilde{\Omega}$, is precisely the standard projection from the universal cover $\widetilde{\Omega} \longrightarrow \Omega$. Hence the developing map is a covering onto its image. $\square$

Returning to the general theory of (G, X)-structures, we introduce the most elementary examples of (G, X)-manifolds, namely, *complete* (G, X)-manifolds.

Definition 2.6. Let M be a (G, X)-manifold with X simply connected. We say that M is complete if and only if the developing map is a covering onto X.

In the case where X is simply connected and the developing map is a covering onto its image, one may recover the manifold from the holonomy alone. In fact, since $\mathrm{dev} : \widetilde{M} \longrightarrow X$ is a covering onto X, by the uniqueness of universal covers, the developing map may be promoted to a global diffeomorphism. Hence $\widetilde{M}/\pi_1(M)$ is diffeomorphic to $X/\mathrm{hol}\,\pi_1(M)$. Thus in the complete case, the holonomy alone defines the structure.

It is worth noting that in general, the holonomy fails to determine the (G, X)-structure on a manifold. For example, Goldman constructed complex projective structures on the torus with the same holonomy representation but different developing maps [Gol87].

That said, there are certain conditions that guarantee a (G, X)-structure is complete. For example if M is a closed manifold and X is a simply connected Riemannian manifold with a G-invariant metric g_X, then every (G, X)-structure on M is complete.

One may pullback g_X on X via the developing map to a Riemannian metric $g_{\widetilde{M}}$ on $\widetilde{M}$ so that the developing map $\mathrm{dev} : \widetilde{M} \longrightarrow X$ is a local isometry. The holonomy invariance of g_X guarantees that pullback metric is invariant under the deck transformations, and thus descends to a Riemannian metric on M. By compactness, the Hopf-Rinow Theorem guarantees this metric is complete, and thus so too is the metric $g_{\widetilde{M}}$ [Lee91, Thm 6.13].

From here one may apply the the fact that a local isometry on a complete Riemannian manifold into a connected Riemannian manifold is a covering map [KN63]. Thus the developing map is a covering from $\widetilde{M}$ to X, and by uniqueness of the universal cover, is necessarily a diffeomorphism. Thus the corresponding (G, X)-structure on M is complete.

In this context, the study of complete (G, X)-structures may be reduced to the study of discrete subgroups Γ of G acting freely on X whose quotient X/Γ is compact.

The existence of a developing pair provides some interesting topological constraints on certain (G, X)-manifolds. For example, one can show that if X is noncompact, then the fundamental group of a closed (G, X)-manifold is necessarily infinite. This follows from the simple observation that if M is compact with a finite fundamental group, then its universal cover $\widetilde{M}$ is also compact. So if M did indeed support a (G, X)-structure, one may choose a developing pair, under which $\operatorname{dev} \widetilde{M}$ is compact, and therefore closed in X. Additionally, since the developing map is a local diffeomorphism, it is in particular an open map, and thus $\operatorname{dev} \widetilde{M}$ is also an open subset of X. By connectedness of X, this means $\operatorname{dev} \widetilde{M} = X$, contradicting the fact that $\widetilde{M}$ is compact. In particular, this prohibits say for example, the projective plane from admitting an affine structure.

As was shown in Example 2.2, to each affine structure on a manifold M, there is a natural affine structure on every covering space of M. This example is easily generalized to each covering space of a (G, X)-structure. That said, if we let $p : C \longrightarrow M$ denote the projection map, and pick points $y \in C$ above $p \in M$, and base their fundamental groups there, we have two developing maps $\operatorname{dev}_C : \widetilde{C} \longrightarrow X$ and $\operatorname{dev}_M : \widetilde{M} \longrightarrow X$. Up to the identification of $\widetilde{C}$ and $\widetilde{M}$, these two maps are equal.

Lemma 2.2. *Let M be a (G, X)-manifold and $p : C \longrightarrow M$ be a covering map. Let C inherit the (G, X)-structure induced by M. Identifying $\widetilde{C}$ with $\widetilde{M}$, the corresponding developing maps are equal.*

Proof. Let α denote a path based at $y \in C$. We may choose coordinate charts $\{V_i\}$ induced from the (G, X)-structure on M to cover α in such a fashion that p

restricted to each such chart is itself a chart in the original atlas for M. This may be done by construction as seen in Example 2.2. That is to say, the charts covering α are of the form $\psi_i := \phi_i \circ p : V_i \longrightarrow X$ where $\phi_i : U_i \longrightarrow X$ are charts from the (G, X)-structure on M.

From this we obtain the developing map applied to α. By Equation 2.18 it will be of the form

$$\mathrm{dev}_C(\alpha) = g_{0,1} g_{1,2} \dots g_{k-1,k} \left(\psi_i \alpha_i(1)\right) \tag{2.25}$$

where each $g_{i-1,i}$ is the induced by the coordinate changes of $\psi_{i-1} \circ \psi_i^{-1}$.

We may project the curve α to the curve $\gamma := p \circ \alpha$ in M based at p. This curve is covered by the charts $\phi_i : U_i \longrightarrow X$ by construction. The developing map applied to the curve γ will yield

$$\mathrm{dev}_M(\gamma) = h_{0,1} h_{1,2} \dots h_{k-1,k} \left(\phi_i \gamma_i(1)\right) \tag{2.26}$$

Equation 2.5 provides us that the coordinate changes of ψ_i and ϕ_i are the same, and thus $h_{i-1,i} = g_{i-1,i}$ for all i. Moreover, $\phi_i \gamma_i(1) = (\phi_i \circ p)\alpha_i(1) = \psi_i \alpha_i(1)$. Using the identification of the universal cover of C with $\widetilde{M}$ via homotopies of paths in C based at y and homotopies of paths in M based at p, we see the induced developing map of M and C on $\widetilde{M}$ are identical. Consequently $\mathrm{dev}_M(\gamma) = \mathrm{dev}_C(\alpha)$. $\square$

2.5 Parallelism and Geodesics on Affine Manifolds

Equipped with the notion of a developing pair we may define familiar notions of affine geometry on affine manifolds. Let I denote an open interval in $\mathbb{R}$. Recall that a curve $\alpha : I \longrightarrow \mathbb{A}^n$ is a *geodesic* if and only if $\alpha''(t) = 0$ for all $t \in I$. Identifying $\mathbb{A}^n$ with $\mathbb{R}^n$, this definition is equivalent to saying that $\alpha(t) = x + tu$ for some $x, u \in \mathbb{R}^n$.

The notion of being a geodesic is invariant under the group of affine transformations. Note if $(A, v) \in \mathrm{Aff}(n, \mathbb{R})$, as in the notation from Equation 2.2, then

$$\frac{d^2}{dt^2}(A, v)(x + tu) = \frac{d^2}{dt^2}A(x + tu) + v = 0 \tag{2.27}$$

This fact allows us to define the notion of a geodesic on an affine manifold.

Definition 2.7. Let M be an affine manifold and pick a developing pair $\mathrm{dev} : \widetilde{M} \longrightarrow \mathbb{A}^n$ and $\mathrm{hol} : \pi_1(M) \longrightarrow \mathrm{Aff}(n, \mathbb{R})$. A curve $\alpha : I \longrightarrow M$ is a called a geodesic if and only if its lift to $\widetilde{M}$ develops to a geodesic in $\mathbb{A}^n$ under the developing map.

At first glance it may appear that Definition 2.7 depends on the choice of lift of α to the universal cover. But if we choose a lift $\widetilde{\alpha}$ of α, any other lift will differ by an element of the fundamental group. That is to say that another lift $\overline{\alpha}$ will be equal to $[\gamma]\widetilde{\alpha}$ where $[\gamma] \in \pi_1(M)$ acts on $\widetilde{M}$ by deck transformations. Thus we have

$$\mathrm{dev}\,(\overline{\alpha}) = \mathrm{dev}\,([\gamma]\widetilde{\alpha}) = \mathrm{hol}\,[\gamma]\mathrm{dev}\,(\widetilde{\alpha}) \tag{2.28}$$

Since $\mathrm{hol}\,[\gamma] \in \mathrm{Aff}(n, \mathbb{R})$, this means by Equation 2.27 that $\mathrm{dev}\,(\overline{\alpha})$ is also a geodesic. Consequently Definition 2.7 is well defined independent from the choice of lift of α to $\widetilde{M}$.

Similarly one may define a notion of parallelism in $\mathbb{A}^n$. Two subsets $X, Y \subset \mathbb{A}^n$ are parallel if and only if they differ by a translation. That is to say there exists a vector $u \in \mathbb{R}^n$ so that $X + u = Y$. The notion of parallelism is yet another invariant of the affine group.

Take an affine automorphism $A : \mathbb{A}^n \longrightarrow \mathbb{A}^n$ with associated linear part $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. For each $y \in Y$, we have that $A(y) = A(x + u) = A(x) + L(u)$ by definition of an affine map. Thus $AY = AX + L(u)$, and consequently AY and AX are parallel as well.

This fact allows us to define parallelism of curves on an affine manifold.

Definition 2.8. Two curves α and β on an affine manifold M are said to be parallel if and only if their lifts develop to parallel curves in affine space.

Again, this notion is well defined for different lifts will differ by elements of the fundamental group acting by deck transformations. By hol-equivariance of the developing map and the fact that affine transformations take parallel sets to parallel sets as seen in the paragraph above, this provides us with a well defined notion of parallel curves in M.

In fact, we may combine these two notions to obtain the notion of *parallel flow* which will be referenced again later in Chapter 4. Recall a complete flow on a manifold M is a collection of diffeomorphisms $\theta_t : M \longrightarrow M$ such that for all

$t, s \in \mathbb{R}$, we have that $\theta_{t+s} = \theta_t \circ \theta_s$ where $\theta_0 = \mathrm{id}_M$. In other words, a complete flow is group homomorphism from $\mathbb{R}$ into the diffeomorphism group of the manifold M. For the purpose of this thesis, all flows are assumed to be complete unless otherwise stated.

Definition 2.9. A flow on an affine manifold M is said to be a parallel flow if and only if its flow lines are parallel and geodesics.

We provide a very simple example of this on the Euclidean torus.

Example 2.7. Consider $\mathbb{R}^2$ and the group of translations Γ generated by $(1, 0)$ and $(0, 1)$. The quotient is a Euclidean torus. The $\mathbb{R}$-action given by translation along a non-zero direction of $u \in \mathbb{R}^2$ taking the points (x, y) to $(x, y) + tu$ descends to the torus. This defines a parallel flow on the Euclidean torus.

Chapter 3: Radiant Manifolds

3.1 Radiant Manifolds

In the exploration of affine manifolds, a very natural sort of affine manifold to take into consideration is a *radiant* manifold. These are manifolds whose affine holonomy preserves a point. Manifolds of this nature have been explored in thorough detail by several authors such as Suhyoung Choi and Thierry Barbot [Cho01] [BC01]. We begin by providing a definition.

Definition 3.1. An n-dimensional affine manifold is *radiant* if and only if its affine holonomy fixes a point in affine space. Equivalently, its affine holonomy is conjugate to a subgroup of $\mathrm{GL}(n, \mathbb{R})$.

The most elementary of example of a closed radiant manifold is the *Hopf-circle*. This is a non-Riemannian affine structure on S^1. Some details of this example are provided below in Example 3.1.

Example 3.1. Let $X = \mathbb{R}^+$ and let Γ be the cyclic group generated by the dilation acting on X taking each x to λx for some positive $\lambda \neq 1$. It is clear that Γ acts both freely and properly on X, and its quotient is a circle. By Lemma 2.1, this provides S^1 with a radiant structure which we refer to as the Hopf-circle.

This structure is in contrast to the standard Euclidean structure on S^1 which is the quotient of the real line acted upon by a cyclic group of translations. The fact they are inequivalent may been seen as the Hopf-circle is geodesically incomplete. In fact, every geodesic on the circle becomes undefined in a finite amount of time.

For example let $\alpha(t) : (0, 1) \longrightarrow X$ be the geodesic $\alpha(t) = 1 - t$. It descends to a geodesic $\gamma(t) : (0, 1) \longrightarrow X/\Gamma$ under the quotient map $p : X \longrightarrow X/\Gamma$. This geodesic, by construction, cannot be extended beyond any $t \geq 1$ for if so then its lift to X beginning at $x = 1$, which is α, would be definable beyond the origin.

This sort of behavior showcases the difference between Euclidean and affine structures on closed manifolds. By the Hopf-Rinow Theorem, closed Euclidean manifolds are all geodesically complete. On the other hand the Hopf-circle provides an example of an affine structure on a compact manifold which is not geodesically complete.

Example 3.2. Let $n \geq 2$ and $X = \mathbb{R}^n \setminus 0$ and Γ be the cyclic group generated by dilation acting on X taking each x to λx, for some positive $\lambda \neq 1$ much like in Example 3.1. Again Γ acts both properly and freely on X, though here the quotient is a trivial S^1-bundle over S^{n-1}. Figure 3.1 below illustrates this for the case of $n = 3$.

Manifolds that arise from this construction are known as *Hopf-manifolds.* The case where $n = 2$ is known as a *Hopf-torus* and is of particular interest in both Nagano and Yagi's and Oliver Baues' classifications of the flat affine structures on the two-torus [NY74] [Bau14]. In Figure 3.2 below, one can see the fundamental

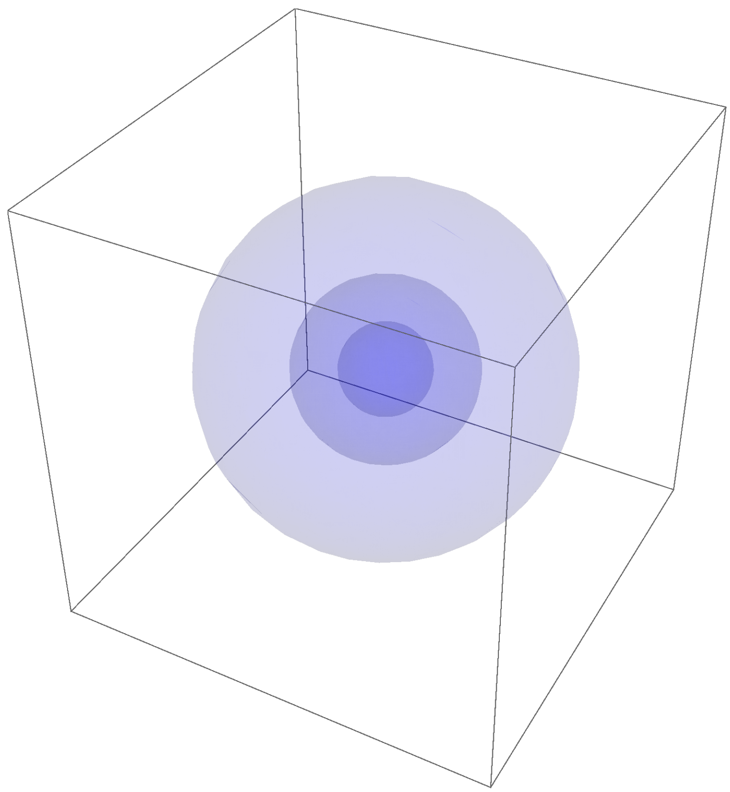

Figure 3.1: Here we illustrate some fundamental domains of the action of Γ on X. Three concentric spheres that all differ by a scaling by λ are drawn. A fundamental domain may be taken to be either region bounded by the spheres along with the bounding spheres themselves. Points on an outer sphere are identified with points on the inner sphere by the scaling factor λ, thus the quotient is seen to be the trivial S^1 bundle over the base S^2. The surrounding box is purely cosmetic.

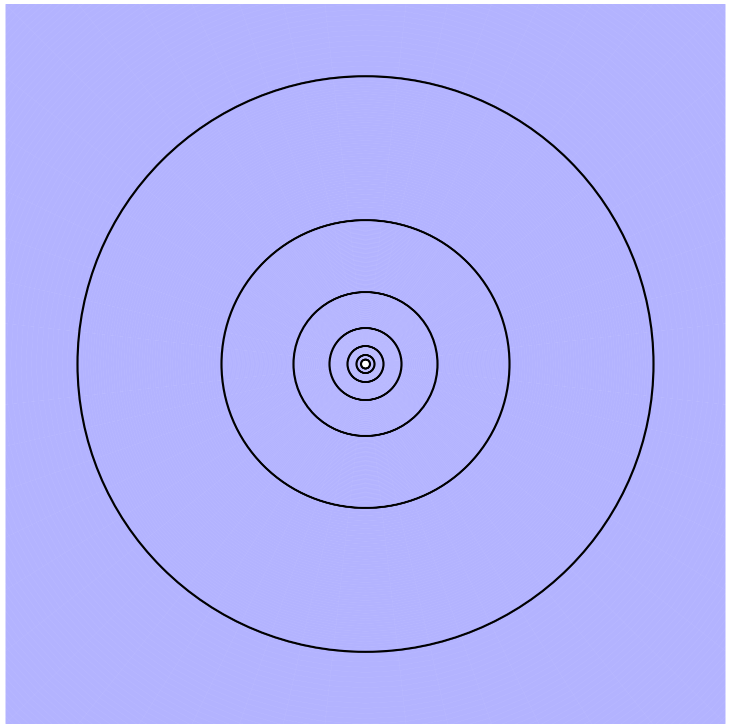

Figure 3.2: Here are some fundamental domains of $\mathbb{R}^2 \setminus 0$ which is acted upon by a cyclic group of dilations. Each annulus forms a fundamental domain for the Hopf-torus where the outer circle gets identified with the inner circle by dilation. By Lemma 2.1, $\mathbb{R}^2 \setminus 0$ is also the developing image of this corresponding structure on the torus.

domains corresponding to the action of dilation on $\mathbb{R}^2 \setminus 0$.

Unlike Example 3.1, the Hopf-torus has many complete geodesics. Simply take any geodesic in $\mathbb{R}^2 \setminus 0$ that does not point towards the origin. It will descend in the quotient to a geodesic which exists for all such time. That said, for each point on the manifold, there exists a direction for which a geodesic in this particular direction cannot be extended beyond a finite amount of time as seen in Figure 3.3. This can be seen by picking any lift of a point on the torus to $x \in \mathbb{R}^2 \setminus 0$. The unique direction connecting the origin and x is the deficient direction for which geodesics pointed towards the origin cannot be defined for all such time. This direction is well-defined as other lifts are of the form $\lambda^n x$.

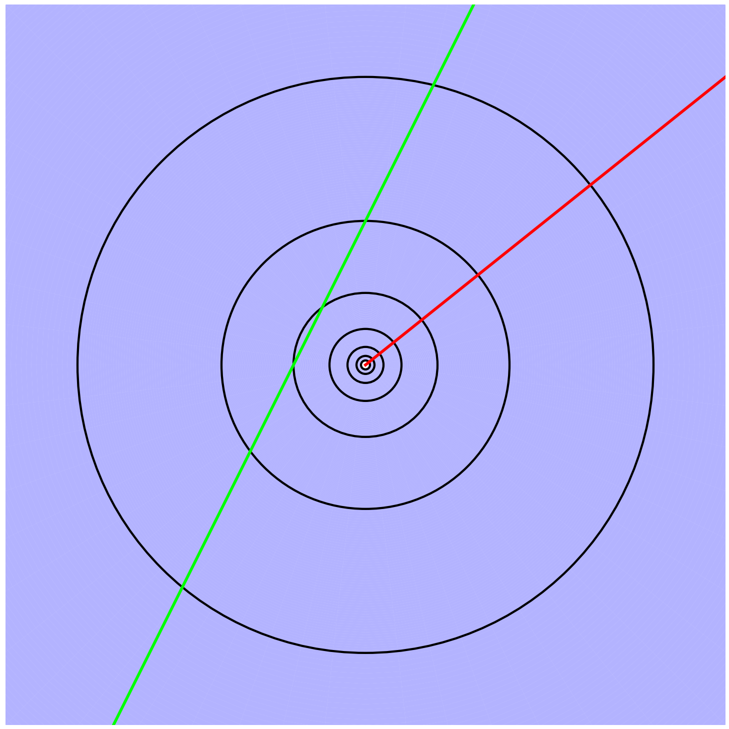

Figure 3.3: As the green geodesic is defined for all time, it will descend to a complete geodesic on the Hopf-torus. This is in contrast with the red geodesic pointed towards the origin that will in a finite amount of time become undefined, and thus the Hopf-torus is geodesically incomplete.

Example 3.3. Let M be a projective manifold of dimension n. Pick a developing pair $\mathrm{dev} : \widetilde{M} \longrightarrow \mathbb{R}P^n$ and a holonomy map $\mathrm{hol} : \pi_1(M) \longrightarrow \mathrm{PGL}(n+1, \mathbb{R})$. We may lift dev to a map into S^n, the double cover of $\mathbb{R}P^n$. We may also lift the holonomy map to the group of lifts of $\mathrm{PGL}(n+1, \mathbb{R})$ which is $\mathrm{GL}(n+1, \mathbb{R})/\mathbb{R}^+$ where $\mathbb{R}^+$ is the subgroup of positive diagonal matrices in $\mathrm{GL}(n+1, \mathbb{R})$. We abusively denote both these maps as dev and hol. As $\mathrm{GL}(n+1, \mathbb{R})/\mathbb{R}^+$ is isomorphic to $\mathrm{SL}^{\pm}(n+1, \mathbb{R})$, the group of matrices of determinant ± 1, this yields the commutative square

$$
\begin{tikzcd}
\widetilde{M} \arrow[r, "{[\gamma]}"] \arrow[d, "\mathrm{dev}"'] & \widetilde{M} \arrow[d, "\mathrm{dev}"] \\
S^n \arrow[r, "\mathrm{hol}[\gamma]"'] & S^n
\end{tikzcd}
\tag{3.1}
$$

where here the holonomy map takes $\pi_1(M) \longrightarrow \mathrm{SL}^{\pm}(n+1, \mathbb{R})$. This is the definition of a projective structure used in Serge Dupont's classification of projective structures whose holonomy factors through a subgroup of the projective group isomorphic to $\mathrm{Aff}^+(1, \mathbb{R})$ [Dup00]. With Equation 3.1, we may form the trivial $\mathbb{R}^+$ bundle on $\widetilde{M}$ and S^n. The trivial $\mathbb{R}^+$ bundle on S^n is diffeomorphic to $\mathbb{R}^{n+1} \setminus 0$.

This induces a local diffeomorphism $F : \mathbb{R}^+ \times \widetilde{M} \longrightarrow \mathbb{R}^{n+1} \setminus 0$. Pick some positive non-trivial scaling factor λ and define the homomorphism $\langle \lambda \rangle \longrightarrow \mathrm{GL}(n + 1, \mathbb{R})$ that takes $\lambda \in \mathbb{R}^+$ to the matrix in $\mathrm{GL}(n + 1, \mathbb{R})$ with λ's along the diagonal.

The group generated by $\langle \lambda \rangle$ and $\pi_1(M)$ acting on $\mathbb{R}^+ \times \widetilde{M}$ by their respective actions is both proper and free. The quotient is $S^1 \times M$, the product of a Hopf-circle and the original projective manifold M. The local diffeomorphism F is equivariant with respect to the induced holonomy homomorphism from $\pi_1(S^1 \times M) \longrightarrow \mathrm{GL}(n+1, \mathbb{R})$ and thus provides $S^1 \times M$ with a radiant affine structure by the remarks in the third paragraph after Equation 2.22. This example is a very specific case of the *radiant suspension* of a projective manifold which is defined in greater generality in Goldman's text [Gol21, p. 128].

Example 3.4. This particular example is found in Thurston and Sullivan's work wherein they describe many interesting affine, inversive, and projective structures on manifolds [ST83]. Construct a projective structure on the torus in the following fashion. Pick three real numbers $a < b < c$ so that $a+b+c=0$. Form the diagonal matrix A consisting of a, b, and c and consider the linear flow defined by e^{tA}. This complete flow on $\mathbb{R}^3 \setminus 0$ descends to a complete flow F_t on $\mathbb{R}P^2$ with three fixed points corresponding to the coordinate axes of $\mathbb{R}^3$ and three invariant projective lines defined by the coordinate planes in $\mathbb{R}^3$. The three fixed points of the flow in $\mathbb{R}P^2$ are a sink, source, and saddle.

Pick an affine patch of the projective plane containing the three fixed points of the flow. Let $\gamma : S^1 \longrightarrow \mathbb{R}P^2$ be an immersed curve contained in this affine patch bounding the source and sink that is everywhere transverse to the flow lines of F_t. Define $\text{dev} : \mathbb{R} \times S^1 \longrightarrow \mathbb{R}P^2$ by the flow applied to the immersed curve, namely map each (t, θ) to $F_t(\gamma(\theta))$. This map a local diffeomorphism as the immersed curve is everywhere transverse to the flow lines. We may lift the developing map to the universal cover of $\mathbb{R} \times S^1$ to get a local diffeomorphism which we also abusively write as $\text{dev} : \mathbb{R}^2 \longrightarrow \mathbb{R}P^2$.

This map obeys an equivariance property. In particular, $\text{dev}(t+1, \theta) = e^A \text{dev}(t, \theta)$ where e^A is acting on $\mathbb{R}P^2$ by its induced projective transformation. On the other hand, $\text{dev}(t, \theta+1) = \text{dev}(t, \theta)$ as this corresponds to the curve $F_t(\gamma(\theta))$ wrapping along θ around back to its original position. Figure 3.4 shows illustrates these properties.

Thus we have a homomorphism $\text{hol} : \pi_1(S^1 \times S^1) \longrightarrow \text{PGL}(3, \mathbb{R})$ for which dev is hol-equivariant. By the remarks in the third paragraph after Equation 2.22, this pair defines a projective structure on the torus.

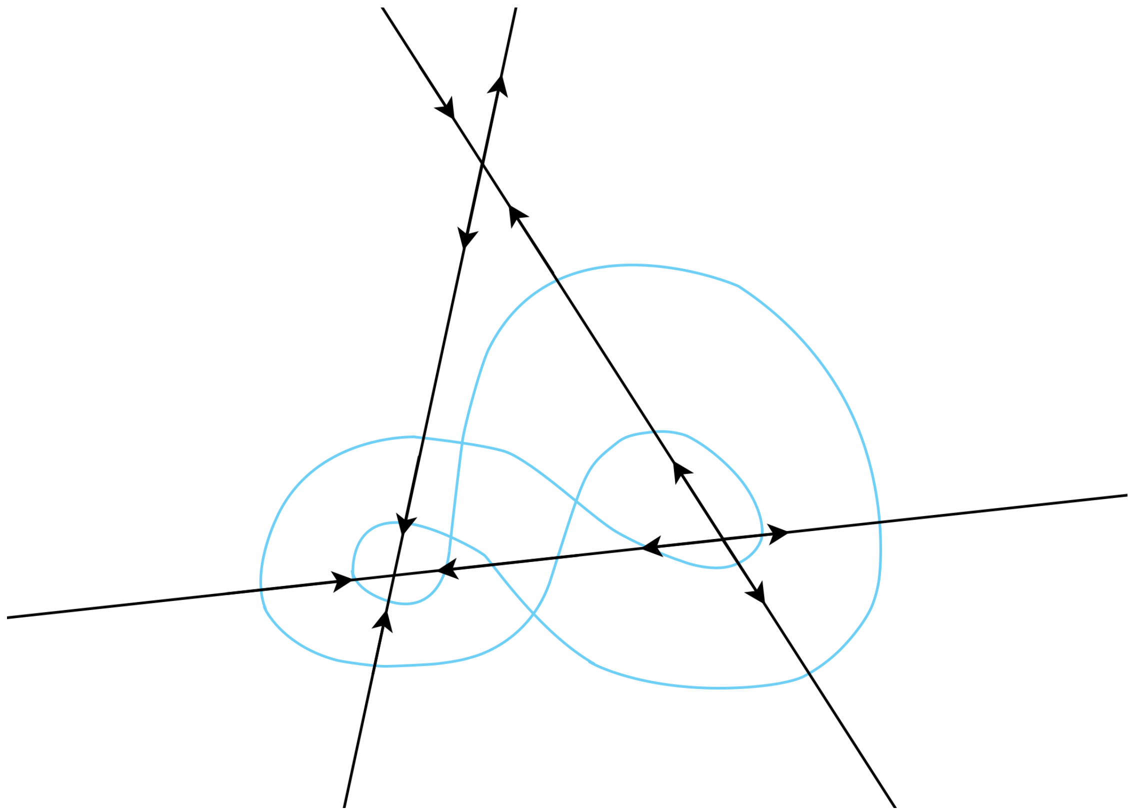

Figure 3.4: The three arrowed projective lines represent the images of the coordinate planes under the projection of $\mathbb{R}^3 \setminus 0 \longrightarrow \mathbb{R}P^2$. They intersect at the source, sink, and saddle of the flow induced by e^{tA}, namely the images of the coordinate axes under the projection. This figure depicts a blue immersed curve everywhere tangent to the flow along with a time one map of the curve under the corresponding flow.

The developing map is not a covering onto its image. In the category of connected smooth manifolds, a local diffeomorphism is a covering map if and only if the unique path lifting property is satisfied [Kap04]. The curve illustrated in Figure 3.5 does not have a lift to $\mathbb{R} \times S^1$. This follows because the projective lines defined by the coordinate planes in $\mathbb{R}^3$ are invariant under the flow of F_t on the projective plane. By starting our path on a point of γ which does not intersect the projective lines, we know its evolution under the flow must also remain outside these projective lines. Because the time component, $t \in \mathbb{R}$, will diverge to infinity as it approaches the invariant projective line, this curve does not lift through the map $\text{dev} : \mathbb{R} \times S^1 \longrightarrow \mathbb{R}P^2$, nor $\text{dev} : \mathbb{R}^2 \longrightarrow \mathbb{R}P^2$.

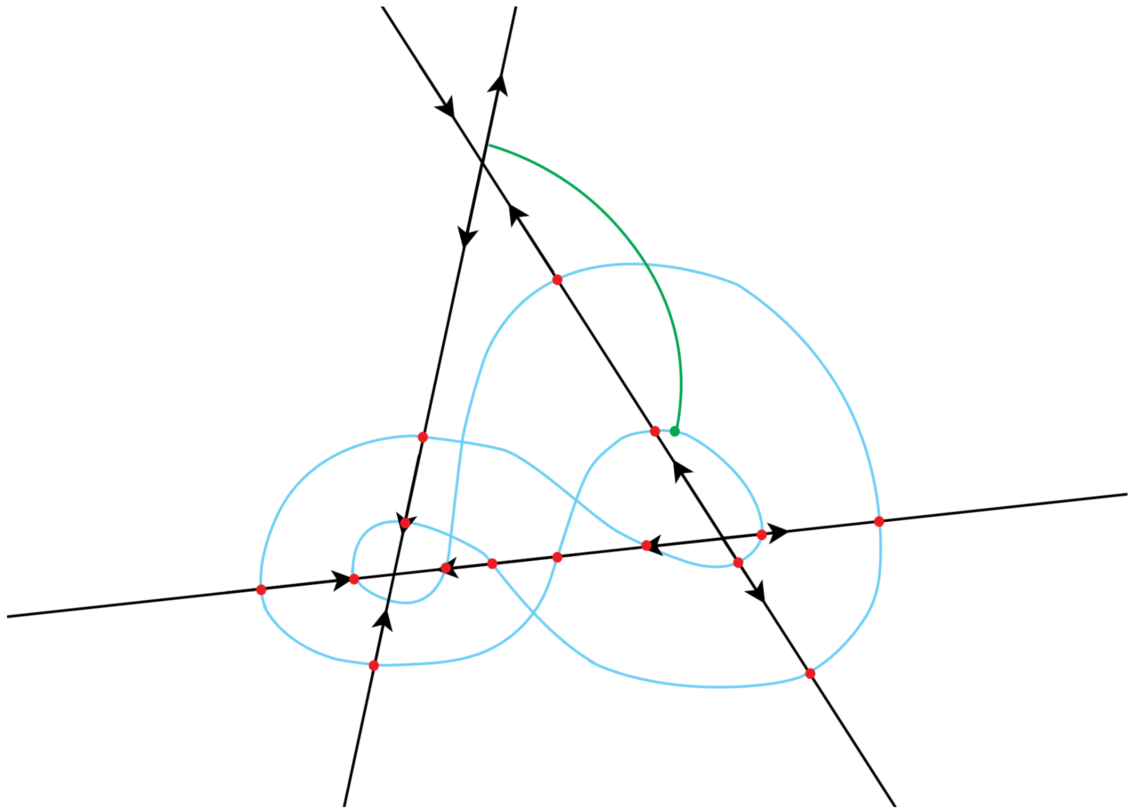

Figure 3.5: The projective arrowed lines are invariant submanifolds of the induced flow on $\mathbb{R}P^2$. Thus the red dots representing the intersection of γ with the projective lines remain on the projective lines under the evolution of the flow. The green curve begins at a point not on an invariant projective line. The lift of the green curve diverges to infinity in $\mathbb{R} \times S^1$ as it is lifted. This follows because the evolution of the segment of γ containing the green point under the flow remains in an invariant region for all time. Thus the green curve does not lift to $\mathbb{R} \times S^1$.

With this projective structure on the torus, we may take the trivial radiant suspension of the torus as in Example 3.3 to yield an affine structure on the three-torus. By construction this affine structure is radiant, and moreover its developing map is not a covering space onto its image which is $\mathbb{R}^3$ minus the three coordinate axes.

3.2 Incompleteness of Radiant Manifolds

By definition, the holonomy of radiant manifolds leaves a point invariant. It is a well known fact that if the radiant manifold is closed, then the invariant point in not inside the developing image. In particular this means that no closed radiant manifold is complete. Here we present a new proof of this fact using the radiant flow induced by the holonomy.

Theorem 3.1. *Fixed points of the holonomy of a closed radiant manifold are not contained within the developing image*

The proof of this fact will be broken into several pieces. First we establish a lemma regarding the ability to pullback vector fields by local diffeomorphisms.

Lemma 3.1. *Let $F : M \longrightarrow N$ be a local diffeomorphism. Let Y be a vector field on N. There exists a unique pullback vector field X on M so that $F_*X = Y$.*

Proof. For each point $m \in M$ choose an open neighborhood $m \in U$ so that F restricted to U is a diffeomorphism onto $F(U)$. Define $X_m := d\left(F^{-1}\right)_{F(m)}\left(Y_{F(m)}\right)$. Smoothness of X follows immediately as locally X is the composition of smooth functions, $X|_U = d(F|_U)^{-1} \circ Y \circ F|_U$. Both uniqueness and the fact $F_*X = Y$ follow readily by construction. □

The next lemma in conjunction with Lemma 3.1 allow us to pullback vector fields invariant under the holonomy through the developing map to $\pi_1(M)$-invariant vector fields on $\widetilde{M}$.

Lemma 3.2. *Let G and H be Lie groups acting on manifolds M and N. Let $\phi : G \longrightarrow H$ be a homomorphism accompanied by a ϕ-equivariant immersion $F : M \longrightarrow N$ in the sense that the following diagram commutes for all $g \in G$.*

$$\begin{array}{ccc} M & \xrightarrow{g} & M \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ N & \xrightarrow[\phi(g)]{} & N \end{array} \tag{3.2}$$

If Y is a $\phi(G)$-invariant vector field on N and X a lift of Y in the sense that $F_*X = Y$, then X is also G-invariant.

Proof. For any $g \in G$ and $m \in M$, we have the following equalities

$$\begin{aligned} dF_{gm}\left(dg_m(X_m)\right) &= d(F \circ g)_m\left(X_m\right) = d(\phi(g) \circ F)_m\left(X_m\right) \\ &= d\left(\phi(g)\right)_{F(m)}\left(dF_m(X_m)\right) = d\left(\phi(g)\right)_{F(m)}\left(Y_{F(m)}\right) \\ &= Y_{\phi(g)F(m)} = Y_{F(gm)} = dF_{gm}\left(X_{gm}\right) \end{aligned} \tag{3.3}$$

The fourth equality follows as $F_*X = Y$ and the fifth because Y is invariant under the action of $\phi(G)$. The first and last equality together with the fact F is an immersion imply that $dg_m(X_m) = X_{gm}$, thus X is invariant under G. □

The next lemma shows that completeness of a vector field is inherited through covering spaces. Within the proof we address the fact that if $F : M \longrightarrow N$ is a map for which $F_*X = Y$ for some vector fields X on M and Y on N, then F takes flow lines of X to flow lines of Y. This fact will be used repeatedly throughout the remainder of this thesis.

Lemma 3.3. *Let $F : C \longrightarrow N$ be a covering space and X and Y be vector fields such that $F_*X = Y$. X is complete if and only if Y is complete.*

Proof. Let X be complete, and $n \in N$. Let β_n be the integral curve through n corresponding to Y. Pick any point $c \in C$ above n. Because X is complete, the integral curve through c corresponding to X is defined for all time. Denote it by α_c. Note that

$$(F \circ \alpha_c)'(t) = dF_{\alpha_c(t)}\left(\alpha_c'(t)\right) = dF_{\alpha_c(t)}\left(X_{\alpha_c(t)}\right) = Y_{(F\circ\alpha_c)(t)} \tag{3.4}$$

Since $(F \circ \alpha_c)(0) = F(c) = n$, this means $F \circ \alpha_c$ is an integral curve through n corresponding to Y and by uniqueness of integral curves, is in fact equal to β_n. Since n was arbitrary, Y is indeed complete.

Conversely, if Y is complete, pick $c \in C$. Let α_c be the integral curve through c corresponding to X. Let β_n be the integral curve through $n := F(c)$. By completeness of Y, β_n is defined for all time. As C is a covering space of N, there exists a unique lift of β_n to $\tilde{\beta}_c$ in C based at c. Similarly to Equation 3.4 we have

$$Y_{\beta_n(t)} = \beta_n'(t) = (F \circ \tilde{\beta}_c)'(t) = dF_{\tilde{\beta}_c(t)}\left(\tilde{\beta}_c{}'(t)\right) \tag{3.5}$$

As $F_*X = Y$, we have $dF_{\tilde{\beta}_c(t)}\left(X_{\tilde{\beta}_c(t)}\right) = Y_{F\circ\tilde{\beta}_c(t)} = Y_{\beta_n(t)}$. Since F is a local diffeomorphism, Equation 3.5 together with the previous sentence imply that

${\tilde{\beta}_c}'(t) = X_{\tilde{\beta}_c(t)}$ so $\tilde{\beta}_c$ is an integral curve of X through $\tilde{\beta}_c(0) = c$ defined for all time. Uniqueness of integral curves implies that $\tilde{\beta}_c = \alpha_c$, and thus X is complete. $\square$

In the proof of Lemma 3.3, the situation where X is complete yields completeness of Y even if F is not a covering space. This means so long as the condition that $F_*X = Y$ is satisfied, F need only be smooth for completeness of X to imply completeness of Y.

Next we establish a topological lemma regarding the ability to promote a local diffeomorphism to a global one.

Lemma 3.4. *Let $F : X \longrightarrow Y$ be a local diffeomorphism where X is connected and let U be an open subset of X such that F restricted to U is a diffeomorphism onto Y. Then $U = X$, so F is a diffeomorphism.*

Proof. By hypothesis U is open. If u_n is some sequence of elements converging to some $x \in X$, we claim $x \in U$.

Define $y_n := F(u_n)$. By continuity, y_n converges to $F(x)$. Since F restricted to U is a diffeomorphism, there exists a unique $u \in U$ so that $F(u) = F(x)$. We claim u_n converges to u.

Let $u \in V$ be any open neighborhood in X about $u \in U$. Because U is open, we may shrink V if necessary to assume $u \in V \subset U$. Because y_n converges to $F(x) = F(u)$ and $F(u) \in F(V)$ is open in Y, as F is an open map, there exists a sufficiently large $N \in \mathbb{N}$ so that $F(u_n) \in F(V)$ for all $n \geq N$. Since F is a diffeomorphism on U, this means u_n converges to both u and x. By uniqueness of limits, this means $u = x$, so U both closed and open in X, thus all of X. $\square$

We now prove a dynamical lemma regarding the radiant flow on $\mathbb{R}^n$ associated to the $\mathrm{GL}(n,\mathbb{R})$-invariant vector field $R = -y^i\partial/\partial y^i$.

Lemma 3.5. *Let $F : N \longrightarrow \mathbb{R}^n$ be a local diffeomorphism where N is connected. Assume that the radiant flow associated to the vector field $R = -y^i\partial/\partial y^i$ can be lifted to N. By this we mean, there exists a complete $\mathbb{R}$-action on N denoted $\widetilde{R}_t$ so that the diagram below commutes for all $t \in \mathbb{R}$.*

$$\begin{tikzcd} N \arrow[r, "\widetilde{R}_t"] \arrow[d, "F"'] & N \arrow[d, "F"] \\ \mathbb{R}^n \arrow[r, "R_t"'] & \mathbb{R}^n \end{tikzcd} \tag{3.6}$$

If $0 \in F(N)$, then F is a diffeomorphism.

Proof. Let $F : N \longrightarrow \mathbb{R}^n$ be such a local diffeomorphism and $\widetilde{R}_t$ be the lifted flow on N. Note $F^{-1}\{0\}$ is a discrete subset of N. Choose countably many disjoint open subsets $\{U_i\}$ about each point in $F^{-1}\{0\}$. As F is a local diffeomorphism, we may choose each U_i in such a fashion that F restricted to each U_i is a diffeomorphism onto an open ball B_i about 0.

Let $\widetilde{R}_\infty U_i$ denote the saturation of U_i with respect to flow on N, namely $\bigcup_{t\in\mathbb{R}} \widetilde{R}_t U_i$. This saturation is an open set and thus an open embedded submanifold of N. We claim that F restricted to $\widetilde{R}_\infty U_i$ is a diffeomorphism onto $\mathbb{R}^n$.

By construction F is equivariant with respect to the $\mathbb{R}$-actions on N and $\mathbb{R}^n$. Moreover $F(\widetilde{U}_i)$ is some open subset about $0 \in \mathbb{R}^n$, so its saturation with respect to the $\mathbb{R}$-action induced by $R = -y^i\partial/\partial y^i$ is all of $\mathbb{R}^n$. In fact, for each $y \in \mathbb{R}^n$, we know that $R_t(y) = e^{-t}y$, so the expansion of an open ball about 0 will indeed contain all of $\mathbb{R}^n$. Consequently, it suffices to show that F restricted to $\widetilde{R}_\infty U_i$ is injective.

To do so, first we show that for all $t > 0$ we have $\widetilde{R}_t U_i \subset U_i$. Let $u_i \in U_i$ and consider the flow line $\widetilde{R}_t u_i$ for $t > 0$. Let $F|_{U_i}$ denote the restriction of F to U_i. Because $F(\widetilde{R}_t u_i) = R_t F(u_i) \in B_i$ for all $t > 0$ and $F|_{U_i} : U_i \longrightarrow B_i$ is a diffeomorphism, this means that $\widetilde{R}_t u_i = F|_{U_i}^{-1}(R_t F(u_i))$ for all $t > 0$. Thus $\widetilde{R}_t u_i \in U_i$ for all $t > 0$ as claimed. Figure 3.6 illustrates the radiant flow restricted to some open subsets of N.

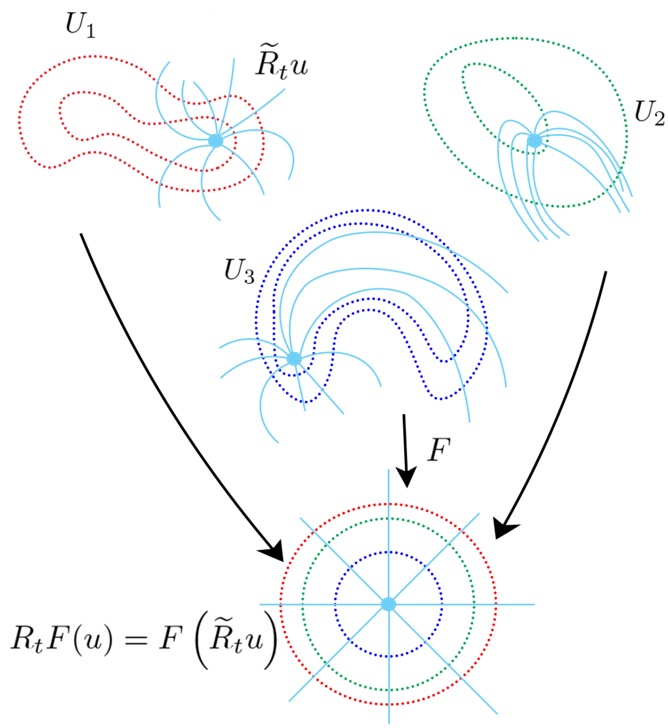

Figure 3.6: Here several neighborhoods U_1, U_2, and U_3 are all mapped to open balls containing the origin of their corresponding color. The flow lines of R are depicted below in light blue and their corresponding lifts are depicted in light blue above. The flow $\widetilde{R}_t$ for $t > 0$ takes U_i to a subset of itself as depicted.

With this established, let $u, v \in U_i$ and choose $s, t \in \mathbb{R}$ so that $F(\widetilde{R}_s u) =$

$F(\widetilde{R}_t v)$. Assume without loss of generality that $s - t \geq 0$. By equivariance of F, this means $F(\widetilde{R}_{s-t} u) = F(v)$. Because $s - t \geq 0$, the above shows that both $\widetilde{R}_{s-t} u$ and v are in U_i and mapped to the same point in B_i. Because F was assumed to be a diffeomorphism on U_i to B_i, this means $\widetilde{R}_{s-t} u = v$. Now if $s - t > 0$, $R_{s-t} F(u) = F(v)$ for $s - t > 0$. The only point satisfying this condition in $\mathbb{R}^n$ is 0, as all other points have non-periodic flow lines, thus both $F(u) = F(v) = 0$, so $u = v$ because there's only one point in U_i mapped to 0 and it is fixed under the lifted flow so $\widetilde{R}_{s-t} u = v$. If $s - t = 0$, then $\widetilde{R}_{s-t} u = v$ implies $u = v$. In either case we see F is injective. Thus F is a diffeomorphism from $\widetilde{R}_\infty U_i$ to $\mathbb{R}^n$ as claimed.

Applying Lemma 3.4 to $F : \widetilde{R}_\infty U_i \longrightarrow \mathbb{R}^n$ yields that F is a diffeomorphism from N onto $\mathbb{R}^n$ as claimed. $\square$

With Lemma 3.5 established, we may easily prove Theorem 3.1.

Proof. Let M be a closed radiant manifold. By definition its affine holonomy is conjugate to a subgroup of the general linear group. Identify $\mathbb{A}^n$ with $\mathbb{R}^n$ by picking an origin and pick a developing pair $\mathrm{dev} : \widetilde{M} \longrightarrow \mathbb{R}^n$ and $\mathrm{hol} : \pi_1(M) \longrightarrow \mathrm{GL}(n, \mathbb{R})$.

The radiant vector field $R = -y^i \partial/\partial y^i$ is invariant under the holonomy. By Lemma 3.1 and Lemma 3.2, there exists a vector field $\widetilde{R}$ on $\widetilde{M}$ that is invariant under the action of $\pi_1(M)$ on $\widetilde{M}$ by deck transformations so that $\mathrm{dev}_* \widetilde{R} = R$.

This vector field descends to a complete vector field on M where completeness is consequence of the fact that M is compact. Thus there is an $\mathbb{R}$-action on M which lifts to an $\mathbb{R}$-action on $\widetilde{M}$ by Lemma 3.3. Since $\mathrm{dev}_* \widetilde{R} = R$, this means that the developing map is equivariant with respect to the $\mathbb{R}$-actions corresponding to $\widetilde{R}$

and R respectively.

Now if $0 \in \operatorname{dev} \widetilde{M}$, we may apply Lemma 3.5 to yield the developing map is a diffeomorphism from $\widetilde{M} \longrightarrow \mathbb{R}^n$, thus the radiant structure on M is complete. Thus M is diffeomorphic to $\mathbb{R}^n / \operatorname{hol} \pi_1(M)$. Because Γ acts freely on $\widetilde{M}$, so too does the holonomy group as dev is a diffeomorphism. But each element of the holonomy preserves $0 \in \mathbb{R}^n$, and thus by freeness, the holonomy is trivial so M is diffeomorphic to $\mathbb{R}^n$ contradicting the fact that M is compact. $\square$

It is worth noting that in the proof of Theorem 3.1, we constructed the lift of the radiant vector field $R = -y^i \partial / \partial y^i$ to the universal cover which was $\pi_1(M)$-invariant and descends to a vector field on the closed manifold M. Since the only zero of R is at the origin, which we have shown is not an element of the developing image, this means there are no zeros of the vector field on the closed manifold. As this vector field is everywhere non-zero, the Euler characteristic of M vanishes. This is related to the long standing Chern Conjecture which stipulates that the Euler class of a closed affine manifold vanishes. Bertram Kostant and Dennis Sullivan prove this in their paper for the case of complete affine manifolds [KS75].

Chapter 4: Parallel Flow

4.1 Holonomy with an Invariant Vector

In this chapter we explore the consequences of having an invariant parallel vector field on a closed affine manifold. Let M be a closed affine $(n+1)$-dimensional manifold whose linear holonomy fixes a vector $v \in \mathbb{R}^{n+1}$. Choose a developing pair $\mathrm{dev} : \widetilde{M} \longrightarrow \mathbb{A}^{n+1}$ and $\mathrm{hol} : \pi_1(M) \longrightarrow \mathrm{Aff}(n+1, \mathbb{R})$. Picking an origin in affine space $\mathbb{A}^{n+1}$ yields a coordinate system onto $\mathbb{R}^{n+1}$. Applying the necessary conjugation we may assume the affine holonomy sits inside the group

$$\mathrm{hol}\,\Gamma \leq \left\{ \begin{pmatrix} 1 & u \\ 0 & A \end{pmatrix} \begin{pmatrix} t \\ v \end{pmatrix} \middle|\, t \in \mathbb{R}, u^T, v \in \mathbb{R}^n, \text{ and } A \in \mathrm{GL}(n, \mathbb{R}) \right\} \tag{4.1}$$

where u is some row vector of length n, v is some column vector of length n, t is some real number, and A is some $n \times n$ invertible matrix with real entries. In the identification between $\mathbb{A}^{n+1}$ and $\mathbb{R}^{n+1}$, we denote the first coordinate by x and the remaining n-coordinates by y^i where $1 \leq i \leq n$. This labeling will be used again for the remainder of the thesis. Note the parallel vector field $\partial/\partial x$ is invariant under the holonomy group in Equation 4.1.

Given that the developing map is a local diffeomorphism, Lemma 3.1 yields the existence of a vector field $\widetilde{X}$ on $\widetilde{M}$ so that $\mathrm{dev}_*\widetilde{X} = \partial/\partial x$ whereas Lemma 3.2 applied to the commutative square

$$\begin{tikzcd} \widetilde{M} \arrow[r, "{[\gamma]}"] \arrow[d, "\mathrm{dev}"'] & \widetilde{M} \arrow[d, "\mathrm{dev}"] \\ \mathbb{R}^{n+1} \arrow[r, "{\mathrm{hol}[\gamma]}"'] & \mathbb{R}^{n+1} \end{tikzcd} \tag{4.2}$$

guarantees that $\widetilde{X}$ is $\pi_1(M)$-invariant as $\partial/\partial x$ is holonomy invariant.

Because $\widetilde{X}$ is a $\pi_1(M)$-invariant non-zero vector field on $\widetilde{M}$, $\widetilde{X}$ descends to a non-zero vector field X on M so in particular closed manifolds of this form have Euler characteristic zero. Much like in the argument provided for the proof of Theorem 3.1, this vector field is complete because M is compact. In addition, the corresponding flow on M is parallel in the sense of Definition 2.9. This may be seen because $\mathrm{dev}_*\widetilde{X} = \partial/\partial x$. Because flow lines of X develop to flow lines of $\partial/\partial x$ under the developing map, M supports a *complete parallel flow*.

4.2 Complete Parallel Flow

In this section we explore the consequences of having a complete parallel flow induced by $\widetilde{X}$ on the universal cover obtained from the pullback under the developing map of the vector field $\partial/\partial x$.

Since $\mathrm{dev}_*\widetilde{X} = \partial/\partial x$, flow lines of $\widetilde{X}$ are sent to flow lines of $\partial/\partial x$ as seen by Equation 3.4. In particular, this means that the developing map is equivariant with respect to the corresponding $\mathbb{R}$-actions on $\widetilde{M}$ and $\mathbb{R}^{n+1}$ induced by $\widetilde{X}$ and $\partial/\partial x$

respectively.

Note that $\mathbb{R}^{n+1}$ may be realized as a principal $\mathbb{R}$-bundle induced by the $\mathbb{R}$-action of translation along the x-coordinate. We claim that this structure may be pulled back via the developing map so that $\widetilde{M}$ may also be realized as a principal $\mathbb{R}$-bundle. Before proceeding, we establish a lemma that guarantees that we may pullback principal bundle structures.

Lemma 4.1. *Let G act on manifolds M and N and $F : M \longrightarrow N$ be a G-equivariant map. If G acts properly on N, then G acts properly on M. Additionally, if G acts freely on N, then G acts freely on M.*

Proof. Let G act properly on N. By the characterization of proper group actions, it suffices to show that if m_i and g_i are sequences in M and G respectively where both m_i and $m_i g_i$ converge, then so does a subsequence of g_i [Lee03, Prop 21.5] .

Let m_i converge to m and $g_i m_i$ converge to m'. Then $n_i := F(m_i)$ converges to $F(m)$ by continuity whereas by equivariance of F, we have that $F(g_i m_i) = g_i F(m_i) = g_i n_i$ converges to $F(m')$. As the action of G on N is proper, this means that a subsequence of g_i converges to g in G, so the action of G on M is proper.

To prove that freeness on N implies freeness on M, note that if $gm = m$ for some $g \in G$ and $m \in M$, then by equivariance of F, $F(m) = F(gm) = gF(m)$, and if g stabilizes $F(m)$, g must be the identity because G acts freely on N. This proves freeness of G on M. $\square$

Note that as a consequence of Lemma 4.1, if G acts both properly and freely on N, then G acts both properly and freely on M. Thus, the principal G-bundle structure inherited from the proper and free action of G on N pullbacks to a principal G-bundle structure on M.

Returning to the situation at hand, the $\mathbb{R}$-action on $\mathbb{R}^{n+1}$ is translation along the x-coordinate of the vectors in $\mathbb{R}^{n+1}$. This $\mathbb{R}$-action is both free and proper on $\mathbb{R}^{n+1}$ and as the developing map is equivariant with respect to both $\mathbb{R}$-actions, the $\mathbb{R}$-action on $\widetilde{M}$ is also free and proper by Lemma 4.1. By the preceding remarks, this means that $\widetilde{M}$ inherits a principal $\mathbb{R}$-bundle structure.

As $\mathbb{R}$ is contractible, this means the principal bundle structure of $\widetilde{M}$ is trivial [Cal13]. In particular it admits a global cross section $N \subset \widetilde{M}$. In terms of diagrams this means that there exists a $\mathbb{R}$-equivariant diffeomorphism $\Phi : \widetilde{M} \longrightarrow \mathbb{R} \times N$ where the diagram in Equation 4.3 commutes. The map $q_1 : \widetilde{M} \longrightarrow \widetilde{M}/\mathbb{R}$ is the associated quotient map to the proper and free action $\mathbb{R}$-action on $\widetilde{M}$ whereas $q_2 : \mathbb{R} \times N \longrightarrow \widetilde{M}/\mathbb{R}$ is the trivial bundle up to an identification of $\widetilde{M}/\mathbb{R}$ with N which exists by prospect of the fact that the q_1 admits a global cross section.

$$\begin{tikzcd} \widetilde{M} \arrow[rr, "\Phi"] \arrow[dr, "q_1"'] & & \mathbb{R} \times N \arrow[dl, "q_2"] \\ & \widetilde{M}/\mathbb{R} & \end{tikzcd} \tag{4.3}$$

Define $\text{dev}' : \mathbb{R} \times N \longrightarrow \mathbb{R}^{n+1}$ via $\text{dev}' = \text{dev} \circ \Phi^{-1}$ and let $\pi_1(M)$ act on $\mathbb{R} \times N$ by $[\gamma](t, n) := (\Phi \circ [\gamma] \circ \Phi^{-1})(t, n)$. By construction, the map dev' obeys the commutative square for all $[\gamma] \in \pi_1(M)$ as in Equation 4.4.

$$
\begin{tikzcd}
\mathbb{R} \times N \arrow[r, "{[\gamma]}"] \arrow[d, "\mathrm{dev}'"'] & \mathbb{R} \times N \arrow[d, "\mathrm{dev}'"] \\
\mathbb{R} \times \mathbb{R}^n \arrow[r, "{\mathrm{hol}[\gamma]}"'] & \mathbb{R} \times \mathbb{R}^n
\end{tikzcd}
\tag{4.4}
$$

We distinguish the first factor of $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ as it is the direction of the parallel flow as defined by $\partial/\partial x$. This new developing pair defines the same geometric structure on M as initially endowed, but this new developing map enjoys the property that it respects the parallel flow on $\widetilde{M} \simeq \mathbb{R} \times N$ inherited from $\partial/\partial x$. This is made precise in Lemma 4.2.

Lemma 4.2. *For any* $(t, n) \in \mathbb{R} \times N$ *and* $s \in \mathbb{R}$, $\mathrm{dev}'(t + s, n) = \mathrm{dev}'(t, n) + (s, 0)$.

Proof. Let $(t, n) \in \mathbb{R} \times N$ and $s \in \mathbb{R}$. Define $\widetilde{m} := \Phi^{-1}(t, n)$. Since Φ, and consequently Φ^{-1}, is $\mathbb{R}$-equivariant we have

$$\Phi^{-1}(t + s, n) = \Phi^{-1}(s \cdot (t, n)) = s \cdot \Phi^{-1}(t, n) = s \cdot \widetilde{m} \tag{4.5}$$

Because the original developing map, dev, maps flow lines of $\widetilde{X}$ to flow lines of $\partial/\partial x$,

$$\begin{aligned} \mathrm{dev}'(t + s, n) &= \left(\mathrm{dev} \circ \Phi^{-1}\right)(t + s, n) = \mathrm{dev}\,(s \cdot \widetilde{m}) = \mathrm{dev}(\widetilde{m}) + (s, 0) \\ &= \mathrm{dev}'(t, n) + (s, 0) \end{aligned} \tag{4.6}$$

$\square$

Returning to the original action of $\pi_1(M)$ on $\widetilde{M}$, recall that $\widetilde{X}$ is invariant under the action of $\pi_1(M)$, so $[\gamma]_*\widetilde{X} = \widetilde{X}$. Consequently, if $\alpha_{\widetilde{x}}$ is a flow line of $\widetilde{X}$

beginning at $\widetilde{x}$, then $[\gamma]\alpha_{\widetilde{x}}$ is a flow line of $\widetilde{X}$ beginning at $[\gamma]\widetilde{x}$. In terms of the corresponding $\mathbb{R}$-action on $\widetilde{M}$, this says the action of $\pi_1(M)$ on $\widetilde{M}$ and the $\mathbb{R}$-action on $\widetilde{M}$ commute. Symbolically, for all $\widetilde{x} \in \widetilde{M}$, we have that $[\gamma]t \cdot \widetilde{x} = t \cdot [\gamma]\widetilde{x}$.

Combining these observations, we have the action of $\pi_1(M)$ on $\mathbb{R}\times N$ as defined in the paragraph after Equation 4.3 also commutes with the $\mathbb{R}$-action on $\mathbb{R} \times N$. This is a simple consequence of the fact that

$$[\gamma]s \cdot (t,n) = \left(\Phi \circ [\gamma] \circ \Phi^{-1}\right)(s \cdot (t,n)) = s \cdot \left(\Phi \circ [\gamma] \circ \Phi^{-1}\right)(t,n) = s \cdot [\gamma](t,n) \tag{4.7}$$

where the second equality follows as the action of s commutes with Φ, Φ^{-1}, and the action of $\pi_1(M)$ on $\widetilde{M}$.

As a consequence of Equation 4.7, this means that the action of $\pi_1(M)$ acts on the trivial fiber bundle structure of $p_2 : \mathbb{R} \times N \longrightarrow N$. More specifically, for each $(t,n) \in \mathbb{R} \times N$, and $[\gamma] \in \pi_1(M)$, $[\gamma]\left(\mathbb{R} \times \{n\}\right) = \mathbb{R} \times \{n'\}$ for some $n' \in N$. This induces an action of $\pi_1(M)$ on the base space N. Thus, for each $[\gamma] \in \pi_1(M)$, the diagram in Equation 4.8 commutes.

$$\begin{array}{ccc} \mathbb{R} \times N & \xrightarrow{[\gamma]} & \mathbb{R} \times N \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_2} \\ N & \xrightarrow[{[\gamma]}]{} & N \end{array} \tag{4.8}$$

A similar statement holds about the holonomy acting on the trivial fiber bundle structure of $p_2 : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Referring to Equation 4.1, each holonomy element is of the form

$$\text{hol }[\gamma] = \begin{pmatrix} 1 & u \\ 0 & A \end{pmatrix} \begin{pmatrix} t \\ v \end{pmatrix} \tag{4.9}$$

and thus takes lines of the form $\mathbb{R}\times\{y\}$ for each $y \in \mathbb{R}^n$ to lines of the form $\mathbb{R}\times\{y'\}$ for some $y' \in \mathbb{R}^n$. Thus the holonomy action on $\mathbb{R}\times\mathbb{R}^n$ descends to an action on $\mathbb{R}^n$ such that the following diagram commutes

$$\begin{tikzcd} \mathbb{R}\times\mathbb{R}^n \arrow[r, "\text{hol}[\gamma]"] \arrow[d, "p_2"'] & \mathbb{R}\times\mathbb{R}^n \arrow[d, "p_2"] \\ \mathbb{R}^n \arrow[r, "\text{hol}[\gamma]"'] & \mathbb{R}^n \end{tikzcd} \tag{4.10}$$

In fact, the induced action of a holonomy element as induced by Equation 4.10 on $\mathbb{R}^n$ is easily seen to be

$$\text{hol }[\gamma]y = Ay + v = (A, v)y \tag{4.11}$$

for any $y \in \mathbb{R}^n$ as seen by referring to Equation 4.9.

It is worth noting, while the action of $\pi_1(M)$ on $\mathbb{R}\times N$ is both proper and free, this is not necessarily the case for the induced action of $\pi_1(M)$ on N. The example below illustrates the fragility of the free and proper conditions.

Example 4.1. Let $\Omega = \mathbb{R}\times D$ where $D \subset \mathbb{R}^2$ is the open unit disk about the origin. Let Γ be the cyclic group generated by the symmetry of Ω taking (t, p) to $(t+1, R_\theta p)$ where R_θ is rotation by an irrational multiple of π. The quotient of Ω/Γ is a solid open torus as seen in Figure 4.1.

Note that Γ takes lines of the form $\mathbb{R}\times\{p\}$ to lines of the form $\mathbb{R}\times\{R_\theta^n p\}$ for some $n \in \mathbb{Z}$ so there is an induced action of Γ on D.

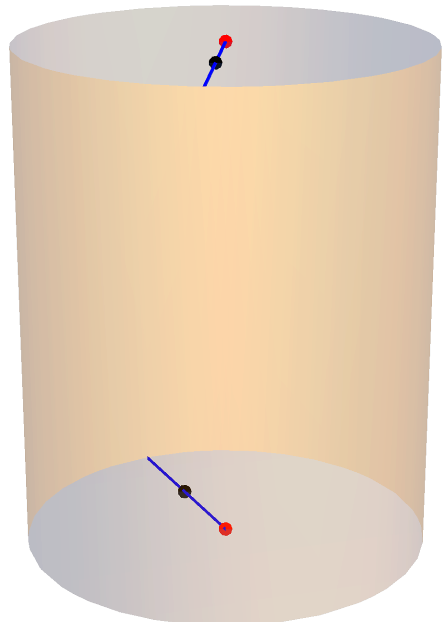

Figure 4.1: A fundamental domain for the action of Γ on Ω. The top and bottom disks of the solid open cylinder are identified via an irrational rotation. In particular, the red and black points are identified respectively. The blue line segments are included to illustrate the irrational rotation and are also identified in the quotient Ω/Γ.

This induced action of Γ on D is neither free nor proper. The group stabilizes $0 \in D$, and worse yet, the orbit of every non-zero point is a dense subset of the circle of its corresponding radius. Figure 4.2 illustrates this.

Recall that $\mathrm{dev}' : \mathbb{R} \times N \longrightarrow \mathbb{R} \times \mathbb{R}^n$ enjoys the property that it is equivariant with respect to the $\mathbb{R}$-actions on $\mathbb{R} \times N$ and $\mathbb{R} \times \mathbb{R}^n$ as in Equation 4.6. Thus dev' descends to a smooth map $\overline{\mathrm{dev}'} : N \longrightarrow \mathbb{R}^n$ that is defined by Equation 4.12 below.

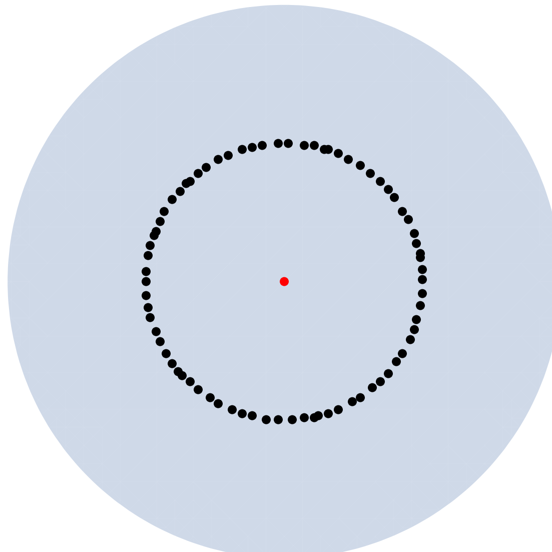

Figure 4.2: Here the induced action of Γ on D is illustrated. The red point corresponds to the line of symmetry of the original cylinder and is fixed under the induced action of Γ so the action is not free. Moreover, the orbit of a black point with radius $r > 0$ inside D under the induced action provides a dense subset of the circle of radius r. Hence the induced action of Γ on D is also not proper.

$$\begin{tikzcd} \mathbb{R} \times N \arrow[r, "\mathrm{dev}'"] \arrow[d, "p_2"'] & \mathbb{R} \times \mathbb{R}^n \arrow[d, "p_2"] \\ N \arrow[r, "\overline{\mathrm{dev}'}"'] & \mathbb{R}^n \end{tikzcd} \tag{4.12}$$

As $\overline{\mathrm{dev}'}$ is the composition of a local diffeomorphism and smooth submersion, it too is a smooth submersion from an n-dimensional manifold to another n-dimensional manifold, so in particular it too is a local diffeomorphism. By construction, $\overline{\mathrm{dev}'}$ is equivariant with respect to the actions of $\pi_1(M)$ on N and the holonomy of $\mathbb{R}^n$. Specifically, for each $[\gamma] \in \pi_1(M)$, the diagram in Equation 4.13

commutes.

$$\begin{tikzcd} N \arrow[r, "{[\gamma]}"] \arrow[d, "\overline{\mathrm{dev}'}"'] & N \arrow[d, "\overline{\mathrm{dev}'}"] \\ \mathbb{R}^n \arrow[r, "\mathrm{hol}[\gamma]"'] & \mathbb{R}^n \end{tikzcd} \tag{4.13}$$

While Equation 4.13 looks as though it defines a geometric structure on $N/\pi_1(M)$, it does not necessarily. As mentioned in Example 4.1, $\pi_1(M)$ does not necessarily act properly and freely on N.

Nevertheless, $\overline{\mathrm{dev}'}$ is still a local diffeomorphism. In fact, to some extent we may relate the developing map dev' and $\overline{\mathrm{dev}'}$. This comes as a consequence of the fact that N embeds as a leaf in the foliation of $\mathbb{R} \times N$.

Lemma 4.3. *Let $U \subset N$ be an open subset for which $\overline{\mathrm{dev}'}$ restricted to U is a diffeomorphism onto its image. Then dev' restricted to $\mathbb{R} \times U \subset \mathbb{R} \times N$ is a diffeomorphism onto its image.*

Proof. Let $U \subset N$ be such an open subset. As dev' is a local diffeomorphism, it is in particular an open map, and the image of $\mathrm{dev}'(\mathbb{R} \times U)$ is an open submanifold of $\mathbb{R} \times \mathbb{R}^n$. Thus to show dev' is a diffeomorphism on $\mathbb{R} \times U$, it suffices to show that dev' restricted to $\mathbb{R} \times U$ is injective.

This follows as if (t, n) and (s, m) are points in $\mathbb{R} \times U$ so that $\mathrm{dev}'(t, n) = \mathrm{dev}'(s, m)$, then $\overline{\mathrm{dev}'}(n) = \overline{\mathrm{dev}'}(m)$. Because $n, m \in U$, and $\overline{\mathrm{dev}'}$ is a diffeomorphism on U, $n = m$. Hence (t, n) and (s, n) lie on the same flow line $\mathbb{R} \times \{n\}$.

By $\mathbb{R}$-equivariance, dev' maps flow lines to flow lines. Because $\mathrm{dev}'(t, n) = \mathrm{dev}'(s, n)$, this necessitates that $t = s$, for otherwise this would necessitate the flow line $\mathbb{R} \times \{n\}$ would get mapped to a circle which is not a flow line of $\partial/\partial x$.

Consequently dev' restricted to $\mathbb{R} \times U$ is injective, and thus a diffeomorphism onto the open subset $\text{dev}'(\mathbb{R} \times U) = \mathbb{R} \times \overline{\text{dev}'}(U)$. $\square$

Note that Lemma 4.3 implies that so long as $\overline{\text{dev}'}$ is a diffeomorphism on some open subset $U \subset N$, this open subset may be saturated by the parallel flow on $\mathbb{R} \times N$ to yield a diffeomorphism of the developing map dev' on $\mathbb{R} \times U$. We call neighborhoods of the form $\mathbb{R} \times U$ *parallel tubular neighborhoods*. Figure 4.3 below illustrates the saturation of the open subset U upon which $\overline{\text{dev}'}$ is a diffeomorphism onto its image.

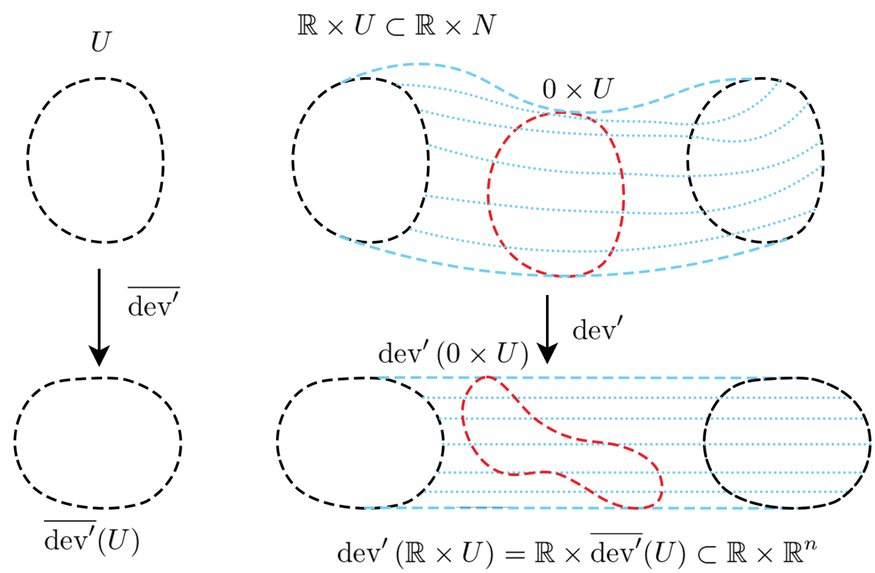

Figure 4.3: On the left is an open neighborhood $U \subset N$ for which $\overline{\text{dev}'}$ is a diffeomorphism onto its image. On the right is the saturation of both these neighborhoods with respect to the $\mathbb{R}$-actions. The map dev' takes the flow lines of $\mathbb{R} \times U$ illustrated in light blue, to the flow lines of $\mathbb{R} \times \mathbb{R}^n$ also illustrated in light blue. The slice $0 \times U$ in red is mapped to some slice of $\mathbb{R} \times \overline{\text{dev}'}(U)$ whose slice is also illustrated in red.

With this established, we provide the proof of one of our main theorems.

Theorem 4.1. *Let M be a closed affine manifold whose linear holonomy fixes a vector in $\mathbb{R}^{n+1}$. Then there exists a complete parallel flow on M which lifts to the universal cover $\widetilde{M}$. In addition, for any point in the universal cover, we may find a neighborhood of the point saturated with respect to the parallel flow such that the developing map restricted to this neighborhood is a diffeomorphism onto its image.*

Proof. If the linear holonomy fixes a vector in $\mathbb{R}^{n+1}$, then up to conjugation, the holonomy lies inside the subgroup defined by Equation 4.1. The vector field $\widetilde{X}$ that projects to $\partial/\partial x$ under the developing map as constructed by Lemma 3.1 and Lemma 3.2 projects to the manifold M as it is $\pi_1(M)$-invariant. Its corresponding complete flow is both parallel as the flow lines develop to flow lines of $\partial/\partial x$ which are parallel.

Associate to the universal cover of M the fiber bundle structure $\mathbb{R} \times N \longrightarrow N$ along with the local diffeomorphism $\overline{\text{dev}'}$ satisfying Equation 4.13. For any point $(t, n) \in \mathbb{R} \times N$, pick a neighborhood $U \subset N$ so that $\overline{\text{dev}'}$ is a diffeomorphism onto its image in $\mathbb{R}^n$. Such a neighborhood exists as $\overline{\text{dev}'}$ is a local diffeomorphism. Saturate this neighborhood with the parallel flow to obtain the neighborhood $\mathbb{R} \times U$ about (t, n). Lemma 4.3 shows the developing map is a diffeomorphism when restricted to this neighborhood thus completing the proof. $\square$

We now begin exploring a natural generalization of the result in Theorem 4.1. Let M be a closed affine $(n + k)$-dimensional manifold where $k \geq 1$ and whose linear holonomy fixes linearly independent vectors $v_1, v_2, \ldots, v_k \in \mathbb{R}^{n+k}$. Similar to the beginning of this section, choose a developing pair $\text{dev} : \widetilde{M} \longrightarrow \mathbb{R}^{n+k}$ and hol : $\pi_1(M) \longrightarrow \mathrm{Aff}(n+k,\mathbb{R})$. Applying the necessary conjugation we may assume each element of the affine holonomy is of the form

$$\mathrm{hol}\,[\gamma] = \begin{pmatrix} I_k & B \\ 0 & A \end{pmatrix} \begin{pmatrix} T \\ v \end{pmatrix} \tag{4.14}$$

where I_k is the $k \times k$ identity matrix, B is some $k \times n$ matrix, 0 denotes the $n \times k$ zero matrix, T and v are column vectors of length k and n respectively, and A denotes an $n \times n$ invertible matrix. In the identification between $\mathbb{A}^{n+k}$ and $\mathbb{R}^{n+k}$, denote the first k-coordinates by x^i and the latter n-coordinates by y^j where $1 \leq i \leq k$ and $1 \leq j \leq n$. The parallel vector fields $\partial/\partial x^i$ are all invariant under the holonomy group as defined in Equation 4.14.

Similar to the previous section, as each of these vector fields is invariant under the holonomy, we obtain k-complete vector fields $\widetilde{X}_i$ on $\widetilde{M}$ via Lemma 3.1, Lemma 3.2, and Lemma 3.3. Each flow corresponding to $\widetilde{X}_i$ is complete by compactness of M and moreover, these flows commute by Lemma 4.4.

Lemma 4.4. *The flows of each $\widetilde{X}_i$ for $1 \leq i \leq k$ defined on $\widetilde{M}$ all commute.*

Proof. As each $\widetilde{X}_i$ is complete, it suffices to show that their brackets commute. Let $\widetilde{x} \in \widetilde{M}$. By definition $\widetilde{X}_i$ is defined such that $\mathrm{dev}_*\widetilde{X}_i = \partial/\partial x^i$. By naturality of the Lie bracket,

$$d\,(\mathrm{dev})_{\widetilde{x}}\left(\left[\widetilde{X}_i, \widetilde{X}_j\right]_{\widetilde{x}}\right) = \left[\mathrm{dev}_*\widetilde{X}_i, \mathrm{dev}_*\widetilde{X}_j\right]_{\mathrm{dev}(\widetilde{x})}$$
$$= \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]_{\mathrm{dev}(\widetilde{x})} = 0 \tag{4.15}$$

Because the developing map is a local diffeormophism, its differential is injective, thus Equation 4.15 implies that $[\widetilde{X}_i, \widetilde{X}_j]_{\widetilde{x}} = 0$. Since $\widetilde{x}$ was arbitrary, this means the bracket of $\widetilde{X}_i$ and $\widetilde{X}_j$ vanishes and thus their flows commute. $\square$

Because their flows commute, this means the flows of the vector fields $\widetilde{X}_i$ define an $\mathbb{R}^k$-action on $\widetilde{M}$ corresponding to the $\mathbb{R}^k$ action on $\mathbb{R}^{n+k}$ induced by the vector fields $\partial/\partial x^i$. The developing map by construction is equivariant with respect to these actions. As the $\mathbb{R}^k$-action on $\mathbb{R}^{n+k}$ is both free and proper, Lemma 4.1 guarantees the action of $\mathbb{R}^k$ on $\widetilde{M}$ is both free and proper thus providing $\widetilde{M}$ with a principal $\mathbb{R}^k$-structure. As $\mathbb{R}^k$ is contractible, this means that the principal $\mathbb{R}^k$-structure on $\widetilde{M}$ is trivial and in particular admits a cross-section N such that $\widetilde{M}$ is isomorphic to the trivial principal $\mathbb{R}^k$-bundle $\mathbb{R}^k \times N \longrightarrow \widetilde{M}/\mathbb{R}^k$.

Via the principal bundle isomorphism between $\widetilde{M}$ with a trivial $\mathbb{R}^k$-bundle $\mathbb{R}^k \times N \longrightarrow \widetilde{M}/\mathbb{R}^k$ we construct the developing map $\mathrm{dev}' : \mathbb{R}^k \times N \longrightarrow \mathbb{R}^k \times \mathbb{R}^n$ as in the paragraph after Equation 4.3 which takes k-planes of the form $\mathbb{R}^k \times \{n\}$ to k-planes of the form $\mathbb{R}^k \times \{y\}$ for $n \in N$ and $y \in \mathbb{R}^n$. Just as was done in Equation

4.4, one may form the commutative square

$$\begin{array}{ccc} \mathbb{R}^k \times N & \xrightarrow{[\gamma]} & \mathbb{R}^k \times N \\ {\scriptstyle \mathrm{dev}'}\downarrow & & \downarrow{\scriptstyle \mathrm{dev}'} \\ \mathbb{R}^k \times \mathbb{R}^n & \xrightarrow[\mathrm{hol}[\gamma]]{} & \mathbb{R}^k \times \mathbb{R}^n \end{array} \tag{4.16}$$

where $\pi_1(M)$ acts on $\mathbb{R}^k \times N$ via conjugation by the principal bundle isomorphism between $\widetilde{M}$ and $\mathbb{R}^k \times N$.

As the action of $\pi_1(M)$ on $\mathbb{R}^k \times N$ commutes with each individual flow corresponding to $\widetilde{X}_i$, it commutes with the entire $\mathbb{R}^k$-action. Thus $\pi_1(M)$ acts on N just as in Equation 4.8.

Referring to Equation 4.14, each holonomy element is of the form

$$\mathrm{hol}\ [\gamma] = \begin{pmatrix} I_k & B \\ 0 & A \end{pmatrix} \begin{pmatrix} T \\ v \end{pmatrix} \tag{4.17}$$

and thus takes k-planes of the form $\mathbb{R}^k \times \{y\}$ to k-planes of the form $\mathbb{R}^k \times \{y'\}$ for some $y' \in \mathbb{R}^n$. Thus both the $\pi_1(M)$ action on $\mathbb{R}^k \times N$ and holonomy action on $\mathbb{R}^k \times \mathbb{R}^n$ both descend to actions on N and $\mathbb{R}^n$.

Arguing in an analogous fashion to the proof of Theorem 4.1, one may quotient out the $\mathbb{R}^k$-actions in Equation 4.16 as done in Equation 4.13 to obtain a local diffeomorphism $\overline{\mathrm{dev}'} : N \longrightarrow \mathbb{R}^n$ so the diagram in Equation 4.18 commutes for all $[\gamma] \in \pi_1(M)$.

$$\begin{array}{ccc} N & \xrightarrow{[\gamma]} & N \\ {\scriptstyle \overline{\mathrm{dev}'}}\downarrow & & \downarrow{\scriptstyle \overline{\mathrm{dev}'}} \\ \mathbb{R}^n & \xrightarrow[\mathrm{hol}[\gamma]]{} & \mathbb{R}^n \end{array} \tag{4.18}$$

where $\pi_1(M)$ and the holonomy are acting as established in the paragraphs before and after Equation 4.17. Analogous to Lemma 4.3 we have the following statement regarding the injectivity of dev'.

Lemma 4.5. *Let $U \subset N$ be an open subset for which $\overline{\mathrm{dev}'}$ restricted to U is a diffeomorphism onto its image. Then dev' restricted to $\mathbb{R}^k \times U \subset \mathbb{R}^k \times N$ is a diffeomorphism onto its image.*

Proof. Let $U \subset N$ be such an open subset. Just as before in Lemma 4.3, it suffices to show that dev' is a diffeomorphism restricted to $\mathbb{R}^k \times U$.

This follows as if (T, n) and (S, m) are points in $\mathbb{R}^k \times U$ so that $\mathrm{dev}'(T, n) = \mathrm{dev}'(S, m)$, then $\overline{\mathrm{dev}'}(n) = \overline{\mathrm{dev}'}(m)$. Because $n, m \in U$, and $\overline{\mathrm{dev}'}$ is a diffeomorphism on U, $n = m$. Hence (T, n) and (S, n) lie on the same k-plane, $\mathbb{R}^k \times \{n\}$.

By $\mathbb{R}^k$-equivariance, dev' maps k-planes to k-planes. Because $\mathrm{dev}'(T, n) = \mathrm{dev}'(S, n)$, this necessitates that $T = S$, for otherwise this would mean the line in $\mathbb{R}^k$ connecting T and S would get mapped to a circle contradicting the fact that dev' maps k-planes to k-planes. $\square$

We conclude this section with a theorem summarizing the generalization of Theorem 4.1.

Theorem 4.2. *Let M be a closed affine manifold whose linear holonomy fixes k-vectors in $\mathbb{R}^{n+k}$. Then there exists k-complete parallel flows on M which lift to the universal cover $\widetilde{M}$. In addition, for any point in the universal cover, we may find a neighborhood of the point saturated with respect to these parallel flows such that* *the developing map restricted to these neighborhoods are diffeomorphisms onto their images.*

Proof. Construct the k-complete commuting flows on $\widetilde{M}$ associated to the lifts $\widetilde{X}_i$ of $\partial/\partial x^i$ through the developing map. Construct the principal bundle isomorphism of $\widetilde{M}$ and $\mathbb{R}^k \times N$ and the corresponding developing pair $\mathrm{dev}' : \mathbb{R}^k \times N \longrightarrow \mathbb{R}^k \times \mathbb{R}^n$ in Equation 4.16. Quotient out by the $\mathbb{R}^k$-action to get the local diffeomorphsim $\overline{\mathrm{dev}'} : N \longrightarrow \mathbb{R}^k$ as in Equation 4.18. Pick a neighborhood U for which $\overline{\mathrm{dev}'}$ is a diffeomorphism restricted to $U \subset N$, and saturate this neighborhood with respect to the $\mathbb{R}^k$-action. Lemma 4.5 guarantees dev' on $\mathbb{R}^k \times U$ is a diffeomorphism onto its image. $\square$

Chapter 5: Affine Manifolds with an Invariant Line

5.1 Incomplete Affine Manifolds with an Invariant Line

We begin this section by recalling a result of Fried, Goldman, and Hirsch wherein they proved the non-existence of certain affine space forms. In particular, they showed that there does not exist complete closed affine manifolds with reducible holonomy [FGH81]. We state their theorem below.

Theorem 5.1. *Let M be a compact complete affine manifold. Then the affine holonomy representation is irreducible.*

Due to their result, it immediately follows that there does not exist a closed complete affine manifold with an invariant line as such a manifold would have a reducible holonomy.

A natural follow up question to the result is whether or not there are incomplete closed affine manifolds with an invariant line whose holonomy representation is reducible. Certainly the Hopf-manifolds as defined in Example 3.2 provide such examples with reducible holonomy, yet they fail to contain their entire invariant line. As it so happens, this is in general an impossibility which is stated as a theorem below, but first we establish some notation.

Let G be the subgroup of affine transformation of $\mathbb{A}^{n+1}$ that preserve some fixed affine line. Pick an origin on this line to identify $\mathbb{A}^{n+1}$ with $\mathbb{R}^{n+1}$. Put coordinates on $\mathbb{R}^{n+1}$ by (x, y) with $x \in \mathbb{R}$ and $y \in \mathbb{R}^n$.

We may rotate $\mathbb{R}^{n+1}$ about this origin to assume that the invariant line is defined by the equation $y = 0$. Thus each affine transformation in consideration must preserve the line $y = 0$. Thus, up to conjugation, we may assume that G is equal to the subgroup of $\mathrm{Aff}(n+1, \mathbb{R})$ as defined below.

$$G = \left\{ \begin{pmatrix} r & v \\ 0 & A \end{pmatrix} \begin{pmatrix} t \\ 0 \end{pmatrix} \;\middle|\; r \neq 0, t \in \mathbb{R}, v^T \in \mathbb{R}^n, \text{ and } A \in \mathrm{GL}(n, \mathbb{R}) \right\} \tag{5.1}$$

With this established, we state our theorem.

Theorem 5.2. *Let Ω be a connected open subset of $\mathbb{R} \times \mathbb{R}^n$ containing the line $y = 0$. There does not exist a subgroup $\Gamma \subset G$ with the discrete topology acting on Ω both properly and freely with a compact quotient.*

Proof. As a preliminary observation, note that no element $\gamma \in \Gamma$ may have $r \neq 1$. For if so, then the action of γ on the invariant line $y = 0$ has a fixed point. This follows as

$$\begin{pmatrix} r & v \\ 0 & A \end{pmatrix} \begin{pmatrix} t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} rx + t \\ 0 \end{pmatrix} \tag{5.2}$$

and because $rx + t = x$ has a solution in x for $r \neq 1$, this would mean Γ would not be acting freely on $\mathbb{R} \times \mathbb{R}^n$. Hence the group Γ must act by translations along the invariant line so we may assume Γ is a subgroup of the affine transformations of the form

$$P = \left\{ \begin{pmatrix} 1 & v \\ 0 & A \end{pmatrix} \begin{pmatrix} t \\ 0 \end{pmatrix} \;\middle|\; t \in \mathbb{R}, v^T \in \mathbb{R}^n, \text{ and } A \in \mathrm{GL}(n, \mathbb{R}) \right\} \tag{5.3}$$

The map from $P \longrightarrow \mathbb{R}$ defined by sending each element in Equation 5.3 to its translational part t is a group homomorphism. Since its image is a subgroup of the real numbers under addition, it is either a dense subgroup of $\mathbb{R}$ or cyclic. Because the group action of Γ is proper on $\mathbb{R} \times \mathbb{R}^n$, and thus the line $y = 0$ in particular, the image of the group homomorphism must be cyclic.

Because Γ acts freely on $\mathbb{R} \times \mathbb{R}^n$, the group homomorphism taking each element of Equation 5.3 to its translational part t is injective. For if γ and γ' are two elements with the same translational part then $\gamma'\gamma^{-1}$ acts trivially on the invariant line and thus by freeness $\gamma = \gamma'$. Hence Γ is generated by a single element of P which we abusively denote by

$$P = \begin{pmatrix} 1 & v \\ 0 & A \end{pmatrix} \begin{pmatrix} t \\ 0 \end{pmatrix} \tag{5.4}$$

where v is some n-length row vector, A is an invertible $n \times n$-matrix, and t is some non-zero real number. The fact that t is non-zero follows because otherwise P would stabilize the point $(0, 0) \in \Omega$ contradicting the hypothesis Γ acts freely on Ω.

We may linearly conjugate P by an element Q of the form in Equation 5.5

below for any $w^T \in \mathbb{R}^n$.

$$Q = \begin{pmatrix} 1 & w \\ \\ 0 & I_n \end{pmatrix} \tag{5.5}$$

This yields that QPQ^{-1} is given by

$$QPQ^{-1} = \begin{pmatrix} 1 & w + v(A - I_n) \\ \\ 0 & A \end{pmatrix} \begin{pmatrix} t \\ \\ 0 \end{pmatrix} \tag{5.6}$$

If 1 is not an eigenvalue of A, then we may find a solution in w to $w + v(A - I_n) = 0$ so that QPQ^{-1} leaves invariant both the n-plane $x = 0$ and line $y = 0$.

If 1 is an eigenvalue of A, let $u \in \mathbb{R}^n$ be an associated eigenvector. Let V denote the subspace of $\mathbb{R} \times \mathbb{R}^n$ generated by the linear combinations of $(1, 0)$ and $(0, u)$. By construction V is invariant under the action of Γ. The affine transformation P applied to $a(1, 0) + b(0, u)$ for some $a, b \in \mathbb{R}$ is provided in Equation 5.7.

$$P \begin{pmatrix} a \\ \\ bu \end{pmatrix} = \begin{pmatrix} a + v(bu) \\ \\ A(bu) \end{pmatrix} + \begin{pmatrix} t \\ \\ 0 \end{pmatrix} = \begin{pmatrix} a + b(vu) \\ \\ bu \end{pmatrix} + \begin{pmatrix} t \\ \\ 0 \end{pmatrix} \tag{5.7}$$

Equation 5.7 shows V is indeed an affine subspace of $\mathbb{R} \times \mathbb{R}^n$ invariant under the action of Γ. As V is a closed Γ-invariant plane in $\mathbb{R} \times \mathbb{R}^n$, $\Omega \cap V$ a closed embedded Γ-invariant surface of Ω, though not necessarily connected. Because the action of Γ is both proper and free on Ω it is both proper and free on $\Omega \cap V$. We may form the quotient manifold $(\Omega \cap V)/\Gamma$. This quotient is compact because $(\Omega \cap V)/\Gamma$ sits inside Ω/Γ as a closed subset of a compact space, and thus is itself compact.

Since Ω is open in $\mathbb{R} \times \mathbb{R}^n$, $\Omega \cap V$ is open in V. Let C be the component of $\Omega \cap V$ that contains the line $y = 0$. Because Γ preserves the invariant line $y = 0$, this means Γ must preserve C.

Returning to the action of P on V, identify V with $\mathbb{R}^2$ via the change of coordinates $\phi : \mathbb{R}^2 \longrightarrow V$ defined by $\phi(a, b) = (a, bu)$. As seen in Equation 5.7, the induced action of P on $\mathbb{R}^2$ is given by

$$P \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & vu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} t \\ 0 \end{pmatrix} \tag{5.8}$$

Under the change of coordinates, C will get mapped to an open subset of $\mathbb{R}^2$ containing the line $b = 0$. Thus up to a change of coordinates, we may assume that C is an open subset of the plane $\mathbb{R}^2$ containing the line $b = 0$, and P acts by the affine transformation as in Equation 5.8.

As the holonomy of this compact affine manifold C/Γ is both abelian and volume preserving, this means the developing map is surjective [FGH81]. By Lemma 2.1, the developing image of this structure is $C \subset \mathbb{R}^2$ which by surjectivity is all of $\mathbb{R}^2$. This though is a contradiction as if $vu \neq 0$, then Γ does not act freely on $\mathbb{R}^2$. In particular it fixes the entire line defined by $b = -t/vu$.

On the other hand if $vu = 0$, then $\mathbb{R}^2/\Gamma$ is an open cylinder, and thus non-compact. Hence 1 is not an eigenvalue of A as claimed.

By the remark after Equation 5.6, we may assume that up to conjugation, P

is of the form

$$P = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} t \\ 0 \end{pmatrix} \tag{5.9}$$

and acts on the Γ-invariant subset $\Omega \subset \mathbb{R} \times \mathbb{R}^n$.

Pick any $(0, v) \in \Omega \subset \{0\} \times \mathbb{R}^n$ with $v \neq 0$ which we may do as Ω is open and contains the point $(0, 0)$. Construct the ray $s(0, v)$ for $s \geq 0$. We may choose an increasing sequence of times s_i so the sequence $s_i(0, v)$ does not converge in Ω. For if $s(0, v)$ leaves Ω, let s_* be the smallest $s > 0$ for which $s(0, v)$ is not in Ω. This time is achieved as Ω is open. We may then simply pick any increasing sequence of s_i that converges to s_* to obtain the sequence $s_i(0, v) \in \Omega$ which does not converge nor have a limit point in Ω.

On the other hand if Ω contains the entire ray $s(0, v)$ for $s \geq 0$, simply pick any increasing sequence s_i diverging to ∞ to obtain a sequence $s_i(0, v)$ in Ω that does not have a limit point in Ω.

This provides us with an infinite closed discrete subset $\{s_i(0, v)\} \subset \Omega$ where no two elements are related via Γ, as they all lie on the same slice of $\Omega \cap (\{0\} \times \mathbb{R}^n)$ and thus cannot be related by Γ as P has non-trivial translational part in the first coordinate of $\mathbb{R} \times \mathbb{R}^n$. Saturate this set by the action of Γ to obtain yet another infinite discrete subset $\Gamma\{(0, s_i v)\} \subset \Omega$. We claim this set is also closed in Ω. For if there is indeed a sequence of points $x_j := P^{n_j}(0, s_j v)$ in Ω that converges to a point in Ω, then we claim n_j is eventually constant.

$$P^{n_j}\begin{pmatrix}0\\ s_jv\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & A^{n_j}\end{pmatrix}\begin{pmatrix}n_jt\\ 0\end{pmatrix}\cdot\begin{pmatrix}0\\ s_jv\end{pmatrix}=\begin{pmatrix}n_jt\\ s_jA^{n_j}v\end{pmatrix} \tag{5.10}$$

As both components of $P^{n_j}(0, s_jv)$ must converge, in particular the sequence of real numbers n_jt must converge. Because t is non-zero, n_jt forms a discrete subset of $\mathbb{R}$. For n_jt to accumulate, this means n_j is eventually constant.

Since $P^{n_j}(0, s_jv)$ converges to a point in Ω, and n_j is eventually constant, this means that $(0, s_jv)$ converges to a point in $P^{-n_j}\Omega = \Omega$. This contradicts the construction of $(0, s_jv)$, as this set has no accumulation points in Ω. Thus $\Gamma\{(0, s_iv)\}$ is a closed discrete Γ-invariant subset of Ω.

Because $\Gamma\{(0, s_iv)\}$ is closed and discrete in Ω, it will descend to a closed discrete subset of Ω/Γ. By construction, since each orbit of $(0, s_iv)$ is distinct, this means the closed discrete subset is infinite. This though contradicts the fact that Ω/Γ is compact. Hence no such manifold can exist as originally claimed. □

5.2 Foliations of Affine Manifolds with an Invariant Line

In this section we explore a natural foliation induced by an affine manifold whose holonomy preserves an invariant line. Much like as was done in Section 5.1, pick a point on the invariant line and identity $\mathbb{A}^{n+1}$ with $\mathbb{R}\times\mathbb{R}^n$. Pick a developing pair $\mathrm{dev} : \widetilde{M} \longrightarrow \mathbb{R}\times\mathbb{R}^n$ whose holonomy lies inside the group G as defined by Equation 5.1.

The group G preserves the foliation of $\mathbb{R} \times \mathbb{R}^n$ by lines of the form $\mathbb{R} \times \{y\}$ where $y \in \mathbb{R}^n$. We may pull back this foliation via the developing map to the fibers $\text{dev}^{-1}\{y\} \subset \widetilde{M}$. The connected components these fibers foliate the universal cover of M [Lee03, p. 513].

In addition, since the holonomy group G preserves the foliation on $\mathbb{R} \times \mathbb{R}^n$ in the sense that G takes leaves to leaves, the fundamental group $\pi_1(M)$ also preserves the foliation on $\widetilde{M}$. For if L is a leaf of the foliation of $\widetilde{M}$, then $\text{dev}(L) \subset \mathbb{R} \times \{y\}$ for some $y \in \mathbb{R}^n$. Because the developing map is hol-equivariant, this means that $\text{dev}([\gamma]L) \subset \text{hol}[\gamma](\mathbb{R} \times \{y\}) = \mathbb{R} \times \{y'\}$ for some $y' \in \mathbb{R}^n$. Thus the action of the fundamental group preserves the induced foliation on $\widetilde{M}$ and descends to a foliation on M. This foliation obeys the property that all the leaves are parallel in the sense of Definition 2.8. One could take this as a definition of a one-dimensional *parallel foliation* in the context of affine manifolds, namely a one-dimensional foliation of a manifold M where the leaves of the foliation develop to parallel lines under the developing map. We conclude this section with an example of this induced foliation on the Hopf-torus.

Example 5.1. Let M be the Hopf-torus as described in Example 3.2. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, consider the developing map $\text{dev} : \mathbb{C} \longrightarrow \mathbb{C}$ given by $\text{dev}(z) = e^z$. The developing map obeys the following equivariance properties, $\text{dev}(z+1) = e^1\text{dev}(z)$ and $\text{dev}(z + 2\pi i) = \text{dev}(z)$. Thus the holonomy map takes the translations induced by 1 and $2\pi i$ on $\mathbb{C}$ are sent to the diagonal 2×2 matrix with e's along its diagonal and the identity map respectively.

Note the holonomy preserves the foliation of $\mathbb{C}$ by lines parallel to the imaginary axes. That is to say that for any $u \in \mathbb{R}$, $\mathrm{hol}\,[\gamma]\,(\mathbb{R} \times \{iu\}) = \mathbb{R} \times \{iu'\}$ for some $u' \in \mathbb{R}$. In fact, if we parametrize each line by t, we may easily solve the equation $e^z = t + iu$. Writing $z = x + iy$ yields that $x = (1/2)\ln(t^2 + u^2)$ and $\tan(y) = u/t$ for all $t \neq 0$. The corresponding foliation in the universal cover, $\mathbb{C}$, is given below in Figure 5.1

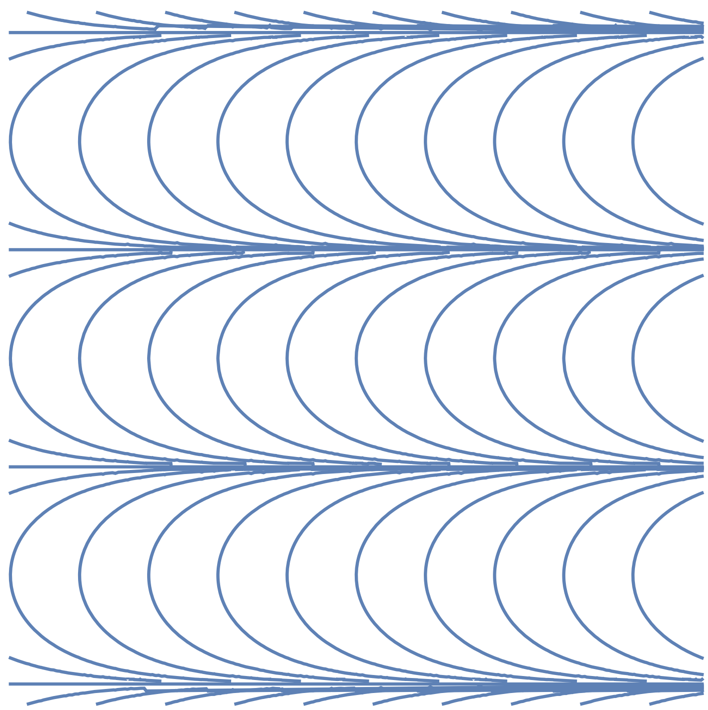

Figure 5.1: One can see this foliation is invariant with respect to the translations in the coordinate axes. Whereas most leaves of the foliations are curved, there are leaves that are lines parallel to the x-axis. These are the leaves corresponding the leaf $\mathbb{R} \times \{0\}$ in affine space which splits into two leaves $\mathbb{R}^+ \times \{0\}$ and $\mathbb{R}^- \times \{0\}$ in $\mathbb{C}^\times$ which is the developing image.

What is interesting about this foliation is that unlike the trivial foliation of the torus by circles or the foliation of the torus by parallel translates of a line of irrational slope, this foliation admits both closed and non-closed leaves. The leaves parallel to the x-axis in the universal cover, $\mathbb{C}$, descend to closed leaves in M under the projection map $p : \mathbb{C} \longrightarrow \mathbb{C}/\pi_1(M)$ as the translations defining M are along the coordinate axes. On the other hand, each one of the curved leaves in $\mathbb{C}$ projects to a leaf of the foliation on the torus that gets arbitrarily close to the closed leaves. Figure 5.2 depicts this behavior.

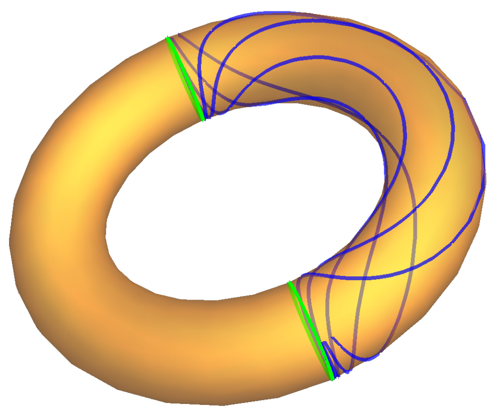

Figure 5.2: Here is the quotient of the foliation as in Figure 5.1 on the Hopf-torus. The green leaves of the foliation correspond to lines that are parallel to the x-axis in $\mathbb{C}$ whereas the blue curves correspond the curved leaves. Note the curved leaves wrap infinitely many times around the direction defined by the green leaf, but never meet the closed leaves.

5.3 Affine Manifolds with an Invariant Line of Translation

In this section we explore the consequences of having a closed affine manifold with an invariant line whose holonomy acts on the invariant line by translations. Similar to how closed radiant manifolds cannot have developing images containing fixed points of the holonomy, we show that closed affine manifolds with a translation invariant line cannot meet this line.

After showing this result we extend it to the case where the holonomy admits an invariant affine k-plane upon which the holonomy acts by translations and reflections. This result provides partial affirmation to the conjecture of Fried and Goldman which stipulates that proper invariant affine subspaces upon which the holonomy acts unipotently lie outside the developing image.

Theorem 5.3. *Let M be an $(n+1)$-dimensional closed affine manifold with $n \geq 1$ whose holonomy admits an invariant line. If the holonomy acts on the invariant line by translations, then the developing image cannot meet this invariant line.*

Before proceeding, we need to first establish a technical lemma in place of Lemma 3.1.

Lemma 5.1. *Let G and H be Lie groups acting on manifolds M and N where G is discrete and acts both properly and freely on M. Let $\phi : G \longrightarrow H$ be a homomorphism accompanied by a ϕ-equivariant submersion $F : M \longrightarrow N$. For each $\phi(G)$-invariant vector field Y on N, we may find a G-invariant vector field on M so that $F_*X = Y$.*

Proof. For each point $m \in M$, because G is discrete and acts properly and freely on M, we may find an open subset $m \in U$ such that $gU \cap U = \emptyset$ unless $g = 1$. As F is a submersion, we may shrink U sufficiently small so that in appropriately chosen coordinate patches, F has a coordinate representation as a linear projection, taking coordinates $(x_1, x_2, \ldots, x_n)$ to $(x_1, x_2, \ldots, x_m)$ with $n \geq m$. Thus we may construct a smooth local lift X_U of Y to U so that $F_* X_U = Y$ on U.

This lift may be pushed forward on each gU to define smooth local lifts $X_{gU} := g_* X_U$ of Y. Each X_{gU} is still a local lift of Y by equivariance of F. Let V be the disjoint union of all the gU, $V := \bigcup_{g \in G} gU$, and define X_V to be X_{gU} on each gU. This is well defined because $gU \cap U = \emptyset$ for all $g \neq 1$. Thus X_V is a local G-equivariant lift of Y on V.

Take the union of all such V_α to cover M. There exists a G-invariant partition of unity subordinate to the covering [Wan17]. By this we mean a collection of non-negative smooth functions $\{\rho_\alpha\} : M \longrightarrow \mathbb{R}$ indexed by the open cover $\{V_\alpha\}$ so that

1. supp $\rho_\alpha \subset V_\alpha$
2. For each $p \in M$, there exists a neighborhood that intersects only finitely many supports supp ρ_α.
3. $\sum_\alpha \rho_\alpha = 1$
4. Each ρ_α is G-invariant.

Using a $\pi_1(M)$-invariant partition of unity subordinate to the cover $\{V_\alpha\}$, we may define the vector field $X := \sum_\alpha \rho_\alpha X_\alpha$ which is by construction G-invariant and smooth. Finally, we have that

$$\begin{aligned} dF_m\left(X_m\right) &= dF_m\left(\sum_\alpha \rho_\alpha(m) X_\alpha\Big|_m\right) \\ &= \sum_\alpha \rho_\alpha(m) dF_m\left(X_\alpha\big|_m\right) = \sum_\alpha \rho_\alpha(m) Y_{F(m)} = Y_{F(m)} \end{aligned} \tag{5.11}$$

Thus X is a G-invariant vector field of M so that $F_* X = Y$ as claimed. □

We now return to the proof of Theorem 5.3.

Proof. To this end, let M be an $(n+1)$-dimensional closed affine manifold whose holonomy admits an invariant line upon which the holonomy acts by translation. Up to conjugacy, this means the holonomy group sits inside the group G defined by matrices of the form as in Equation 5.12

$$G = \left\{ \left(\begin{array}{cc} 1 & v \\ 0 & A \end{array} \right) \left(\begin{array}{c} t \\ 0 \end{array} \right) \,\middle|\, t \in \mathbb{R}, v^T \in \mathbb{R}^n, \text{ and } A \in \mathrm{GL}(n, \mathbb{R}) \right\} \tag{5.12}$$

Pick a developing pair $\mathrm{dev} : \widetilde{M} \longrightarrow \mathbb{R} \times \mathbb{R}^n$ and $\mathrm{hol} : \pi_1(M) \longrightarrow G$ where G is defined as above in Equation 5.12.

By the result of Theorem 4.1 there exists a complete parallel flow on $\widetilde{M}$ defined by the lift of the invariant vector field $\partial/\partial x$ through the developing map. Thus we may assume the developing pair obeys the commutative diagram as found in Equation 4.4. That is, we may write $\widetilde{M}$ as a trivial principal $\mathbb{R}$-bundle, $\mathbb{R} \times N$, and assume the developing map takes fibers of the bundle $\mathbb{R} \times \{n\}$ to lines of the form

$\mathbb{R} \times \{y\}$ where $y \in \mathbb{R}^n$.

That said, we may form the quotient of the commutative square in Equation 4.4 by each $\mathbb{R}$-action to obey the commutative square as in Equation 4.13. Recall here the action of the fundamental group and holonomy are the induced actions as defined by Equation 4.8 and Equation 4.10 respectively. Because the holonomy by hypothesis has no translational part in the $\mathbb{R}^n$ component of $\mathbb{R} \times \mathbb{R}^n$, this means the holonomy acts purely linearly on $\mathbb{R}^n$ in the diagram defined by Equation 4.13.

Up until this point we have yet to assume the developing image $\mathrm{dev}'(\mathbb{R} \times N)$ meets the invariant line $y = 0$. For if it does, this means that the image $\overline{\mathrm{dev}'}(N)$ contains $0 \in \mathbb{R}^n$.

Precompose $\overline{\mathrm{dev}'} : N \longrightarrow \mathbb{R}^n$ with the trivial bundle map $p : \mathbb{R} \times N \longrightarrow N$ to obtain the square below in Equation 5.13 that commutes for each $[\gamma] \in \pi_1(M)$.

$$\begin{tikzcd} \mathbb{R} \times N \arrow[r, "{[\gamma]}"] \arrow[d, "\overline{\mathrm{dev}'} \circ p"'] & \mathbb{R} \times N \arrow[d, "\overline{\mathrm{dev}'} \circ p"] \\ \mathbb{R}^n \arrow[r, "\mathrm{hol}[\gamma]"'] & \mathbb{R}^n \end{tikzcd} \tag{5.13}$$

Recall the holonomy acts purely linearly on $\mathbb{R}^n$ as seen in Equation 5.12. Thus it preserves the radiant vector field $R = -y^i \partial/\partial y^i$. By Lemma 3.1 and Lemma 3.2, we may lift R to an auxiliary $\pi_1(M)$-invariant vector field $\overline{R}$ on N via the local diffeomorphism $\overline{\mathrm{dev}'} : N \longrightarrow \mathbb{R}^n$ where $\overline{\mathrm{dev}'}_* \overline{R} = R$. While R is complete, we do not at the moment have a means to guarantee that $\overline{R}$ is complete. To do so, we must lift $\overline{R}$ to the universal cover $\mathbb{R} \times N$, through the trivial bundle $p : \mathbb{R} \times N \longrightarrow N$.

As $\pi_1(M)$ acts on $\mathbb{R} \times N$ both properly and freely, and the projection $\mathbb{R} \times N \longrightarrow$

N is $\pi_1(M)$-equivariant, Lemma 5.1 provides a $\pi_1(M)$-invariant vector field on $\mathbb{R}\times N$ that lifts the $\pi_1(M)$-invariant $\overline{R}$ on N. Denote this vector field by $\widetilde{R}$.

By similar arguments to Section 3.2, $\widetilde{R}$ descends to a vector field on M which is by compactness complete, and consequentially has a flow defined for all time. By Lemma 3.3, the corresponding flow on $\mathbb{R} \times N$ is also complete. The remark after Lemma 3.3 guarantees $\overline{R}$ is complete.

As $\overline{R}$ is a complete lift of the radiant vector field R on $\mathbb{R}^n$, and $0 \in \overline{\mathrm{dev}'}(N)$, this means by Lemma 3.5 that $\overline{\mathrm{dev}'} : N \longrightarrow \mathbb{R}^n$ may be promoted from a local diffeomorphism to a global diffeomorphism.

Saturating N by the $\mathbb{R}$-action induced by the parallel flow by means of Theorem 4.1 yields that $\mathrm{dev}' : \mathbb{R} \times N \longrightarrow \mathbb{R} \times \mathbb{R}^n$ is a diffeomorphism. Thus the affine structure on M is complete and M is diffeomorphic to the quotient of $\mathbb{R} \times \mathbb{R}^n$ by a subgroup of G as defined in Equation 5.12. This though contradicts Theorem 5.1 as the holonomy is by hypothesis reducible. $\square$

From this we obtain two immediate corollaries regarding Theorem 5.3.

Corollary 5.1. *Let M be an $(n+1)$-dimensional closed affine manifold with $n \geq 1$ whose holonomy admits an invariant line. If the holonomy acts on the invariant line by translations and reflections, then the developing image cannot meet this invariant line.*

Proof. Let M be such a manifold as in the hypothesis. This means the holonomy lies in the group

$$H = \left\{ \begin{pmatrix} \pm 1 & v \\ 0 & A \end{pmatrix} \begin{pmatrix} t \\ 0 \end{pmatrix} \;\middle|\; t \in \mathbb{R}, v^T \in \mathbb{R}^n, \text{ and } A \in \mathrm{GL}(n, \mathbb{R}) \right\} \tag{5.14}$$

which is a finite extension of the group G as defined in Equation 5.12. Thus we may lift to the double cover manifold $C \longrightarrow M$ to assume the holonomy lies inside G in Equation 5.12. By Theorem 5.3, the developing map of C must miss the invariant line of the holonomy. Since the developing maps of M and C are identical by Lemma 2.2, the developing map of M misses the invariant line as claimed. $\square$

As a final Corollary to Theorem 5.3, we have the following result.

Corollary 5.2. *Let M be an $(n+k)$-dimensional closed affine manifold with $k \geq 1$ whose holonomy admits an invariant k-plane. If the holonomy acts on this invariant k-plane by reflections and translations, the developing image cannot meet the invariant k-plane.*

Proof. Assume the holonomy acts on the invariant k-plane by translations alone so the holonomy elements are of the form in Equation 4.17. By the result of Theorem 4.2, there exists k-complete parallel flows on $\widetilde{M}$ defined by the lifts of the invariant vector fields $\partial/\partial x^1, \partial/\partial x^2, \ldots, \partial/\partial x^k$. Thus we may assume the developing pair obeys the commutative diagram as found in Equation 4.16. One quotients the diagram by the $\mathbb{R}^k$-action to obtain the commutative square as in Equation 4.18 then apply the same arguments as done in the proof of Theorem 5.3 to show that $\overline{\mathrm{dev}'}(N)$ avoids $0 \in \mathbb{R}^n$, for otherwise one has that $\mathrm{dev}' : \mathbb{R}^k \times N \longrightarrow \mathbb{R}^k \times \mathbb{R}^n$

is a diffeomorphism, and thus M is complete with reducible holonomy violating Theorem 5.1.

With this result established the case where the holonomy lies in the finite extension by reflections follows immediately from the argument provided in Corollary 5.2. $\square$

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